QUILLEN'S ADJUNCTION THEOREM FOR DERIVED FUNCTORS, REVISITED

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Abstract. The aim of this paper is to present a very simple original, purely formal, proof of Quillen's adjunction theorem for derived functors [8], and of some more recent variations and generalizations of this theorem [3], [9]. This is obtained by proving an abstract adjunction theorem for "absolute" derived functors. In contrast with all known proofs, the explicit construction of the derived functors is not used.

We recall that for every category $\mathcal{C}$ and every class of arrows W in $\mathcal{C}$, there exist a category $W^{-1}\mathcal{C}$, called *localization of* $\mathcal{C}$ *by* W, and a functor $P : \mathcal{C} \to W^{-1}\mathcal{C}$, the *localization functor*, that carries arrows in W into isomorphisms in $W^{-1}\mathcal{C}$, and universal for this property: for every category $\mathcal{D}$ and every functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(W)$ is contained in the class of isomorphisms in $\mathcal{D}$, there is a unique functor $\widetilde{F} : W^{-1}\mathcal{C} \to \mathcal{D}$ such that $F = \widetilde{F}P$.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "P"'] \arrow[dr, "F"] & \\ W^{-1}\mathcal{C} \arrow[r, dashed, "\widetilde{F}"'] & \mathcal{D} \end{tikzcd}$$

The category $W^{-1}\mathcal{C}$ is obtained from $\mathcal{C}$ by formally inverting the arrows in W; it has the same objects as $\mathcal{C}$ and the morphisms are equivalence classes of "composable zigzags" of morphisms in $\mathcal{C}$, the arrows going in the "wrong direction" belonging to W (see [4]). The category $W^{-1}\mathcal{C}$ is not necessarily *locally small i.e.* the *class* of arrows from an object to another is not in general a *set*. This set theoretic problem can be ignored using Grothendieck's notion of universe [1]. In this paper, in order to simplify, we introduce the following definition.

Definition. A *localizer* is a pair $(\mathcal{C}, W)$, where $\mathcal{C}$ is a category and W a class of arrows in $\mathcal{C}$ such that the localized category $W^{-1}\mathcal{C}$ is locally small.

Examples. If $\mathcal{C}$ is a *small* category, then for every set W of arrows in $\mathcal{C}$, the pair $(\mathcal{C}, W)$ is a localizer. If $\mathcal{C}$ is a Quillen model category and W the class of weak equivalences in C, then $(\mathcal{C}, W)$ is a localizer [8, Ch. I, 1.13, Th. 1′].

Let $(\mathcal{C}, W)$ be a localizer, $P : \mathcal{C} \to W^{-1}\mathcal{C}$ the localization functor and $F : \mathcal{C} \to \mathcal{D}$ an arbitrary functor. We recall that a *right derived functor* of F is a pair $(\mathsf{R}F, \alpha)$,

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where $\mathsf{R}F : W^{-1}\mathcal{C} \longrightarrow \mathcal{D}$ is a functor and $\alpha : F \longrightarrow \mathsf{R}F \circ P$ a natural transformation,

$$\begin{tikzcd} \mathcal{C} \arrow[d, "P"'] \arrow[dr, "F"] & \\ W^{-1}\mathcal{C} \arrow[r, "\mathsf{R}F"'] \arrow[ur, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

satisfying the following universal property. For every functor $G : W^{-1}\mathcal{C} \longrightarrow \mathcal{D}$, and every natural transformation $\gamma : F \longrightarrow G \circ P$, there is a unique natural transformation $\delta : \mathsf{R}F \longrightarrow G$ such that $\gamma = (\delta \star P)\,\alpha$.

$$\begin{tikzcd} F \arrow[d, "\alpha"'] \arrow[dr, "\gamma"] & \\ \mathsf{R}F \circ P \arrow[r, "\delta \star P"'] & G \circ P \end{tikzcd}$$

This condition means exactly that the functor $\mathsf{R}F$ (together with the natural transformation α) is a *left* Kan extension of F along the localization functor P. The pair $(\mathsf{R}F, \alpha)$ is an *absolute* right derived functor of F if for every functor $H : \mathcal{D} \longrightarrow \mathcal{E}$, the pair $(H \circ \mathsf{R}F, H \star \alpha)$ is a right derived functor of $H \circ F$. An absolute right derived functor of F is in particular a right derived functor of F (take $H = 1_{\mathcal{D}}$).

Example. If $\mathcal{C}$ is a Quillen model category and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor that carries weak equivalences between fibrant objects in $\mathcal{C}$ into isomorphisms in $\mathcal{D}$, then there exists an *absolute* right derived functor $(\mathsf{R}F, \alpha)$ of F. In order to prove this, we observe that Quillen's existence theorem for derived functors [8, Ch. I, 4.2, Prop. 1] implies that F has a right derived functor $(\mathsf{R}F, \alpha)$ constructed as follows. For every object X in $\mathcal{C}$, choose a fibrant resolution $i_X : X \longrightarrow X'$; then $\mathsf{R}F(X) = F(X')$ and $\alpha_X = F(i_X) : F(X) \longrightarrow F(X') = \mathsf{R}F(X)$. As for every functor $H : \mathcal{D} \longrightarrow \mathcal{E}$, the functor $H \circ F$ carries weak equivalences between fibrant objects in $\mathcal{C}$ into isomorphisms in $\mathcal{E}$, the same construction gives a right derived functor of $H \circ F$ equal to $(H \circ \mathsf{R}F, H \star \alpha)$, which proves the statement.

Let $(\mathcal{C}, W)$ and $(\mathcal{C}', W')$ be two localizers, $P : \mathcal{C} \longrightarrow W^{-1}\mathcal{C}$ and $P' : \mathcal{C}' \longrightarrow W'^{-1}\mathcal{C}'$ the localization functors, and $F : \mathcal{C} \longrightarrow \mathcal{C}'$ an arbitrary functor. A *total right derived functor* (resp. *absolute total right derived functor*) of F is a pair $(\underline{\mathsf{R}}F, \alpha)$, where $\underline{\mathsf{R}}F : W^{-1}\mathcal{C} \longrightarrow W'^{-1}\mathcal{C}'$ is a functor and $\alpha : P' \circ F \longrightarrow \underline{\mathsf{R}}F \circ P$ a natural transformation , which is a right derived functor (resp. an absolute right derived functor) of $P' \circ F$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{C}' \arrow[d, "P'"] \\ W^{-1}\mathcal{C} \arrow[r, "\underline{\mathsf{R}}F"'] & W'^{-1}\mathcal{C}' \end{tikzcd}$$

Example. If $\mathcal{C}$ and $\mathcal{C}'$ are two Quillen model categories and $F : \mathcal{C} \longrightarrow \mathcal{C}'$ a functor that carries weak equivalences between fibrant objects in $\mathcal{C}$ into weak equivalences in $\mathcal{C}'$, then there exists an *absolute* total right derived functor of F. To prove this, we observe that if P' is the localization functor from $\mathcal{C}$ to its localization by the weak equivalences, then the functor $P' \circ F$ carries weak equivalences between fibrant objects in $\mathcal{C}$ into isomorphisms, and the statement is a particular case of the previous example.

The notions of *left derived functor*, of *absolute left derived functor*, of *total left derived functor*, and of *absolute total left derived functor* are defined in a dual way.

Theorem. *Let $(\mathcal{C}, W)$ and $(\mathcal{C}', W')$ be two localizers,*

$$P : \mathcal{C} \longrightarrow W^{-1}\mathcal{C} \qquad \textit{and} \qquad P' : \mathcal{C}' \longrightarrow W'^{-1}\mathcal{C}'$$

the localization functors,

$$F : \mathcal{C} \longrightarrow \mathcal{C}' \quad , \qquad G : \mathcal{C}' \longrightarrow \mathcal{C}$$

a pair of adjoint functors, and

$$\varepsilon : F \circ G \longrightarrow 1_{\mathcal{C}'} \quad , \qquad \eta : 1_{\mathcal{C}} \longrightarrow G \circ F$$

the unit and counit of the adjunction. We suppose that the functor F (resp. G) *has an absolute total left* (resp. *right*) *derived functor* $(\underline{\mathsf{L}}F, \alpha)$ (resp. $(\underline{\mathsf{R}}G, \beta)$).

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "P"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & \mathcal{C}' \arrow[d, "P'"] \\ W^{-1}\mathcal{C} \arrow[r, "\underline{\mathsf{L}}F"'] & W'^{-1}\mathcal{C}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}' \arrow[r, "G"] \arrow[d, "P'"'] \arrow[dr, phantom, "\swarrow_{\beta}"] & \mathcal{C} \arrow[d, "P"] \\ W'^{-1}\mathcal{C}' \arrow[r, "\underline{\mathsf{R}}G"'] & W^{-1}\mathcal{C} \end{tikzcd}$$

Then the pair of functors

$$\underline{\mathsf{L}}F : W^{-1}\mathcal{C} \longrightarrow W'^{-1}\mathcal{C}' \quad , \qquad \underline{\mathsf{R}}G : W'^{-1}\mathcal{C}' \longrightarrow W^{-1}\mathcal{C}$$

is a pair of adjoint functors, and we can choose the unit and counit of the adjunction

$$\underline{\varepsilon} : \underline{\mathsf{L}}F \circ \underline{\mathsf{R}}G \longrightarrow 1_{W'^{-1}\mathcal{C}'} \quad , \qquad \underline{\eta} : 1_{W^{-1}\mathcal{C}} \longrightarrow \underline{\mathsf{R}}G \circ \underline{\mathsf{L}}F$$

in such a way that the two following squares commute.

$$\begin{tikzcd} \underline{\mathsf{L}}F \circ P \circ G \arrow[r, "\underline{\mathsf{L}}F \star \beta"] \arrow[d, "\alpha \star G"'] & \underline{\mathsf{L}}F \circ \underline{\mathsf{R}}G \circ P' \arrow[d, "\underline{\varepsilon} \star P'"] \\ P' \circ F \circ G \arrow[r, "P' \star \varepsilon"'] & P' \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathsf{R}}G \circ P' \circ F & \underline{\mathsf{R}}G \circ \underline{\mathsf{L}}F \circ P \arrow[l, "\underline{\mathsf{R}}G \star \alpha"'] \\ P \circ G \circ F \arrow[u, "\beta \star F"] & P \arrow[l, "P \star \eta"] \arrow[u, "\underline{\eta} \star P"'] \end{tikzcd}$$

Proof. As $(\underline{\mathsf{R}}G, \beta)$ (resp. $(\underline{\mathsf{L}}F, \alpha)$) is an absolute total right (resp. left) derived functor of G (resp. of F), *i.e.* an absolute right (resp. left) derived functor of $P \circ G$ (resp. of $P' \circ F$), the pair $(\underline{\mathsf{L}}F \circ \underline{\mathsf{R}}G, \underline{\mathsf{L}}F \star \beta)$ (resp. $(\underline{\mathsf{R}}G \circ \underline{\mathsf{L}}F, \underline{\mathsf{R}}G \star \alpha)$) is a right (resp. left) derived functor of $\underline{\mathsf{L}}F \circ P \circ G$ (resp. of $\underline{\mathsf{R}}G \circ P' \circ F$). The universal property of right (resp. left) derived functors says that for every functor $H' : W'^{-1}\mathcal{C}' \to W'^{-1}\mathcal{C}'$ (resp. $H : W^{-1}\mathcal{C} \to W^{-1}\mathcal{C}$) and every natural transformation $\gamma' : \underline{\mathsf{L}}F \circ P \circ G \to H' \circ P'$ (resp. $\gamma : H \circ P \to \underline{\mathsf{R}}G \circ P' \circ F$), there exists a unique natural transformation

$$\delta' : \underline{\mathsf{L}}F \circ \underline{\mathsf{R}}G \longrightarrow H' \qquad (\text{ resp. } \quad \delta : H \longrightarrow \underline{\mathsf{R}}G \circ \underline{\mathsf{L}}F \ )$$

such that

$$\gamma' = (\delta' \star P')(\underline{\mathsf{L}}F \star \beta) \qquad (\text{ resp. } \quad \gamma = (\underline{\mathsf{R}}G \star \alpha)(\delta \star P) \ ) \quad .$$

This universal property applied to the functor $H' = 1_{W'^{-1}\mathcal{C}'}$ (resp. $H = 1_{W^{-1}\mathcal{C}}$) and the natural transformation

$$\gamma' = (P'\star\varepsilon)(\alpha\star G) :\ \underline{\mathsf{L}}F\circ P\circ G \xrightarrow{\alpha\star G} P'\circ F\circ G \xrightarrow{P'\star\varepsilon} P' = 1_{W'^{-1}\mathcal{C}'}\circ P'$$

(resp. $\gamma = (\beta\star F)(P\star\eta) :\ 1_{W^{-1}\mathcal{C}}\circ P = P \xrightarrow{P\star\eta} P\circ G\circ F \xrightarrow{\beta\star F} \underline{\mathsf{R}}G\circ P'\circ F$),

$$\begin{tikzcd}
\mathcal{C}' \arrow[d, "P'"'] \arrow[dr, "\underline{\mathsf{L}}F\circ P\circ G"] \arrow[dr, phantom, "\Swarrow\underline{\mathsf{L}}F\star\beta"'] & \\
W'^{-1}\mathcal{C}' \arrow[r, "\underline{\mathsf{L}}F\circ\underline{\mathsf{R}}G"'] & W'^{-1}\mathcal{C}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}' \arrow[d, "P'"'] \arrow[dr, "\underline{\mathsf{L}}F\circ P\circ G"] \arrow[dr, phantom, "\Swarrow(P'\star\varepsilon)(\alpha\star G)"'] & \\
W'^{-1}\mathcal{C}' \arrow[r, "1_{W'^{-1}\mathcal{C}'}"'] & W'^{-1}\mathcal{C}'
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "P"'] \arrow[dr, "\underline{\mathsf{R}}G\circ P'\circ F"] \arrow[dr, phantom, "\underline{\mathsf{R}}G\star\alpha\Nearrow"'] & \\
W^{-1}\mathcal{C} \arrow[r, "\underline{\mathsf{R}}G\circ\underline{\mathsf{L}}F"'] & W^{-1}\mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[d, "P"'] \arrow[dr, "\underline{\mathsf{R}}G\circ P'\circ F"] \arrow[dr, phantom, "(\beta\star F)(P\star\eta)\Nearrow"'] & \\
W^{-1}\mathcal{C} \arrow[r, "1_{W^{-1}\mathcal{C}}"'] & W^{-1}\mathcal{C}
\end{tikzcd}$$

implies the existence of a unique natural transformation

$$\underline{\varepsilon} : \underline{\mathsf{L}}F\circ\underline{\mathsf{R}}G \longrightarrow 1_{W'^{-1}\mathcal{C}'} \qquad \text{(resp.}\quad \underline{\eta} : 1_{W^{-1}\mathcal{C}} \longrightarrow \underline{\mathsf{R}}G\circ\underline{\mathsf{L}}F\ \text{)}$$

such that

$$(P'\star\varepsilon)(\alpha\star G) = (\underline{\varepsilon}\star P')(\underline{\mathsf{L}}F\star\beta) \qquad \text{(resp.}\quad (\beta\star F)(P\star\eta) = (\underline{\mathsf{R}}G\star\alpha)(\underline{\eta}\star P)\ \text{)}.$$

It remains to prove that

$$(\underline{\mathsf{R}}G\star\underline{\varepsilon})(\underline{\eta}\star\underline{\mathsf{R}}G) = 1_{\underline{\mathsf{R}}G} \qquad\text{and}\qquad (\underline{\varepsilon}\star\underline{\mathsf{L}}F)(\underline{\mathsf{L}}F\star\underline{\eta}) = 1_{\underline{\mathsf{L}}F} \quad .$$

$$\underline{\mathsf{R}}G \xrightarrow{\underline{\eta}\star\underline{\mathsf{R}}G} \underline{\mathsf{R}}G\circ\underline{\mathsf{L}}F\circ\underline{\mathsf{R}}G \xrightarrow{\underline{\mathsf{R}}G\star\underline{\varepsilon}} \underline{\mathsf{R}}G$$

$$\underline{\mathsf{L}}F \xrightarrow{\underline{\mathsf{L}}F\star\underline{\eta}} \underline{\mathsf{L}}F\circ\underline{\mathsf{R}}G\circ\underline{\mathsf{L}}F \xrightarrow{\underline{\varepsilon}\star\underline{\mathsf{L}}F} \underline{\mathsf{L}}F$$

The uniqueness part of the universal property of derived functors, implies that it is enough to prove that

$$\Big[\big((\underline{\mathsf{R}}G\star\underline{\varepsilon})(\underline{\eta}\star\underline{\mathsf{R}}G)\big)\star P'\Big]\beta = \beta \qquad\text{and}\qquad \alpha\Big[\big((\underline{\varepsilon}\star\underline{\mathsf{L}}F)(\underline{\mathsf{L}}F\star\underline{\eta})\big)\star P\Big] = \alpha \quad .$$

Let us prove the first of these two equalities:

$$\begin{aligned}
&\Big[\big((\underline{\mathsf{R}}G\star\underline{\varepsilon})(\underline{\eta}\star\underline{\mathsf{R}}G)\big)\star P'\Big]\beta = (\underline{\mathsf{R}}G\star\underline{\varepsilon}\star P')(\underline{\eta}\star\underline{\mathsf{R}}G\circ P')\beta = \\
&\quad = (\underline{\mathsf{R}}G\star\underline{\varepsilon}\star P')(\underline{\mathsf{R}}G\circ\underline{\mathsf{L}}F\star\beta)(\underline{\eta}\star P\circ G) = \Big[\underline{\mathsf{R}}G\star\big((\underline{\varepsilon}\star P')(\underline{\mathsf{L}}F\star\beta)\big)\Big](\underline{\eta}\star P\circ G) \\
&\quad = \Big[\underline{\mathsf{R}}G\star\big((P'\star\varepsilon)(\alpha\star G)\big)\Big](\underline{\eta}\star P\circ G) = (\underline{\mathsf{R}}G\circ P'\star\varepsilon)(\underline{\mathsf{R}}G\star\alpha\star G)(\underline{\eta}\star P\circ G) \\
&\quad = (\underline{\mathsf{R}}G\circ P'\star\varepsilon)\Big[\big((\underline{\mathsf{R}}G\star\alpha)(\underline{\eta}\star P)\big)\star G\Big] = (\underline{\mathsf{R}}G\circ P'\star\varepsilon)\Big[\big((\beta\star F)(P\star\eta)\big)\star G\Big] \\
&\quad = (\underline{\mathsf{R}}G\circ P'\star\varepsilon)(\beta\star F\circ G)(P\star\eta\star G) = \beta(P\circ G\star\varepsilon)(P\star\eta\star G) \\
&\quad = \beta\Big[P\star\big((G\star\varepsilon)(\eta\star G)\big)\Big] = \beta \quad .
\end{aligned}$$

The second equality is proved dually. □

Corollary. (Quillen's adjunction theorem for derived functors.) *Let $\mathcal{C}$ and $\mathcal{C}'$ be two Quillen model categories and*

$$F : \mathcal{C} \longrightarrow \mathcal{C}' \quad , \qquad G : \mathcal{C}' \longrightarrow \mathcal{C}$$

a pair of adjoint functors. We suppose that F (resp. *G*) *carries weak equivalences between cofibrant objects in $\mathcal{C}$* (resp. *between fibrant objects in $\mathcal{C}'$*) *into weak equivalences in $\mathcal{C}'$* (resp. *in $\mathcal{C}$*). *Then F* (resp. *G*) *has a total left* (resp. *right*) *derived functor $(\underline{\mathsf{L}}F, \alpha)$* (resp. *$(\underline{\mathsf{R}}G, \beta)$*)*, and the functor $\underline{\mathsf{L}}F$ is a left adjoint of the functor $\underline{\mathsf{R}}G$.*

Proof. The example preceding the theorem and its dual imply that these derived functors exist and are *absolute* derived functors. Therefore the corollary is a particular case of our theorem. □

Remarks. In his book [8], Quillen proves his adjunction theorem under the additional hypothesis that F preserves cofibrations and G preserves fibrations [*loc. cit.* Ch. 1, 4.5 Th. 3], but our proof shows that these hypotheses are not necessary. In the more recent books of Hovey [6] and of Hirschhorn [5], the Quillen's adjunction theorem is proved with even stronger hypotheses by requiring (F, G) to be a *Quillen adjunction* between *closed* model categories, with *functorial factorizations.* The variants of the adjunction theorem in a more general setting then Quillen model categories, obtained by Dwyer, Hirschhorn, Kan and Smith [3], or by Radulescu-Banu [9], in his paper on *Anderson-Brown-Cisinski model categories*, alias *catégories dérivables* [2], are also implied by "the adjunction theorem for absolute derived functors", proved here. This theorem will be used, in a forthcoming paper with Bruno Kahn [7], to prove a generalization of the adjunction theorem of Radulescu-Banu, implying all known adjunction theorems for derived functors.

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