Blowup constructions for Lie groupoids and a Boutet de Monvel type calculus

by Claire Debord and Georges Skandalis

Université Clermont Auvergne
LMBP, UMR 6620 - CNRS
Campus des Cézeaux,
3, Place Vasarely
TSA 60026 CS 60026
63178 Aubière cedex, France
claire.debord@math.univ-bpclermont.fr

Université Paris Diderot, Sorbonne Paris Cité
Sorbonne Universités, UPMC Paris 06, CNRS, IMJ-PRG
UFR de Mathématiques, CP **7012** - Bâtiment Sophie Germain
5 rue Thomas Mann, 75205 Paris CEDEX 13, France
skandalis@math.univ-paris-diderot.fr

Abstract

We present natural and general ways of building Lie groupoids, by using the classical procedures of blowups and of deformations to the normal cone. Our constructions are seen to recover many known ones involved in index theory. The deformation and blowup groupoids obtained give rise to several extensions of C^*-algebras and to full index problems. We compute the corresponding K-theory maps. Finally, the blowup of a manifold sitting in a transverse way in the space of objects of a Lie groupoid leads to a calculus which is quite similar to the Boutet de Monvel calculus for manifolds with boundary.

Contents

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1 Introduction

Let $G \rightrightarrows M$ be a Lie groupoid. The Lie groupoid G comes with its natural pseudodifferential calculus. For example

- if the groupoid G is just the pair groupoid $M \times M$, the associated calculus is the ordinary (pseudo)differential calculus on M;
- if the groupoid G is a family groupoid $M \times_B M$ associated with a fibration $p : M \to B$, the associated (pseudo)differential operators are families of operators acting on the fibers of p (those of [3]);
- if the groupoid G is the holonomy groupoid of a foliation, the associated (pseudo)differential operators are longitudinal operators as defined by Connes in [6];
- if the groupoid G is the monodromy groupoid *i.e.* the groupoid of homotopy classes (with fixed endpoints) of paths in a (compact) manifold M, the associated (pseudo)differential operators are the $\pi_1(M)$-invariant operators on the universal cover of M...

The groupoid G defines therefore a class of partial differential equations.

Our study will focus here on the corresponding index problems on M. The index takes place naturally in the K-theory of the C^*-algebra of G.

Let then V be a submanifold of M. We will consider V as bringing a singularity into the problem: it forces operators of G to "slow down" near V, at least in the normal directions. Inside V, they should only propagate along a sub-Lie-groupoid $\Gamma \rightrightarrows V$ of G.
This behavior is very nicely encoded by a groupoid $SBlup_{r,s}(G,\Gamma)$ obtained by using a blow-up construction of the inclusion $\Gamma \to G$.

The blowup construction (Blup) and the deformation to the normal cone (DNC) are well known constructions in algebraic geometry as well as in differential geometry. Let X be a submanifold of a manifold Y. Denote by N_X^Y the normal bundle.

- The *deformation to the normal cone* of X in Y is a smooth manifold $DNC(Y,X)$ obtained by naturally gluing $N_X^Y \times \{0\}$ with $Y \times \mathbb{R}^*$.
- The *blowup* of X in Y is a smooth manifold $Blup(Y,X)$ where X is inflated to the projective space $\mathbb{P}N_X^Y$. It is obtained by gluing $Y \setminus X$ with $\mathbb{P}N_X^Y$ in a natural way. We will mainly consider its variant the *spherical blowup* $SBlup(Y,X)$ (which is a manifold with boundary) in which the sphere bundle $\mathbb{S}N_X^Y$ replaces the projective bundle $\mathbb{P}N_X^Y$.

The first use of deformation groupoids in connection with index theory appeared in [8]. A. Connes showed there that the analytic index on a compact manifold M can be described using a groupoid, called the "tangent groupoid". This groupoid was obtained as a deformation to the normal cone of the diagonal inclusion of M into the pair groupoid $M \times M$.

Since Connes' construction, deformation groupoids were used by many authors in various contexts.

- This idea of Connes was extended in [19] by considering the same construction of a deformation to the normal cone for smooth immersions which are groupoid morphisms. The groupoid obtained was used in order to define the wrong way functoriality for immersions of foliations ([19, section 3]). An analogous construction for submersions of foliations was also given in a remark ([19, remark 3.19]).
- In [31, 34] Monthubert-Pierrot and Nistor-Weinstein-Xu considered the deformation to the normal cone of the inclusion $G^{(0)} \to G$ of the space of units of a smooth groupoid G. This generalization of Connes' tangent groupoid was called the adiabatic groupoid of G and denoted by G_{ad}. It was shown that this adiabatic groupoid still encodes the analytic index associated with G.
- Many other important articles use this idea of deformation groupoids. We will briefly discuss some of them in the sequel of the paper. It is certainly out of the scope of the present paper to review them all...

Let us briefly present the objectives of our paper.

The groupoids $DNC(G,\Gamma)$ and $SBlup_{r,s}(G,\Gamma)$.

In the present paper, we give a systematic construction of deformation to the normal cone groupoids and define the blowup deformations of groupoids. Our constructions of deformation and blowup groupoids recover all the ones discussed above.
More precisely, we use the functoriality of these two constructions (*Blup* and *DNC*) and note that any smooth subgroupoid $\Gamma \rightrightarrows V$ of a Lie groupoid $G \rightrightarrows M$ gives rise to a deformation to the normal cone Lie groupoid $DNC(G,\Gamma) \rightrightarrows DNC(M,V)$ and to a blowup Lie groupoid $Blup_{r,s}(G,\Gamma) \rightrightarrows Blup(M,V)$.
We will be mainly interested here to the restriction $DNC_+(G,\Gamma)$ to $\mathbb{R}_+$ of the deformation groupoid $DNC(G,\Gamma)$ and to a variant of $Blup_{r,s}(G,\Gamma)$ which is the *spherical blowup Lie groupoid* $SBlup_{r,s}(G,\Gamma) \rightrightarrows SBlup(M,V)$.

Notation 1.1. We will use the following notation:
If E is a real vector bundle over a manifold (or over a locally compact space) M, the corresponding projective bundle $\mathbb{P}(E)$ is the bundle over M whose fiber over a point x of M is the projective space $\mathbb{P}(E_x)$. The bundle $\mathbb{P}(E)$ is simply the quotient of $E \setminus M$ by the natural action of $\mathbb{R}^*$ by dilation. The quotient of $E \setminus M$ under the action of $\mathbb{R}_+^*$ by dilation is the (total space of the) sphere bundle $\mathbb{S}(E)$.

By construction, both these groupoids are the union of an open subgroupoid, their "regular part", and a "singular" one which is a closed subgroupoid.

- The regular part of the groupoid $DNC_+(G,\Gamma)$ is the direct product $G \times \mathbb{R}_+^*$ (where $\mathbb{R}_+^*$ is just a space);
- its "singular part" is the closed subgroupoid $\mathcal{N}_\Gamma^G \rightrightarrows N_M^V$ which is the normal bundle of the inclusion $\Gamma \to G$ endowed with a Lie groupoid structure over N_M^V. This normal groupoid is a $\mathcal{VB}$ groupoid (in the sense of Pradines *cf.* [35, 22]).

In the same way, the spherical blowup groupoid $SBlup_{r,s}(G,\Gamma)$ is the disjoint union of

- an open subgroupoid which is the restriction $G_{\mathring{M}}^{\mathring{M}}$ of G to $\mathring{M} = M \setminus V$;
- a boundary which is a groupoid $\mathcal{S}N_\Gamma^G \rightrightarrows \mathbb{S}N_V^M$ which is fibered over Γ that is a *spherical bundle groupoid* over Γ.

The spherical blowup groupoid naturally encodes evolution along G which is constrained to fix $V = \Gamma^{(0)}$ and evolve along Γ on it. When looking at the corresponding index problems, we are naturally led to consider natural exact sequences and the corresponding KK-elements.
It turns out, that the corresponding index problems for $DNC_+(G,\Gamma)$ are somewhat easier to handle, but in fact, in most cases equivalent to those of $SBlup_{r,s}(G,\Gamma)$. Both of them are particular cases of connecting maps and full index problems associated to a saturated open subset in a Lie groupoid. We will therefore just need to apply the results of [13] in order to handle them.

Connecting maps and index maps

Connecting maps. The decomposition of $SBlup_{r,s}(G,\Gamma)$ and $DNC_+(G,\Gamma)$ gives rise to exact sequences of C^*-algebras that we wish to "compute":

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}N_\Gamma^G) \longrightarrow 0 \qquad E_{SBlup}^{\partial}$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \qquad E_{DNC_+}^{\partial}$$

Full index maps. Denote by $\Psi^*(DNC_+(G,\Gamma))$ and $\Psi^*(SBlup_{r,s}(G,\Gamma))$ the C^*-algebra of order 0 pseudodifferential operators on the Lie groupoids $DNC_+(G,\Gamma)$ and $SBlup_{r,s}(G,\Gamma)$ respectively. The above decomposition of groupoids give rise to extensions of groupoid C^*-algebras of pseudodifferential type

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow \Psi^*(SBlup_{r,s}(G,\Gamma)) \xrightarrow{\sigma_{full}} \Sigma_{SBlup}(G,\Gamma) \longrightarrow 0 \qquad E_{SBlup}^{\widetilde{\mathrm{ind}}}$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow \Psi^*(DNC_+(G,\Gamma)) \xrightarrow{\sigma_{full}} \Sigma_{DNC_+}(G,\Gamma) \longrightarrow 0 \qquad E_{DNC_+}^{\widetilde{\mathrm{ind}}}$$

where $\Sigma_{DNC_+}(G,\Gamma)$ and $\Sigma_{SBlup}(G,\Gamma)$ are called the *full symbol algebra*, and the morphisms σ_{full} the *full symbol* maps.

The full symbol maps. The full symbol algebras are naturally fibered products (see [13, §4]):

$$\Sigma_{SBlup}(G,\Gamma) = C(\mathbb{S}\mathfrak{A}^* SBlup_{r,s}(G,\Gamma)) \times_{C(\mathbb{S}\mathfrak{A}^* \mathcal{S}N_\Gamma^G)} \Psi^*(\mathcal{S}N_\Gamma^G)$$

and

$$\Sigma_{DNC_+}(G,\Gamma) = C(\mathbb{S}\mathfrak{A}^* DNC_+(G,\Gamma)) \times_{C(\mathbb{S}\mathfrak{A}^* \mathcal{N}_\Gamma^G)} \Psi^*(\mathcal{N}_\Gamma^G).$$

Thus, the full symbol maps have two components:

- The usual commutative symbol map. They are morphisms:

  $\Psi^*(DNC_+(G,\Gamma)) \to C(\mathbb{S}\mathfrak{A}^* DNC_+(G,\Gamma))$ and $\Psi^*(SBlup_{r,s}(G,\Gamma)) \to C(\mathbb{S}\mathfrak{A}^* SBlup_{r,s}(G,\Gamma))$.

  The commutative symbol takes its values in the algebra of continuous fonctions on the sphere bundle of the algebroid of the Lie groupoids (with boundary) $DNC_+(G,\Gamma))$ and $SBlup_{r,s}(G,\Gamma)$.

- The restriction to the boundary:

  $$\sigma_\partial : \Psi^*(SBlup_{r,s}(G,\Gamma)) \to \Psi^*(\mathcal{S}N_\Gamma^G) \text{ and } \Psi^*(DNC_+(G,\Gamma)) \to \Psi^*(\mathcal{N}_\Gamma^G)\ .$$

Associated KK-elements. Assume that the groupoid Γ is amenable. Then the groupoids $\mathcal{N}_\Gamma^G$ and $\mathcal{S}N_\Gamma^G$ are also amenable, and exact sequences E^∂_{SBlup} and $E^\partial_{DNC_+}$ give rise to connecting elements $\partial^{G,\Gamma}_{SBlup} \in KK^1(C^*(\mathcal{S}N_\Gamma^G), C^*(G^{\mathring{M}}_{\mathring{M}}))$ and $\partial^{G,\Gamma}_{DNC_+} \in KK^1(C^*(\mathcal{N}_\Gamma^G), C^*(G \times \mathbb{R}_+^*))$ (*cf.* [20]). Also, the full symbol C^*-algebras $\Sigma_{SBlup}(G,\Gamma)$ and $\Sigma_{DNC_+}(G,\Gamma)$ are nuclear and we also get KK-elements $\widetilde{\mathrm{ind}}^{G,\Gamma}_{SBlup} \in KK^1(\Sigma_{SBlup}(G,\Gamma), C^*(G^{\mathring{M}}_{\mathring{M}}))$ and $\widetilde{\mathrm{ind}}^{G,\Gamma}_{DNC_+} \in KK^1(\Sigma_{DNC_+}(G,\Gamma), C^*(G \times \mathbb{R}_+^*))$.
If Γ is not amenable, these constructions can be carried over in E-theory (of maximal groupoid C^*-algebras).

Connes-Thom elements. We will establish the following facts.

a) There is a natural Connes-Thom element $\beta \in KK^1(C^*(SBlup_{r,s}(G,\Gamma)), C^*(DNC_+(G,\Gamma)))$. This element restricts to very natural elements $\beta' \in KK^1(C^*(G^{\mathring{M}}_{\mathring{M}}), C^*(G \times \mathbb{R}_+^*))$ and $\beta'' \in KK^1(C^*(\mathcal{S}N_\Gamma^G), C^*(\mathcal{N}_\Gamma^G))$.

   These elements extend to elements $\beta_\Psi \in KK^1(\Psi^*(SBlup_{r,s}(G,\Gamma)), \Psi^*(DNC_+(G,\Gamma)))$ and $\beta_\Sigma \in KK^1(\Sigma_{SBlup}(G,\Gamma)), \Sigma_{DNC_+}(G,\Gamma)))$.

   We have $\partial^{G,\Gamma}_{SBlup} \otimes \beta' = -\beta'' \otimes \partial^{G,\Gamma}_{DNC_+}$ (*cf.* facts 5.1 and 5.2) and $\widetilde{\mathrm{ind}}^{G,\Gamma}_{SBlup} \otimes \beta' = -\beta_\Sigma \otimes \widetilde{\mathrm{ind}}^{G,\Gamma}_{DNC_+}$ (fact 5.3).

b) If $M \setminus V$ meets all the orbits of G, then β' is KK-invertible. Therefore, in that case, $\partial^{G,\Gamma}_{DNC_+}$ determines $\partial^{G,\Gamma}_{SBlup}$ and $\widetilde{\mathrm{ind}}^{G,\Gamma}_{DNC_+}$ determines $\widetilde{\mathrm{ind}}^{G,\Gamma}_{SBlup}$.

c) We will say that V is $\mathfrak{A}G$*-small* if the transverse action of G on V is nowhere 0, *i.e.* if for every $x \in V$, the image by the anchor of the algebroid $\mathfrak{A}G$ of G is not contained in T_xV (*cf.* definition 5.5). In that case, β', β'', β_Σ are KK-invertible: the connecting elements $\partial^{G,\Gamma}_{DNC_+}$ and $\partial^{G,\Gamma}_{SBlup}$ determine each other, and the full index maps $\widetilde{\mathrm{ind}}^{G,\Gamma}_{DNC_+}$ and $\widetilde{\mathrm{ind}}^{G,\Gamma}_{SBlup}$ determine each other

Computation. If $\Gamma = V$, then $C^*(\mathcal{N}_V^G)$ is KK-equivalent to $C_0(N_V^G)$ using a Connes-Thom isomorphism ([7]) and the element $\partial^{G,\Gamma}_{DNC_+}$ is the Kasparov product of the inclusion of N_V^G in the algebroid $\mathfrak{A}G = N_M^G$ of G (using a tubular neighborhood) and the index element $\mathrm{ind}_G \in KK(C_0(\mathfrak{A}^*G), C^*(G))$ of the groupoid G (prop. 5.11.d). Of course, if V is $\mathfrak{A}G$-small, we obtain the analogous result for $\partial^{G,\Gamma}_{SBlup}$.
The computation of the corresponding full index is also obtained in the same way in prop. 5.11.e).

A Boutet de Monvel type calculus.

Let H be a Lie groupoid. In [11], extending ideas of Aastrup, Melo, Monthubert and Schrohe [1], we studied the *gauge adiabatic groupoid* H_{ga}: the crossed product of the adiabatic groupoid of H by the natural action of $\mathbb{R}_+^*$. We constructed a bimodule $\mathscr{E}_H$ giving a Morita equivalence between the algebra of order 0 pseudodifferential operators on H and a natural ideal in the convolution C^*-algebra $C^*(H_{ga})$ of this gauge adiabatic groupoid.
The gauge adiabatic groupoid H_{ga} is in fact a blowup groupoid, namely $SBlup_{r,s}(H\times(\mathbb{R}\times\mathbb{R}),H^{(0)})$ (restricted to the clopen subset $H^{(0)}\times\mathbb{R}_+$ of $SBlup(H^{(0)}\times\mathbb{R},H^{(0)})=H^{(0)}\times(\mathbb{R}_-\sqcup\mathbb{R}_+)$).

Let now $G\rightrightarrows M$ be a Lie groupoid and let V be a submanifold M which is *transverse* to the action of G (see def. 2.1). We construct a Poisson-trace bimodule: it is a $C^*(SBlup_{r,s}(G,V))-\Psi^*(G_V^V)$ bimodule $\mathscr{E}_{PT}(G,V)$, which is a full $\Psi^*(G_V^V)$ Hilbert module. When G is the direct product of G_V^V with the pair groupoid $\mathbb{R}\times\mathbb{R}$ the Poisson-trace bimodule coincide with $\mathscr{E}_{G_V^V}$ constructed in [11]. In the general case, thanks to a convenient (spherical) blowup construction, we construct a linking space between the groupoids $SBlup_{r,s}(G,V)$ and $(G_V^V)_{ga}=SBlup_{r,s}(G_V^V\times(\mathbb{R}\times\mathbb{R}),V)$. This linking space defines a $C^*(SBlup_{r,s}(G,V))-C^*((G_V^V)_{ga})$ bimodule $\mathscr{E}(G,V)$ which is a Morita equivalence of groupoids when V meets all the orbits of G. The Poisson-trace -bimodule is then the composition of $\mathscr{E}_{G_V^V}$ with $\mathscr{E}(G,V)$.

Denote by $\Psi^*_{BM}(G;V)$ the *Boutet de Monvel type* algebra consisting of matrices $R=\begin{pmatrix}\Phi & P\\ T & Q\end{pmatrix}$ with $\Phi\in\Psi^*(SBlup_{r,s}(G,V))$, $P\in\mathscr{E}_{PT}(G,V)$, $T\in\mathscr{E}^*_{PT}(G,V)$ and $Q\in\Psi^*(G_V^V)$, and $C^*_{BM}(G;V)$ its ideal - where $\Phi\in C^*(SBlup_{r,s}(G,V))$. This algebra has obvious similarities with the one involved in the Boutet de Monvel calculus for manifolds with boundary [4]. We will examine its relationship with these two algebras in a forthcoming paper.
We still have two natural symbol maps: the classical symbol $\sigma_c:\Psi^*_{BM}(G,V)\to C(\mathbb{S}\mathfrak{A}^*G)$ given by $\sigma_c\begin{pmatrix}\Phi & P\\ T & Q\end{pmatrix}=\sigma_c(\Phi)$ and the boundary symbol r_V which is restriction to the boundary.
We have an exact sequence:

$$0\to C^*(G^{\mathring{M}\sqcup V}_{\mathring{M}\coprod V})\to\Psi^*_{BM}(G;V)\xrightarrow{\sigma_{BM}}\Sigma_{BM}(G,V)\to 0$$

where $\Sigma_{BM}(G,V)=\Psi^*_{BM}(G;V)/C^*(G^{\mathring{M}\sqcup V}_{\mathring{M}\sqcup V})$ and σ_{BM} is defined using both σ_c and r_V.
We may note that $\Psi^*(SBlup_{r,s}(G,V))$ identifies with the full hereditary subalgebra of $\Psi^*_{BM}(G,V)$ consisting of elements of the form $\begin{pmatrix}\Phi & 0\\ 0 & 0\end{pmatrix}$. We thus obtain Boutet de Monvel type index theorems for the connecting map of this exact sequence - as well as for the corresponding relative K-theory.

The paper is organized as follows:

- In section 2 we recall some classical facts, constructions and notation involving groupoids.
- In section 3 we review two geometric constructions: deformation to the normal cone and blowup, and their functorial properties.
- In section 4, using this functoriality, we study deformation to the normal cone and blowup in the Lie groupoid context. We outline examples which recover groupoids constructed previously by several authors.
- In section 5, applying the results obtained in [13], we compute the connecting maps and index maps of the groupoids constructed in section 4.
- In section 6, we describe the above mentioned Boutet de Monvel type calculus.

- The present paper is the second part of the article that appeared on the arXiv (arXiv:1705.09588). Since this paper was quite long and addressed a large variety of situations, we decided to split it into two pieces hoping to make it easier to read. The first part is [13].

  Our constructions involved a large amount of notation, that we tried to choose as coherent as possible. We found it however helpful to list several items of the notation introduced in [13] and the one introduced here in an index at the end of the paper.

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2 Transversality and Morita equivalence of groupoids

2.1 Some notation

Let $G \overset{r,s}{\rightrightarrows} G^{(0)}$ be a groupoid with source s, range r and space of units $G^{(0)}$. For any maps $f : A \to G^{(0)}$ and $g : B \to G^{(0)}$, define

$$G^f = \{(a, x) \in A \times G;\ r(x) = f(a)\},\ G_g = \{(x, b) \in G \times B;\ s(x) = g(b)\}$$

and

$$G^f_g = \{(a, x, b) \in A \times G \times B;\ r(x) = f(a),\ s(x) = g(b)\}\ .$$

In particular for $A, B \subset G^{(0)}$, we put $G^A = \{x \in G;\ r(x) \in A\}$ and $G_A = \{x \in G;\ s(x) \in A\}$; we also put $G^B_A = G_A \cap G^B$.

2.2 Transversality

Let us recall the following definition (see *e.g.* [38] for details):

Definition 2.1. Let $G \overset{r,s}{\rightrightarrows} M$ be a Lie groupoid with set of objects $G^{(0)} = M$ and Lie algebroid $\mathfrak{A}G$ with anchor map $\natural$. Let V be a manifold. A smooth map $f : V \to M$ is said to be *transverse* to (the action of the groupoid) G if for every $x \in V$, $df_x(T_xV) + \natural_{f(x)}\mathfrak{A}_{f(x)}G = T_{f(x)}M$.
An equivalent condition is that the map $(\gamma, y) \mapsto r(\gamma)$ defined on the fibered product $G_f = G \underset{s,f}{\times} V$ is a submersion from G_f to M.
A submanifold V of M is *transverse* to G if the inclusion $V \to M$ is transverse to G - equivalently, if for every $x \in V$, the composition $q_x = p_x \circ \natural_x : \mathfrak{A}_xG \to (N^M_V)_x = T_xM/T_xV$ is onto.

Remark 2.2. Let V be a (locally) closed submanifold of M transverse to a groupoid $G \overset{r,s}{\rightrightarrows} M$. Denote by N^M_V the (total space) of the normal bundle of V in M. Upon arguing locally, we can assume that V is compact.
By the transversality assumption the anchor $\natural : \mathfrak{A}G_{|V} \to TM_{|V}$ induces a surjective bundle morphism $\mathfrak{A}G_{|V} \to N^M_V$. Choosing a subbundle W' of the restriction $\mathfrak{A}G_{|V}$ such that $W' \to N^M_V$ is an isomorphism and using an exponential map, we thus obtain a submanifold $W \subset G$ such that $r : W \to M$ is a diffeomorphism onto an open neighborhood of V in M and s is a submersion from W onto V. Replacing W by a an open subspace, we may assume that $r(W)$ is a tubular neighborhood of V in M, diffeomorphic to N^M_V. The map $W \times_V G^V_V \times_V W \to G$ defined by $(\gamma_1, \gamma_2, \gamma_3) \mapsto \gamma_1 \circ \gamma_2 \circ \gamma_3^{-1}$ is a diffeomorphism and a groupoid isomorphism from the pull back groupoid (see next section) $(G^V_V)^s_s = W \times_V G^V_V \times_V W$ onto the open subgroupoid $G^{r(W)}_{r(W)}$ of G.

2.3 Pull back

If $f : V \to M$ is transverse to a Lie groupoid $G \overset{r,s}{\rightrightarrows} M$, then G_f^f is a submanifold of $V \times G \times V$ naturally equipped with a structure of Lie groupoid $G_f^f \rightrightarrows V$. It is called the *pull back groupoid.*

If $f_i : V_i \to M$ are transverse to G (for $i = 1, 2$) then we obtain a Lie groupoid $G_{f_1 \sqcup f_2}^{f_1 \sqcup f_2} \rightrightarrows V_1 \sqcup V_2$. The *linking manifold* $G_{f_2}^{f_1}$ is a clopen submanifold. We denote by $C^*(G_{f_2}^{f_1})$ the closure in $C^*(G_{f_1 \sqcup f_2}^{f_1 \sqcup f_2})$ of the space of functions (half densities) with support in $G_{f_2}^{f_1}$; it is a $C^*(G_{f_1}^{f_1}) - C^*(G_{f_2}^{f_2})$ bimodule.

Fact 2.3. *The bimodule $C^*(G_{f_2}^{f_1})$ is full if all the G-orbits meeting $f_2(V_2)$ meet also $f_1(V_1)$.*

2.4 Morita equivalence

Two Lie groupoids $G_1 \overset{r,s}{\rightrightarrows} M_1$ and $G_2 \overset{r,s}{\rightrightarrows} M_2$ are *Morita equivalent* if there exists a groupoid $G \overset{r,s}{\rightrightarrows} M$ and smooth maps $f_i : M_i \to M$ transverse to G such that the pull back groupoids $G_{f_i}^{f_i}$ identify to G_i and $f_i(M_i)$ meets all the orbits of G.
Equivalently, a Morita equivalence is given by a linking manifold X with extra data: surjective smooth submersions $r : X \to G_1^{(0)}$ and $s : X \to G_2^{(0)}$ and compositions $G_1 \times_{s,r} X \to X$, $X \times_{s,r} G_2 \to X$, $X \times_{r,r} X \to G_2$ and $X \times_{s,s} X \to G_1$ with natural associativity conditions (see [32] for details).
In the above situation, X is the manifold $G_{f_2}^{f_1}$ and the extra data are the range and source maps and the composition rules of the groupoid $G_{f_1 \sqcup f_2}^{f_1 \sqcup f_2} \rightrightarrows M_1 \sqcup M_2$ (see [32]).
If the map $r : X \to G_1^{(0)}$ is surjective but $s : X \to G_2^{(0)}$ is not necessarily surjective, then G_1 is Morita equivalent to the restriction of G_2 to the open saturated subspace $s(X)$. We say that G_1 is *sub-Morita* equivalent to G_2.

2.5 Remarks on possible singularity

About corners We wish to emphasize a remark already made in [13]:
Many manifolds and groupoids that occur in our constructions have boundaries or corners. In fact all the groupoids we consider sit naturally inside Lie groupoids *without boundaries* as restrictions to closed *saturated* subsets. This means that we consider subgroupoids $G_V^V = G_V$ of a Lie groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$ where V is a closed subset of $G^{(0)}$. Such groupoids, have a natural algebroid, adiabatic deformation, pseudodifferential calculus, *etc.* that are restrictions to V and G_V of the corresponding objects on $G^{(0)}$ and G. We chose to give our definitions and constructions for Lie groupoids for the clarity of the exposition. The case of a longitudinally smooth groupoid over a manifold with corners is a straightforward generalization using a convenient restriction.

About non-Haudorffness Our groupoids need not be Hausdorff. Precisely, for $G \rightrightarrows G^{(0)}$, the manifold G may be a non-Haudorff manifold, but $G^{(0)}$ will always be assumed to be Haudorff. Of course a non Hausdorff manifold is locally Hausdorff.

3 Two classical geometric constructions: Blowup and deformation to the normal cone

One of the main objects in our study is a Lie groupoid G based on a groupoid restricted to a half space. This corresponds to the inclusion of a hypersurface V of $G^{(0)}$ into G and gives rise to the "gauge adiabatic groupoid" ${}_{ga}G$. The construction of ${}_{ga}G$ is in fact a particular case of the blowup construction corresponding to the inclusion of a Lie subgroupoid into a groupoid. In this section, we will explain this general construction. We will give a more detailed description in the case of an inclusion $V \to G$ when V is a submanifold of $G^{(0)}$.

Let Y be a manifold and X a locally closed submanifold (the same constructions hold if we are given an injective immersion $X \to Y$). Denote by N_X^Y the (total space) of the normal bundle of X in Y.

3.1 Deformation to the normal cone

The deformation to the normal cone $DNC(Y, X)$ is obtained by gluing $N_X^Y \times \{0\}$ with $Y \times \mathbb{R}^*$. The smooth structure of $DNC(Y, X)$ is described by use of any exponential map $\theta : U' \to U$ which is a diffeomorphism from an open neighborhood U' of the 0-section in N_X^Y to an open neighborhood U of X. The map θ is required to satisfy $\theta(x, 0) = x$ for all $x \in X$ and $p_x \circ d\theta_x = p'_x$ where $p_x : T_xY \to (N_X^Y)_x = (T_xY)/(T_xX)$ and $p'_x : T_xN_X^Y \simeq (N_X^Y)_x \oplus (T_xX) \to (N_X^Y)_x$ are the projections. The manifold structure of $DNC(Y, X)$ is described by the requirement that:

a) the inclusion $Y \times \mathbb{R}^* \to DNC(Y, X)$ and

b) the map $\Theta : \Omega' = \{((x, \xi), \lambda) \in N_X^Y \times \mathbb{R};\ (x, \lambda\xi) \in U'\} \to DNC(Y, X)$ defined by $\Theta((x, \xi), 0) = ((x, \xi), 0)$ and $\Theta((x, \xi), \lambda) = (\theta(x, \lambda\xi), \lambda) \in Y \times \mathbb{R}^*$ if $\lambda \neq 0$.

are diffeomorphisms onto open subsets of $DNC(Y, X)$.
It is easily shown that $DNC(Y, X)$ has indeed a smooth structure satisfying these requirements and that this smooth structure does not depend on the choice of θ. (See for example [5] for a detailed description of this structure).
In other words, $DNC(Y, X)$ is obtained by gluing $Y \times \mathbb{R}^*$ with Ω' by means of the diffeomorphism $\Theta : \Omega' \cap (N_X^Y \times \mathbb{R}^*) \to U \times \mathbb{R}^*$.
Let us recall the following facts which are essential in our construction.

Definition 3.1. The gauge action of $\mathbb{R}^*$. The group $\mathbb{R}^*$ acts on $DNC(Y, X)$ by $\lambda.(w, t) = (w, \lambda t)$ and $\lambda.((x, \xi), 0) = ((x, \lambda^{-1}\xi), 0)$ $\big($with $\lambda, t \in \mathbb{R}^*$, $w \in Y$, $x \in X$ and $\xi \in (N_X^Y)_x\big)$.

Remarks 3.2. a) Since some natural Lie groupoids are non Hausdorff, we may have to consider non Hausdorff manifolds. In that case, the usual properness condition is replaced by local properness (*cf.* [13, Remark 2.5]). This means that every point has a neighborhood invariant under the action, on which the action is proper.

b) The gauge action is easily seen to be free and (locally) proper on the open subset $DNC(Y, X) \setminus X \times \mathbb{R}$. Indeed, for $(x, \xi, t) \in \Omega' \subset N_X^Y \times \mathbb{R}$ the gauge action is given by $\lambda.(x, \xi, t) = (x, \lambda^{-1}\xi, \lambda t)$ under the map Θ^{-1}.

Definition 3.3. Functoriality. Given a commutative diagram of smooth maps

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f_X"'] & Y \arrow[d, "f_Y"] \\ X' \arrow[r, hook] & Y' \end{tikzcd}$$

where the horizontal arrows are inclusions of submanifolds, we naturally obtain a smooth map $DNC(f) : DNC(Y, X) \to DNC(Y', X')$. This map is defined by $DNC(f)(y, \lambda) = (f_Y(y), \lambda)$ for $y \in Y$ and $\lambda \in \mathbb{R}_*$ and $DNC(f)(x, \xi, 0) = (f_X(x), f_N(\xi), 0)$ for $x \in X$ and $\xi \in (N_X^Y)_x = T_xY/T_xX$ where $f_N : N_x \to (N_{X'}^{Y'})_{f_X(x)} = T_{f_X(x)}Y'/T_{f_X(x)}X'$ is the linear map induced by the differential $(df_Y)_x$ at x. This map is of course equivariant with respect to the gauge action of $\mathbb{R}^*$.

Remarks 3.4. Let us make a few remarks concerning the DNC construction.

a) The map equal to identity on $X \times \mathbb{R}^*$ and sending $X \times \{0\}$ to the zero section of N_X^Y leads to an embedding of $X \times \mathbb{R}$ into $DNC(Y, X)$, we may often identify $X \times \mathbb{R}$ with its image in $DNC(Y, X)$. As $DNC(X, X) = X \times \mathbb{R}$, this corresponds to the fonctoriality of DNC for the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X \arrow[d] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

b) We have a natural smooth map $\pi : DNC(Y,X) \to Y \times \mathbb{R}$ defined by $\pi(y,\lambda) = (y,\lambda)$ (for $y \in Y$ and $\lambda \in \mathbb{R}^*$) and $\pi((x,\xi),0) = (x,0)$ (for $x \in X \subset Y$ and $\xi \in (N_X^Y)_x$ a normal vector). This corresponds to the fonctoriality of DNC for the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ Y \arrow[r, hook] & Y \end{tikzcd}$$

c) To see that the smooth structure on $DNC(Y,X)$ is well defined and establish functoriality, one may also note that the following maps are smooth:

- the map $\pi : DNC(Y,X) \to Y \times \mathbb{R}$ defined above;
- given a smooth function $f : Y \to \mathbb{R}$ whose restriction to X is 0, the map $F_f : DNC(Y,X) \to \mathbb{R}$ defined by $F_f(y,\lambda) = \dfrac{f(y)}{\lambda}$ (for $y \in Y$ and $\lambda \in \mathbb{R}^*$) and $F_f(x, p_x(\xi), 0) = df_x(\xi)$ for $x \in X$ and $\xi \in T_xY$ where $p_x : T_xY \to (N_X^Y)_x = T_xY/T_xX$ is the quotient map (note that df_x vanishes on T_xX).

These maps describe the smooth structure of $DNC(Y,X)$. Indeed given a manifold Z, a map $g : Z \to DNC(Y,X)$ is smooth if and only if $\pi \circ g$ and the maps $F_f \circ g$ are smooth. Actually, a finite number of those give rise to an immersion $DNC(Y,X) \to Y \times \mathbb{R} \times \mathbb{R}^k$ (at least locally - if we do not assume X to be compact).

d) If Y_1 is an open subset of Y_2 such that $X \subset Y_1$, then $DNC(Y_1,X)$ is an open subset of $DNC(Y_2,X)$ and $DNC(Y_2,X)$ is the union of the open subsets $DNC(Y_1,X)$ and $Y_2 \times \mathbb{R}^*$. This reduces to the case when Y_1 is a tubular neighborhood - and therefore to the case where Y is (diffeomorphic to) the total space of a real vector bundle over X. In that case one gets $DNC(Y,X) = Y \times \mathbb{R}$ and the gauge action of $\mathbb{R}^*$ on $DNC(Y,X) = Y \times \mathbb{R}$ is given by $\lambda.((x,\xi),t) = ((x,\lambda^{-1}\xi),\lambda t)$ (with $\lambda \in \mathbb{R}^*$, $t \in \mathbb{R}$, $x \in X$ and $\xi \in Y_x$).

e) More generally, let E be (the total space of) a real vector bundle over Y. Then $DNC(E,X)$ identifies with the total space of the pull back vector bundle $\hat\pi^*(E)$ over $DNC(Y,X)$, where $\hat\pi$ is the composition of $\pi : DNC(Y,X) \to Y \times \mathbb{R}$ (remark b) with the projection $Y \times \mathbb{R} \to Y$. The gauge action of $\mathbb{R}^*$ is $\lambda.(w,\xi) = (\lambda.w, \lambda^{-1}\xi)$ for $w \in DNC(Y,X)$ and $\xi \in E_{\hat\pi(w)}$.

f) Let X_1 be a (locally closed) smooth submanifold of a smooth manifold Y_1 and let $f : Y_2 \to Y_1$ be a smooth map transverse to X_1. Put $X_2 = f^{-1}(X_1)$. Then the normal bundle $N_{X_2}^{Y_2}$ identifies with the pull back of $N_{X_1}^{Y_1}$ by the restriction $X_2 \to X_1$ of f. It follows that $DNC(Y_2,X_2)$ identifies with the fibered product $DNC(Y_1,X_1) \times_{Y_1} Y_2$.

g) More generally, let Y, Y_1, Y_2 be smooth manifolds and $f_i : Y_i \to Y$ be smooth maps. Assume that f_1 is transverse to f_2. Let $X \subset Y$ and $X_i \subset Y_i$ be (locally closed) smooth submanifolds. Assume that $f_i(X_i) \subset X$ and that the restrictions $g_i : X_i \to X$ of f_i are transverse also. We thus have a diagram

$$\begin{tikzcd} X_1 \arrow[r, "g_1"] \arrow[d, hook] & X \arrow[d, hook] & X_2 \arrow[l, "g_2"'] \arrow[d, hook] \\ Y_1 \arrow[r, "f_1"] & Y & Y_2 \arrow[l, "f_2"'] \end{tikzcd}$$

Then the maps $DNC(f_i) : DNC(Y_i,X_i) \to DNC(Y,X)$ are transverse and the deformation to the normal cone of fibered products $DNC(Y_1 \times_Y Y_2, X_1 \times_X X_2)$ identifies with the fibered product $DNC(Y_1,X_1) \times_{DNC(Y,X)} DNC(Y_2,X_2)$.

Note that construction f) is the particular case $X = Y = Y_1$ of our construction here.

Notation 3.5. For every locally closed subset T of $\mathbb{R}$ containing 0 and with the notation of remark 3.4.b below, we define :

$$DNC^T(Y,X) = Y \times (T \setminus \{0\}) \cup N_X^Y \times \{0\} = \pi^{-1}(Y \times T)$$

It is the *restriction* of $DNC(Y,X)$ to T. We put $DNC_+(Y,X) = DNC^{\mathbb{R}_+}(Y,X) = Y \times \mathbb{R}_+^* \cup N_X^Y \times \{0\}$.

3.2 Blowup constructions

The blowup $Blup(Y,X)$ is a smooth manifold which is a union of $Y \setminus X$ with the (total space) $\mathbb{P}(N_X^Y)$ of the projective space of the normal bundle N_X^Y of X in Y. We will also use the "spherical version" $SBlup(Y,X)$ of $Blup(Y,X)$ which is a manifold with boundary obtained by gluing $Y \setminus X$ with the (total space of the) sphere bundle $\mathbb{S}(N_X^Y)$. We have an obvious smooth onto map $SBlup(Y,X) \to Blup(Y,X)$ with fibers 1 or 2 points. These spaces are of course similar and we will often give details in our constructions to the one of them which is the most convenient for our purposes.

We may view $Blup(Y,X)$ as the quotient space of a submanifold of the deformation to the normal cone $DNC(Y,X)$ under the gauge action of $\mathbb{R}^*$.
Recall that the group $\mathbb{R}^*$ acts on $DNC(Y,X)$ by $\lambda.(w,t) = (w, \lambda t)$ and $\lambda.((x,\xi),0) = ((x, \lambda^{-1}\xi), 0)$ (with $\lambda, t \in \mathbb{R}^*$, $w \in Y$, $x \in X$ and $\xi \in (N_X^Y)_x$). According to remark 3.2.b), this action is free and (locally) proper on the open subset $DNC(Y,X) \setminus X \times \mathbb{R}$.

Definition 3.6. We put

$$Blup(Y,X) = \Big(DNC(Y,X) \setminus X \times \mathbb{R}\Big)/\mathbb{R}^*$$

and

$$SBlup(Y,X) = \Big(DNC_+(Y,X) \setminus X \times \mathbb{R}_+\Big)/\mathbb{R}_+^*.$$

Remark 3.7. With the notation of section 3.1, $Blup(Y,X)$ is thus obtained by gluing $Y \setminus X = ((Y \setminus X) \times \mathbb{R}^*)/\mathbb{R}^*$, with $(\Omega' \setminus (X \times \mathbb{R}))/\mathbb{R}^*$ using the map Θ which is equivariant with respect to the gauge action of $\mathbb{R}^*$.
Choose a euclidean metric on N_X^Y. Let $\mathbb{S} = \{((x,\xi),\lambda) \in \Omega';\ \|\xi\| = 1\}$ and τ the involution of $\mathbb{S}$ given by $((x,\xi),\lambda) \mapsto ((x,-\xi),-\lambda)$. The map Θ induces a diffeomorphism of $\mathbb{S}/\tau$ with an open neighborhood $\tilde{\Omega}$ of $\mathbb{P}(N_X^Y)$ in $Blup(Y,X)$.

Since $\hat{\pi} : DNC(Y,X) \to Y$ is invariant by the gauge action of $\mathbb{R}^*$, we obtain a natural smooth map $\tilde{\pi} : Blup(Y,X) \to Y$ whose restriction to $Y \setminus X$ is the identity and whose restriction to $\mathbb{P}(N_X^Y)$ is the canonical projection $\mathbb{P}(N_X^Y) \to X \subset Y$. This map is easily seen to be proper.

Remark 3.8. Note that, according to remark 3.4.e), $DNC(Y,X)$ canonically identifies with the open subset $Blup(Y \times \mathbb{R}, X \times \{0\}) \setminus Blup(Y \times \{0\}, X \times \{0\})$ of $Blup(Y \times \mathbb{R}, X \times \{0\})$. Thus, since the map $Blup(Y \times \mathbb{R}, X \times \{0\}) \to Y \times \mathbb{R}$ is proper, one may think at $Blup(Y \times \mathbb{R}, X \times \{0\})$ as a "local compactification" of $DNC(Y,X)$.

Example 3.9. In the case where Y is a real vector bundle over X, $Blup(Y,X)$ identifies non canonically with an open submanifold of the bundle of projective spaces $\mathbb{P}(Y \times \mathbb{R})$ over X. Indeed, in that case $DNC(Y,X) = Y \times \mathbb{R}$; choose a euclidian structure on the bundle Y. Consider the smooth involution Φ from $(Y \setminus X) \times \mathbb{R}$ onto itself which to (x,ξ,t) associates $(x, \dfrac{\xi}{\|\xi\|^2}, t)$ (for $x \in X,\ \xi \in Y_x,\ t \in \mathbb{R}$). This map transforms the gauge action of $\mathbb{R}^*$ on $DNC(Y,X)$ into the action of $\mathbb{R}^*$ by dilations on the vector bundle $Y \times \mathbb{R}$ over X and thus defines a diffeomorphism of $Blup(Y,X)$ into its image which is the open set $\mathbb{P}(Y \times \mathbb{R}) \setminus X$ where X embeds into $\mathbb{P}(Y \times \mathbb{R})$ by mapping $x \in X$ to the line $\{(x,0,t),\ t \in \mathbb{R}\}$.

Remark 3.10. Since we will apply this construction to morphisms of groupoids that need not be proper, we have to relax properness as in [13, Remark 2.2]: we will say that $f : Y \to X$ is *locally proper* if every point in X has a neighborhood V such that the restriction $f^{-1}(V) \to V$ of f is proper. In particular, if Y is a non Hausdorff manifold and X is a locally closed submanifold of Y, then the map $Blup(Y \times \mathbb{R}, X \times \{0\}) \to Y \times \mathbb{R}$ is locally proper

Functoriality

Definition 3.11 (Functoriality)**.** Let

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f_X"'] & Y \arrow[d, "f_Y"] \\ X' \arrow[r, hook] & Y' \end{tikzcd}$$

be a commutative diagram of smooth maps, where the horizontal arrows are inclusions of closed submanifolds. Let $U_f = DNC(Y, X) \setminus DNC(f)^{-1}(X' \times \mathbb{R})$ be the inverse image by $DNC(f)$ of the complement in $DNC(Y', X')$ of the subset $X' \times \mathbb{R}$. We thus obtain a smooth map $Blup(f) : Blup_f(Y, X) \to Blup(Y', X')$ where $Blup_f(Y, X) \subset Blup(Y, X)$ is the quotient of U_f by the gauge action of $\mathbb{R}^*$.

In particular,

a) If $X \subset Y_1$ are (locally) closed submanifolds of a manifold Y_2, then $Blup(Y_1, X)$ is a submanifold of $Blup(Y_2, X)$.

b) Also, if Y_1 is an open subset of Y_2 such that $X \subset Y_1$, then $Blup(Y_1, X)$ is an open subset of $Blup(Y_2, X)$ and $Blup(Y_2, X)$ is the union of the open subsets $Blup(Y_1, X)$ and $Y_2 \setminus X$. This reduces to the case when Y_1 is a tubular neighborhood.

Fibered products

Let X_1 be a (locally closed) smooth submanifold of a smooth manifold Y_1 and let $f : Y_2 \to Y_1$ be a smooth map transverse to X_1. Put $X_2 = f^{-1}(X_1)$. Recall from remark 3.4.f that in this situation $DNC(Y_2, X_2)$ identifies with the fibered product $DNC(Y_1, X_1) \times_{Y_1} Y_2$. Thus $Blup(Y_2, X_2)$ identifies with the fibered product $Blup(Y_1, X_1) \times_{Y_1} Y_2$.

4 Constructions of groupoids

4.1 Normal groupoids, deformation groupoids and blowup groupoids

4.1.1 Definitions

Let Γ be a closed Lie subgroupoid of a Lie groupoid G. Using functoriality (*cf.* Definition 3.11) of the DNC and $Blup$ construction we may construct a deformation and a blowup groupoid.

a) The normal bundle N_Γ^G carries a Lie groupoid structure with objects $N_{\Gamma^{(0)}}^{G^{(0)}}$. We denote by $\mathcal{N}_\Gamma^G \rightrightarrows N_{\Gamma^{(0)}}^{G^{(0)}}$ this groupoid.

b) The manifold $DNC(G, \Gamma)$ is naturally a Lie groupoid (unlike what was asserted in remark 3.19 of [19]). Its unit space is $DNC(G^{(0)}, \Gamma^{(0)})$; its source and range maps are $DNC(s)$ and $DNC(r)$; the space of composable arrows identifies with $DNC(G^{(2)}, \Gamma^{(2)})$ and its product with $DNC(m)$ where m denotes both products $G^{(2)} \to G$ and $\Gamma^{(2)} \to \Gamma$.

c) The subset $\widetilde{DNC}(G, \Gamma) = U_r \cap U_s$ of $DNC(G, \Gamma)$ consisting of elements whose image by $DNC(r)$ and $DNC(s)$ is not in $G_1^{(0)} \times \mathbb{R}$ is an open subgroupoid of $DNC(G, \Gamma)$: it is the restriction of $DNC(G, \Gamma)$ to the open subspace $DNC(G^{(0)}, G_1^{(0)}) \setminus G_1^{(0)} \times \mathbb{R}$.

d) The group $\mathbb{R}^*$ acts on $DNC(G,\Gamma)$ via the gauge action by groupoid morphisms. Its action on $\widetilde{DNC}(G,\Gamma)$ is (locally) proper. Therefore the open subset $Blup_{r,s}(G,\Gamma) = \widetilde{DNC}(G,\Gamma)/\mathbb{R}^*$ of $Blup(G,\Gamma)$ inherits a groupoid structure as well: its space of units is $Blup(G_2^{(0)}, G_1^{(0)})$; its source and range maps are $Blup(s)$ and $Blup(r)$ and the product is $Blup(m)$.

e) In the same way, we define the groupoid $SBlup_{r,s}(G,\Gamma)$. It is the quotient of the restriction $\widetilde{DNC}_+(G,\Gamma)$ of $\widetilde{DNC}(G,\Gamma)$ to $\mathbb{R}_+$ by the action of $\mathbb{R}_+^*$.

f) The singular part of $SBlup_{r,s}(G,\Gamma)$, *i.e.* its restriction to the boundary $\mathbb{S}N_V^M$ is the *spherical normal groupoid* $\mathcal{S}N_\Gamma^G$. It is the quotient by the action of $\mathbb{R}_+^*$ of the restriction of $\mathcal{N}_\Gamma^G \rightrightarrows N_V^M$ to the open subset $N_V^M \setminus V$ of its objects.

An analogous result about the groupoid structure on $Blup_{r,s}(G,\Gamma)$ in the case of $\Gamma^{(0)}$ being a hypersurface of $G^{(0)}$ can be found in [15, Theorem 2.8] (*cf.* also [14]).

4.1.2 Algebroid and anchor

The (total space of the) Lie algebroid $\mathfrak{A}\Gamma$ is a closed submanifold (and a subbundle) of $\mathfrak{A}G$. The Lie algebroid of $DNC(G,\Gamma)$ is $DNC(\mathfrak{A}G,\mathfrak{A}\Gamma)$. Its anchor map is $DNC(\natural_G) : DNC(\mathfrak{A}G,\mathfrak{A}\Gamma) \to DNC(TG^{(0)}, T\Gamma^{(0)})$.
The groupoid $DNC(G,\Gamma)$ is the union of its open subgroupoid $G \times \mathbb{R}^*$ with its closed Lie subgroupoid $\mathcal{N}_\Gamma^G$. The algebroid of $G \times \mathbb{R}^*$ is $\mathfrak{A}G \times \mathbb{R}^*$ and the anchor is just the map $\natural_G \times \mathrm{id} : \mathfrak{A}G \times \mathbb{R}^* \to T(G^{(0)} \times \mathbb{R}_+^*)$.

4.1.3 Stability under Morita equivalence

Let $G_1 \rightrightarrows G_1^{(0)}$ and $G_2 \rightrightarrows G_2^{(0)}$ be Lie groupoids, $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ Lie subgroupoids. A *Morita equivalence of the pair* $(\Gamma_1 \subset G_1)$ with the pair $(\Gamma_2 \subset G_2)$ is given by a pair $(X \subset Y)$ where Y is a linking manifold which is a Morita equivalence between G_1 and G_2 and $X \subset Y$ is a submanifold of Y such that the maps r, s and products of Y (see page 8) restrict to a Morita equivalence X between Γ_1 and Γ_2.
Then, by functoriality,

- $DNC(Y,X)$ is a Morita equivalence between $DNC(G_1,\Gamma_1)$ and $DNC(G_2,\Gamma_2)$,
- $DNC_+(Y,X)$ is a Morita equivalence between $DNC_+(G_1,\Gamma_1)$ and $DNC_+(G_2,\Gamma_2)$,
- $Blup_{r,s}(Y,X)$ is a Morita equivalence between $Blup_{r,s}(G_1,\Gamma_1)$ and $Blup_{r,s}(G_2,\Gamma_2)$,
- $SBlup_{r,s}(Y,X)$ is a Morita equivalence between $SBlup_{r,s}(G_1,\Gamma_1)$ and $SBlup_{r,s}(G_2,\Gamma_2)$...

Note that if Y and X are sub-Morita equivalences, the above linking spaces are also sub-Morita equivalences.

4.1.4 Groupoids on manifolds with boundary

Let M be a manifold and V an hypersurface in M and suppose that V cuts M into two manifolds with boundary $M = M_- \cup M_+$ with $V = M_- \cap M_+$. Then by considering a tubular neighborhood of V in M, $DNC(M,V) = M \times \mathbb{R}^* \cup \mathcal{N}_V^M \times \{0\}$ identifies with $M \times \mathbb{R}$, the quotient $\widetilde{DNC}(M,V)/\mathbb{R}_+^*$ identifies with two copies of M and $SBlup(M,V)$ identifies with the disjoint union $M_- \sqcup M_+$. Under this last identification, the class under the gauge action of a normal vector in $\mathcal{N}_V^M \setminus V \times \{0\}$ pointing in the direction of M_+ is an element of $V \subset M_+$.

Let M_b be manifold with boundary V. A *piece of Lie groupoid* is the restriction $G = \widetilde{G}_{M_b}^{M_b}$ to M_b of a Lie groupoid $\widetilde{G} \rightrightarrows M$ where M is a neighborhood of M_b and $\widetilde{G}$ is a groupoid without boundary.

With the above notation, since V is of codimension 1 in M, $SBlup(M,V) = M_b \sqcup M_-$ where $M_- = M \setminus \overset{\circ}{M}$ is the complement in M of the interior $\overset{\circ}{M} = M_b \setminus V$ of M_b in M.
Let then $\Gamma \rightrightarrows V$ be a Lie subgroupoid of $\widetilde{G}$.
We may construct $SBlup_{r,s}(\widetilde{G},\Gamma)$ and consider its restriction to the open subset M_b of $SBlup(M,V)$. We thus obtain a longitudinally smooth groupoid that will be denoted $SBlup_{r,s}(G,\Gamma)$.
Note that the groupoid $SBlup_{r,s}(G,\Gamma) \rightrightarrows M_b$ is the restriction to M_b of a Lie groupoid $\mathcal{G} \rightrightarrows M$ for which M_b is saturated. Indeed $SBlup_{r,s}(G,\Gamma)$ is an open subgroupoid of $SBlup_{r,s}(\widetilde{G},\Gamma) \rightrightarrows M_b \sqcup M_-$ which is a piece of the Lie groupoid $\widetilde{DNC}(\widetilde{G},\Gamma)/\mathbb{R}_+^* \rightrightarrows \widetilde{DNC}(M,V)/\mathbb{R}_+^* \simeq M \sqcup M$. We may then let $\mathcal{G}$ be the restriction of $\widetilde{DNC}(M,V)/\mathbb{R}_+^*$ to one of the copies of M.

In this way, we may treat by induction a finite number of boundary components and in particular groupoids on manifolds with (embeded) corners.

Remarks 4.1. a) Let us highlight that we do *not* assume V to be saturated for G. In particular the boundary V can happen to be transverse to the groupoid $\widetilde{G}$. In that case G is in fact a manifold with corners. The blowup construction will change $\widetilde{G}$ in such a way that V becomes a saturated subset.

b) If M is a manifold with boundary V and $G = M \times M$ is the pair groupoid, then $SBlup_{r,s}(G,V)$ is in fact the groupoid associated with the 0 calculus in the sense of Mazzeo (*cf.* [23, 28, 25]), *i.e.* the canonical pseudodifferential calculs associated with $SBlup_{r,s}(G,V)$ is the Mazzeo-Melrose's 0-calculus. Indeed, the sections of the algebroid of $SBlup_{r,s}(G,V)$ are exactly the vector fields of M vanishing at the boundary V, *i.e.* those generating the 0-calculus.

c) In a recent paper [33], an alternative description of $SBlup_{r,s}(G,V)$ is given under the name of *edge modification* for G along the "$\mathfrak{A}G$-tame manifold" V, thus in particular V is transverse to G. This is essentially the gluing construction described in 4.3.4 below.

4.2 Description of the normal groupoids

In this section, we study the normal groupoid $\mathcal{N}_\Gamma^G$ *i.e.* the restriction of $DNC(G,\Gamma)$ to its singular part N_V^M, as well as the projective normal groupoid $\mathcal{P}N_\Gamma^G$ the restriction of $Blup_{r,s,}(G,\Gamma)$ to its singular part $\mathbb{P}N_V^M$. The groupoid $\mathcal{N}_\Gamma^G$ is a $\mathcal{VB}$ groupoid in the sense of Pradines [35, 22]. In the particular case where $\Gamma = V$ is just a space, the groupoids $\mathcal{N}_\Gamma^G$ and $\mathcal{P}N_\Gamma^G$ are bundles of linear and projective groupoids over the base V in a sense defined bellow. In that case, a Thom-Connes isomorphism computes the KK-theory of $C^*(\mathcal{N}_\Gamma^G)$ (prop. 4.5).

4.2.1 Linear groupoids

Let E be a vector space over a field $\mathbb{K}$ and let F be a vector sub-space. Let $r, s : E \to F$ be linear retractions of the inclusion $F \to E$.

The linear groupoid. The space E is endowed with a groupoid structure $\mathcal{E}$ with base F. The range and source maps are r and s and the product is $(x,y) \mapsto (x \cdot y) = x + y - s(x)$ for (x,y) composable, *i.e.* such that $s(x) = r(y)$. One can easily check:

- Since r and s are linear retractions: $r(x \cdot y) = r(x)$ and $s(x \cdot y) = s(y)$.
- If (x,y,z) are composable, then $(x \cdot y) \cdot z = x + y + z - (r+s)(y) = x \cdot (y \cdot z)$.
- The inverse of x is $(r+s)(x) - x$.

Remarks 4.2. a) Note that, given E and linear retractions r and s on F, $\mathcal{E} \rightrightarrows F$ is the only possible linear groupoid structure([1]) on E . Indeed, for any $x \in E$ one must have $x \cdot s(x) = x$

[1] A linear groupoid is a groupoid G such that $G^{(0)}$ and G are vector spaces and all structure maps (unit, range, source, product) are linear.

and $r(x)\cdot x = x$. By linearity, it follows that for every composable pair $(x,y) = (x,s(x)) + (0, y - s(x))$ we have $x \cdot y = x \cdot s(x) + 0 \cdot (y - s(x)) = x + y - s(x)$.

b) The morphism $r - s : E/F \to F$ gives an action of E/F on F by addition. The groupoid associated to this action is in fact $\mathcal{E}$.

c) Given a linear groupoid structure on a vector space E, we obtain the "dual" linear groupoid structure $\mathcal{E}^*$ on the dual space E^* given by the subspace $F^\perp = \{\xi \in E^*;\ \xi|_F = 0\}$ and the two retractions $r^*, s^* : E^* \to F^\perp$ with kernels $(\ker r)^\perp$ and $(\ker s)^\perp$: for $\xi \in E^*$ and $x \in E$, $r^*(\xi)(x) = \xi(x - r(x))$ and similarly $s^*(\xi)(x) = \xi(x - s(x))$.

The projective groupoid. The multiplicative group $\mathbb{K}^*$ acts on $\mathcal{E}$ by groupoid automorphisms. This action is free on the restriction $\widetilde{\mathcal{E}} = \mathcal{E} \setminus (\ker r \cup \ker s)$ of the groupoid $\mathcal{E}$ to the subset $F \setminus \{0\}$ of $\mathcal{E}^{(0)} = F$.
The projective groupoid is the quotient groupoid $\mathcal{P}E = \widetilde{\mathcal{E}}/\mathbb{K}^*$. It is described as follows.
As a set $\mathcal{P}E = \mathbb{P}(E) \setminus (\mathbb{P}(\ker r) \cup \mathbb{P}(\ker s))$ and $\mathcal{P}^{(0)} = \mathbb{P}(F) \subset \mathbb{P}(E)$. The source and range maps $r, s : \mathcal{P}E \to \mathbb{P}(F)$ are those induced by $r, s : E \to F$. The product of $x, y \in \mathcal{P}E$ with $s(x) = r(y)$ is the line $x \cdot y = \{u + v - s(u);\ u \in x,\ v \in y;\ s(u) = r(v)\}$. The inverse of $x \in \mathcal{P}E$ is $(r + s - id)(x)$.

Remarks 4.3. a) When F is just a vector line, $\mathcal{P}E$ is a group. Let us describe it:

we have a canonical morphism $h : \mathcal{P}E \to \mathbb{K}^*$ defined by $r(u) = h(x)s(u)$ for $u \in x$. The kernel of h is $\mathbb{P}(\ker(r - s)) \setminus \mathbb{P}(\ker r)$. Note that $F \subset \ker(r - s)$ and therefore $\ker(r - s) \not\subset \ker r$, whence $\ker r \cap \ker(r - s)$ is a hyperplane in $\ker(r - s)$. The group $\ker h$ is then easily seen to be isomorphic to $\ker(r) \cap \ker(s)$. Indeed, choose a non zero vector w in F; then the map which assigns to $u \in \ker(r) \cap \ker(s)$ the line with direction $w + u$ gives such an isomorphism onto $\ker h$.

Then:

- If $r = s$, $\mathcal{P}E$ is isomorphic to the abelian group $\ker(r) = \ker(s)$.
- If $r \neq s$, choose x such that r and s do not coincide on x and let P be the plane $F \oplus x$. The subgroup $\mathbb{P}(P) \setminus \{\ker r \cap P, \ker s \cap P\}$ of $\mathcal{P}E$ is isomorphic through h with $\mathbb{K}^*$. It thus defines a section of h. In that case $\mathcal{P}E$ is the group of dilations $(\ker(r) \cap \ker(s)) \rtimes \mathbb{K}^*$.

b) In the general case, let $d \in \mathbb{P}(F)$. Put $E_d^d = r^{-1}(d) \cap s^{-1}(d)$.

- The stabilizer $(\mathcal{P}E)_d^d$ is the group $\mathcal{P}E_d^d = \mathbb{P}(E_d^d) \setminus (\mathbb{P}(\ker r) \cup \mathbb{P}(\ker s))$ described above.
- The orbit of a line d is the set of $r(x)$ for $x \in \mathcal{P}E$ such that $s(x) = d$. It is therefore $\mathbb{P}(d + r(\ker s))$.

c) the following are equivalent:

(i) $(r, s) : E \to F \times F$ is onto;

(ii) $r(\ker s) = F$;

(iii) $(r - s) : E/F \to F$ is onto;

(iv) the groupoid $\mathcal{P}E$ has just one orbit.

d) When $r = s$, the groupoid $\mathcal{P}E$ is the product of the abelian group E/F by the space $\mathbb{P}(F)$.

When $r \neq s$, the groupoid $\widetilde{\mathcal{E}}$ is Morita equivalent to $\mathcal{E}$ since $F \setminus \{0\}$ meets all the orbits of $\mathcal{E}$.

If $\mathbb{K}$ is a locally compact field and $r \neq s$, the smooth groupoid $\mathcal{P}E$ is Morita equivalent to the groupoid crossed-product $\widetilde{\mathcal{E}} \rtimes \mathbb{K}^*$.

In all cases, when $\mathbb{K}$ is a locally compact field, $\mathcal{P}E$ is amenable.

The spherical groupoid. If the field is $\mathbb{R}$, we may just take the quotient by $\mathbb{R}_+^*$ instead of $\mathbb{R}^*$. We then obtain similarly the *spherical groupoid* $\mathcal{S}E = \mathbb{S}(E) \setminus (\mathbb{S}(\ker r) \cup \mathbb{S}(\ker s))$ where $\mathcal{S}^{(0)}(E) = \mathbb{S}(F) \subset \mathbb{S}(E)$.
The involutive automorphism $u \mapsto -u$ of E leads to a $\mathbb{Z}/2\mathbb{Z}$ action, by groupoid automorphisms on $\mathcal{S}E$. Since this action is free (and proper!), it follows that the quotient groupoid $\mathcal{P}E$ and the crossed product groupoid crossed product $\mathcal{S}E \rtimes \mathbb{Z}/2\mathbb{Z}$ are Morita equivalent. Thus $\mathcal{S}E$ is also amenable.
As for the projective case, if $(r,s) : E \to F \times F$ is onto, the groupoid $\mathcal{S}E$ has just one orbit. The stabilizer of $d \in \mathbb{S}(F)$ identifies with the group $(\ker r \cup \ker s) \rtimes \mathbb{R}_+^*$, and therefore the groupoid $\mathcal{S}E$ is Morita equivalent to the group $(\ker r \cup \ker s) \rtimes \mathbb{R}_+^*$.

4.2.2 Bundle groupoids

We may of course perform the constructions of section 4.2 (with say $\mathbb{K} = \mathbb{R}$) when E is a (real) vector bundle over a space V, F is a subbundle and r, s are bundle maps. We obtain respectively vector bundle groupoids, projective bundle groupoids and spherical bundle groupoids: $\mathcal{E}$, $(\mathcal{P}E, r, s)$ and $(\mathcal{S}E, r, s)$ which are respectively families of linear, projective and spherical groupoids.

Remarks 4.4. a) A vector bundle groupoid is just given by a bundle morphism $\alpha = (r - s) : E/F \to F$. It is isomorphic to the semi direct product $F \rtimes_\alpha E/F$.

b) All the groupoids defined here are amenable, since they are continuous fields of amenable groupoids (*cf.* [2, Prop. 5.3.4]).

The analytic index element $\mathrm{ind}_G \in KK(C_0(\mathfrak{A}^*G), C^*(G))$ of a vector bundle groupoid G is a KK-equivalence.
The groupoid G is a vector bundle E over a locally compact space X, $G^{(0)}$ is a vector subbundle F and G is given by a linear bundle map $(r - s) : E/F \to F$.

Proposition 4.5 (A Thom-Connes isomorphism)**.** *Let E be a vector bundle groupoid. Then $C^*(E)$ is KK-equivalent to $C_0(E)$. More precisely, the index $\mathrm{ind}_E : KK(C_0(\mathfrak{A}^*E), C^*(E))$ is invertible.*

Proof. Put $F = E^{(0)}$ and $H = E/F$. Then H acts on $C_0(F)$ and $C^*(E) = C_0(F) \rtimes H$.
We use the equivariant KK-theory of Le Gall (*cf.* [21]) $KK_H(A, B)$.
The Thom element of the complex bundle $H \oplus H$ defines an invertible element

$$t_H \in KK_H(C_0(X), C_0(H \oplus H)).$$

We deduce that, for every pair A, B of H algebras, the morphism

$$\tau_{C_0(H)} : KK_H(A, B) \to KK_H(A \otimes_{C_0(X)} C_0(H), B \otimes_{C_0(X)} C_0(H))$$

is an isomorphism. Its inverse is $x \mapsto t_H \otimes \tau_{C_0(H)}(x) \otimes t_H^{-1}$.
Denote by A_0 the $C_0(X)$ algebra A endowed with the trivial action of H. We have an isomorphism of H-algebras $u_A : C_0(H) \otimes_{C(X)} A \simeq C_0(H) \otimes_{C(X)} A_0$.
It follows that the restriction map $KK_H(A, B)$ to $KK_X(A, B)$ (associated to the groupoid morphism $X \to H$) is an isomorphism - compatible of course with the Kasparov product.
Let $v_A \in KK_H(A_0, A)$ be the element whose image in $KK_X(A_0, A)$ is the identity. The descent of $j_H(v_A) \in KK(C_0(H^*) \otimes_{C(X)} A, A \rtimes H)$ is a KK-equivalence. The proposition follows by letting $A = C_0(F)$. ☐

4.2.3 $\mathcal{VB}$ groupoids

Let us come to the general situation where Γ is a submanifold and a subgroupoid of G.
Recall from [35, 22] that a $\mathcal{VB}$ groupoid is a groupoid which is a vector bundle over a groupoid G. More precisely:

Definition 4.6. Let $G \overset{r_G,s_G}{\rightrightarrows} G^{(0)}$ be a groupoid. A $\mathcal{VB}$ groupoid over G is a vector bundle $p : E \to G$ with a groupoid structure $E \overset{r_E,s_E}{\rightrightarrows} E^{(0)}$ such that all the groupoid maps are linear vector bundle morphisms. This means that $E^{(0)} \subset E$ is a vector subbundle of the restriction of E to $G^{(0)}$ and that r_E, s_E, $x \mapsto x^{-1}$ and the composition are linear bundle maps. We also assume that the bundle maps $r_E : E \to r_G^*(E^{(0)})$ and $s_E : E \to s_G^*(E^{(0)})$ are surjective.

Note that we can associate a projective groupoid and a spherical groupoid to any $\mathcal{VB}$ groupoid $p : E \to \Gamma$: let $\widetilde{E}$ be the restriction of E to its open subset $\widetilde{E^{(0)}}$ which is the complement in $E^{(0)}$ of the zero-section $\Gamma^{(0)}$, and put $\mathcal{P}E = \widetilde{E}/\mathbb{R}^*$ and $\mathcal{S}E = \widetilde{E}/\mathbb{R}_+^*$.
The projection $\mathcal{N}_\Gamma^G \to \Gamma$ is a groupoid morphism and it is easily seen that $\mathcal{N}_\Gamma^G$ is a $\mathcal{VB}$ groupoid over Γ.
In fact every $\mathcal{VB}$ groupoid $E \to \Gamma$ can be seen as a normal groupoid: the normal groupoid to the inclusion $\Gamma \to E$.

4.3 Examples of deformation groupoids and blowup groupoids

We examine some particular cases of inclusions of groupoids $G_1 \subset G_2$. The various constructions of deformation to the normal cone and blow-up allow us to recover many well known groupoids.
As already noted in the introduction, our constructions immediately extend to the case where we restrict to a closed saturated subset of a smooth groupoid, in particular for manifolds with corners.

4.3.1 Inclusion $F \subset E$ of vector spaces

Let E be a real vector space - considered as a group - and F a vector subspace of E. The inclusion of groups $F \to E$ gives rise to a groupoid $DNC(E,F)$. Using any supplementary subspace of F in E, we may identify the groupoid $DNC(E,F)$ with $E \times \mathbb{R} \rightrightarrows \mathbb{R}$. Its C^*-algebra identifies then with $C_0(E^* \times \mathbb{R})$.
More generally, if F is a vector-subbundle of a vector bundle E over a manifold M (considered as a family of groups indexed by M), then the groupoid $DNC(E,F) \rightrightarrows M \times \mathbb{R}$ identifies with $E \times \mathbb{R}$ and its C^*-algebra is $C_0(E^* \times \mathbb{R})$.
Let $p_E : E \to M$ be a vector bundle over a manifold M and let V be a submanifold of M. Let $p_F : F \to V$ be a subbundle of the restriction of E to V. We use a tubular construction and find an open subset U of M which is a vector bundle $\pi : Q \to V$. Using π, we may extend F to a subbundle F_U of the restriction to F on U. Using that, we may identify $DNC(E,F)$ with the open subset $E \times \mathbb{R}^* \cup p_E^{-1}(U) \times \mathbb{R}$ of $E \times \mathbb{R}$. Its C^*-algebra identifies then with $C_0(E^* \times \mathbb{R}^* \cup p_{E^*}^{-1}(U) \times \mathbb{R})$.

4.3.2 Inclusion $G^{(0)} \subset G$: adiabatic groupoid

The deformation to the normal cone $DNC(G, G^{(0)})$ is the adiabatic groupoid G_{ad} ([31, 34]), it is obtained by using the deformation to the normal cone construction for the inclusion of $G^{(0)}$ as a Lie subgroupoid of G. The normal bundle $N_{G^{(0)}}^G$ is the total space of the Lie algebroid $\mathfrak{A}(G)$ of G. Note that its groupoid structure coincides with its vector bundle structure. Thus,

$$DNC(G, G^{(0)}) = G \times \mathbb{R}^* \cup \mathfrak{A}(G) \times \{0\} \rightrightarrows G^{(0)} \times \mathbb{R} .$$

We often denote $DNC(G, G^{(0)})$ by G_{ad} and G_{ad}^+, $G_{ad}^{[0,1]}$, $G_{ad}^{[0,1)}$ its restriction respectively to $\mathbb{R}_+$, to $[0,1]$ and to $[0,1)$.
Note that $Blup(G^{(0)}, G^{(0)}) = \emptyset = Blup_{r,s}(G, G^{(0)})$.
The particular case where G is the pair groupoid $M \times M$ is the original construction of the "tangent groupoid" of Alain Connes ([8]).

4.3.3 Gauge adiabatic groupoid

Start with a Lie groupoid $G \rightrightarrows V$.

Let $G \times (\mathbb{R} \times \mathbb{R}) \overset{\tilde{r},\tilde{s}}{\rightrightarrows} V \times \mathbb{R}$ be the product groupoid of G with the pair groupoid over $\mathbb{R}$.
First notice that since $V \times \{0\}$ is a codimension 1 submanifold in $V \times \mathbb{R}$, $SBlup(V \times \mathbb{R}, V \times \{0\})$ is canonically isomorphic to $V \times (\mathbb{R}_- \sqcup \mathbb{R}_+)$. Then $SBlup_{\tilde{r},\tilde{s}}(G \times (\mathbb{R} \times \mathbb{R}), V \times \{(0,0)\})_{V\times\mathbb{R}_+}^{V\times\mathbb{R}_+}$ is the semi-direct product groupoid $G_{ad}(V \times \mathbb{R}_+) \rtimes \mathbb{R}_+^*$:

$$SBlup_{\tilde{r},\tilde{s}}(G \times (\mathbb{R} \times \mathbb{R}), V \times \{(0,0)\})_{V\times\mathbb{R}_+}^{V\times\mathbb{R}_+} = G_{ad}^+ \rtimes \mathbb{R}^* \rightrightarrows V \times \mathbb{R}_+ \ .$$

In other words, $SBlup_{\tilde{r},\tilde{s}}(G \times (\mathbb{R} \times \mathbb{R}), V \times \{(0,0)\})_{V\times\mathbb{R}_+}^{V\times\mathbb{R}_+}$ is the gauge adiabatic groupoid used in [11]; we often denote it by G_{ga} .
Indeed, as $G \times (\mathbb{R} \times \mathbb{R})$ is a vector bundle over G, $DNC(G \times (\mathbb{R} \times \mathbb{R}), V \times \{(0,0)\}) \simeq DNC(G,V) \times \mathbb{R}^2$ (remark 3.4.e). Under this identification, the gauge action of $\mathbb{R}^*$ is given by $\lambda.(w,t,t') = (\lambda.w, \lambda^{-1}t, \lambda^{-1}t')$. The maps $DNC(\tilde{s})$ and $DNC(\tilde{r})$ are respectively $(w,t,t') \mapsto (DNC(s)(w), t')$ and $(w,t,t') \mapsto (DNC(r)(w), t)$. It follows that $SBlup_{\tilde{r},\tilde{s}}(G \times (\mathbb{R} \times \mathbb{R}), V \times \{(0,0)\})$ is the quotient by the diagonal action of $\mathbb{R}_+^*$ of the open subset $DNC(G,V) \times (\mathbb{R}^*)^2$ of $DNC_+(G,V) \times \mathbb{R}^2$.
According to the description of the groupoid of a group action on a groupoid given in [13, section 2.3] it is isomorphic to $DNC(G,V)_+ \rtimes \mathbb{R}_+^* \times \{-1,+1\}^2$ where $\{-1,+1\}^2$ is the pair groupoid over $\{-1,+1\}$.

4.3.4 Inclusion of a transverse submanifold of the unit space

Let G be a Lie groupoid with set of objects $M = G^{(0)}$ and let V be a submanifold of M. We now study the special case of normal and blowup groupoids $DNC(G,V)$ and $Blup_{r,s}(G,V)$ (as well as $SBlup_{r,s}(G,V)$) associated to the groupoid morphism $V \to G$.
Put $\mathring{M} = M \setminus V$. Let $N = N_V^G$ and $N' = N_V^M$ be the normal bundles. We identify N' with a subbundle of N by means of the inclusion $M \subset G$. The submersions $r, s : G \to M$ give rise to bundle morphisms $r^N, s^N : N \to N'$ that are sections of the inclusion $N' \to N$. By construction, using remark 4.2.a), the groupoid $DNC(G,V)$ is the union of $G \times \mathbb{R}^*$ with the family of linear groupoids $\mathcal{N}_{r^N,s^N}(N)$. It follows that $Blup_{r,s}(G,V)$ is the union of $G_{\mathring{M}}^{\mathring{M}}$ with the family $(\mathcal{P}N, r^N, s^N)$ of projective groupoids.
If V is transverse to G, the bundle map $r^N - s^N : N = N_V^G \to N' = N_V^M$ is surjective; it follows that

- $\mathcal{L}_{r^N,s^N}(N)$ identifies with the pull-back groupoid $\big(\mathfrak{A}(G_V^V)\big)_q^q$ where $q : N' \to V$ is the projection,
- $(\mathcal{P}N, r^N, s^N)$ with the pull-back groupoid $\big(\mathfrak{A}(G_V^V) \rtimes \mathbb{R}^*\big)_\rho^\rho$ where $\rho : \mathbb{P}(N') \to V$ is the projection,
- $(\mathcal{S}N, r^N, s^N)$ with the pull-back groupoid $\big(\mathfrak{A}(G_V^V) \rtimes \mathbb{R}_+^*\big)_p^p$ where $p : \mathbb{S}(N') \to V$ is the projection.

Let us give a local description of these groupoids in the neighborhood of the transverse submanifold V.
Put $\mathring{G} = G_{M\setminus V}^{M\setminus V}$. Upon arguing locally, we can assume that V is compact.
By Remark 2.2, V admits a tubular neighborhood $W \simeq N_V^M$ such that G_W^W is the pull back of G_V^V by the retraction $q : W \to V$.
The normal groupoid $DNC(G_W^W, V)$ identifies with the pull back groupoid $(DNC(G_V^V, V))_q^q$ of the adiabatic deformation $DNC(G_V^V, V) = (G_V^V)_{ad}$ by the map $q : N_V^M \to V$.
The (spherical) blowup groupoid $SBlup_{r,s}(G_W^W, V)$ identifies with the pull back groupoid $(DNC_+(G_V^V, V) \rtimes \mathbb{R}_+^*)_p^p$ of the gauge adiabatic deformation $DNC_+(G_V^V, V) \rtimes \mathbb{R}_+^* = (G_V^V)_{ga}$ by the map $p : \mathbb{S}N_V^M \to V$.

In order to get $SBlup_{r,s}(G,V)$, we then may glue $(DNC_+(G_V^V,V)\rtimes\mathbb{R}_+^*)_p^p$ with $\mathring{G}$ in their common open subset $\big((G_V^V)_q^q\big)_{W\setminus V}^{W\setminus V}\simeq G_{W\setminus V}^{W\setminus V}$.

4.3.5 Inclusion $G_V^V\subset G$ for a transverse hypersurface V of G: b-groupoid

If V is a hypersurface of M, the blowup $Blup(M\times M,V\times V)$ is just the construction of Melrose of the b-space. Its open subspace $Blup_{r,s}(M\times M,V\times V)$ is the associated groupoid of Monthubert [29, 30]. Moreover, if G is a groupoid on M and V is transverse to G we can form the restriction groupoid $G_V^V\subset G$ which is a submanifold of G. The corresponding blow up construction $Blup_{r,s}(G,G_V^V)$ identifies with the fibered product $Blup_{r,s}(M\times M,V\times V)\times_{M\times M}G$ (*cf.* remark 3.4.f).

Iterating (at least locally) this construction, we obtain the b-groupoid of Monthubert for manifolds with corners - *cf.* [29, 30].

Remark 4.7. The groupoid $Blup_{r,s}(G,V)$ corresponds to inflating all the distances when getting close to V.

The groupoid $Blup_{r,s}(G,G_V^V)$ is a kind of *cylindric deformation groupoid* which is obtained by pushing the boundary V at infinity but keeping the distances along V constant.

Remark 4.8. Intermediate examples between these two are given by a subgroupoid $\Gamma\rightrightarrows V$ of G_V^V. In the case where $G=M\times M$, such a groupoid Γ is nothing else than the holonomy groupoid $Hol(V,\mathcal{F})$ of a regular foliation $\mathcal{F}$ of V (with trivial holonomy groups). The groupoid $SBlup_{r,s}(M\times M,Hol(V,\mathcal{F}))$ is a holonomy groupoid of a singular foliation of M: the sections of its algebroid. Its leaves are $M\setminus V$ and the leaves of $(V,\mathcal{F})$. The corresponding calculus, when M is a manifold with a boundary V is Rochon's generalization ([36]) of the ϕ calculus of Mazzeo and Melrose ([24]).

Iterating (at least locally) this construction, we obtain the holonomy groupoid associated to a stratified space in [10].

4.3.6 Inclusion $G_V^V\subset G$ for a saturated submanifold V of G: shriek map for immersion

Suppose now that V is saturated thus $G_V^V=G_V=G^V$.
In such a situation the groupoid G_V^V acts on the normal bundle $N_{G_V^V}^G=r^*(N_V^{G^{(0)}})$ and $DNC(G,G_V^V)\rightrightarrows DNC(G^{(0)},V)$ coincides with the normal groupoid of the immersion $\varphi:G_V^V\to G$. This construction was defined in the case of foliation groupoids in [19, section 3] and was used in order to define $\varphi_!$ as its associated KK-element.

4.3.7 Inclusion $G_1\subset G_2$ with $G_1^{(0)}=G_2^{(0)}$

This is the case for the tangent and adiabatic groupoid discussed above. Let us mention two other kinds of this situation[2] that can be encountered in the literature:

a) The case of a subfoliation $\mathcal{F}_1$ of a foliation $\mathcal{F}_2$ on a manifold M: shriek map for submersion. As pointed out in remark 3.19 of [19] the corresponding deformation groupoid $DNC(G_2,G_1)$ gives an alternative construction of the element $\varphi_!$ where $\varphi:M/\mathcal{F}_1\to M/\mathcal{F}_2$ is a submersion of leaf spaces.

b) The case of a subgroup of a Lie group.

  - If K is a maximal compact subgroup of a reductive Lie group G, the connecting map associated to the exact sequence of $DNC(G,K)$ is the Dirac extension mapping the twisted K-theory of K to the K-theory of $C_r^*(G)$ (see [16]).
  - In the case where Γ is a dense (non amenable) countable subgroup of a compact Lie group K, the groupoid $DNC(K,\Gamma)$ was used in [17] in order to produce a Hausdorff groupoid for which the Baum-Connes map is not injective.

[2]Note that in this case $Blup(G_2^{(0)},G_1^{(0)})=\emptyset$, whence $Blup_{r,s}(G_2,G_1)=\emptyset$.

4.3.8 Wrong way functoriality

Let $f : G_1 \to G_2$ be a morphism of Lie groupoids. If f is an (injective) immersion the construction of $DNC_+(G_2, G_1)$ gives rise to a short exact sequence

$$0 \longrightarrow C^*(G_2 \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G_2, G_1)) \longrightarrow C^*(\mathcal{N}_{G_1}^{G_2}) \longrightarrow 0.$$

and consequently to a connecting map from the K-theory of the C^*-algebra of the groupoid $\mathcal{N}_{G_1}^{G_2}$, which is a $\mathcal{VB}$ groupoid over G_1, to the K-theory of $C^*(G_2)$. This wrong way functoriality map will be discussed in the next section.

More generally let $\mathcal{G} = G_1^{(0)} \times G_2 \times G_1^{(0)}$ be the product of G_2 by the pair groupoid of $G_1^{(0)}$. Assume that the map $x \mapsto (r(x), f(x), s(x))$ is an immersion from $G_1 \to \mathcal{G}$.
The above construction gives a map from $K_*(C^*(\mathcal{N}_{G_1}^{\mathcal{G}}))$ to $K_*(C^*(\mathcal{G}))$ which is isomorphic to $K_*(C^*(G_2))$ since the groupoids G_2 and $\mathcal{G}$ are canonically Morita equivalent.

5 The C^*-algebra of a deformation and of a blowup groupoid, full symbol and index map

Let $G \rightrightarrows M$ be a Lie groupoid and $\Gamma \rightrightarrows V$ a Lie subgroupoid of G. The groupoids $DNC_+(G, \Gamma)$ and $SBlup_{r,s}(G, \Gamma)$ that we constructed admit the closed saturated subsets $N_V^M \times \{0\}$ and $\mathbb{S}N_V^M$ respectively. We apply results of [13] in order to compute various KK-elements involved in index theory for such situation.
In order to shorten the notation we put $\mathring{M} = M \setminus V$.
The *full symbol algebras* are the quotient C^*-algebras:

- $\Sigma_{DNC_+}(G, \Gamma) = \Psi^*(DNC_+(G, \Gamma))/C^*(G \times \mathbb{R}+^*)$;
- $\Sigma_{SBlup}(G, \Gamma) = \Psi^*(SBlup_{r,s}(G, \Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}})$.

They give rise to the exact sequences

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{r,s}(G, \Gamma)) \longrightarrow C^*(\mathcal{S}N_\Gamma^G) \longrightarrow 0 \qquad E_{SBlup}^\partial$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G, \Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \qquad E_{DNC_+}^\partial$$

of groupoid C^*-algebras as well as index type ones

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow \Psi^*(SBlup_{r,s}(G, \Gamma)) \longrightarrow \Sigma_{SBlup}(G, \Gamma) \longrightarrow 0 \qquad E_{SBlup}^{\widetilde{\mathrm{ind}}}$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow \Psi^*(DNC_+(G, \Gamma)) \longrightarrow \Sigma_{DNC_+}(G, \Gamma) \longrightarrow 0 \qquad E_{DNC_+}^{\widetilde{\mathrm{ind}}}$$

We will compare the exact sequences given by DNC and by $SBlup$.
If V is $\mathfrak{A}G$*-small* (see notation 5.5 below), we will show that, in a sense, DNC and $SBlup$ give rise to equivalent exact sequences - both for the “connecting” ones and for the “index” ones.
We will then compare these elements with a coboundary construction.
We will compute these exact sequences when $\Gamma = V \subset M$. Finally, we will study a refinement of these constructions using relative K-theory.

5.1 “DNC” versus “Blup”

Let $\Gamma \rightrightarrows V$ be a submanifold and a subgroupoid of a Lie-groupoid $G \rightrightarrows M$. We will further assume that the groupoid Γ is amenable. We still put $\mathring{M} = M \setminus V$ and let $\mathring{\mathcal{N}}_\Gamma^G$ be the restriction of the groupoid $\mathcal{N}_\Gamma^G$ to the open subset $\mathring{N}_V^M = N_V^M \setminus V$ of its unit space N_V^M.

5.1.1 The connecting element

As the groupoid Γ is amenable we have exact sequences both for the reduced and for the maximal C^*-algebras:

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}N_\Gamma^G) \longrightarrow 0 \qquad E^{\partial}_{SBlup}$$

and

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \qquad E^{\partial}_{DNC_+}$$

By amenability, these exact sequences admit completely positive cross sections and therefore define elements $\partial^{G,\Gamma}_{SBlup} \in KK^1(C^*(\mathring{\mathcal{N}}_\Gamma^G/\mathbb{R}_+^*), C^*(G_{\mathring{M}}^{\mathring{M}}))$ and $\partial^{G,\Gamma}_{DNC_+} \in KK^1(C^*(\mathcal{N}_\Gamma^G), C^*(G\times\mathbb{R}_+^*))$.
With the notation of section 4.1, write DNC_+ for DNC restricted to $\mathbb{R}_+$ and $\widetilde{DNC}_+$ for $\widetilde{DNC}$ restricted to $\mathbb{R}_+$.
By [13, section 5.3], we have a diagram where the vertical arrows are KK^1-equivalences and the squares commute in KK-theory.

$$\begin{tikzcd}
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] \arrow[d, dash, "\beta'"] & C^*(SBlup_{r,s}(G,\Gamma)) \arrow[r] \arrow[d, dash, "\beta"] & C^*(\mathring{\mathcal{N}}_\Gamma^G/\mathbb{R}_+^*) \arrow[r] \arrow[d, dash, "\beta''"] & 0 & E^{\partial}_{SBlup} \\
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] & C^*(\widetilde{DNC}_+(G,\Gamma)) \arrow[r] & C^*(\mathring{\mathcal{N}}_\Gamma^G) \arrow[r] & 0 & E^{\partial}_{\widetilde{DNC}_+}
\end{tikzcd}$$

Denote by $\partial^{G,\Gamma}_{\widetilde{DNC}_+}$ the connecting element associated to $E^{\partial}_{\widetilde{DNC}_+}$. We thus have, according to [13, prop. 5.3]:

Fact 5.1. $\partial^{G,\Gamma}_{SBlup} \otimes \beta' = -\beta'' \otimes \partial^{G,\Gamma}_{\widetilde{DNC}_+} \in KK^1(C^*(\mathcal{S}N_\Gamma^G), C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*))$.

We also have a commutative diagram where the vertical maps are inclusions:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \arrow[r] \arrow[d, "j'"] & C^*(\widetilde{DNC}_+(G,\Gamma)) \arrow[r] \arrow[d, "j"] & C^*(\mathring{\mathcal{N}}_\Gamma^G) \arrow[r] \arrow[d, "j''"] & 0 \\
0 \arrow[r] & C^*(G \times \mathbb{R}_+^*) \arrow[r] & C^*(DNC_+(G,\Gamma)) \arrow[r] & C^*(\mathcal{N}_\Gamma^G) \arrow[r] & 0
\end{tikzcd} \qquad (5.1)$$

We thus find

Fact 5.2. $(j'')^*(\partial^{G,\Gamma}_{DNC_+}) = j'_*(\partial^{G,\Gamma}_{\widetilde{DNC}_+}) \in KK^1(C^*(\mathring{\mathcal{N}}_\Gamma^G), C^*(G\times\mathbb{R}_+^*))$.

5.1.2 The full symbol index

We now compare the elements $\widetilde{\mathrm{ind}}^{G,\Gamma}_{SBlup} \in KK^1(\Sigma_{SBlup}(G,\Gamma), C^*(G_{\mathring{M}}^{\mathring{M}}))$ and
$\widetilde{\mathrm{ind}}^{G,\Gamma}_{DNC_+} \in KK^1(\Sigma_{DNC_+}(G,\Gamma), C^*(G\times\mathbb{R}_+^*))$ defined by the semi-split exact sequences:

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow \Psi^*(SBlup_{r,s}(G,\Gamma)) \longrightarrow \Sigma_{SBlup}(G,\Gamma) \longrightarrow 0 \qquad E^{\widetilde{\mathrm{ind}}}_{SBlup}$$

and

$$0 \longrightarrow C^*(G\times\mathbb{R}_+^*) \longrightarrow \Psi^*(DNC_+(G,\Gamma)) \longrightarrow \Sigma_{DNC_+}(G,\Gamma) \longrightarrow 0 \qquad E^{\widetilde{\mathrm{ind}}}_{DNC_+}$$

Put $\Sigma_{\widetilde{DNC}_+}(G,\Gamma)=\Psi^*(\widetilde{DNC}_+(G,\Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*)$. By [13, prop. 5.4], we have a diagram where the vertical arrows are KK^1-equivalences and the squares commute in KK-theory.

$$\begin{tikzcd} 0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] \arrow[d, dash, "\beta'"] & \Psi^*(SBlup_{r,s}(G,\Gamma)) \arrow[r] \arrow[d, dash, "\beta_\Psi"] & \Sigma_{SBlup}(G,\Gamma) \arrow[r] \arrow[d, dash, "\beta_\Sigma"] & 0 \\ 0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*) \arrow[r] & \Psi^*(\widetilde{DNC}_+(G,\Gamma)) \arrow[r] & \Sigma_{\widetilde{DNC}_+}(G,\Gamma) \arrow[r] & 0 \end{tikzcd}$$

We let $\widetilde{\mathrm{ind}}_{\widetilde{DNC}_+}^{G,\Gamma}\in KK^1(\Sigma_{\widetilde{DNC}_+}(G,\Gamma),C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*))$ be the connecting map induced by the second exact sequence. We thus have:

Fact 5.3. $\widetilde{\mathrm{ind}}_{SBlup}^{G,\Gamma}\otimes\beta'=-\beta_\Sigma\otimes\widetilde{\mathrm{ind}}_{\widetilde{DNC}_+}^{G,\Gamma}\in KK^1(\Sigma_{SBlup}(G,\Gamma),C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*))$.

We also have a commutative diagram where the vertical maps are inclusions:

$$\begin{tikzcd} 0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*) \arrow[r] \arrow[d, "j'"] & \Psi^*(\widetilde{DNC}_+(G,\Gamma)) \arrow[r] \arrow[d, "j_\Psi"] & \Sigma_{\widetilde{DNC}_+}(G,\Gamma) \arrow[r] \arrow[d, "j_\Sigma"] & 0 \\ 0 \arrow[r] & C^*(G\times\mathbb{R}_+^*) \arrow[r] & \Psi^*(DNC_+(G,\Gamma)) \arrow[r] & \Sigma_{DNC_+}(G,\Gamma) \arrow[r] & 0 \end{tikzcd} \tag{5.2}$$

We thus find:

Fact 5.4. $j_\Sigma^*(\widetilde{\mathrm{ind}}_{DNC_+}^{G,\Gamma})=j'_*(\widetilde{\mathrm{ind}}_{\widetilde{DNC}_+}^{G,\Gamma})\in KK^1(\Sigma_{\widetilde{DNC}_+}(G,\Gamma),C^*(G\times\mathbb{R}_+^*))$.

5.1.3 When V is $\mathfrak{A}G$-small

If V is small in each G orbit, *i.e.* if the Lebesgue measure (in the manifold G^x) of G_V^x is 0 for every x, it follows from prop. 5.6 below that the inclusion $i: C^*(G_{\mathring{M}}^{\mathring{M}})\hookrightarrow C^*(G)$ is an isomorphism. Also, if $\mathring{M}$ meets all the orbits of G, the inclusion i is a Morita equivalence. In these cases $\partial_{DNC_+}^{G,\Gamma}$ determines $\partial_{SBlup}^{G,\Gamma}$.

Definition 5.5. We will say that V is *$\mathfrak{A}G$-small* if for every $x\in V$, the composition $\mathfrak{A}G_x\xrightarrow{\natural_x}T_xM\longrightarrow(N_V^M)_x$ is not the zero map

If V is $\mathfrak{A}G$-small, then the orbits of the groupoid $\mathcal{N}_\Gamma^G$ are never contained in the 0 section, *i.e.* they meet the open subset $\mathring{N}_V^M$, and in fact the set $V\times\{0\}$ is small in every orbit of the groupoid $DNC(G,\Gamma)$. It follows that the map j is an isomorphism - as well of course as j' and j'' of diagram (5.1). In that case, $\partial_{DNC_+}^{G,\Gamma}$ and $\partial_{SBlup}^{G,\Gamma}$ correspond to each other under these isomorphisms.

Proposition 5.6. *(cf. [18, 12]) Let $\mathcal{G}\rightrightarrows Y$ be a Lie groupoid and let $X\subset Y$ be a (locally closed) submanifold. Assume that, for every $x\in X$, the composition $\mathfrak{A}\mathcal{G}_x\xrightarrow{\natural_x}T_xY\longrightarrow(N_X^Y)_x$ is not the zero map. Then the inclusion $C^*(G_{Y\setminus X}^{Y\setminus X})\to C^*(G)$ is an isomorphism.*

Proof. For every $x\in V$, we can find a neighborhood U of $x\in M$, a section X of $\mathfrak{A}G$ such that, for every $y\in U$, $\natural_y(X(y))\neq 0$ and, if $y\in U\cap V$, $\natural_y(X(y))\notin T_y(V)$. Denote by $\mathcal{F}$ the foliation of U associated with the vector field X. It follows from [18] that $C_0(U\setminus V)C^*(U,\mathcal{F})=C^*(U,\mathcal{F})$. As $C^*(U,\mathcal{F})$ acts in a non degenerate way on the Hilbert-$C^*(G)$ module $C^*(G^U)$, we deduce that $C_0(U\setminus V)C^*(G^U)=C^*(G_U)$. We conclude using a partition of the identity argument that $C_c(M\setminus V)C^*(G)=C_c(M)C^*(G)$, whence $C_0(M\setminus V)C^*(G)=C_0(M)C^*(G)=C^*(G)$. $\square$

Proposition 5.7. *We assume that Γ is amenable and that V is $\mathfrak{A}G$-small.*
Then, the inclusions $j_\Sigma : \Sigma_{\widetilde{DNC}_+}(G,\Gamma) \to \Sigma_{DNC_+}(G,\Gamma)$, $j_\Psi : \Psi^*(\widetilde{DNC}_+(G,\Gamma)) \to \Psi^*(DNC_+(G,\Gamma))$ and $j_{symb} : C_0(\mathbb{S}\mathfrak{A}^*(\widetilde{DNC}_+(G,\Gamma))) \to C_0(\mathbb{S}\mathfrak{A}^*(DNC_+(G,\Gamma)))$ are KK-equivalences.

Proof. We have a diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & C^*(\widetilde{DNC}_+(G,\Gamma)) \arrow[r] \arrow[d, "j"] & \Psi^*(\widetilde{DNC}_+(G,\Gamma)) \arrow[r] \arrow[d, "j_\Psi"] & C_0(\mathbb{S}\mathfrak{A}^*(\widetilde{DNC}_+(G,\Gamma)) \arrow[r] \arrow[d, "j_{symb}"] & 0 \\
0 \arrow[r] & C^*(DNC_+(G,\Gamma)) \arrow[r] & \Psi^*(DNC_+(G,\Gamma)) \arrow[r] & C_0(\mathbb{S}\mathfrak{A}^*(DNC_+(G,\Gamma)) \arrow[r] \arrow[d] & 0 \\
 & & & C_0(\mathbb{S}\mathfrak{A}^*G_{|V} \times \mathbb{R}_+) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

As j is an equality, we find an exact sequence

$$0 \longrightarrow \Psi^*(\widetilde{DNC}_+(G,\Gamma)) \xrightarrow{j_\Psi} \Psi^*(DNC_+(G,\Gamma)) \longrightarrow C_0(\mathbb{S}\mathfrak{A}^*G_{|V} \times \mathbb{R}_+) \longrightarrow 0.$$

As $j' : C^*(G^{\mathring{M}}_{\mathring{M}} \times \mathbb{R}_+^*) \to C^*(G \times \mathbb{R}_+^*)$ is also an equality, we find (using diagram (5.2)) an exact sequence

$$0 \longrightarrow \Sigma_{\widetilde{DNC}_+}(G,\Gamma)) \xrightarrow{j_\Sigma} \Sigma_{DNC_+}(G,V) \longrightarrow C_0(\mathbb{S}\mathfrak{A}^*G_{|V} \times \mathbb{R}_+) \longrightarrow 0.$$

As the algebra $C_0(\mathbb{S}\mathfrak{A}^*G_{|V} \times \mathbb{R}_+)$ is contractible, we deduce that j_{symb} and then j_Ψ and j_Σ are KK-equivalences. □

As a summary of these considerations, we find:

Theorem 5.8. *Let $G \rightrightarrows M$ be a Lie groupoid and $\Gamma \rightrightarrows V$ a Lie subgroupoid of G. Assume that Γ is amenable and put $\mathring{M} = M \setminus V$. Let $i : C^*(G^{\mathring{M}}_{\mathring{M}}) \to C^*(G)$ be the inclusion. Put $\hat{\beta}'' = j''_*(\beta'') \in KK^1(C^*(\mathcal{S}N_\Gamma^G), C^*(\mathcal{N}_\Gamma^G))$ and $\hat{\beta}_\Sigma = (j_\Sigma)_*(\beta_\Sigma) \in KK^1(\Sigma_{SBlup}(G,\Gamma), \Sigma_{DNC_+}(G,V))$.*

a) *We have equalities*

- $\partial^{G,\Gamma}_{SBlup} \otimes [i] = \hat{\beta}'' \otimes \partial^{G,\Gamma}_{DNC_+} \in KK^1(C^*(\mathcal{S}N_\Gamma^G), C^*(G))$ *and*
- $\widetilde{\mathrm{ind}}^{G,\Gamma}_{SBlup} \otimes [i] = \hat{\beta}_\Sigma \otimes \widetilde{\mathrm{ind}}^{G,\Gamma}_{DNC_+} \in KK^1(C^*(\Sigma_{SBlup}(G,\Gamma), C^*(G))$

b) *If V is $\mathfrak{A}G$-small, then i is an isomorphism and the elements $\hat{\beta}''$ and $\hat{\beta}_\Sigma$ are invertible.*

□

5.2 The KK-element associated with DNC

The connecting element $\partial^{G,\Gamma}_{DNC_+}$ can be expressed in the following way: let $\mathcal{G}$ be the restriction of $DNC(G,\Gamma)$ to $[0,1]$, *i.e.* $\mathcal{G} = DNC^{[0,1]}(G,\Gamma) = \mathcal{N}_\Gamma^G \times \{0\} \cup G \times (0,1]$. We have a semi-split exact sequence:

$$0 \to C^*(G \times (0,1]) \to C^*(\mathcal{G}) \xrightarrow{\mathrm{ev}_0} C^*(\mathcal{N}_\Gamma^G) \to 0\ .$$

As $C^*(G \times (0,1])$ is contractible, ev_0 is a KK-equivalence. Let $ev_1 : C^*(\mathcal{G}) \to C^*(G)$ be evaluation at 1 and let $\delta_\Gamma^G = [ev_0]^{-1} \otimes [ev_1] \in KK(C^*(\mathcal{N}_\Gamma^G), C^*(G))$. Let $[Bott] \in KK^1(\mathbb{C}, C_0(\mathbb{R}_+^*))$ be the Bott element. We find

Fact 5.9. $\partial^{G,\Gamma}_{DNC_+} = \delta^G_\Gamma \underset{\mathbb{C}}{\otimes} [Bott]$.

Consider now the groupoid $\mathcal{G}^{[0,1]}_{ad}$. It is a family of groupoids indexed by $[0,1]\times[0,1]$:

- its restriction to $\{s\}\times[0,1]$ for $s\neq 0$ is $G^{[0,1]}_{ad}$;
- its restriction to $\{0\}\times[0,1]$ is $(\mathcal{N}^G_\Gamma)^{[0,1]}_{ad}$;
- its restriction to $[0,1]\times\{s\}$ for $s\neq 0$ is $\mathcal{G} = DNC_{[0,1]}(G,\Gamma)$;
- its restriction to $[0,1]\times\{0\}$ is the algebroid $\mathfrak{A}\mathcal{G} = DNC_{[0,1]}(\mathfrak{A}G,\mathfrak{A}\Gamma)$ of $\mathcal{G}$.

For every locally closed subset $X\subset[0,1]\times[0,1]$, denote by $\mathcal{G}^X_{ad}$ the restriction of $\mathcal{G}^{[0,1]}_{ad}$ to X.
For every closed subset $X\subset[0,1]\times[0,1]$, denote by $q_X : C^*(\mathcal{G}^{[0,1]}_{ad}) \to C^*(\mathcal{G}^X_{ad})$ the restriction map.
We thus have the following commutative diagram:

$$\begin{tikzcd}
C^*(\mathcal{N}^G_\Gamma) \arrow[rr, dotted, bend left, "\delta^G_\Gamma"] & DNC_{[0,1]}(G,\Gamma) \arrow[l] \arrow[r] & C^*(G) \\
C^*((\mathcal{N}^G_\Gamma)^{[0,1]}_{ad}) \arrow[u] \arrow[d] & C^*(\mathcal{G}^{[0,1]}_{ad}) \arrow[l, "q_{\{0\}\times[0,1]}"'] \arrow[r, "q_{\{1\}\times[0,1]}"] \arrow[u, "q_{[0,1]\times\{1\}}"] \arrow[d, "q_{[0,1]\times\{0\}}"] \arrow[ul, "q_{(0,1)}"] \arrow[ur, "q_{(1,1)}"'] \arrow[dl, "q_{(0,0)}"'] \arrow[dr, "q_{(1,0)}"] & C^*(G^{[0,1]}_{ad}) \arrow[u] \arrow[d] \\
C^*(N^{\mathfrak{A}G}_{\mathfrak{A}\Gamma}) \arrow[u, dotted, bend left=60, "\mathrm{ind}_{\mathcal{N}^G_\Gamma}"] \arrow[rr, dotted, bend right, "\delta^{\mathfrak{A}G}_{\mathfrak{A}\Gamma}"'] & C^*(DNC_{[0,1]}(\mathfrak{A}G,\mathfrak{A}\Gamma)) \arrow[l] \arrow[r] & C_0(\mathfrak{A}^*G) \arrow[uu, dotted, bend right=60, "\mathrm{ind}_G"']
\end{tikzcd}$$

For every locally closed subset $T\subset[0,1]$, the C^*-algebras $C^*(\mathcal{G}^{(0,1]\times T}_{ad})$ and $C^*(\mathcal{G}^{T\times(0,1]}_{ad})$ are null homotopic as well as $C^*(\mathcal{G}^{[0,1]^2\setminus\{0,0)\}}_{ad})$. It follows that $q_{\{0\}\times[0,1]}$, $q_{[0,1]\times\{0\}}$ and $q_{\{(0,0)\}}$ are KK-equivalences.
Now $[q_{(0,0)}]^{-1}\otimes[q_{(0,1)}] = \mathrm{ind}_{\mathcal{N}^G_\Gamma}$ and it follows that $[q_{(0,0)}]^{-1}\otimes[q_{(1,1)}] = \mathrm{ind}_{\mathcal{N}^G_\Gamma}\otimes\delta^G_\Gamma$.
In the same way, $[q_{(0,0)}]^{-1}\otimes[q_{(1,0)}] = \delta^{\mathfrak{A}G}_{\mathfrak{A}\Gamma}$ and it follows that $[q_{(0,0)}]^{-1}\otimes[q_{(1,1)}] = \delta^{\mathfrak{A}G}_{\mathfrak{A}\Gamma}\otimes\mathrm{ind}_G$.
Finally, it follows from example 4.3.1 that $\delta^{\mathfrak{A}G}_{\mathfrak{A}\Gamma}$ is associated with a morphism $\varphi : C_0(\mathfrak{A}^*(\mathcal{N}^G_V)) \hookrightarrow C_0(\mathfrak{A}^*G)$ corresponding to an inclusion of $\mathfrak{A}^*(\mathcal{N}^G_\Gamma)$ in $\mathfrak{A}^*G$ as a tubular neighborhood.
We thus have established:

Fact 5.10. $\mathrm{ind}_{\mathcal{N}^G_\Gamma}\otimes\delta^G_\Gamma = [\varphi]\otimes\mathrm{ind}_G$.

Similar groupoids and commutative diagrams for the special case of $\mathcal{V}$ being the normal bundle of the inclusion of a manifold M into some $\mathbb{R}^n$, $G=\mathcal{V}\times\mathcal{V}$ and $\Gamma = \mathcal{V}\underset{M}{\times}\mathcal{V}$ appeared in section 6.1 of [9] in order to give a K-theoretical proof using groupoids of the Atiyah-Singer index theorem.

5.3 The case of a submanifold of the space of units

Let G be a Lie groupoid with objects M and let $\Gamma = V\subset M$ be a closed submanifold of M. In this section, we push further the computations the connecting maps and indices *i.e.* the connecting maps of the exact sequences E^∂_{SBlup}, $E^\partial_{DNC_+}$, $E^{\widetilde{\mathrm{ind}}}_{SBlup}$ and $E^{\widetilde{\mathrm{ind}}}_{DNC_+}$.

5.3.1 Connecting map and index map

From [13, propositions 4.1, 4.6, 4.7] and fact 5.10, we find

Proposition 5.11. a) *The index element* $\mathrm{ind}_{\mathcal{N}_V^G} \in KK(C_0(\mathfrak{A}^*N_V^G), C^*(\mathcal{N}_V^G))$ *is invertible.*

b) *The inclusion* $j : \Sigma_{N_V^M \times \{0\}}(DNC_+(G,V)) \hookrightarrow \Sigma_{DNC_+}(G,V)$ *is invertible in* KK*-theory.*

c) *The* C^**-algebra* $\Sigma_{DNC_+}(G,V)$ *is naturally* KK^1*-equivalent with the mapping cone* C_χ *of the map* $\chi : C_0(\mathfrak{A}^*G \times \mathbb{R}_+^*) \to C_0(DNC_+(M,V))$ *defined by* $\chi(f)(x) = \begin{cases} f(x,0) & \text{if } x \in M \times \mathbb{R}_+^* \\ 0 & \text{if } x \in N_V^M. \end{cases}$

d) *The connecting element* $\partial_{DNC_+}^{G,V} \in KK^1(C^*(\mathcal{N}_V^G), C^*(G \times \mathbb{R}_+^*)) = KK(C^*(\mathcal{N}_V^G), C^*(G))$ *is* $\delta_V^G = \mathrm{ind}_{\mathcal{N}_V^G}^{-1} \otimes [\varphi] \otimes \mathrm{ind}_G$ *where* $\varphi : C_0(\mathfrak{A}^*N_V^G) \to C_0(\mathfrak{A}^*G)$ *is the inclusion using the tubular neighborhood construction.*

e) *Under the* KK^1 *equivalence of c), the full index element*

$$\widetilde{\mathrm{ind}}_{DNC_+}^{G,V} \in KK^1(\Sigma_{DNC_+}(G,V), C^*(G \times \mathbb{R}_+^*)) = KK^1(\mathsf{C}_\chi, C^*(G))$$

is $q^*([Bott] \underset{\mathbb{C}}{\otimes} \mathrm{ind}_G)$ *where* $q : \mathsf{C}_\chi \to C_0(\mathfrak{A}^*G \times \mathbb{R}_+^*)$ *is evaluation at* 0. $\square$

The element $[\chi] \in KK(C_0(\mathfrak{A}^*G \times \mathbb{R}_+^*), C_0(DNC_+(M,V)))$ is the Kasparov product of the "Euler element" of the bundle $\mathfrak{A}^*G$ which is the class in $KK(C_0(\mathfrak{A}^*G), C_0(M)) = KK(C_0(\mathfrak{A}^*G \times \mathbb{R}_+^*), C_0(M \times \mathbb{R}_+^*))$ of the map $x \mapsto (x,0)$ with the inclusion $C_0(M \times \mathbb{R}_+^*) \to C_0(DNC_+(M,V))$. It follows that $[\chi]$ is often the zero element of $KK(C_0(\mathfrak{A}^*G \times \mathbb{R}_+^*), C_0(DNC_+(M,V)))$. In particular, this is the case when the Euler class of the bundle $\mathfrak{A}^*G$ vanishes. In that case, the algebra $\Sigma_{DNC_+}(G,V)$ is KK-equivalent to $C_0(\mathfrak{A}^*G) \oplus C_0(DNC_+(M,V))$.

If V is $\mathfrak{A}G$ small, then, by theorem 5.8, $\partial_{SBlup}^{G,V}$ and $\widetilde{\mathrm{ind}}_{SBlup}^{G,V}$ are immediately deduced from proposition 5.11.

Remark 5.12. Let M_b be a manifold with boundary and $V = \partial M_b$. Put $\mathring{M} = M_b \setminus V$. Let G be a piece of Lie groupoid on M_b in the sense of section 4.1.4. Thus G is the restriction of a Lie groupoid $\widetilde{G} \rightrightarrows M$, where M is a neighborhood of M_b. Recall that in this situation, $SBlup(M,V) = M_b \sqcup M_-$, where $M = M_b \cup M_-$ and $M \cap M_- = V$, and we let $SBlup_{r,s}(G,V) \rightrightarrows M_b$ be the restriction of $SBlup_{r,s}(\widetilde{G},V)$ to M_b.

Let us denote by $\mathring{\mathcal{N}}_V^G$ the open subset of $N_V^{\widetilde{G}}$ made of (normal) tangent vectors whose image under the differential of the source and range maps of $\widetilde{G}$ are non vanishing elements of N_V^M pointing in the direction of M_b. The groupoid $SBlup_{r,s}(G,V)$ is the union $\mathring{\mathcal{N}}_V^G/\mathbb{R}_+^* \cup G_{\mathring{M}}^{\mathring{M}}$.

We have exact sequences

$$0 \to C^*(G_{\mathring{M}}^{\mathring{M}}) \to C^*(SBlup_{r,s}(G,V)) \to C^*(\mathring{\mathcal{N}}_V^G/\mathbb{R}_+^*) \to 0$$

$$0 \to C^*(G_{\mathring{M}}^{\mathring{M}}) \to \Psi^*(SBlup_{r,s}(G,V)) \to \Sigma_{SBlup}(G,V) \to 0.$$

As V is of codimension 1, we find that V is $\mathfrak{A}\widetilde{G}$-small if and only if it is transverse to $\widetilde{G}$. In that case, Proposition 5.11 computes the KK-theory of $C^*(\mathring{\mathcal{N}}_V^G/\mathbb{R}_+^*)$ and of $\Sigma_{SBlup}(G,V)$ and the KK-class of the connecting maps of these exact sequences.

In particular, we obtain a six term exact sequence

$$\begin{tikzcd}
K_0(C(M_b)) \arrow[r] & K_0(\Sigma_{SBlup}(G,V)) \arrow[r] & K_1(C_0(\mathfrak{A}^*G_{\mathring{M}}^{\mathring{M}})) \arrow[d, "\chi"] \\
K_0(C_0(\mathfrak{A}^*G_{\mathring{M}}^{\mathring{M}})) \arrow[u, "\chi"] & K_1(\Sigma_{SBlup}(G,V)) \arrow[l] & K_0(C(M_b)) \arrow[l]
\end{tikzcd}$$

and the index map $K_*(\Sigma_{SBlup}(G,V)) \to K_{*+1}(G_{\mathring{M}}^{\mathring{M}})$ is the composition of $K_*(\Sigma_{SBlup}(G,V)) \to K_{*+1}(C_0(\mathfrak{A}^*G_{\mathring{M}}^{\mathring{M}}))$ with the index map of the groupoid $G_{\mathring{M}}^{\mathring{M}}$.

This holds, in particular, if $G = M_b \times M_b$ since the boundary $V = \partial M_b$ is transverse to $\widetilde{G} = M \times M$. Note that in that case, $\chi = 0$ (in $KK(C_0(T^*\mathring{M}), C_0(M_b)))$ so that we obtain a (non canonically) split short exact sequence:

$$0 \longrightarrow K_*(C_0(M_b)) \longrightarrow K_*(\Sigma_{SBlup}(G,V)) \longrightarrow K_{*+1}(C_0(\mathfrak{A}^*G_{\mathring{M}}^{\mathring{M}})) \longrightarrow 0.$$

5.3.2 The index map via relative K-theory

It follows now from [13, prop. 4.8]:

Proposition 5.13. *Let $\psi_{DNC} : C_0(DNC_+(M,V)) \to \Psi^*(DNC_+(G,V))$ be the inclusion map which associates to a (smooth) function f the order 0 (pseudo)differential operator multiplication by f and $\sigma_{full} : \Psi^*(DNC_+(G,V)) \to \Sigma_{DNC_+}(G,V)$ the full symbol map. Put $\mu_{DNC} = \sigma_{full} \circ \psi_{DNC}$. Then the relative K-group $K_*(\mu_{DNC})$ is naturally isomorphic to $K_{*+1}(C_0(\mathfrak{A}^*G))$. Under this isomorphism,* $\mathrm{ind}_{rel} : K_*(\mu_{DNC}) \to K_*(C^*(G \times \mathbb{R}_+^*)) = K_{*+1}(C^*(G))$ *identifies with* ind_G. □

Let us say also just a few words on the relative index map for $SBlup_{r,s}(G,V)$, *i.e.* for the map $\mu_{SBlup} : C_0(SBlup_+(M,V)) \to \Sigma_{SBlup}(G,V)$ which is the composition of the inclusion $\psi_{SBlup} : C_0(SBlup(M,V) \to \Psi^*(SBlup_{r,s}(G,V))$ with the full index map $\sigma_{full} : \Psi^*(SBlup_{r,s}(G,V)) \to \Sigma_{SBlup}(G,V))$, and the corresponding relative index map $\mathrm{ind}_{rel} : K_*(\mu_{SBlup}) \to K_*(C^*(G_{\mathring{M}}^{\mathring{M}}))$. Equivalently we wish to compute the relative index map $\mathrm{ind}_{rel} : K_*(\mu_{\widetilde{DNC}}) \to K_{*+1}(C^*(G_{\mathring{M}}^{\mathring{M}}))$, where $\mu_{\widetilde{DNC}} : C_0(\widetilde{DNC}_+(M,V)) \to \Sigma_{\widetilde{DNC}_+}(G,V)$. We restrict to the case when V is $\mathfrak{A}G$ small.
We have a diagram

$$0 \longrightarrow C_0(\widetilde{DNC}_+(M,V)) \longrightarrow C_0(DNC_+(M,V)) \longrightarrow C_0(V \times \mathbb{R}_+) \longrightarrow 0$$

and it follows that the inclusion $C_0(\widetilde{DNC}_+(M,V)) \to C_0(DNC_+(M,V))$ is a KK-equivalence.
Since the inclusions $\Psi^*(\widetilde{DNC}_+(G,V)) \to \Psi^*(DNC_+(G,V))$ and $\Sigma_{\widetilde{DNC}_+}(G,V) \to \Sigma_{DNC_+}(G,V)$ are also KK-equivalences (prop. 5.7), it follows that the inclusion $\mathsf{C}_{\mu_{\widetilde{DNC}}} \to \mathsf{C}_{\mu_{DNC}}$ of mapping cones is a KK-equivalence - and therefore the relative K-groups $K_*(\mu_{\widetilde{DNC}})$ and $K_*(\mu_{DNC})$ are naturally isomorphic. Using this, together with the Connes-Thom isomorphism, we deduce:

Corollary 5.14. *We assume that V is $\mathfrak{A}G$ small*

a) *The relative K-group $K_*(\mu_{\widetilde{DNC}})$ is naturally isomorphic to $K_{*+1}(C_0(\mathfrak{A}^*G))$. Under this isomorphism,* $\mathrm{ind}_{rel} : K_*(\mu_{\widetilde{DNC}}) \to K_*(C^*(G \times \mathbb{R}_+^*)) = K_{*+1}(C^*(G))$ *identifies with* ind_G.

b) *The relative K-group $K_*(\mu_{SBlup})$ is naturally isomorphic to $K_*(C_0(\mathfrak{A}^*G))$. Under this isomorphism,* $\mathrm{ind}_{rel} : K_*(\mu_{SBlup}) \to K_*(C^*(G))$ *identifies with* ind_G. □

6 A Boutet de Monvel type calculus

In this section, we consider the $SBlup$ construction in the special case of a transverse submanifold of the unit space of a groupoid. We use the bimodule that we constructed in [11] in order to obtain an algebra resembling the algebra of 2×2 matrices in the Boutet de Monvel pseudodifferential calculus (of order 0) on manifolds with boundary.

From now on, we suppose that V is a *transverse submanifold* of M with respect to the Lie groupoid $G \rightrightarrows M$. In particular V is $\mathfrak{A}G$-small - of course, we assume that (in every connected component of V), the dimension of V is strictly smaller than the dimension of M.

6.1 The Poisson-trace bimodule

As V is transverse to G, the groupoid G_V^V is a Lie groupoid, so that we can construct its "gauge adiabatic groupoid" $(G_V^V)_{ga}$ (see section 4.3.3).
In [11], we constructed a bi-module relating the C^*-algebra of the groupoid $(G_V^V)_{ga}$ and the C^*-algebra of pseudodifferential operators of G_V^V.

In this section,

- We first show that the groupoid $(G_V^V)_{ga}$, is (sub-) Morita equivalent to $SBlup_{r,s}(G,V)$ (*cf.* also section 4.3.4 for a local construction).
- Composing the resulting bimodules, we obtain the "Poisson-trace" bimodule that relates the C^*-algebras $C^*(SBlup_{r,s}(G,V))$ and $\Psi^*(G_V^V)$.

6.1.1 The $SBlup_{r,s}(G,V)-(G_V^V)_{ga}$-bimodule $\mathscr{E}(G,V)$

Define the map $j : M \sqcup (V\times\mathbb{R}) \to M$ by letting $j_0 : M \to M$ be the identity and $j_1 : V\times\mathbb{R}\to M$ the composition of the projection $V\times\mathbb{R}\to V$ with the inclusion. Let $\mathcal{G} = G_j^j$. As V is assumed to be transverse, the map j is also transverse, and therefore $\mathcal{G}$ is a Lie groupoid.
It is the union of four clopen subsets

- the groupoids $G_{j_0}^{j_0} = G = \mathcal{G}_M^M$ and $G_{j_1}^{j_1} = G_V^V\times(\mathbb{R}\times\mathbb{R}) = \mathcal{G}_{V\times\mathbb{R}}^{V\times\mathbb{R}}$.
- the *linking spaces* $G_{j_1}^{j_0} = \mathcal{G}_{V\times\mathbb{R}}^M = G_V\times\mathbb{R}$ and $G_{j_0}^{j_1} = \mathcal{G}_M^{V\times\mathbb{R}} = G^V\times\mathbb{R}$.

By functoriality, we obtain a sub-Morita equivalence of $SBlup_{r,s}(G_V^V\times\mathbb{R}\times\mathbb{R},V)$ and $SBlup_{r,s}(G,V)$ (see section 4.1.3).
Let us describe this sub-Morita equivalence in a slightly different way:
Let also $\Gamma = V\times\{0,1\}^2$, sitting in $\mathcal{G}$:

$$V\times\{(0,0)\}\subset G = G_{j_0}^{j_0}\ ; \qquad V\times\{(0,1)\}\subset G_V\times\{0\}\subset G_{j_1}^{j_0}\ ;$$
$$V\times\{(1,0)\}\subset G^V\times\{0\}\subset G_{j_0}^{j_1}\ ; \qquad V\times\{(1,1)\}\subset G_V^V\times\{(0,0)\}\subset G_{j_1}^{j_1}\ .$$

It is a subgroupoid of $\mathcal{G}$. The blowup construction applied to $\Gamma\subset\mathcal{G}$ gives then a groupoid $SBlup_{r,s}(\mathcal{G},\Gamma)$ which is the union of:

$$SBlup_{r,s}(G,V)\ ; \qquad SBlup_{r,s}(G_V\times\mathbb{R},V)\ ;$$
$$SBlup_{r,s}(G^V\times\mathbb{R},V)\ ; \qquad SBlup_{r,s}(G_V^V\times\mathbb{R}\times\mathbb{R},V)\ .$$

Recall that $SBlup(V\times\mathbb{R},V\times\{0\})\simeq V\times(\mathbb{R}_-\sqcup\mathbb{R}_+)$. Thus $SBlup_{r,s}(\mathcal{G},\Gamma)$ is a groupoid with objects $SBlup(M,V)\sqcup V\times\mathbb{R}_-\sqcup V\times\mathbb{R}_+$.

The restriction of $SBlup_{r,s}(\mathcal{G},\Gamma)$ to $V\times\mathbb{R}_+$ coincides with the restriction of $SBlup_{r,s}(G_V^V\times\mathbb{R}\times\mathbb{R},V)$ to $V\times\mathbb{R}_+$: it is the gauge adiabatic $(G_V^V)_{ga}$ groupoid of G_V^V (*cf.* section 4.3.3).

Put $SBlup_{r,s}(G_V\times\mathbb{R},V)_+ = SBlup_{r,s}(\mathcal{G},\Gamma)_{V\times\mathbb{R}_+}^{SBlup(M,V)}$. It is a linking space between the groupoids $SBlup_{r,s}(G,V)$ and $(G_V^V)_{ga}$. Put also $SBlup_{r,s}(G^V\times\mathbb{R},V)_+ = SBlup_{r,s}(\mathcal{G},\Gamma)^{V\times\mathbb{R}_+}_{SBlup(M,V)}$.

With the notation used in fact 2.3, we define the $C^*(SBlup_{r,s}(G,V)) - C^*((G_V^V)_{ga})$-bimodule $\mathscr{E}(G,V)$ to be $C^*(SBlup_{r,s}(G_V\times\mathbb{R},V)_+)$. It is the closure of $C_c(SBlup_{r,s}(G_V\times\mathbb{R},V)_+)$ in $C^*(SBlup_{r,s}(\mathcal{G},\Gamma))$. It is a full Hilbert-$C^*(SBlup_{r,s}(G,V)) - C^*((G_V^V)_{ga})$-module.
The Hilbert-$C^*((G_V^V)_{ga})$-module $\mathscr{E}(G,V)$ is full and $\mathcal{K}(\mathscr{E}(G,V))$ is the ideal $C^*(SBlup_{r,s}(G_\Omega^\Omega,V))$ where $\Omega = r(G_V)$ is the union of orbits which meet V.

Notice that $\Omega = M\setminus V\sqcup V\times\mathbb{R}^*$ and $F = \mathbb{S}N_V^M\sqcup V\sqcup V$ gives a partition by respectively open and closed saturated subsets of the units of $SBlup_{r,s}(\mathcal{G},\Gamma)$. Furthermore $SBlup_{r,s}(\mathcal{G},\Gamma)_\Omega^\Omega = \mathcal{G}_\Omega^\Omega$ and

$C^*(\mathcal{G}_\Omega^\Omega) = C^*(\mathcal{G})$ according to proposition 5.6. This decomposition gives rise to an exact sequence of C*-algebras.

$$0 \longrightarrow C^*(\mathcal{G}) \longrightarrow C^*(SBlup_{r,s}(\mathcal{G},\Gamma)) \longrightarrow C^*(\mathcal{S}N_\Gamma^{\mathcal{G}}) \longrightarrow 0$$

This exact sequence gives rise to an exact sequence of bimodules:

$$\begin{tikzcd} 0 \arrow[r] & C^*(G) \arrow[r] \arrow[d, no head, "\mathring{\mathscr{E}}(G,V)"] & C^*(SBlup_{r,s}(G,V)) \arrow[r] \arrow[d, no head, "\mathscr{E}(G,V)"] & C^*(\mathcal{S}N_V^G) \arrow[r] \arrow[d, no head, "\mathscr{E}^\partial(G,\Gamma)"] & 0 \\ 0 \arrow[r] & C^*(G_{V\times\mathbb{R}_+^*}^{V\times\mathbb{R}_+^*}) \arrow[r] & C^*((G_V^V)_{ga}) \arrow[r] & C^*(\mathfrak{A}G_V^V \rtimes \mathbb{R}_+^*) \arrow[r] & 0 \end{tikzcd}$$

where $\mathring{\mathscr{E}}(G,V) = C^*(\mathcal{G}_{V\times\mathbb{R}_+^*}^{M\setminus V})$ and $\mathscr{E}^\partial(G,\Gamma) = C^*\big((\mathcal{S}N_\Gamma^{\mathcal{G}})_V^{\mathbb{S}N_V^M}\big) = \mathscr{E}(G,V)/\mathring{\mathscr{E}}(G,V)$.

6.1.2 The Poisson-trace bimodule $\mathscr{E}_{PT}$

In [11], we constructed, for every Lie groupoid H a $C^*(H_{ga}) - \Psi^*(H)$-bimodule $\mathscr{E}_H$.
Recall that the Hilbert $\Psi^*(H)$-module $\mathscr{E}_H$ is full and that $\mathcal{K}(\mathscr{E}_H) \subset C^*(H_{ga})$ is the kernel of a natural $*$-homomorphism $C^*(H_{ga}) \to C_0(H^{(0)} \times \mathbb{R})$. We also showed that the bimodule $\mathscr{E}_H$ gives rise to an exact sequence of bimodule as above:

$$\begin{tikzcd} 0 \arrow[r] & C^*(H \times \mathbb{R}_+^* \times \mathbb{R}_+^*) \arrow[r] \arrow[d, no head, "\mathring{\mathscr{E}}_H"] & C^*(H_{ga}) \arrow[r] \arrow[d, no head, "\mathscr{E}_H"] & C^*(\mathfrak{A}H \rtimes \mathbb{R}_+^*) \arrow[r] \arrow[d, no head, "\mathscr{E}_H^\partial"] & 0 \\ 0 \arrow[r] & C^*(H) \arrow[r] & \Psi^*(H) \arrow[r] & C_0(\mathbb{S}\mathfrak{A}^*H) \arrow[r] & 0 \end{tikzcd}$$

Putting together the bimodule $\mathscr{E}(G,V)$ and $\mathscr{E}_{G_V^V}$ we obtain a $C^*(SBlup_{r,s}(G,V))-\Psi^*(G_V^V)$ bimodule $\mathscr{E}(G,V) \otimes_{C^*((G_V^V)_{ga})} \mathscr{E}_{G_V^V}$ that we call the *Poisson-trace* bimodule and denote by $\mathscr{E}_{PT}(G,V)$ - or just $\mathscr{E}_{PT}$. It leads to the exact sequence of bimodule:

$$\begin{tikzcd} 0 \arrow[r] & C^*(G) \arrow[r] \arrow[d, no head, "\mathring{\mathscr{E}}_{PT}(G,V)"] & C^*(SBlup_{r,s}(G,V)) \arrow[r] \arrow[d, no head, "\mathscr{E}_{PT}(G,V)"] & C^*(\mathcal{S}N_V^G) \arrow[r] \arrow[d, no head, "\mathscr{E}_{PT}^\partial(G,V)"] & 0 \\ 0 \arrow[r] & C^*(G_V^V) \arrow[r] & \Psi^*(G_V^V) \arrow[r] & C_0(\mathbb{S}\mathfrak{A}^*G_V^V) \arrow[r] & 0 \end{tikzcd}$$

The Poisson-trace bimodule is a full Hilbert $\Psi^*(G_V^V)$-module and $\mathcal{K}(\mathscr{E}_{PT}(G,V))$ is a two sided ideal of $C^*(SBlup_{r,s}(G,V))$. Denote by $\mathscr{E}_{PT}(G,V)^*$ its dual module, *i.e.* the $\Psi^*(G_V^V)-C^*(SBlup_{r,s}(G,V))$-bimodule $\mathcal{K}(\mathscr{E}_{PT}(G,V),\Psi^*(G_V^V))$.

6.2 A Boutet de Monvel type algebra

The C^*-algebra $C^*_{BM}(G,V) = \mathcal{K}\Big(C^*(SBlup_{r,s}(G,V)) \oplus \mathscr{E}_{PT}(G,V)^*\Big)$ is an algebra made of matrices of the form $\begin{pmatrix} K & P \\ T & Q \end{pmatrix}$ where $K \in C^*(SBlup_{r,s}(G,V))$, $P \in \mathscr{E}_{PT}(G,V)$, $T \in \mathscr{E}_{PT}(G,V)^*$, $Q \in \Psi^*(G_V^V)$.
We have an exact sequence (where $\mathring{M} \sqcup V \neq M$ denotes the topological disjoint union of $\mathring{M}$ with V):

$$0 \to C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V}) \to C^*_{BM}(G,V) \xrightarrow{r_V^{C^*}} \Sigma_{bound}^{C^*}(G,V) \to 0,$$

where the quotient $\Sigma_{bound}^{C^*}(G,V)$ is the algebra of the *Boutet de Monvel type boundary symbols.* It is the algebra of matrices of the form $\begin{pmatrix} k & p \\ t & q \end{pmatrix}$ where $k \in C^*(\mathcal{S}N_V^G)$, $q \in C(\mathbb{S}\mathfrak{A}^*G_V^V)$, $p, t^* \in$

$\mathscr{E}^V_{PT}(G,V) := \mathscr{E}_{PT}(G,V) \otimes_{\Psi^*(G^V_V)} C(\mathbb{S}\mathfrak{A}^* G^V_V)$. The map $r_V^{C^*}$ is of the form

$$r_V^{C^*}\begin{pmatrix} K & P \\ T & Q \end{pmatrix} = \begin{pmatrix} r_V^{\asymp}(K) & r_V^{\propto}(P) \\ r_V^{\reflectbox{$\propto$}}(T) & \sigma_V(Q) \end{pmatrix}$$

where:

- the quotient map σ_V is the ordinary order 0 principal symbol map on the groupoid G^V_V;
- the quotient maps $r_V^{\asymp}, r_V^{\propto}, r_V^{\reflectbox{$\propto$}}$ are restrictions to the boundary N_V^M:

$$r_V^{\asymp} : C^*(SBlup_{r,s}(G,V)) \to C^*(\mathbb{S}N_V^G) = C^*(SBlup_{r,s}(G,V))/C^*(G^{\mathring{M}}_{\mathring{M}}),$$

$$r_V^{\propto} : \mathscr{E}_{PT}(G,V) \to \mathscr{E}^V_{PT}(G,V) = \mathscr{E}_{PT}(G,V)/C^*(G^{\mathring{M}}_V),$$

  and $r_V^{\reflectbox{$\propto$}}(T) = r_V^{\propto}(T^*)^*$.

The map $r_V^{C^*}$ is called the *zero order symbol map of the Boutet de Monvel type calculus.*

6.3 A Boutet de Monvel type pseudodifferential algebra

We denote by $\Psi^*_{BM}(G,V)$ the algebra of matrices $\begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix}$ with $\Phi \in \Psi^*(SBlup_{r,s}(G,V))$, $P \in \mathscr{E}_{PT}(G,V)$, $T \in \mathscr{E}_{PT}(G,V)^*$ and $Q \in \Psi^*(G^V_V)$.
Such an operator $R = \begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix}$ has two symbols:

- the *classical symbol* $\sigma_c : \Psi^*_{BM}(G,V) \to C_0(\mathbb{S}\mathfrak{A}^* SBlup_{r,s}(G,V))$ given by $\sigma_c\begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix} = \sigma_c(\Phi)$;
- the *boundary symbol* $r_V^{BM} : \Psi^*_{BM}(G,V) \to \Sigma^{\Psi^*}_{bound}(G,V)$ defined by

$$r_V\begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix} = \begin{pmatrix} r_V^{\psi}(\Phi) & r_V^{\propto}(P) \\ r_V^{\reflectbox{$\propto$}}(T) & \sigma_V(Q) \end{pmatrix}$$

  where $r_V^{\psi} : \Psi^*(SBlup_{r,s}(G,V)) \to \Psi^*(\mathcal{S}N_V^G)$ is the restriction.

Here $\Sigma^{\Psi^*}_{bound}(G,V)$ denotes the algebra of matrices of the form $\begin{pmatrix} \phi & p \\ t & q \end{pmatrix}$ with $\phi \in \Psi^*(\mathcal{S}N_V^G)$, $p, t^* \in \mathscr{E}^V_{PT}(G,V)$ and $q \in C(\mathbb{S}\mathfrak{A}^* G^V_V)$.
The *full symbol* map is the morphism

$$\sigma_{BM} : \Psi^*_{BM}(G,V) \to \Sigma_{BM}(G,V) := C_0(\mathbb{S}\mathfrak{A}^* SBlup_{r,s}(G,V)) \times_{C_0(\mathbb{S}\mathfrak{A}^* \mathcal{S}N_V^G)} \Sigma^{\Psi^*}_{bound}(G,V)$$

defined by $\sigma_{BM}(R) = (\sigma_c(R), r_V(R))$.
We have an exact sequence:

$$0 \to C^*(G^{\mathring{M} \sqcup V}_{\mathring{M} \sqcup V}) \to \Psi^*_{BM}(G,V) \xrightarrow{\sigma_{BM}} \Sigma_{BM}(G,V) \to 0. \qquad E_{BM}$$

We may note that $\Psi^*(SBlup_{r,s}(G,V))$ (*resp.* $\Psi^*(\mathcal{S}N_V^G)$) identifies with the full hereditary subalgebra of $\Psi^*_{BM}(G,V)$ (*resp.* of $\Sigma_{BM}(G,V)$) consisting of elements of the form $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$.

6.4 K-theory of the symbol algebras and index maps

In this section we examine the index map corresponding to the Boutet de Monvel type calculus and in particular to the exact sequence E_{BM}. We compute the K-theory of the symbol algebra Σ_{BM} and the connecting element $\widetilde{\mathrm{ind}}_{BM} \in KK^1(\Sigma_{BM}, C^*(G))$ (3).
We then extend this computation by including bundles into the picture *i.e.* by computing a relative K-theory map.

6.4.1 K-theory of Σ_{BM} and computation of the index

As the Hilbert $\Psi^*(G_V^V)$ module $\mathscr{E}_{PT}(G,V)$ is full,

- the subalgebra $\left\{ \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix};\ K \in C^*(SBlup_{r,s}(G,V)) \right\}$ is a full hereditary subalgebra of $C^*_{BM}(G,V)$;
- the subalgebra $\left\{ \begin{pmatrix} \Phi & 0 \\ 0 & 0 \end{pmatrix};\ \Phi \in \Psi^*(SBlup_{r,s}(G,V)) \right\}$ is a full hereditary subalgebra of $\Psi^*_{BM}(G,V)$;
- the subalgebra $\left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix};\ x \in \Sigma_{SBlup}(G,V) \right\}$ is a full hereditary subalgebra of $\Sigma_{BM}(G,V)$;
- the subalgebra $\left\{ \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix};\ k \in C^*(\mathcal{S}N_V^G) \right\}$ is a full hereditary subalgebra of $\Sigma^{C^*}_{bound}(G,V)$;
- the subalgebra $\left\{ \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix};\ \phi \in \Psi^*(\mathcal{S}N_V^G) \right\}$ is a full hereditary subalgebra of $\Sigma^{\Psi^*}_{bound}(G,V)$.

We have a diagram of exact sequences where the vertical inclusions are Morita equivalences:

$$\begin{tikzcd}
0 \arrow[r] & C^*(G_{\mathring{M}}^{\mathring{M}}) \arrow[r] \arrow[d, hook] & \Psi^*(SBlup_{r,s}(G,V)) \arrow[r, "\sigma_{full}"] \arrow[d, hook] & \Sigma_{SBlup}(G,V) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V}) \arrow[r] & \Psi^*_{BM}(G,V) \arrow[r, "\sigma_{BM}"] & \Sigma_{BM}(G,V) \arrow[r] & 0
\end{tikzcd}$$

We thus deduce immediately from theorem 5.8 and prop. 5.11:

Corollary 6.1. *The algebra $\Sigma_{BM}(G,V))$ is KK-equivalent with the mapping cone C_χ and, under this K-equivalence, the index $\widetilde{\mathrm{ind}}_{BM}$ is $q^*([Bott]\otimes_{\mathbb{C}}\mathrm{ind}_G)$ where $q : \mathsf{C}_\chi \to C_0(\mathfrak{A}^*G\times\mathbb{R}_+^*)$ is evaluation at* 0.

6.4.2 Index in relative K-theory

One may also consider more general index problems, which are concerned with generalized boundary value problems in the sense of [37, 26, 27]: those are concerned with index of *fully elliptic* operators of the form $R = \begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix}$, where we are given hermitian complex vector bundles $E_\pm$ over $SBlup(M,V)$ and $F_\pm$ over V, and

- Φ is an order 0 pseudodifferential operator of the Lie groupoid $SBlup_{r,s}(G,V)$ from sections of E_+ to sections of E_-;
- P is an order 0 "Poisson type" operator from sections of F_+ to sections of E_-;
- T is an order 0 "trace type" operator from sections of E_+ to sections of F_-;

[3] We use the Morita equivalence of $C^*(G)$ with $C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V})$.

- Q is an order 0 pseudodifferential operator of the Lie groupoid G_V^V from sections of F_+ to sections of F_-.

In other words, writing $E_\pm$ as associated with projections $p_\pm \in M_N(C^\infty(SBlup(M,V)))$ and $F_\pm$ as associated with projections $q_\pm \in M_N(C^\infty(V))$, then $R \in (p_- \oplus q_-)M_N(\Psi^*_{BM}(G,V))(p_+ \oplus q_+)$.
Full ellipticity for R means just that the *full symbol* of R is invertible, *i.e.* that there is a quasi-inverse $R' \in (p_+ \oplus q_+)M_N(\Psi^*_{BM}(G,V))(p_- \oplus q_-)$, such that $(p_+ \oplus q_+) - R'R \in M_N(C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V}))$ and $(p_- \oplus q_-) - RR' \in M_N(C^*(G_{\mathring{M}\sqcup V}^{\mathring{M}\sqcup V}))$.

In other words, we wish to compute the morphism $\mathrm{ind}_{rel} : K_*(\mu_{BM}) \to K_*(C^*_{BM}(G,V))$ where μ_{BM} is the natural morphism $\mu_{BM} : C_0(SBlup(M,V)) \oplus C_0(V) \to \Sigma_{BM}(G,V)$.

Let us outline here this computation. We start with a remark.

Remark 6.2. Let $H \rightrightarrows V$ be a Lie groupoid. The bimodule $\mathscr{E}_H^\partial$ is a Morita equivalence of an ideal $C^*(\mathfrak{A}H \rtimes \mathbb{R}_+^*)$ with $C_0(\mathbb{S}\mathfrak{A}^*H)$ and therefore defines an element $\zeta_H \in KK(C_0(\mathbb{S}\mathfrak{A}^*H), C^*(\mathfrak{A}H \rtimes \mathbb{R}_+^*))$. Let $\mu_H : C_0(V) \to C_0(\mathbb{S}\mathfrak{A}^*H)$ be the inclusion (given by the map $\mathbb{S}\mathfrak{A}^*H \to V$). The composition $\mu_H^*(\zeta_H)$ is the zero element in $KK(C_0(V), C^*(\mathfrak{A}H \rtimes \mathbb{R}_+^*))$. Indeed $\mu_H^*(\zeta_H)$ can be decomposed as

- the Morita equivalence $C_0(V) \sim C_0((V \times \mathbb{R}_+^*) \rtimes \mathbb{R}_+^*)$,
- the inclusion $C_0(V \times \mathbb{R}_+^*) \rtimes \mathbb{R}_+^* \subset C_0(V \times \mathbb{R}_+) \rtimes \mathbb{R}_+^*$,
- the inclusion $C_0(V \times \mathbb{R}_+) \rtimes \mathbb{R}_+^* \to C_0(\mathfrak{A}^*H) \rtimes \mathbb{R}_+^*$ corresponding to the map $(x,\xi) \mapsto (x, \|\xi\|)$ from $\mathfrak{A}^*H$ to $V \times \mathbb{R}_+$.

Now, the Toeplitz algebra $C_0(\mathbb{R}_+) \rtimes \mathbb{R}_+^*$ is K-contractible.

From this remark, we immediately deduce:

Proposition 6.3. *The inclusion $C_0(V) \to \Sigma_{BM}(G,V)$ is the zero element in KK-theory.* □

We have a diagram

$$\begin{tikzcd} C_0(SBlup(M,V)) \oplus C_0(V) \arrow[r, "\mu_{SBlup}\oplus\mu_V"] \arrow[d, equal] & \Sigma_{SBlup}(G,V) \oplus C_0(\mathbb{S}\mathfrak{A}^*G_V^V) \arrow[d] \\ C_0(SBlup(M,V)) \oplus C_0(V) \arrow[r, "\psi_{BM}"] & \Sigma_{BM}(G,V) \end{tikzcd}$$

The mapping cone $\mathsf{C}_{\check{\mu}_{SBlup}}$ of the morphism $\check{\mu}_{SBlup} : C_0(SBlup(M,V)) \oplus 0 \to \Psi^*_{BM}(G,V)$ is Morita-equivalent to the mapping cone of the morphism $\mu_{SBlup} : C_0(SBlup_+(M,V)) \to \Sigma_{SBlup}(G,V))$ and therefore it is KK-equivalent to $C_0(\mathfrak{A}^*G \times \mathbb{R})$ by Cor. 5.14.
We then deduce:

Theorem 6.4. a) *The relative K-theory of μ_{BM} is naturally isomorphic to $K_*(\mathfrak{A}^*G) \oplus K_{*+1}(C_0(V))$.*

b) *Under this equivalence, the relative index map identifies with* ind_G *on $K_*(\mathfrak{A}^*G)$ and the zero map on $K_{*+1}(C_0(V)$.* □

List of Symbols

Fiber bundles

N_V^M	The normal bundle of a submanifold V of a manifold M
$\mathbb{P}(E), \mathbb{S}(E), \mathbb{S}(E^*)$	The projective, sphere and dual sphere bundles associated to a real vector bundle E over M, whose fiber over $x \in M$ are respectively the projective space $\mathbb{P}(E_x)$, the sphere $\mathbb{S}(E_x)$ and the cosphere $\mathbb{S}(E_x^*)$, page 4

Groupoids, deformation and blowup spaces

$G \overset{r,s}{\rightrightarrows} G^{(0)}$	A Lie groupoid with source s, range r and space of units $G^{(0)}$
$\mathfrak{A}G$	The Lie algebroid of the groupoid G
G^A, G_B, G_B^A	If A and B are subsets of $G^{(0)}$, $G^A = \{x \in G;\ r(x) \in A\}$, $G_B = \{x \in G;\ s(x) \in B\}$ and $G_A^B = G_A \cap G^B$, page 7
G^f, G_g, G_g^f	If $f : A \to G^{(0)}$ and $g : B \to G^{(0)}$ are maps, $G^f = \{(a,x) \in A \times G;\ r(x) = f(a)\}$, $G_g = \{(x,b) \in G \times B;\ s(x) = g(b)\}$ and $G_g^f = G^f \cap G_g$, page 7
G_{ad}, $G_{ad}^{[0,1]}$, $G_{ad}^{[0,1)}$	The adiabatic groupoid of G and its restriction respectively to $G^{(0)} \times [0,1]$ and to $G^{(0)} \times [0,1)$, page 17
$DNC(Y,X)$	The deformation to the normal cone of the inclusion of a submanifold X in a manifold Y, $DNC(Y,X) = Y \times \mathbb{R}^* \cup N_X^Y$, page 9
$DNC_+(Y,X)$	The restriction $DNC_{\mathbb{R}_+}(Y,X)$, page 11
$Blup(Y,X)$	The blowup of the inclusion of a submanifold X in a manifold Y, $Blup(Y,X) = Y \setminus X \cup \mathbb{P}(N_X^Y)$, page 11
$SBlup(Y,X)$	The spherical blowup of the inclusion of a submanifold X in a manifold Y, $SBlup(Y,X) = Y \setminus X \cup \mathbb{S}(N_X^Y)$, page 11
$Blup_f(Y,X)$	The subspace of $Blup(Y,X)$ on which $Blup(f) : Blup_f(Y,X) \to Blup(Y',X')$ can be defined for a smooth map $f : Y \to Y'$ (with $f(X) \subset X'$), page 12
$DNC(G,\Gamma) \rightrightarrows DNC(G_2^{(0)}, G_1^{(0)})$	The deformation groupoid where Γ is a closed Lie subgroupoid of a Lie groupoid G, page 12
$\widetilde{DNC}(G,\Gamma)$, $\widetilde{DNC}_+(G,\Gamma)$	The open subgroupoid of $DNC(G,\Gamma)$ consisting of elements whose image by $DNC(r)$ and $DNC(s)$ is not in $G_1^{(0)} \times \mathbb{R}$ and its restriction to $\mathbb{R}_+$, page 13
$Blup_{r,s}(G,\Gamma) \rightrightarrows Blup(G^{(0)}, \Gamma^{(0)})$	The blowup groupoid $Blup_r(G,\Gamma) \cap Blup_s(G,\Gamma)$ where Γ is a closed Lie subgroupoid of a Lie groupoid G, it is the quotient of $\widetilde{DNC}(G,\Gamma)$ under the gauge action, page 13
$SBlup_{r,s}(G,\Gamma)$	The spherical version of $Blup_{r,s}(G,\Gamma)$, it is quotient of $\widetilde{DNC}_+(G,\Gamma)$ under the restricted gauge action, page 13

C*-Algebras

$C^*(G)$	The (either maximal or reduced) C^*-algebra of the groupoid G
$\Psi^*(G)$	The C^*-algebra of order ≤ 0 pseudodifferential operators on G vanishing at infinity on $G^{(0)}$
C_f	The mapping cone of a morphism $f : A \to B$ of C^*-algebra
$\Sigma^W(G)$	The quotient $\Psi^*(G)/C^*(G_W)$
$\Sigma_{DNC_+}(G,\Gamma)$, $\Sigma_{\widetilde{DNC}_+}(G,\Gamma)$	Respectively the algebras $\Psi^*(DNC_+(G,\Gamma))/C^*(G \times \mathbb{R}+^*)$ and $\Psi^*(\widetilde{DNC}_+(G,\Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*)$, page 20
$\Sigma_{SBlup}(G,\Gamma)$	The algebra $\Psi^*(SBlup_{r,s}(G,\Gamma))/C^*(G_{\mathring{M}}^{\mathring{M}})$, page 20

KK-elements

$[f]$	The KK-element, in $KK(A,B)$ associated to a morphism of C*-algebra $f : A \to B$
ind_G	The KK-element $[ev_0]^{-1} \otimes [ev_1]$, which belongs to $KK(C_0(\mathfrak{A}^*G), C^*(G))$, associated to the deformation groupoid $G_{ad}^{[0,1]} = G \times (0,1] \cup \mathfrak{A}(G) \times \{0\} \rightrightarrows G^{(0)} \times [0,1]$
$\widetilde{\mathrm{ind}}_G$	The connecting element, which belongs to $KK^1(C(\mathbb{S}\mathfrak{A}^*\mathcal{G}), C^*(\mathcal{G}))$ associated to the short exact sequence $0 \to C^*(G) \to \Psi^*(G) \to C(\mathbb{S}\mathfrak{A}^*G) \to 0$
∂_G^W	The connecting element, which belongs to $KK^1(C^*(G\|_F), C^*(G\|_W))$, associated to the short exact sequence $0 \longrightarrow C^*(G\|_W) \longrightarrow C^*(G) \longrightarrow C^*(G\|_F) \longrightarrow 0$ where W is a saturated open subset of $G^{(0)}$ and $F = G^{(0)} \setminus W$
$\partial_{SBlup}^{G,\Gamma}$, $\partial_{DNC_+}^{G,\Gamma}$, $\partial_{\widetilde{DNC}_+}^{G,\Gamma}$	Respectively the element $\partial_{SBlup_{r,s}(G,\Gamma)}^{\mathring{M}}$, $\partial_{DNC_+(G,\Gamma)}^{M\times\mathbb{R}_+^*}$ and $\partial_{\widetilde{DNC}_+(G,\Gamma)}^{\mathring{M}\times\mathbb{R}_+^*}$, page 21
$\widetilde{\mathrm{ind}}_{full}^W(G)$	The connecting element, which belongs to $KK^1(\Sigma^W(G), C^*(G_W))$ associated to the short exact sequence $0 \longrightarrow C^*(G_W) \longrightarrow \Psi^*(G) \longrightarrow \Sigma^W(G) \longrightarrow 0$
$\widetilde{\mathrm{ind}}_{SBlup}^{G,\Gamma}$, $\widetilde{\mathrm{ind}}_{DNC_+}^{G,\Gamma}$, $\widetilde{\mathrm{ind}}_{\widetilde{DNC}_+}^{G,\Gamma}$	Respectively the elements $\widetilde{\mathrm{ind}}_{full}^{\mathring{M}}(SBlup_{r,s}(G,\Gamma))$, $\widetilde{\mathrm{ind}}_{full}^{M\times\mathbb{R}_+^*}(DNC_+(G,\Gamma))$ and $\widetilde{\mathrm{ind}}_{full}^{\mathring{M}\times\mathbb{R}_+^*}(\widetilde{DNC}_+(G,\Gamma))$

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