Lie groupoids, exact sequences, Connes-Thom elements, connecting maps and index maps

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Abstract

We study various exact sequences associated with a closed saturated subset in the space of units of a Lie groupoid: the corresponding exact sequence of groupoid C*-algebras, the associated index maps and full index maps. Moreover we study Connes-Thom type isomorphisms of Lie groupoid C*-algebras.

Contents

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1 Introduction

Let $G \overset{r,s}{\rightrightarrows} M$ be a Lie groupoid and let $F \subset M$ be a closed saturated submanifold (or just a closed saturated subset). Put $W = M \setminus F$. The groupoid $G = \mathbb{G}_W \sqcup G_F$ is in this case the disjoint union of the open subgroupoid $\mathring{G} = G_W$ of elements whose source (and range) is in W and the closed subgroupoid G_F of elements whose source (and range) is in F.
In the present paper, we construct and study elements in the K-theory of the groupoid C^*-algebra $C^*(G_W)$ that take into account the submanifold F as being "boundary components" and/or "behavior at infinity".
Such a situation is encountered in many natural cases: the tangent groupoid of Connes [7] and its generalizations (*cf.* [25, 28]). They appear in many geometric situations as manifolds with boundary or with corners [22, 23, 24, 19], stratified manifolds [10, 11], Lie manifolds (*cf.* [27] and literature there).
Our main examples are deformation groupoids and blowup groupoids which are studied in [14].

Connecting maps and index maps

These groupoids give rise to several exact sequences of C^*-algebras, the corresponding connecting maps and to index problems that will be the main object of our study here.

Connecting maps. The partition $G = G_W \sqcup G_F$ leads to the following exact sequence of C^*-algebras that we wish to "compute":

$$0 \longrightarrow C^*(G_W) \longrightarrow C^*(G) \longrightarrow C^*(G_F) \longrightarrow 0 \qquad E^\partial$$

Full index maps. Denote by $\Psi^*(G)$ the C^*-algebra of order 0 pseudodifferential operators on the Lie groupoid G. The above decomposition of groupoids give rise to extensions of groupoid C^*-algebras of pseudodifferential type

$$0 \longrightarrow C^*(G_W) \longrightarrow \Psi^*(G) \xrightarrow{\sigma_{full}} \Sigma^W(G) \longrightarrow 0 \qquad E^{\widetilde{\text{ind}}}$$

where $\Sigma^W(G)$ is called the *full symbol algebra*, and the morphism σ_{full} the *full symbol* maps.

The full symbol maps. The full symbol algebra is naturally a fibered product:

$$\Sigma^W(G) = C(\mathbb{S}\mathfrak{A}^*G) \times_{C(\mathbb{S}\mathfrak{A}^*G_F)} \Psi^*(G_F).$$

Thus, the full symbol map has two components:

- The usual commutative symbol of the groupoid G which is a morphism: $\Psi^*(G) \to C(\mathbb{S}\mathfrak{A}^*G)$. The commutative symbol takes its values in the algebra of continuous fonctions on the cosphere bundle of the algebroid $\mathfrak{A}G$ of the Lie groupoid G.
- The restriction to the boundary: $\sigma_\partial : \Psi^*(G) \to \Psi^*(G_F)$. This map is sometimes called the boundary symbol or the non commutative symbol. Note indeed that, in general, $\Psi^*(G)$ is not commutative.

Associated KK-elements. Assume that the groupoid G_F is amenable. Then the exact sequence E^∂ gives rise to a connecting element $\partial^{G,F} \in KK^1(C^*(G_F), C^*(G_W))$ (*cf.* [16]).
Also, the full symbol C^*-algebra $\Sigma^W(G)$ is nuclear and the exact sequence $E^{\widetilde{\text{ind}}}$ determines a KK-element $\widetilde{\text{ind}}^{G,F} \in KK^1(\Sigma^W(G), C^*(G_W))$.
If G_F is not amenable, these constructions can be carried over in E-theory (of maximal groupoid C^*-algebras).

Full index and relative K-theory. We obtain a finer construction by using relative K-theory. It is a general fact that relative K-theory gives more precise index theorems than connecting maps (*cf. e.g.* [4, 29, 20, 21, 3]). In particular, the relative K-theory point of view allows to take into account symbols from a vector bundle to another one.
Let $\psi : C_0(M) \to \Psi^*(G)$ be the natural inclusion and consider the morphism $\mu = \sigma_{full} \circ \psi : C_0(M) \to \Sigma^W(G)$. We construct a morphism $\mathrm{ind}_{rel}^{G,F} : K_*(\mu) \to K_*(C^*(G_W))$ and show that the full index map $\widetilde{\mathrm{ind}}^{G,F}$ factors through it.

Fredholm realization. Following [1, 2], we also address the following question. When can an elliptic symbol σ for the groupoid G, which defines an element $\sigma \in K_1(C_0(\mathbb{S}\mathfrak{A}^*G))$, be lifted to an element in $\Psi^*(G)$ which is invertible modulo $C^*(G_W)$?
A particular case of interest is when G_W is the pair groupoid $W \times W$. In this case we are just asking whether this symbol can be lifted to a Fredholm operator?
Our constructions allow us to answer naturally this question (prop. 4.3).

Computation in a particular case. In some natural cases, the index of G_F is an invertible element of $KK(C_0(\mathfrak{A}^*G_F), C^*(G_F))$. This happens in particular for the adiabatic groupoids and gauge adiabatic groupoids.
In that case, we are able to compute the elements $\partial^{G,F}$ and $\widetilde{\mathrm{ind}}^{G,F}$ in terms of the index element of the groupoid G.
Moreover, under this hypothesis, the relative K-group $K_*(\mu)$ is canonically isomorphic to $K_*(C_0(\mathfrak{A}^*G_W))$; and through this isomorphism $\mathrm{ind}_{rel}^{G,F}$ identifies with the index map of the groupoid G_W.

Connes-Thom elements We will finally examine the following situation which turns out to be quite useful in our applications [14].
Let $\mathbb{R}_+^*$ act on a Lie groupoid G smoothly, freely and properly by groupoid automorphisms. The quotient space $G/\mathbb{R}$ is naturally a Lie groupoid. Then we have Connes-Thom isomorphisms ([6]) $\beta^G \in KK^1(C^*(G/\mathbb{R}_+^*), C^*(G))$ and $\beta_\Psi^G \in KK^1(\Psi^*(G/\mathbb{R}_+^*), \Psi^*(G))$.
If W is an open $\mathbb{R}_+^*$ invariant saturated subset of $G^{(0)}$, then we also construct a Connes-Thom isomorphism $\beta_\Sigma^{(G,W)} \in KK^1(\Sigma^{W/\mathbb{R}_+^*}((G/\mathbb{R}_+^*), \Sigma^W(G))$.
Moreover, the connecting elements $\partial^{G,F}$ and $\partial^{G/\mathbb{R}_+^*,F/\mathbb{R}_+^*}$ as well as the full index elements $\widetilde{\mathrm{ind}}^{G,F}$ and $\widetilde{\mathrm{ind}}^{G/\mathbb{R}_+^*,F/\mathbb{R}_+^*}$ correspond to each other under these Connes-Thom isomorphisms, as well as the corresponding relative K-theory maps.

The paper is organized as follows:

- In section 2 we recall some classical facts, constructions and notation involving groupoids.
- Section 3 is a brief reminder of a quite classical facts about connecting elements associated to short exact sequences of C^*-algebras.
- Section 4 contains our main results. Given a Lie groupoid and an open saturated subset of its unit space, we consider connecting maps and full index maps, compare them, compute them in some cases... In particular, we study a Fredholm realizability problem generalizing works of Albin and Melrose ([1, 2]) and index maps using relative K-theory.
- In section 5 we study a proper action of $\mathbb{R}_+^*$ on a Lie groupoid G with an open saturated subset wich is $\mathbb{R}_+^*$-invariant. We compare the connecting maps and the index maps of G with those of $G/\mathbb{R}_+^*$, using Connes' analogue of the Thom isomorphism.

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The present paper gathers the general algebraic constructions of natural KK-elements involved in the study of groupoids of the article [14] that appeared on the arXiv (arXiv:1705.09588). Since [14] was quite long and addressed a large variety of situations, we decided to split it into two pieces hoping to make it easier to read. The second piece is focussed on some geometrical constructions of deformation to the normal cone and blowup type together with application of results presented here in order to study some corresponding index problems. We have kept in the next section some general material of [14] needed in both papers.

2 Some quite classical constructions involving groupoids

2.1 Some classical notation

Throughout the paper, if E is a real vector bundle over a locally compact space M, we denote by $\mathbb{S}E$ the corresponding sphere bundle: it is fiber the bundle over M whose fiber over a point x of M is the space of half lines in E_x. The (total space of) the bundle $\mathbb{S}E$ is simply the quotient of $E \setminus M$ by the natural action of $\mathbb{R}_+^*$ by dilation.

Let G be a Lie groupoid. We denote by $G^{(0)}$ its space of objects and $r, s : G \to G^{(0)}$ the range and source maps.
The algebroid of G is denoted by $\mathfrak{A}G$, and its anchor by $\natural : \mathfrak{A}G \to TG^{(0)}$. Recall that (the total space of) $\mathfrak{A}G$ is the normal bundle $N_{G^{(0)}}^G$ and the anchor map is induced by $(dr - ds)$.
We denote by $\mathfrak{A}^*G$ the dual bundle of $\mathfrak{A}G$ and by $\mathbb{S}\mathfrak{A}^*G$ the sphere bundle of $\mathfrak{A}^*G$.

- We denote by $C^*(G)$ its (full or reduced) C^*-algebra. We denote by $\Psi^*(G)$ the C^*-algebra of order ≤ 0 (classical, *i.e.* polyhomogeneous) pseudodifferential operators on G vanishing at infinity on $G^{(0)}$ (if $G^{(0)}$ is not compact). More precisely, it is the norm closure in the multiplier algebra of $C^*(G)$ of the algebra of classical pseudodifferential operators on G with compact support in G.

  We have an exact sequence of C^*-algebras $0 \to C^*(G) \to \Psi^*(G) \to C_0(\mathbb{S}\mathfrak{A}^*G) \to 0$.

  As mentioned in the introduction, our constructions involve connecting maps associated to short exact sequences of groupoid C^*-algebras, therefore they make sens a priori for the full C^*-algebras, and give rise to E-theory elements ([8]). Nevertheless, in many interesting situations, the quotient C^*-algebra will be the C^*-algebra of an amenable groupoid, thus the corresponding exact sequence is semi-split as well as for the reduced and the full C^*-algebras and it defines moreover a KK-element. In these situations $C^*(G)$ may either be the reduced or the full C^*-algebra of the groupoid G and we have preferred to leave the choice to the reader.

- For $A, B \subset G^{(0)}$, we put $G^A = \{x \in G;\ r(x) \in A\}$ and $G_A = \{x \in G;\ s(x) \in A\}$; we also put $G_A^B = G_A \cap G^B$.

  Notice that A is a *saturated* subset of $G^{(0)}$ if and only if $G_A = G^A = G_A^A$.

- We denote by G_{ad} the adiabatic groupoid of G, ([25, 28]), it is obtained by using the deformation to the normal cone construction for the inclusion of $G^{(0)}$ as a Lie subgroupoid of G. Thus:
$$G_{ad} = G \times \mathbb{R}^* \cup \mathfrak{A}G \times \{0\} \rightrightarrows G^{(0)} \times \mathbb{R}\ .$$
  If X is a locally closed saturated subset of $M \times \mathbb{R}$, we will denote sometimes by $G_{ad}(X)$ the restriction $(G_{ad})_X^X$ of G_{ad} to X: it is a locally compact groupoid.

  In the sequel of the paper, we let $G_{ad}^{[0,1]} = G_{ad}(G^{(0)} \times [0,1])$ and $G_{ad}^{[0,1)} = G_{ad}(G^{(0)} \times [0,1))$ *i.e.*
$$G_{ad}^{[0,1]} = G \times (0,1] \cup \mathfrak{A}G \times \{0\} \rightrightarrows G^{(0)} \times [0,1] \quad \text{and} \quad G_{ad}^{[0,1)} = G \times (0,1) \cup \mathfrak{A}G \times \{0\} \rightrightarrows G^{(0)} \times [0,1).$$

Remark 2.1. Many manifolds and groupoids that occur in our constructions have boundaries or corners. In fact all the groupoids we consider sit naturally inside Lie groupoids *without boundaries* as restrictions to closed *saturated* subsets. This means that we consider subgroupoids $G_V^V = G_V$ of a Lie groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$ where V is a closed saturated subset of $G^{(0)}$. Such groupoids, have a natural algebroid, adiabatic deformation, pseudodifferential calculus, *etc.* that are restrictions to V and G_V of the corresponding objects on $G^{(0)}$ and G. We chose to give our definitions and constructions for Lie groupoids for the clarity of the exposition. The case of a longitudinally smooth groupoid over a manifold with corners is a straightforward generalization using a convenient restriction.

2.2 Morita equivalence

Two Lie groupoids $G_1 \overset{r,s}{\rightrightarrows} M_1$ and $G_2 \overset{r,s}{\rightrightarrows} M_2$ are *Morita equivalent* if there exists a *linking manifold* X with extra data: surjective smooth submersions $r : X \to G_1^{(0)}$ and $s : X \to G_2^{(0)}$ and compositions $G_1 \times_{s,r} X \to X$, $X \times_{s,r} G_2 \to X$, $X \times_{r,r} X \to G_2$ and $X \times_{s,s} X \to G_1$ with natural associativity conditions (see [26] for details).
If the map $r : X \to G_1^{(0)}$ is surjective but $s : X \to G_2^{(0)}$ is not necessarily surjective, then G_1 is Morita equivalent to the restriction of G_2 to the open saturated subspace $s(X)$. We say that G_1 is *sub-Morita* equivalent to G_2.

2.3 Semi-direct products

Action of a groupoid on a space. Recall that an action of a groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$ on a space V is given by a map $p : V \to G^{(0)}$ and the action $G \times_{s,p} V \to V$ denoted by $(g, x) \mapsto g.x$ with the requirements $p(g.x) = r(g)$, $g.(h.x) = (gh).x$ and $u.x = x$ if $u = p(x)$.

In that case, we may form the crossed product groupoid $V \rtimes G$:

- as a set $V \rtimes G$ is the fibered product $V \times_{p,r} G$;
- the unit space $(V \rtimes G)^{(0)}$ is V. The range and source maps are $r(x, g) = x$ and $s(x, g) = g^{-1}.x$;
- the composition is given by $(x, g)(y, h) = (x, gh)$ (with $g.y = x$).

If G is a Lie groupoid, M is a manifold and if all the maps defined are smooth, then $V \rtimes G$ is a Lie groupoid.

Action of a group on a groupoid. Let Γ be a Lie group acting on a Lie groupoid $G \overset{r,s}{\rightrightarrows} M$ by Lie groupoid automorphisms. The set $G \times \Gamma$ is naturally a Lie groupoid $G \rtimes \Gamma \overset{r_\rtimes, s_\rtimes}{\rightrightarrows} M$ we put $r_\rtimes(g, \gamma) = r(g)$, $s_\rtimes(g, \gamma) = \gamma^{-1}(s(g))$ and, when (g_1, γ_1) and (g_2, γ_2) are composable, their product is $(g_1, \gamma_1)(g_2, \gamma_2) = (g_1\, \gamma_1(g_2), \gamma_1\gamma_2)$.

Note that the semi-direct product groupoid $G \rtimes \Gamma$ is canonically isomorphic to the quotient $\mathbb{G}/\Gamma$ of the product $\mathbb{G} = G \times (\Gamma \times \Gamma)$ of G by the pair groupoid $\Gamma \times \Gamma$ where the Γ action on $\mathbb{G}$ is the diagonal one: $\gamma \cdot (g, \gamma_1, \gamma_2) = (\gamma(g), \gamma\gamma_1, \gamma\gamma_2)$.

Free and proper action of a group on a groupoid. When the action of Γ on G (and therefore on its closed subset $M = G^{(0)}$) is free and proper, we may define the quotient groupoid $G/\Gamma \overset{r,s}{\rightrightarrows} M/\Gamma$.

In that case, the groupoid G/Γ acts on M and the groupoid G identifies with the action groupoid $M \rtimes (G/\Gamma)$. Indeed, let $p : M \to M/\Gamma$ and $q : G \to G/\Gamma$ be the quotient maps. If $x \in M$ and $h \in G/\Gamma$ are such that $s(h) = p(x)$, then there exists a unique $g \in G$ such that $q(g) = h$ and $s(g) = x$; we put then $h.x = r(g)$. It is then immediate that $\varphi : G \to M \times_{p,r} (G/\Gamma)$ given by $\varphi(g) = (r(g), q(g))$ is a groupoid isomorphism.

The groupoid G/Γ is Morita equivalent to $G \rtimes \Gamma$: indeed one easily identifies $G \rtimes \Gamma$ with the pull back groupoid $(G/\Gamma)_q^q$ where $q : M \to M/\Gamma$ is the quotient map.

Note also that in this situation the action of Γ on G leads to an action of Γ on the Lie algebroid $\mathfrak{A}G$ and $\mathfrak{A}(G/\Gamma)$ identifies with $\mathfrak{A}G/\Gamma$.

Remark 2.2. As the Lie groupoids we are considering need not be Hausdorff, the properness condition has to be relaxed. We will just assume that the action is *locally proper*, *i.e.* that every point in G has a Γ-invariant neighborhood on which the action of Γ is proper.

Action of a groupoid on a groupoid. Recall that an action of a groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$ on a groupoid $H \overset{r_H,s_H}{\rightrightarrows} H^{(0)}$ is by groupoid automorphisms (*cf.* [5]): if G acts on $H^{(0)}$ through a map $p_0 : H^{(0)} \to G^{(0)}$, we have $p = p_0 \circ r_H = p_0 \circ s_H$ and $g.(xy) = (g.x)(g.y)$.

In that case, we may form the crossed product groupoid $H \rtimes G = \mathbb{G}$:

- as a set $H \rtimes G$ is the fibered product $H \times_{p,r} G$;
- the unit space $\mathbb{G}^{(0)}$ of $\mathbb{G} = H \rtimes G$ is $H^{(0)}$. The range and source maps are $r_{\mathbb{G}}(x, g) = r_H(x)$ and $s_{\mathbb{G}}(x, g) = g^{-1}.s_H(x)$;
- the composition is given by $(x, g)(y, h) = (x(g.y), gh)$.

If G and H are Lie groupoids and if all the maps defined are smooth, then $\mathbb{G} = H \rtimes G$ is a Lie groupoid.

2.4 Index maps for Lie groupoids

Recall (*cf.* [25, 28]) that if G is any Lie groupoid, the index map is an element in $KK(C_0(\mathfrak{A}^*G), C^*(G))$ which can be constructed thanks to the adiabatic groupoid $G_{ad}^{[0,1]}$ of G as

$$\mathrm{ind}_G = [ev_0]^{-1} \otimes [ev_1]$$

where

$$ev_0 : C^*(G_{ad}^{[0,1]}) \to C^*(G_{ad}(0)) \simeq C_0(\mathfrak{A}^*G) \quad \text{and} \quad ev_1 : C^*(G_{ad}^{[0,1]}) \to C^*(G_{ad}(1)) \simeq C^*(G)$$

are the evaluation morphisms (recall that $[ev_0]$ is invertible).
It follows quite immediately that the element $\widetilde{\mathrm{ind}}_G \in KK^1(C(\mathbb{S}\mathfrak{A}^*G), C^*(G))$ corresponding to the pseudodifferential exact sequence

$$0 \to C^*(G) \to \Psi^*(G) \to C(\mathbb{S}\mathfrak{A}^*G) \to 0 \qquad E_{\Psi^*(G)}$$

is the composition $\widetilde{\mathrm{ind}}_G = \mathrm{ind}_G \otimes q_{\mathfrak{A}^*G}$ where $q_{\mathfrak{A}^*G} \in KK^1(C(\mathbb{S}\mathfrak{A}^*G), C_0(\mathfrak{A}^*G))$ corresponds to the pseudodifferential exact sequence for $\mathfrak{A}G$ which is

$$0 \to C_0(\mathfrak{A}^*G) \to C(B\mathfrak{A}^*G) \to C(\mathbb{S}\mathfrak{A}^*G) \to 0 \qquad E_{\Psi^*(\mathfrak{A}G)}$$

This connecting element is immediately seen to be the element of $KK(C_0(\mathbb{S}\mathfrak{A}^*G \times \mathbb{R}_+^*), C_0(\mathfrak{A}^*G))$ associated to the inclusion of $\mathbb{S}\mathfrak{A}^*G \times \mathbb{R}_+^*$ as the open subset $\mathfrak{A}^*G \setminus G^{(0)}$ - where $G^{(0)}$ sits in $\mathfrak{A}^*G$ as the zero section.

3 A (well known) remark on exact sequences

We will use the quite immediate (and well known) result:

Lemma 3.1. *Consider a commutative diagram of semi-split exact sequences of C^*-algebras*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_1 & \longrightarrow & A_1 & \xrightarrow{q_1} & B_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_J} & & \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \\ 0 & \longrightarrow & J_2 & \longrightarrow & A_2 & \xrightarrow{q_2} & B_2 & \longrightarrow & 0 \end{array}$$

a) *We have $\partial_1 \otimes [f_J] = [f_B] \otimes \partial_2$ where ∂_i denotes the element in $KK^1(B_i, J_i)$ associated with the exact sequence*

$$0 \longrightarrow J_i \longrightarrow A_i \longrightarrow B_i \longrightarrow 0.$$

b) *If two of the vertical arrows are KK-equivalences, then so is the third one.*

Notation 3.2. When $f : A \to B$ is a morphism of C^*-algebra, we will denote the corresponding mapping cone by $\mathsf{C}_f = \{(x, h) \in A \oplus B[0, 1) \ ; \ h(0) = f(x)\}$.

Proof. a) See *e.g.* [9]. Let C_{q_i} be the mapping cone of q_i and $j_i : B_i(0, 1) \to \mathsf{C}_{q_i}$ and $e_i : J_i \to \mathsf{C}_{q_i}$ the natural (excision) morphisms. The *excision morphism* e_i is K-invertible and $\partial_i = [j_i] \otimes [e_i]^{-1}$.

b) For every separable C^*-algebra D, by applying the "five lemma" to the diagram

$$\begin{array}{ccccccccccc} ... & \longrightarrow & KK^*(D, J_1) & \longrightarrow & KK^*(D, A_1) & \longrightarrow & KK^*(D, B_1) & \longrightarrow & KK^{*+1}(D, J_1) & \longrightarrow & ... \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ ... & \longrightarrow & KK^*(D, J_2) & \longrightarrow & KK^*(D, A_2) & \longrightarrow & KK^*(D, B_2) & \longrightarrow & KK^{*+1}(D, J_2) & \longrightarrow & ... \end{array}$$

we find that all vertical arrows are invertible. Applying this to $D = J_2$ (*resp.* A_2, B_2) we find a one sided inverse to $[f_J]$ (*resp.* f_A, f_B). Applying this again to $D = J_1$ (*resp.* A_1, B_1), it follows that this inverse is two-sided. □

4 Saturated open subsets, connecting maps and full index map

Let $G \rightrightarrows M$ be a Lie groupoid and F be a closed subset of M saturated for G. Put $W = M \setminus F$. Denote by G_W the open subgroupoid $G_W = G_W^W$ of G and G_F its complement. If F is not a submanifold, then G_F is not a Lie groupoid, but as explained in remark 2.1, we still can define $\Psi^*(G_F)$ (it is the quotient $\Psi^*(G)/\Psi^*(G_W)$) the symbol map, *etc.*

Define the *full symbol algebra* $\Sigma^W(G)$ to be the quotient $\Psi^*(G)/C^*(G_W)$ and the *full symbol map* to be the quotient map $\Psi^*(G) \to \Psi^*(G)/C^*(G_W)$. Looking at the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^*(G_W) & \longrightarrow & C^*(G) & \longrightarrow & C^*(G_F) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} & & \\ 0 & \longrightarrow & \Psi^*(G_W) & \longrightarrow & \Psi^*(G) & \longrightarrow & \Psi^*(G_F) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*G_W) & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*G) & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*G_F) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

we see that $\Sigma^W(G)$ is the fibered product $C_0(\mathbb{S}\mathfrak{A}^*G) \oplus_{C_0(\mathbb{S}\mathfrak{A}^*G_F)} \Psi^*(G_F)$. The full symbol map is thus composed of

- the usual commutative symbol map: the map $\Psi^*(G) \to C_0(\mathbb{S}\mathfrak{A}^*G)$;
- the restriction to the singular part: the map $\Psi^*(G) \to \Psi^*(G_F)$.

In this section we will be interested in the description of elements $\partial_G^W \in KK^1(C^*(G_F), C^*(G_W))$ and $\widetilde{\mathrm{ind}}_{full}^W(G) \in KK^1(\Sigma^W(G), C^*(G_W))$ associated to the exact sequences

$$0 \longrightarrow C^*(G_W) \longrightarrow C^*(G) \longrightarrow C^*(G_F) \longrightarrow 0 \qquad E_\partial$$

and

$$0 \longrightarrow C^*(G_W) \longrightarrow \Psi^*(G) \longrightarrow \Sigma^W(G) \longrightarrow 0. \qquad E_{\widetilde{\mathrm{ind}}_{full}}$$

To that end, it will be natural to assume that the restriction G_F of G to F is amenable - so that the above sequences are exact and semi-split for the reduced as well as the full groupoid algebra.

4.1 Connecting map and index

Assume that the groupoid G_F is amenable. We have a diagram

$$\begin{array}{rccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
E_\partial : & 0 & \longrightarrow & C^*(G_W) & \longrightarrow & C^*(G) & \longrightarrow & C^*(G_F) & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow j & & \\
E_{\widetilde{\mathrm{ind}}_{full}} : & 0 & \longrightarrow & C^*(G_W) & \longrightarrow & \Psi^*(G) & \longrightarrow & \Sigma^W(G) & \longrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & C_0(\mathbb{S}\mathfrak{A}^*G) & = & C_0(\mathbb{S}\mathfrak{A}^*G) & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
\end{array}$$

It follows from that Lemma 3.1 that we have the equality $\partial_G^W = j^*(\widetilde{\mathrm{ind}}_{full}^W(G))$.

4.2 Connecting maps

Proposition 4.1. *Let $\partial_G^W \in KK^1(C^*(G_F), C^*(G_W))$ be the element associated with the exact sequence*

$$0 \longrightarrow C^*(G_W) \longrightarrow C^*(G) \longrightarrow C^*(G_F) \longrightarrow 0.$$

Similarly, let $\partial_{\mathfrak{A}G}^W \in KK^1(C_0((\mathfrak{A}^*G)_{|F}), C_0((\mathfrak{A}^*G)_{|W}))$ be associated with the exact sequence

$$0 \longrightarrow C_0((\mathfrak{A}^*G)_{|W}) \longrightarrow C_0(\mathfrak{A}^*G) \longrightarrow C_0((\mathfrak{A}^*G)_{|F}) \longrightarrow 0.$$

We have $\partial_{\mathfrak{A}G}^W \otimes \mathrm{ind}_{G_W} = \mathrm{ind}_{G_F} \otimes \partial_G^W$.

Proof. Indeed, we just have to apply twice Lemma 3.1 using the adiabatic deformation $G_{ad}^{[0,1]}$ and the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0((\mathfrak{A}^*G)_{|W})) & \longrightarrow & C_0(\mathfrak{A}^*G) & \longrightarrow & C_0((\mathfrak{A}^*G)_{|F})) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle ev_0} & & \uparrow{\scriptstyle ev_0} & & \uparrow{\scriptstyle ev_0} & & \\ 0 & \longrightarrow & C^*(G_{ad}(W\times[0,1]) & \longrightarrow & C^*(G_{ad}^{[0,1]}) & \longrightarrow & C^*(G_{ad}(F\times[0,1])) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle ev_1} & & \downarrow{\scriptstyle ev_1} & & \downarrow{\scriptstyle ev_1} & & \\ 0 & \longrightarrow & C^*(G_W) & \longrightarrow & C^*(G) & \longrightarrow & C^*(G_F) & \longrightarrow & 0 \end{array}$$

□

4.3 A general remark on the index

In the same way as the index $\mathrm{ind}_G \in KK(C_0(\mathfrak{A}^*G), C^*(G))$ constructed using the adiabatic groupoid is more primitive and to some extent easier to handle than $\widetilde{\mathrm{ind}}_G \in KK^1(C_0(\mathbb{S}\mathfrak{A}^*G), C^*(G))$ constructed using the exact sequence of pseudodifferential operators, there is in this "relative" situation a natural more primitive element.

Denote by $\mathfrak{A}_W G = G_{ad}(F \times [0,1) \cup W \times \{0\})$ the restriction of G_{ad} to the saturated locally closed subset $F \times [0,1) \cup W \times \{0\}$. Note that, since we assume that G_F is amenable, and since $\mathfrak{A}G$ is also amenable (it is a bundle groupoid), the groupoid $\mathfrak{A}_W G$ is amenable.

Similarly to [10, 11], we define the *noncommutative algebroid* of G relative to F to be $C^*(\mathfrak{A}_W G)$. Note that by definition we have:

$$C^*(G_{ad}^{[0,1)})/C^*(G_{ad}(W \times (0,1)) = C^*(G_{ad}(F \times [0,1) \cup W \times \{0\}) = C^*(\mathfrak{A}_W G)$$

We have an exact sequence

$$0 \to C^*(G_W \times (0,1]) \longrightarrow C^*(G_{ad}(F \times [0,1) \cup W \times [0,1])) \xrightarrow{ev_0} C^*(\mathfrak{A}_W G) \to 0,$$

where $ev_0 : C^*(G_{ad}(F \times [0,1) \cup W \times [0,1])) \to C^*(G_{ad}(F \times [0,1) \cup W \times \{0\}) = C^*(\mathfrak{A}_W G)$ is the restriction morphism. As $C^*(G_W \times (0,1])$ is contractible the KK-class $[ev_0] \in KK(C^*(G_{ad}(F \times [0,1) \cup W \times [0,1])), C^*(\mathfrak{A}_W G))$ is invertible. Let $ev_1 : C^*(G_{ad}(F \times [0,1) \cup W \times [0,1])) \to C^*(G_W)$ be the usual evaluation at 1. We put:

$$\mathrm{ind}_G^W = [ev_0]^{-1} \otimes [ev_1] \in KK(C^*(\mathfrak{A}_W G), C^*(G_W)) \ .$$

Recall from [12, Rem 4.10] and [13, Thm. 5.16] that there is a natural action of $\mathbb{R}$ on $\Psi^*(G)$ such that $\Psi^*(G) \rtimes \mathbb{R}$ is an ideal in $C^*(G_{ad}^{[0,1)})$ (using a homeomorphism of $[0,1)$ with $\mathbb{R}_+$). This ideal is the kernel of the composition $C^*(G_{ad}^{[0,1)}) \xrightarrow{\mathrm{ev}_0} C_0(\mathfrak{A}^*G) \to C(M)$.

Recall that the restriction to $C^*(G)$ of the action of $\mathbb{R}$ is inner. It follows that $C^*(G_W) \subset \Psi^*(G)$ is invariant by the action of $\mathbb{R}$ - and $C^*(G_W) \rtimes \mathbb{R} = C^*(G_W) \otimes C_0(\mathbb{R}) = C^*(G_{ad}(W \times (0,1)))$.

We thus obtain an action of $\mathbb{R}$ on $\Sigma^W(G) = \Psi^*(G)/C^*(G_W)$ and an inclusion $i : \Sigma^W(G) \rtimes \mathbb{R} \hookrightarrow C^*(\mathfrak{A}_W G)$.

Proposition 4.2. *The element* $\widetilde{\mathrm{ind}}_{full}^W \in KK^1(\Sigma^W(G), C^*(G_W))$ *corresponding to the exact sequence*

$$0 \longrightarrow C^*(G_W) \longrightarrow \Psi^*(G) \longrightarrow \Sigma^W(G) \longrightarrow 0. \qquad E_{\widetilde{\mathrm{ind}}_{full}}$$

is the Kasparov product of:

- *the Connes-Thom element* $[th] \in KK^1(\Sigma^W(G), \Sigma^W(G) \rtimes \mathbb{R})$ *([6, 15]);*
- *the inclusion* $i : \Sigma^W(G) \rtimes \mathbb{R} \hookrightarrow C^*(\mathfrak{A}_W G)$*;*

- *the index* $\mathrm{ind}_G^W = [ev_0]^{-1} \otimes [ev_1]$ *defined above.*

Proof. By naturality of the Connes Thom element, it follows that

$$\widetilde{\mathrm{ind}}^W_{full} \otimes [B] = -[th] \otimes [\partial]$$

where $\partial \in KK^1(\Sigma^W(G) \rtimes \mathbb{R}, C^*(G_W \times (0,1)))$ is the KK^1-element corresponding with the exact sequence

$$0 \longrightarrow C^*(G_W) \rtimes \mathbb{R} \longrightarrow \Psi^*(G) \rtimes \mathbb{R} \longrightarrow \Sigma^W(G) \rtimes \mathbb{R} \longrightarrow 0$$

and $[B] \in KK^1(C^*(G_W), C^*(G_W) \rtimes \mathbb{R})$ is the Connes-Thom element. Note that, since the action is inner, $[B]$ identifies with the Bott element in $KK^1(C^*(G_W), C^*(G_W) \otimes C_0(\mathbb{R}))$ under the natural isomorphism $C^*(G_W) \rtimes \mathbb{R} \simeq C^*(G_W) \otimes C_0(\mathbb{R})$.
By the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(G_W) \rtimes \mathbb{R} & \longrightarrow & \Psi^*(G) \rtimes \mathbb{R} & \longrightarrow & \Sigma^W(G) \rtimes \mathbb{R} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & C^*(G_W \times (0,1)) & \longrightarrow & C^*(G_{ad}^{[0,1)}) & \longrightarrow & C^*(\mathfrak{A}_W G) & \longrightarrow & 0
\end{array}$$

we deduce that $[\partial] = i^*[\partial']$ where ∂' corresponds to the second exact sequence.
Finally, we have a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C^*(G_W \times (0,1)) & \longrightarrow & C^*(G_{ad}^{[0,1)}) & \longrightarrow & C^*(\mathfrak{A}_W G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C^*(G_W \times (0,1]) & \longrightarrow & C^*(G_{ad}(F \times [0,1) \cup W \times [0,1])) & \xrightarrow{ev_0} & C^*(\mathfrak{A}_W G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle ev_1} & & & & \\
 & & C^*(G_W) & = & C^*(G_W) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where exact sequences are semisplit. Now the connecting element corresponding to the exact sequence

$$0 \longrightarrow C^*(G_W \times (0,1)) \longrightarrow C^*(G_{ad}(F \times [0,1) \cup W \times [0,1])) \xrightarrow{ev_0 \oplus ev_1} C^*(\mathfrak{A}_W G) \oplus C^*(G_W) \longrightarrow 0$$

is $[\partial'] \oplus [B]$ and it follows that

$$[ev_0] \otimes [\partial'] + [ev_1] \otimes [B] = 0.$$

As $\widetilde{\mathrm{ind}}^W_{full} \otimes [B] = -[th] \otimes [i] \otimes [\partial']$ and $[\partial'] = -[ev_0]^{-1} \otimes [ev_1] \otimes [B]$, the result follows from invertibility of the Bott element. $\square$

4.4 Relative K-theory and full index

It is actually better to consider the index map in a relative K-theory setting. Indeed, the starting point of the index problem is a pair of bundles $E_\pm$ over M together with a pseudodifferential operator P from sections of E_+ to sections of E_- which is invertible modulo $C^*(G_W)$. Consider the morphism

$\psi : C_0(M) \to \Psi^*(G)$ which associates to a (smooth) function f the order 0 (pseudo)differential operator multiplication by f and $\sigma_{full} : \Psi^*(G) \to \Sigma^W(G)$ the full symbol map.
Put $\mu = \sigma_{full} \circ \psi$.
By definition, for any $P \in \Psi^*(G)$, the triple $(E_\pm, \sigma_{full}(P))$ is an element in the relative K-theory of the morphism μ. The index $\cdot \otimes \widetilde{\mathrm{ind}}^W_{full}(G)$ considered in the previous section is the composition of the morphism $K_1(\Sigma^W(G)) \to K_0(\mu)$ [1] with the index map $\mathrm{ind}_{rel} : K_0(\mu) \to K_0(C^*(G_W))$ which to $(E_\pm, \sigma_{full}(P))$ associates the class of P.
The morphism ind_{rel} can be thought of as the composition of the obvious morphism $K_0(\mu) \to K_0(\sigma_{full}) \simeq K_0(\ker(\sigma_{full})) = K_0(C^*(G_W))$.

4.5 Fredholm realization

Let σ be a classical elliptic symbol which defines an element in $K_1(C_0(\mathbb{S}\mathfrak{A}^*G))$. A natural question is: when can this symbol be lifted to a pseudodifferential element which is invertible modulo $C^*(G_W)$? In particular, if G_W is the pair groupoid $W \times W$ *i.e.* if we are dealing with Lie manifolds in the sense of [27], this question reads: when can this symbol be extended to a Fredholm operator? Particular cases of this question were studied in [1, 2].
This is handled by the following proposition.

Proposition 4.3. *Let σ be an invertible element in $M_n(C_0(\mathbb{S}\mathfrak{A}^*G)^+)$ (where $C_0(\mathbb{S}\mathfrak{A}^*G)^+$ is obtained by adjoining a unit to $C_0(\mathbb{S}\mathfrak{A}^*G)$ - if $G^{(0)}$ is not compact). Then the following are equivalent.*

(i) *There exists $p \in \mathbb{N}$ and an invertible element $x \in M_{n+p}(\Sigma^W(G)^+)$ such that $q(x) = \sigma \oplus 1_p$.*

(ii) *The class $[\sigma]$ of σ in $K^1(C_0(\mathbb{S}\mathfrak{A}^*G)^+)$ is in the image of the morphism $q : \Sigma^W(G) \to C_0(\mathbb{S}\mathfrak{A}^*G)$.*

(iii) *The image of $[\sigma]$ by the connecting map of the exact sequence*

$$0 \longrightarrow C^*(G_F) \longrightarrow \Sigma^W(G) \xrightarrow{q} C_0(\mathbb{S}\mathfrak{A}^*G) \longrightarrow 0 \qquad (E_1)$$

vanishes.

(iv) *The image of the restriction $[\sigma_F] \in K^1(C_0(\mathbb{S}\mathfrak{A}^*G_F))$ by the connecting map of the exact sequence*

$$0 \longrightarrow C^*(G_F) \longrightarrow \Psi^*(G_F) \xrightarrow{q} C_0(\mathbb{S}\mathfrak{A}^*G_F) \longrightarrow 0 \qquad (E_2)$$

vanishes.

(v) *The index of σ in $K_0(C^*(G))$ is in the image of $K_0(C^*(G_W))$ through the K-theory map associated with the inclusion $C^*(G_W) \to C^*(G)$.*

Proof. The equivalence (i)⇔(ii) is a classical fact in topological K-theory. Indeed, (i)⇒(ii) is obvious. Conversely, if the image of σ via the connecting map of E vanishes, then the class of σ in $K_1(C_0(\mathbb{S}\mathfrak{A}^*G))$ is in the image of $K_1(\Sigma^W(G))$. This means that there exists $p \in \mathbb{N}$ and an invertible element $x \in M_{n+p}(\Sigma^W(G)^+)$ such that $q(x)$ and $\sigma \oplus 1_p$ are in the same path connected component of $GL_{n+p}(C_0(\mathbb{S}\mathfrak{A}^*G)^+)$.
Now the morphism $q : M_{n+p}(\Sigma^W(G)^+) \to M_{n+p}(C_0(\mathbb{S}\mathfrak{A}^*G)^+)$ is open and therefore the image of the connected component $GL_{n+p}(\Sigma^W(G)^+)_{(0)}$ of 1_{n+p} in $GL_{n+p}(\Sigma^W(G)^+)$ is an open (and therefore also closed) subgroup of $GL_{n+p}(C_0(\mathbb{S}\mathfrak{A}^*G)^+)$.
It follows immediately that $q\Big(GL_{n+p}(\Sigma^W(G)^+)_{(0)}\Big) = GL_{n+p}(C_0(\mathbb{S}\mathfrak{A}^*G)^+)_{(0)}$.
Finally $(\sigma \oplus 1_p)x^{-1}$ is in the image of $GL_{n+p}(\Sigma^W(G)^+)_{(0)}$, therefore $\sigma \oplus 1_p$ can be lifted to an invertible element of $M_n(\Sigma^W(G)^+)$.

[1] Recall that if $f : A \to B$ is a morphism of C^*-algebras, we have a natural morphism $u : K_{*+1}(B) \to K_*(f)$ corresponding to the inclusion of the suspension of B in the cone of f.

(ii)⇔(iii) follows immediately from the six-term exact sequence associated to exact sequence (E_1).

Considering the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(G_F) & \longrightarrow & \Sigma^W(G) & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*G) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^*(G_F) & \longrightarrow & \Psi^*(G_F) & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*G_F) & \longrightarrow & 0 \end{array}$$

we find that $\partial_{E_1}([\sigma]) = \partial_{E_2}([\sigma_F]) = \mathrm{ind}([\sigma_F])$. And thus (iii)⇔(iv).

Now, the index of $[\sigma_F]$ in $C^*(G_F)$ is the image of the index of $[\sigma]$ in $C^*(G)$. Therefore (iv)⇔(v) follows from the six-term exact sequence associated to exact sequence

$$0 \to C^*(G_W) \to C^*(G) \to C^*(G_F) \to 0$$

□

Of course, there is an analogous statement for a class in $K_0(C_0(\mathbb{S}\mathfrak{A}^*G))$ rather than K_1.

4.6 Full symbol algebra and index when ind_{G_F} is invertible

In this section, we will make a quite strong assumption on the groupoid G_F: we will assume that the index element $\mathrm{ind}_{G_F} \in KK(C_0((\mathfrak{A}^*G)_{|F}), C^*(G_F))$ is invertible. Under this assumption we may compute the K-theory of the C^*-algebras $C^*(G_F)$ and $\Sigma^W(G)$. We will show that the connecting element $\partial^{G,F}$ and the full index element $\widetilde{\mathrm{ind}}^{G,F}$ both factor through the index element $\mathrm{ind}_{G_W} \in KK(C_0(\mathfrak{A}^*G_W), C^*(G_W))$ of the Lie groupoid G_W.
This assumption means that the groupoid G_F satisfies the Baum-Connes conjecture and that the classifying space for proper actions is G_F itself.
It is satisfied in practice when G_F is a bundle of simply connected solvable Lie groups - such as adiabatic groupoids or "gauge adiabatic groupoids" (see [12]). It is useful in our applications in [14].
From prop. 4.1 we immediately find the following result.

Proposition 4.4. *If the index element* $\mathrm{ind}_{G_F} \in KK(C_0((\mathfrak{A}^*G)_{|F}), C^*(G_F))$ *is invertible, then the element* ∂_G^W *is the composition* $\mathrm{ind}_{G_F}^{-1} \otimes \partial_{\mathfrak{A}G}^W \otimes \mathrm{ind}_{G_W}$. □

Let us now pass to the computation of the full index.

Using the notation introduced in section 4.4, we denote by

$$\Psi_F^*(G) = \psi(C_0(M)) + \Psi^*(G_W)$$

the subalgebra of $\Psi^*(G)$ made of pseudodifferential operators which become trivial (*i.e.* multiplication operators) on F. Let $\Sigma_F(G)$ be the algebra of the corresponding symbols:

$$\Sigma_F(G) = \Psi_F^*(G)/C^*(G_W) = \mu(C_0(M)) + C_0(\mathbb{S}\mathfrak{A}^*G_W) \ .$$

It is the subalgebra of $C_0(\mathbb{S}\mathfrak{A}^*G)$ of symbols $a(x,\xi)$ with $x \in M$ and $\xi \in (\mathbb{S}\mathfrak{A}^*G)_x$ whose restriction on F does not depend on ξ. Its spectrum is the quotient $(\mathbb{S}\mathfrak{A}^*G)_{|F}$ of $\mathbb{S}\mathfrak{A}^*G$ where we identify (x,ξ) with (x,η) for every $x \in F$ and any $\xi, \eta \in (\mathbb{S}\mathfrak{A}^*G)_x$.

Lemma 4.5. *Assume that the index element* $\mathrm{ind}_{G_F} \in KK(C_0((\mathfrak{A}^*G)_{|F}), C^*(G_F))$ *is invertible, i.e. that the* C^**-algebra of the adiabatic groupoid* $C^*(G_{ad}(F \times [0,1)))$ *is* K*-contractible.*

a) *The inclusion* $j_\psi : C_0(F) \to \Psi^*(G_F)$ *is a* KK*-equivalence.*

b) *The inclusion* $j_\sigma : \mathbb{C}_0((\mathbb{S}\mathfrak{A}^*G)_{|F}) = \Sigma_F(G) = \Psi_F^*(G)/C^*(G_W) \to \Sigma^W(G)$ *is also a* KK*-equivalence.*

Proof. a) Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0((\mathfrak{A}^*G)_{|F}) & \longrightarrow & C_0((B\mathfrak{A}^*G)_{|F}) & \longrightarrow & C_0((\mathbb{S}\mathfrak{A}^*G)_{|F}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle ev_0} & & \uparrow{\scriptstyle ev_0} & & \uparrow{\scriptstyle ev_0} & & \\
0 & \longrightarrow & C^*(G_{ad}(F \times [0,1])) & \longrightarrow & \Psi^*(G_{ad}(F \times [0,1])) & \longrightarrow & C((\mathbb{S}\mathfrak{A}^*G)_{|F} \times [0,1]) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle ev_1} & & \downarrow{\scriptstyle ev_1} & & \downarrow{\scriptstyle ev_1} & & \\
0 & \longrightarrow & C^*(G_F) & \longrightarrow & \Psi^*(G_F) & \longrightarrow & C((\mathbb{S}\mathfrak{A}^*G)_{|F}) & \longrightarrow & 0
\end{array}$$

where the horizontal exact sequences are the pseudodifferential exact sequences $E_{\Psi^*(\mathfrak{A}G)_F}$, $E_{\Psi^*(G_{ad}(F\times[0,1]))}$ and $E_{\Psi^*(G_F)}$. Since ind_{G_F} is invertible $ev_1 : C^*(G_{ad}(F \times [0,1])) \to C^*(G_F)$ is a KK-equivalence. Hence, the left and right vertical arrows are all KK-equivalences, and therefore so are the middle ones. The inclusion $C_0(F)$ in $C_0((B\mathfrak{A}^*G)_{|F})$ is a homotopy equivalence and therefore the inclusions $C_0(F) \to \Psi^*(G_{ad}(F \times [0,1]))$ and $C_0(F) \to \Psi^*(G_F)$ are KK-equivalences.

b) Apply Lemma 3.1 to the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Psi^*(G_W) & \longrightarrow & \Psi^*_F(G) & \longrightarrow & C_0(F) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle J_\psi} & & \downarrow{\scriptstyle j_\psi} & & \\
0 & \longrightarrow & \Psi^*(G_W) & \longrightarrow & \Psi^*(G) & \longrightarrow & \Psi^*(G_F) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(G_W) & \longrightarrow & \Psi^*_F(G) & \longrightarrow & \Sigma_F(G) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle J_\psi} & & \downarrow{\scriptstyle j_\sigma} & & \\
0 & \longrightarrow & C^*(G_W) & \longrightarrow & \Psi^*(G) & \longrightarrow & \Sigma^W(G) & \longrightarrow & 0
\end{array}$$

we find that J_Ψ and j_σ are K-equivalences. ∎

The diagram in lemma 4.5.b) shows that $\partial_F = j_\sigma^*(\widetilde{\mathrm{ind}}^W_{full}(G))$ where $\partial_F \in KK^1(\Sigma_F(G), C^*(G_W))$ is the KK-element associated with the exact sequence

$$0 \longrightarrow C^*(G_W) \longrightarrow \Psi^*_F(G) \longrightarrow \Sigma_F(G) \longrightarrow 0.$$

So, let's compute the KK-theory of $\Sigma_F(G)$ and the connecting element ∂_F.

Consider the vector bundle $\mathfrak{A}G$ as a Lie groupoid (with objects M). It is its own Lie algebroid.

- Its C^*-algebra $C^*(\mathfrak{A}G)$ identifies with $C_0(\mathfrak{A}^*G)$ and $C^*(\mathfrak{A}G_W)$ with $C_0(\mathfrak{A}^*G_W)$.
- The spectrum of the commutative C^*-algebra $\Psi^*(\mathfrak{A}G)$ identifies with the total space $B\mathfrak{A}^*G$ of the bundle of closed balls in $\mathfrak{A}^*G$, whence $\Psi^*(\mathfrak{A}G)$ identifies with $C_0(B\mathfrak{A}^*G)$; it is homotopy equivalent to $C_0(M)$. The inclusion $C^*(\mathfrak{A}G) \to \Psi^*(\mathfrak{A}G)$ identifies with the inclusion $C_0(\mathring{B}\mathfrak{A}^*G) \to C_0(B\mathfrak{A}^*G)$, where $\mathring{B}\mathfrak{A}^*G$ is the total space of the bundle of open balls in $\mathfrak{A}^*G$.
- The spectrum of $\Psi^*_F(\mathfrak{A}G)$ is denoted by $B_F\mathfrak{A}^*G$; it is the quotient of $B\mathfrak{A}^*G$ where we identify two points (x, ξ) and (x, η) for $x \in F$; it is also homotopy equivalent to $C_0(M)$.
- Since the algebroid of $\mathfrak{A}G$ is $\mathfrak{A}G$ itself, $\Sigma_F(\mathfrak{A}G) = \Sigma_F(G)$; the spectrum of this commutative C^*-algebra is $\mathbb{S}_F\mathfrak{A}^*G$ which is the image of $\mathbb{S}\mathfrak{A}^*G$ in $B_F\mathfrak{A}^*G$.

We further note.

a) Let $k : C_0(\mathfrak{A}^*G_W) \to C_0(M)$ be given by $k(f)(x) = \begin{cases} f(x,0) & \text{if } x \in W \\ 0 & \text{if } x \in F \end{cases}$. We find a commutative diagram
$$\begin{array}{ccc} C_0(\mathring{B}\mathfrak{A}^*G_W) & \longrightarrow & C_0(B_F\mathfrak{A}^*G) \\ \downarrow & & \downarrow \\ C_0(\mathfrak{A}^*G_W) & \xrightarrow{k} & C_0(M) \end{array}$$
where the vertical arrows are homotopy equivalences.

b) The exact sequence $0 \to C^*(\mathfrak{A}G_W) \to \Psi^*_F(\mathfrak{A}G) \to \Sigma_F(\mathfrak{A}G) \to 0$, reads $0 \to C_0(\mathring{B}\mathfrak{A}^*G_W) \to C_0(B_F\mathfrak{A}^*G) \to C_0(S_F\mathfrak{A}^*G) \to 0$.

We deduce using successively (b) and (a):

Proposition 4.6. a) *The algebra $C_0(S_F\mathfrak{A}^*G)$ is KK^1-equivalent with the mapping cone of the inclusion $C_0(\mathring{B}\mathfrak{A}^*G_W) \to C_0(B_F\mathfrak{A}^*G)$.*

b) *This mapping cone is homotopy equivalent to the mapping cone of the morphism k.*

□

Note finally that we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0((\mathfrak{A}^*G)_{|W}) & \longrightarrow & \Psi^*_F(\mathfrak{A}G) & \longrightarrow & \Sigma_F(\mathfrak{A}G) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle ev_0} & & \uparrow{\scriptstyle ev_0} & & \uparrow{\scriptstyle ev_0} & & \\ 0 & \longrightarrow & C^*(G_{ad}(W\times[0,1])) & \longrightarrow & \Psi^*_{F\times[0,1]}(G_{ad}) & \longrightarrow & \Sigma_{F\times[0,1]}(G_{ad}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle ev_1} & & \downarrow{\scriptstyle ev_1} & & \downarrow{\scriptstyle ev_1} & & \\ 0 & \longrightarrow & C^*(G_W) & \longrightarrow & \Psi^*_F(G) & \longrightarrow & \Sigma_F(\mathfrak{A}G) & \longrightarrow & 0 \end{array}$$

The right vertical arrows are KK-equivalences, and therefore we find $\partial \otimes [ev_0]^{-1} \otimes [ev_1] = \partial_F$, where ∂ is the connecting element of the first horizontal exact sequence. To summarize, we have proved:

Proposition 4.7. *Assume that the index element $\mathrm{ind}_{G_F} \in KK(C_0((\mathfrak{A}^*G)_{|F}), C^*(G_F))$ is invertible.*

a) *The inclusion $j_\sigma : \Sigma_F(G) \to \Sigma^W(G)$ is a KK-equivalence.*

b) *The analytic index $\widetilde{\mathrm{ind}}^W_{full}(G) \in KK^1(\Sigma^W(G), C^*(G_W))$ corresponding to the exact sequence*

$$0 \longrightarrow C^*(G_W) \longrightarrow \Psi^*(G) \longrightarrow \Sigma^W(G) \longrightarrow 0$$

is the Kasparov product of

- *the element $[j_\sigma]^{-1} \in KK(\Sigma^W(G), \Sigma_F(G))$;*
- *the connecting element $\partial \in KK^1(\Sigma_F(\mathfrak{A}G), C_0((\mathfrak{A}^*G)_{|W}))$ associated with the exact sequence of (abelian) C^*-algebras*

$$0 \longrightarrow C_0((\mathfrak{A}^*G)_{|W}) \longrightarrow \Psi^*_F(\mathfrak{A}G) \longrightarrow \Sigma_F(\mathfrak{A}G) \longrightarrow 0;$$

- *the analytic index element ind_{G_W} of G_W, i.e. the element*

$$[ev_0]^{-1} \otimes [ev_1] \in KK(C_0((\mathfrak{A}^*G)_{|W}), C^*(G_W)).$$

□

Let us now compute the group $K_*(\mu)$ and the morphism ind_{rel} when the index element $\mathrm{ind}_{G_F} \in KK(C_0((\mathfrak{A}^*G)_{|F}),$ is invertible.

Proposition 4.8. *Assume that the index element* $\mathrm{ind}_{G_F} \in KK(C_0((\mathfrak{A}^*G)_{|F}), C^*(G_F))$ *is invertible. Then* $K_*(\mu)$ *is naturally isomorphic to* $K_*(C_0(\mathfrak{A}^*G_W))$. *Under this isomorphism,* ind_{rel} *identifies with* ind_{G_W}.

Proof. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(W) & \longrightarrow & C_0(M) & \longrightarrow & C_0(F) & \longrightarrow & 0\\ && \Big\downarrow{\scriptstyle i} && \Big\downarrow{\scriptstyle \mu} && \Big\downarrow{\scriptstyle j_\Psi} && \\ 0 & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*G_W) & \longrightarrow & \Sigma^W(G) & \longrightarrow & \Psi^*(G_F) & \longrightarrow & 0 \end{array}$$

As j_Ψ is an isomorphism in K-theory, the map $K_*(i) \to K_*(\mu)$ induced by the first commutative square of this diagram is an isomorphism. As $K_*(i) = K_*(\mathsf{C}_i)$ and $\mathsf{C}_i \simeq C_0(\mathfrak{A}^*G_W)$, we obtained the desired isomorphism $K_*(C_0(\mathfrak{A}^*G_W)) \simeq K_*(\mu)$.
Comparing the diagrams

$$\begin{array}{ccccc} C_0(W) & \longrightarrow & C_0(M) & \longrightarrow & \Psi^*(G)\\ \Big\downarrow{\scriptstyle i} && \Big\downarrow{\scriptstyle \mu} && \Big\downarrow{\scriptstyle \sigma_{full}}\\ C_0(\mathbb{S}\mathfrak{A}^*G_W) & \longrightarrow & \Sigma^W(G) & = & \Sigma^W(G) \end{array} \quad \text{and} \quad \begin{array}{ccccc} C_0(W) & \longrightarrow & \Psi^*(G_W) & \longrightarrow & \Psi^*(G)\\ \Big\downarrow{\scriptstyle i} && \Big\downarrow{\scriptstyle \sigma_W} && \Big\downarrow{\scriptstyle \sigma_{full}}\\ C_0(\mathbb{S}\mathfrak{A}^*G_W) & = & C_0(\mathbb{S}\mathfrak{A}^*G_W) & \longrightarrow & \Sigma^W(G) \end{array}$$

we find that the composition $K_*(i) \overset{\sim}{\longrightarrow} K_*(\mu) \longrightarrow K_*(\sigma_{full})$ coincides with the index
$K_*(i) \longrightarrow K_*(\sigma_W) \overset{\sim}{\longrightarrow} K_*(\sigma_{full})$. □

Remark 4.9. We wrote the relative index map in terms of morphisms of K-groups. One can also write everything in terms KK-theory, by replacing relative K-theory by mapping cones, *i.e.* construct the relative index as the element of $KK(\mathsf{C}_\mu, C^*(G_W))$ given as $\psi_{\mathsf{C}}^*([e]^{-1})$ where $e : C^*(G_W) \to \mathsf{C}_{\sigma_{full}}$ is the (KK-invertible) "excision map" associated with the (semi-split) exact sequence $0 \to C^*(G_W) \to \Psi^*(G) \overset{\sigma_{full}}{\longrightarrow} \Sigma_F(G) \to 0$ and $\psi_{\mathsf{C}} : \mathsf{C}_\mu \to \mathsf{C}_{\sigma_{full}}$ is the morphism associated with ψ. See also [30] where a relative KK-theory is defined.

5 Connes-Thom elements and quotient of a groupoid by a $\mathbb{R}_+^*$ action

Let $\mathbb{R}_+^*$ act smoothly and properly on a Lie groupoid G. In this section, we relate the connecting map and the full index map of the groupoid $G/\mathbb{R}_+^*$ with the ones of G. This is of course done using Connes' analogue of the Thom isomorphism ([6, 15]).

5.1 Proper action of $\mathbb{R}_+^*$ on a manifold

Remark 5.1 (Connes-Thom elements)**.** Let $\mathbb{R}_+^*$ act smoothly (freely and) properly on a manifold M. We have a canonical invertible KK-element $\alpha = (H, D) \in KK^1(C_0(M), C_0(M/\mathbb{R}_+^*))$.

- The Hilbert module H is obtained as a completion of $C_c(M)$ with respect to the $C_0(M/\mathbb{R}_+^*)$ valued inner product $\langle \xi | \eta \rangle (p(x)) = \displaystyle\int_0^{+\infty} \overline{\xi(t.x)}\, \eta(t.x) \frac{dt}{t}$ for $\xi, \eta \in C_c(M)$, where $p : M \to M/\mathbb{R}_+^*$ is the quotient map.
- The operator D is $\dfrac{1}{t}\dfrac{\partial}{\partial t}$.

The inverse element $\beta \in KK^1(C_0(M/\mathbb{R}_+^*), C_0(M))$ is constructed in the following way: $C_0(M/\mathbb{R}_+^*)$ sits in the multipliers of $C_0(M)$. One may define a continuous function $f : M \to [-1, 1]$ such that, uniformly on compact sets of M, $\lim_{t\to\pm\infty} f(e^t.x) = \pm 1$. The pair $(C_0(M), f)$ is then an element in $KK^1(C_0(M/\mathbb{R}_+^*), C_0(M))$. To construct f, one may note that, by properness, we actually have a section $\varphi : M/\mathbb{R}_+^* \to M$ and we may thus construct a homeomorphism $\mathbb{R}_+^* \times M/\mathbb{R}_+^* \to M$ defined by $(t, x) \mapsto t.\varphi(x)$. Then put $f(e^t, x) = t(1 + t^2)^{-1/2}$.
As an extension of C^*-algebras the element β is given by considering $P = (M \times \mathbb{R}_+)/\mathbb{R}_+^*$ (where $\mathbb{R}_+^*$ acts -properly - diagonally). Then M sits as an open subset $(M \times \mathbb{R}_+^*)/\mathbb{R}_+^*$ and we have an exact sequence $0 \to C_0(M) \to C_0(P) \to C_0(M/\mathbb{R}_+^*) \to 0$.

5.2 Proper action of $\mathbb{R}_+^*$ on a groupoid

Let now $\mathbb{R}_+^*$ act smoothly (locally *cf.* remark 2.2) properly on a Lie groupoid $G \rightrightarrows M$. The groupoid $G/\mathbb{R}_+^*$ acts on M, and the element α is G invariant - and β is almost G invariant in the sense of [18]. In other words, we obtain elements $\alpha \in KK^1_{G/\mathbb{R}_+^*}(C_0(M), C_0(M/\mathbb{R}_+^*))$ and $\beta \in KK^1_{G/\mathbb{R}_+^*}(C_0(M/\mathbb{R}_+^*), C_0(M))$ which are inverses of each other in Le Gall's equivariant KK-theory for groupoids.
Using the descent morphism of Kasparov ([17]) and Le Gall ([18]), we obtain elements

$$j_{G/\mathbb{R}_+^*}(\alpha) \in KK^1(C^*(G), C^*(G/\mathbb{R}_+^*)) \ \text{ and } \ j_{G/\mathbb{R}_+^*}(\beta) \in KK^1(C^*(G/\mathbb{R}_+^*), C^*(G))$$

that are also inverses of each other.
Note also that the element $\beta_G = j_{G/\mathbb{R}_+^*}(\beta)$ is the connecting element of the extension of groupoid C^*-algebras $0 \to C^*(G) \longrightarrow C^*(\mathbb{G}) \xrightarrow{ev_0} C^*(G/\mathbb{R}_+^*) \to 0$, where $\mathbb{G} = (G \times \mathbb{R}_+)/\mathbb{R}_+^*$ and ev_0 comes from the evaluation at $G \times \{0\}$. Using the pseudodifferential operators on the groupoid $\mathbb{G}$, we obtain a KK-element $\beta_G^\Psi \in KK^1(\Psi^*(G/\mathbb{R}_+^*), \Psi^*(G))$. We find a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^*(G) & \longrightarrow & C^*(\mathbb{G}) & \longrightarrow & C^*(G/\mathbb{R}_+^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Psi^*(G) & \longrightarrow & \Psi^*(\mathbb{G}) & \longrightarrow & \Psi^*(G/\mathbb{R}_+^*) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*G) & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*\mathbb{G}) & \longrightarrow & C_0(\mathbb{S}\mathfrak{A}^*(G/\mathbb{R}_+^*)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The third horizontal exact sequence corresponds to the proper action of $\mathbb{R}_+^*$ on $\mathbb{S}\mathfrak{A}^*G$. In fact $\mathbb{S}\mathfrak{A}^*\mathbb{G}$ is homeomorphic (using a cross section) to $(\mathbb{S}\mathfrak{A}^*(G/\mathbb{R}_+^*)) \times \mathbb{R}_+$. As the connecting elements of the first and third horizontal (semi-split) exact sequences are invertible, it follows that $C^*(\mathbb{G})$ and $C_0(\mathbb{S}\mathfrak{A}^*\mathbb{G})$ are K-contractible, whence so is $\Psi^*(G)$ and therefore β_G^Ψ is a KK-equivalence. Hence we have obtained:

Proposition 5.2. *If $\mathbb{R}_+^*$ acts smoothly (locally) properly on the Lie groupoid G, the connecting elements $\beta_G \in KK^1(C^*(G/\mathbb{R}_+^*), C^*(G))$, $\beta_{\mathbb{S}\mathfrak{A}^*G} \in KK^1(C_0(\mathbb{S}\mathfrak{A}^*(G/\mathbb{R}_+^*), C_0(\mathbb{S}\mathfrak{A}^*G))$ and $\beta_G^\Psi \in KK^1(\Psi^*(G/\mathbb{R}_+^*), \Psi^*(G))$ are KK-equivalences.*

5.3 Closed saturated subsets and connecting maps

If W is an open saturated subset in M for the actions of G and of $\mathbb{R}_+^*$ and $F = M \setminus W$, one compares the corresponding elements. We then obtain a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(G_W^W/\mathbb{R}_+^*) & \longrightarrow & C^*(G/\mathbb{R}_+^*) & \longrightarrow & C^*(G_F/\mathbb{R}_+^*) & \longrightarrow & 0 \\
 & & \Big| \beta' & & \Big| \beta & & \Big| \beta'' & & \\
0 & \longrightarrow & C^*(G_W^W) & \longrightarrow & C^*(G) & \longrightarrow & C^*(G_F) & \longrightarrow & 0
\end{array}$$

where the horizontal arrows are morphisms and the vertical ones KK^1-equivalences.
Using the deformation groupoid $\mathcal{G} = G_F^F \times [0,1) \cup G \times \{0\}$ which is the restriction of the groupoid $G \times [0,1) \rightrightarrows M \times [0,1]$ to the closed saturated subset $F \times [0,1) \cup M \times \{0\}$, we obtain:

Proposition 5.3. *If G_F is amenable, $\partial^W_{G/\mathbb{R}_+^*} \otimes \beta' = -\beta'' \otimes \partial^W_G \in KK(C^*(G_F/\mathbb{R}_+^*), C^*(G_W^W))$ where $\partial^W_G \in KK^1(C^*(G_F), C^*(G_W^W))$ and $\partial^W_{G/\mathbb{R}_+^*} \in KK^1(C^*(G_F/\mathbb{R}_+^*), C^*(G_W^W/\mathbb{R}_+^*))$ denote the KK-elements associated with the above exact sequences.*

Proof. Indeed, the connecting map of a semi-split exact sequence $0 \to J \to A \xrightarrow{p} A/J \to 0$ is obtained as the KK-product of the morphism $A/J(0,1) \to \mathsf{C}_p$ with the KK-inverse of the morphism $J \to \mathsf{C}_p$. The $-$ sign comes from the fact that we have naturally elements of $KK(C^*(G_F/\mathbb{R}_+^* \times (0,1)^2), C^*(G_W^W))$ which are equal but with opposite orientations of $(0,1)^2$. $\square$

Note also that the same holds for Ψ^* in place of C^*.

5.4 Connes-Thom invariance of the full index

Let W be as above: an open subset of M saturated for G and invariant under the action of $\mathbb{R}_+^*$. One compares the corresponding $\widetilde{\mathrm{ind}}_{full}$ elements. Indeed, we have a diagram

$$\begin{array}{cccccccccc}
 & 0 & \longrightarrow & C^*(G_W^W/\mathbb{R}_+^*) & \longrightarrow & \Psi^*(G/\mathbb{R}_+^*) & \longrightarrow & \Sigma^{W/\mathbb{R}_+^*}(G/\mathbb{R}_+^*) & \longrightarrow & 0 \\
 & & & \Big| \beta^{G_W} & & \Big| \beta_\Psi^G & & \Big| \beta_\Sigma^{(G,W)} & & \\
E_{\widetilde{\mathrm{ind}}_{full}} : & 0 & \longrightarrow & C^*(G_W^W) & \longrightarrow & \Psi^*(G) & \longrightarrow & \Sigma^W(G) & \longrightarrow & 0
\end{array}$$

where the horizontal arrows are morphisms and the vertical ones KK^1-elements. As β^{G_W} and β_Ψ^G are invertible, we deduce as in prop. 5.3:

Proposition 5.4. a) *The element $\beta_\Sigma^{(G,W)}$ is invertible.*

b) *We have $\beta_\Sigma^{(G,W)} \otimes \widetilde{\mathrm{ind}}^W_{full}(G) = -\widetilde{\mathrm{ind}}^{W/\mathbb{R}_+^*}_{full}(G/\mathbb{R}_+^*) \otimes \beta^{G_W}$.* $\square$

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