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Krieger Factors Isomorphic to Their Tensor Square and Pure Point Spectrum Flows

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A factor M, isomorphic to its tensor square, whose Sakai flip $\sigma \in \mathrm{Aut}(M \otimes M)$ is approximately inner, has a flow of weights with pure point spectrum.

INTRODUCTION

Which factors of type III_0 are isomorphic to their tensor square? By Krieger's theorem [18] and a result of Hamachi, Oka, and Osikawa [15], a Krieger factor (i.e., a crossed product of an abelian von Neumann algebra by an ergodic automorphism) whose flow of weights has pure point spectrum is isomorphic to its tensor square. In Section 2, we obtain a converse of this statement: if a factor M whose Sakai flip is approximately inner satisfies $M \otimes M \cong M$, then its flow of weights has pure point spectrum (Theorem 2.1).

The L^∞-point spectrum of the flow of weights of a factor is its Connes' invariant T. In the first section we prove that for every ITPFI factor (of type III_0) there exists an ITPFI_2 factor (of type III_0) with the same Connes' invariant T (Proposition 1.1 and Theorem 1.4).

In Section 3, we give examples of ITPFI_2 factors related to the hierarchy defined by W. Krieger in [18].

Note that Proposition 1.1 and Theorem 2.1 are related to questions 9 and 6 asked by E. J. Woods (see [22, Sect. 6]). Also, question 5 is answered by Example 1.8.

We would like to thank S. Baaj, who pointed out a gap in a previous form of the proof of Theorem 2.1 to us.

Notation. (1) Let $(L_k)_{k\geqslant 1}$ be a sequence of integers $\geqslant 1$ and $(\lambda_k)_{k\geqslant 1}$ be a sequence of real numbers in (0, 1). Let ϕ_k be the state on $M_2(C)$ given by

$$\phi_k(x) = \frac{1}{1+\lambda_k} \mathrm{Tr}\left(x \begin{pmatrix} 1 & 0 \\ 0 & \lambda_k \end{pmatrix} \right) \qquad (x \in M_2(C)).$$

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Let $M(L_k, \lambda_k)$ denote the ITPFI$_2$ factor $\bigotimes_{k \geqslant 1} (M_2(C), \phi_k)^{\otimes L_k}$. Recall that $M(L_k, \lambda_k)$ is of type I iff $\sum_{k \geqslant 1} L_k \lambda_k < \infty$ (cf. [1, Lemma 2.14]).

(2) If M is a factor, $T(M)$ and $S(M)$ denote Connes' invariants T and S [3, Definitions 1.3.1 and 3.1.1].

Recall that $T(M)$ is the point spectrum of the flow of weights of M (cf., for instance, [16]).

(3) Recall that an ergodic finite measure preserving flow is pure point spectrum with point spectrum T, where T is a countable subgroup of $\hat{\mathbb{R}}$, iff it is isomorphic to the action of $\mathbb{R}$ by rotation on $\hat{T}$, given by the transpose of the inclusion of T in $\hat{\mathbb{R}}$ (cf. [9, Chap. 12]).

All the von Neumann algebras appearing here act on a separable hilbert space. Also the measures encountered are standard.

I. On Connes' Invariant T for ITPFI$_2$ Factors

Let N be an ITPFI factor of type III. In this section we prove that there exists an ITPFI$_2$ factor M of type III$_0$ with $T(M) = T(N)$ (Theorem 1.4). If we do not impose that M be of type III$_0$, this is done in Proposition 1.1. If $T(N)$ is a dense subgroup of $\mathbb{R}$, then automatically M is of type III$_0$. Examples of ITPFI$_2$ factors of type III$_0$ with Connes' invariant T equal to $\{0\}$ or $\theta\mathbb{Z}$ are constructed in the proof of Theorem 1.4 and Examples 1.8. In particular every countable subgroup of $\mathbb{R}$ is Connes' invariant T of an ITPFI$_2$ factor (Remark 1.2(a)).

Moreover in 1.8 we give an example of an ITPFI$_2$ factor M of type III$_0$ such that $M \otimes M$ is of type III$_\lambda$, $\lambda \neq 0$. This gives an answer to problem 5 (Section 6) of [22].

Uncountable subgroups of $\mathbb{R}$ occur as invariant T for ITPFI$_2$ factors (cf. [21]). We give here a sufficient condition, in terms of the eigenvalue list, for an ITPFI$_2$ to have an uncountable invariant T (Proposition 1.3).

We begin with

1.1. Proposition. *Let N be an ITPFI factor. Then there exists an ITPFI$_2$ factor M with $T(M) = T(N)$.*

Proof. If $T(N) = \{0\}$, take M to be the Araki–Woods factor of type III$_1$.

If not, let $2\pi/\operatorname{Log} \lambda \in T(N)$, $0 < \lambda < 1$, and let $N = \bigotimes_{k \geqslant 0} (M_{n_k}(C), \psi_k)$ where ψ_k has eigenvalues

$$\left\{ \frac{\lambda^{q_{k,j}}}{\Lambda_k}; 1 \leqslant j \leqslant n_k, q_{k,j} \in \mathbb{N} \right\} \qquad \left(\Lambda_k = \sum_{j=1}^{n_k} \lambda^{q_{k,j}} \right).$$

By Corollaire 1.3.9 of [3],

$$\frac{\theta}{\operatorname{Log}\lambda} \in T(N) \Leftrightarrow \sum_{k \geqslant 0} \left(1 - \frac{1}{\Lambda_k} \left| \sum_{j=1}^{n_k} \lambda^{q_{k,j}} e^{i\theta q_{k,j}} \right| \right) < \infty$$

$$\Leftrightarrow \sum_{k \geqslant 0} \left(1 - \frac{1}{\Lambda_k^2} \left| \sum_{j=1}^{n_k} \lambda^{q_{k,j}} e^{i\theta q_{k,j}} \right|^2 \right) < \infty$$

as for $z \in \mathbb{C}$, $|z| < 1$, $1 - |z| \leqslant 1 - |z|^2 \leqslant 2(1 - |z|)$. We have:

$$\frac{\theta}{\operatorname{Log}\lambda} \in T(N) \Leftrightarrow \sum_{k \geqslant 0} \sum_{j < l} \frac{\lambda^{q_{k,j} + q_{k,l}}}{\Lambda_k^2} (1 - \cos\theta(q_{k,j} - q_{k,l})).$$

For $n \geqslant 0$, let L_n be the integral part of

$$L'_n = \sum_{(k,j,l) \in E_n} \frac{\lambda^{q_{k,j} + q_{k,l} - n}}{\Lambda_k^2}$$

where

$$E_n = \{(k, j, l);\, j < l \text{ and } |q_{k,j} - q_{k,l}| = n\} \qquad (L_n \in \mathbb{N} \cup \{+\infty\}).$$

Put $M = M(L_n, \lambda^n)$.

As above, we have by Corollaire 1.3.9 of [3],

$$\frac{\theta}{\operatorname{Log}\lambda} \in T(M) \Leftrightarrow \sum_{n \geqslant 0} L_n \left(1 - \frac{1}{1+\lambda^n} |1 + \lambda^n e^{in\theta}| \right) < \infty$$

$$\Leftrightarrow \sum_{n \geqslant 0} L_n \lambda^n (1 - \cos n\theta) < \infty$$

$$\Leftrightarrow \sum_{n \geqslant 0} L'_n \lambda^n (1 - \cos n\theta) < \infty \quad \left(\text{as } \sum_{n \geqslant 0} \lambda^n < \infty\right).$$

Hence,

$$\frac{\theta}{\operatorname{Log}\lambda} \in T(N) \Leftrightarrow \frac{\theta}{\operatorname{Log}\lambda} \in T(M). \quad \blacksquare$$

1.2. *Remark.* (a) In [8, Theorem 8.3 and Corollary 2.9], A. Connes and E. J. Woods prove that a Krieger factor whose flow of weights is pure point spectrum is an ITPFI. Therefore every countable subgroup of $\mathbb{R}$ is Connes' invariant T of an ITPFI factor. Using Proposition 1.1 we deduce that every countable subgroup of $\mathbb{R}$ is Connes' invariant T of an ITPFI_2 factor. This fact is equivalent to the following related with simultaneous approximations:

Let T be a countable subgroup of $\mathbb{R}$. Then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of non-negative real numbers and $\lambda\in(0,1)$ such that:

$$\frac{\theta}{\operatorname{Log}\lambda}\in T \quad \text{iff} \quad \sum_{n\geqslant 0} a_n(1-\cos n\theta)<\infty$$
$$\text{iff} \quad \sum_{n\geqslant 0} a_n\, d(n\theta, 2\pi\mathbb{Z})^2<\infty$$

where $d(r, 2\pi\mathbb{Z})$ denotes the distance between r and $2\pi\mathbb{Z}$.

(b) In [21], E. J. Woods constructed ITPFI_2 factors with uncountable invariant T. However, note that if T is Connes' invariant T of an ITPFI_2 factor, then T is a K_σ-group. Indeed T admits the equation

$$\frac{\theta}{\operatorname{Log}\lambda}\in T\Leftrightarrow \sum_{n\geqslant 0} a_n(1-\cos n\theta)<\infty.$$

For $p, N\in\mathbb{N}$, put $K_{p,N}=\{\theta\in\mathbb{R};\, |\theta|+\sum_{n=0}^{N} a_n(1-\cos n\theta)\leqslant p\}$. Then $K_{p,N}$ is a compact subset of $\mathbb{R}$ and $K_p=\bigcap_N K_{p,N}$ is compact. Moreover $T=(1/\operatorname{Log}\lambda)(\bigcup_p K_p)$.

The following statement gives a sufficient condition for an ITPFI_2 factor to have an uncountable invariant T.

Let $(r_k)_{k\geqslant 1}$ be an increasing sequence of positive real numbers and let $(L_k)_{k\geqslant 1}$ be a sequence of positive integers. Let $M=M(L_k, \exp(-r_k))$ be the ITPFI_2 factor corresponding to the sequence $(L_k, e^{-r_k})_{k\geqslant 1}$.

1.3. Proposition. *If* $\sum_{k\geqslant 1} L_k e^{-r_k}(r_k/r_{k+1})^2<\infty$, *then* $T(M)$ *is uncountable.*

Proof. Let $E=\{k;\, r_{k+1}/r_k\geqslant 3\}$. We have $\sum_{k\notin E} L_k e^{-r_k}<\infty$. Therefore, replacing $\mathbb{N}$ by E (write $E=\{k_n;\, n\in\mathbb{N}\}$ $(k_1<k_2<\cdots)$, then $r_{k_{n+1}}\geqslant r_{k_n+1}\geqslant 3r_{k_n}$), we may assume $r_{k+1}/r_k\geqslant 3$, for all k.

Let Ξ denote the set of sequences of zeros and ones. For all $\varepsilon\in\Xi$, we define inductively the sequence α_k^ε by the formulae $\varepsilon_k\leqslant\alpha_k^\varepsilon<\varepsilon_k+1$, $\sum_{j=1}^{k-1}(r_k\alpha_j^\varepsilon/r_j)+\alpha_k^\varepsilon\in\mathbb{N}$. For $\varepsilon\in\Xi$, put $\theta_\varepsilon=\sum_{k\geqslant 1}(\alpha_k^\varepsilon/r_k)$. One has:

$$d(r_k\theta_\varepsilon, \mathbb{Z})\leqslant \sum_{j\geqslant k+1}\frac{r_k\alpha_j^\varepsilon}{r_j}<\sum_{j\geqslant k+1}\frac{r_k(\varepsilon_j+1)}{r_j}\leqslant\sum_{j\geqslant k+1}\frac{2r_k}{r_j}$$
$$\leqslant 2\frac{r_k}{r_{k+1}}\sum_{j\geqslant k+1}\frac{r_{k+1}}{r_j}\leqslant 2\frac{r_k}{r_{k+1}}\sum_{j\geqslant k+1}3^{k+1-j}\leqslant 3\,\frac{r_k}{r_{k+1}}.$$

Hence, $\sum_{k\geqslant 1} L_k e^{-r_k}\, d(r_k\theta_\varepsilon, \mathbb{Z})^2<\infty$, and $2\pi\theta_\varepsilon\in T(M)$. It remains only to check that $\theta_\varepsilon\neq\theta_{\varepsilon'}$ if $\varepsilon\neq\varepsilon'$. If $\varepsilon_j=\varepsilon'_j$ for $j<k$ and $\varepsilon_k\neq\varepsilon'_k$, for instance, $\varepsilon_k=0$ and $\varepsilon'_k=1$, we have

$$\theta_{\varepsilon'}-\theta_\varepsilon=\sum_{j\geqslant k}\frac{\alpha_j^{\varepsilon'}-\alpha_j^{\varepsilon}}{r_j}=\frac{1}{r_k}+\sum_{j\geqslant k+1}\frac{\alpha_j^{\varepsilon'}-\alpha_j^{\varepsilon}}{r_j}\geqslant\frac{1}{r_k}-\sum_{j\geqslant k+1}\frac{|\alpha_j^{\varepsilon'}-\alpha_j^{\varepsilon}|}{r_j}$$

$$\geqslant\frac{1}{r_k}\left(1-\sum_{j\geqslant k+1}\frac{|\alpha_j^{\varepsilon'}-\alpha_j^{\varepsilon}|}{3^{j-k}}\right)>\frac{1}{r_k}\left(1-\sum_{j\geqslant k+1}\frac{2}{3^{j-k}}\right)=0.$$

So, $\theta_{\varepsilon'}>\theta_\varepsilon$.

Ξ being uncountable, $T(M)$ is uncountable. ∎

Now we give a stronger version of Proposition 1.1.

1.4. THEOREM. *Let T be a countable subgroup of $\mathbb{R}$. Then, there exists an ITPFI$_2$ factor M of type III$_0$, with $T(M)=T$.*

Proof. It remains only to construct examples of ITPFI$_2$ factors of type III$_0$ whose invariant T is $\{0\}$ or θZ, $\theta\in\mathbb{R}$.

Let α and β be two positive real numbers. Define the sequence $(r_k)_{k\geqslant 1}$ by $r_1=\alpha$, $r_{k+1}=2r_k+\alpha_k$, where

$$\alpha_k=\alpha \quad \text{if } k \text{ is even}$$
$$=\beta \quad \text{if } k \text{ is odd}.$$

Let $(L_k)_{k\geqslant 1}$ be the sequence given by $L_k=[e^{r_k}/k]$.[1]

Let $M=M(L_k, e^{-r_k})$.

We first check that M is of type III$_0$.

Let $(\Omega,\mu)=\prod_{k\geqslant 1}(\{0,1,\dots,L_k\},\mu_k)$, where μ_k is defined by

$$\mu_k(j)=\frac{L_k!}{(L_k-j)!\,j!}\,\frac{e^{-r_kj}}{(1+e^{-r_k})^{L_k}}.$$

Let $A=\{\omega\in\Omega\mid\omega_k=0 \text{ or } 1, \text{ for all } k\}$. We have

$$\mu(A)=\prod_{k\geqslant 1}\frac{1+L_ke^{-r_k}}{(1+e^{-r_k})^{L_k}}\geqslant\prod_{k\geqslant 1}(1+L_ke^{-r_k})\exp(-L_ke^{-r_k}).$$

So

$$\text{Log}\,\mu(A)\geqslant\sum_{k\geqslant 1}\text{Log}(1+L_ke^{-r_k})-L_ke^{-r_k}\geqslant\sum_{k\geqslant 1}\frac{L_ke^{-r_k}}{1+L_ke^{-r_k}}-L_ke^{-r_k}$$

$$\geqslant-\sum_{k\geqslant 1}\frac{(L_ke^{-r_k})^2}{1+L_ke^{-r_k}}\geqslant-\sum_{k\geqslant 1}\frac{1/k^2}{1+1/k}=-1>-\infty.$$

So, $\mu(A)>0$.

[1] If $x\in\mathbb{R}$, $[x]$ denotes its integral part.

Therefore the flow of weights of M is given by the action of $\mathbb{R}$ by translation on $(A \times \mathbb{R}, \mu | A \times dx)/\mathcal{R}$,[2] where $\mathcal{R}$ is the equivalence relation of $A \times \mathbb{R}$, defined by $(x, t)\, \mathcal{R}(y, s)$ if there exists K such that $x_k = y_k$ for $k > K$ and

$$\sum_{j=1}^{K} x_j r_j - t = \sum_{j=1}^{K} y_j r_j - s$$

(cf. Appendix of [12]). As $r_{k+1} > 2r_k$ for all k, the map $A \times [0, \alpha) \to A \times \mathbb{R}/\mathcal{R}$ is one-to-one. In particular, this flow is not periodic (or trivial) and M is not of type III_λ, $\lambda \neq 0$. As $\sum_{k \geqslant 1} L_k e^{-r_k} = +\infty$, M is of type III; hence III_0.

Moreover, if θ is a real number and $\theta \in T(M)$, then $\sum_{k=1}^{\infty} L_k e^{-r_k} (1 - \cos(\theta r_k)) < +\infty$. As $\sum_{k \geqslant 1} e^{-r_k} < +\infty$, we get $\sum_{k \geqslant 1} (1/k)(1 - \cos(\theta r_k)) < +\infty$. We derive that $\sum_{k \geqslant 2} (1/k)(1 - \cos(\theta r_{k-1})) < +\infty$. Using the inequalities,

$$1 - \cos 2x = 2(1 - \cos x)(1 + \cos x) \leqslant 4(1 - \cos x)$$

and $1 - \cos(x - y) \leqslant 2((1 - \cos x) + (1 - \cos y))$, we get

$$\sum_{k \geqslant 2} \frac{1}{k}(1 - \cos \theta \alpha_{k-1}) = \sum_{k \geqslant 2} \frac{1}{k}(1 - \cos \theta(r_k - 2r_{k-1})) < +\infty.$$

Therefore $\theta \in (2\pi/\alpha)\, \mathbb{Z}$ and $\theta \in (2\pi/\beta)\, \mathbb{Z}$.

Taking α and β rationally independent, we get $T(M) = \{0\}$.

Taking $\alpha = \beta$, we get $T(M) \subseteq (2\pi/\alpha)\, \mathbb{Z}$. In this case $r_k = (2^k - 1)\, \alpha$, hence $2\pi/\alpha \in T(M)$, and thus $T(M) = (2\pi/\alpha)\, \mathbb{Z}$. ∎

The sequence $(L_k, e^{-r_k})_{k \geqslant 1}$, appearing in this proof, has the following "non-interacting level" property:

Let L_k be a sequence of positive integers and let r_k be a sequence of positive real numbers. Let $(\Omega, \mu) = \prod_{k \geqslant 1} (\{0, ..., L_k\}, \mu_k)$ where μ_k is defined by

$$\mu_k(j) = \frac{L_k!}{(L_k - j)!\, j!} \frac{e^{-r_k j}}{(1 + e^{-r_k})^{L_k}}.$$

1.5. Definition. The sequence (L_k, e^{-r_k}) is said to satisfy the (weak) non-interacting level condition if there exist $\varepsilon > 0$ and sequences a_k, b_k of integers such that:

(i) $\mu(\{\omega = (\omega_k)_{k \geqslant 1} \in \Omega \,|\, a_k \leqslant \omega_k \leqslant b_k, \text{ for all } k \geqslant 1\}) > 0$.

(ii) For all $k \geqslant 1$, $r_k \geqslant \varepsilon + \sum_{n=1}^{k-1} (b_n - a_n)\, r_n$.

[2] This quotient denotes the ergodic decomposition.

Using the flow of weights as in the proof of Theorem 1.4 (cf. also [12, Remark 4.2]) it is very easy to get:

1.6. PROPOSITION. *If the sequence* (L_k, e^{-r_k}) *satisfies the (weak) non-interacting level condition and if* $\sum_{k\geqslant 1} L_k e^{-r_k} = +\infty$, *then* $M(L_k, e^{-r_k})$ *is of type* III_0.

1.7. *Remarks.* (1) Proposition 1.6 is a generalization of Lemma 10.4 of [1] (see also comment after Lemma 10.4).

(2) If $r_{k+1} > 2r_k$ and $\sum_{k\geqslant 1} (L_k e^{-r_k})^2 < \infty$, then the sequence (L_k, e^{-r_k}) satisfies the (weak) non-interacting level condition (see proof of Theorem 1.4).

(3) Using Tchebyshev's inequality, it is easy to see that if the sequence (L_k, e^{-r_k}) satisfies $\sum_{k\geqslant 1} L_k e^{-r_k}(r_k/r_{k+1})^2 < \infty$ (see Proposition 1.3), then it fulfills the (weak) non-interacting level condition.

If M is the ITPFI$_2$ factor considered in Theorem 1.4, then for every $n \geqslant 1$, $M^{\otimes n}$ is of type III$_0$ (cf. Remark 1.7 (2)). This is not the case in the following:

1.8. EXAMPLES. Let α, β, and t be three positive real numbers and j be equal to 2 or 3. Define the sequence $(r_k)_{k\geqslant 1}$ by $r_1 = \alpha$ and $r_{k+1} = (p_k + j) r_k + \alpha_k$, where $p_k = p$ if $p! \leqslant k < (p+1)!$ and

$$\begin{aligned} \alpha_k &= \alpha \qquad \text{if } k \text{ is even} \\ &= \beta \qquad \text{if } k \text{ is odd.} \end{aligned}$$

Let $(L_k)_{k\geqslant 1}$ be the sequence of integers given by $L_k = [te^{+r_k}]$.

Put $M_{j,t} = M(L_k, e^{-r_k})$. For the behaviour of $M_{j,t}$, we need first the following.

1.9. LEMMA. (i) *If* $\alpha = \beta$, $2\pi/\alpha \in T(M_{j,t})$ *for all* j *and* t.

(ii) *For all* j, t, s, $M_{j,t} \otimes M_{j,s} = M_{j,t+s}$.

(iii) *If* $j = 2$ *and* $t < 1$ *or* $j = 3$ *and* $t \leqslant 1$, *then* $M_{j,t}$ *is of type* III_0.

(iv) *If* $j = 2$ *and* $t \geqslant 1$ *or* $j = 3$ *and* $t > 1$, *then* $e^{-\alpha} \in S(M_{j,t})$ *and* $e^{-\beta} \in S(M_{j,t})$.

Proof. (i) is obvious, as all e^{-r_k} are of the form $(e^{-\alpha})^{q_k}$ with q_k integer.

(ii) is obvious, as $\sum_{k\geqslant 1} e^{-r_k} < \infty$.

(iii) Let $(\Omega, \mu) = \prod_{k\geqslant 1} (\{0, 1,..., L_k\}, \mu_k)$, with μ_k defined by

$$\mu_k(l) = \frac{L_k!}{(L_k - l)!\, l!} \frac{e^{-r_k l}}{(1 + e^{-r_k})^{L_k}}.$$

As for $0 \leqslant x \leqslant 1$, $e^x(1+x/n)^{-n}(1-1/n) \leqslant 1$, we get for $t \leqslant 1$

$$\mu_k(p_k+j) = \frac{L_k(L_k-1)\cdots(L_k-p_k-j+1)}{(p_k+j)!} \frac{e^{-r_k(p_k+j)}}{(1+e^{-r_k})^{L_k}}$$
$$\leqslant \frac{t^{p_k+j}}{(p_k+j)!\,(1+e^{-r_k})^{L_k}} \prod_{i=1}^{p_k+j-1} \left(1-\frac{i}{L_k}\right)$$
$$\leqslant \frac{t^{p_k+j}}{(p_k+j)!} \exp(-L_k e^{-r_k}).$$

But, as

$$\frac{\mu_k(l+1)}{\mu_k(l)} = \frac{L_k-l}{l+1} e^{-r_k} \leqslant \frac{L_k e^{-r_k}}{l+1},$$

we have

$$\mu_k(\{p_k+j,..., L_k\}) \leqslant \mu_k(p_k+j) \left(1 + \sum_{k \geqslant 1} \frac{(L_k e^{-r_k})^i}{(p_k+j+1)\cdots(p_k+j+i)}\right)$$
$$\leqslant \mu_k(p_k+j)\, exp(L_k e^{-r_k}) \leqslant \frac{t^{p_k+j}}{(p_k+j)!}.$$

Now,

$$\sum_{k \geqslant 1} \frac{t^{p_k+j}}{(p_k+j)!} = \sum_{p \geqslant 1} \sum_{k=p!}^{(p+1)!-1} \frac{t^{p+j}}{(p+j)!} = \sum_{p \geqslant 1} \frac{t^{p+j}((p+1)!-p!}{(p+j)!}$$
$$= \sum_{p \geqslant 1} \frac{t^{p+j} p \cdot p!}{(p+j)!}.$$

If $j=2$ and $t<1$, we get

$$\sum_{k \geqslant 1} \mu_k(\{p_k+2,..., L_k\}) \leqslant \sum_{p \geqslant 1} \frac{t^{p+2}p}{(p+1)(p+2)} < \infty.$$

If $j=3$ and $t \leqslant 1$, we have

$$\sum_{k \geqslant 1} \mu_k(\{p_k+3,..., L_k\}) \leqslant \sum_{p \geqslant 1} \frac{t^{p+3}p}{(p+1)(p+2)(p+3)} < \infty.$$

Therefore in both cases, $\mu(A)>0$, where $A \subset \Omega$ is defined by $A = \{(\omega_k)_{k \geqslant 1} \in \Omega \mid 0 \leqslant \omega_k \leqslant p_k+j-1$, for all $k\}$. Thus the non-interacting level condition is satisfied. Now use Proposition 1.6.

(iv) We have

$$\mu_k(l) \geqslant \frac{(L_k-l+1)^l e^{-r_k l}}{l! \exp(L_k e^{-r_k})} \geqslant \left(\frac{L_k-l+1}{L_k+1}\right)^l \frac{t^l}{l!\, e^t}.$$

As

$$\left(\frac{L_k-l+1}{L_k+1}\right)^l \geqslant \exp\left(-\frac{l^2}{L_k-l+1}\right) \geqslant \exp\left(-\frac{2l^2}{L_k}\right) \quad \text{if } l \leqslant \frac{L_k}{2},$$

we get (for large k)

$$\mu_k(p_k+j) \geqslant \exp\left(-\frac{2(p_k+j)^2}{L_k}\right)\frac{t^{p_k+j}}{(p_k+j)!\,e^t} \geqslant \frac{t^{p_k+j}}{2(p_k+j)!\,e^t}.$$

Let $(K_k, \Phi_k) = (M_2(C), \phi_k)^{\otimes L_k} \otimes (M_2(C), \phi_{k+1})^{\otimes L_{k+1}}$ where ϕ_k is a state on $M_2(C)$, with eigenvalues $\{1-/(1+e^{-r_k}), e^{-r_k}/(1+e^{-r_k})\}$. The eigenvalues of Φ_k are

$$\frac{\exp(-lr_k - l'r_{k+1})}{(1+e^{-r_k})^{L_k}\,(1+e^{-r_{k+1}})^{L_{k+1}}} = \lambda_{l,l'},$$

with multiplicity

$$m_{l,l'} = \frac{L_k!}{(L_k-l)!\,l!}\cdot\frac{L_{k+1}!}{(L_{k+1}-l')!\,l'!}.$$

We have

$$m_{0,1} = L_{k+1} \qquad \text{and} \qquad m_{p_k+j,0} \leqslant \frac{(L_k)^{p_k+j}}{(p_k+j)!}.$$

Let $K_k^1 = \{\lambda, \text{ eigenvalue of } \Phi_k; \lambda = \lambda_{p_k+j,0}\}$ and $K_k^2 = \{\lambda, \text{ eigenvalue of } \Phi_k; \lambda = \lambda_{0,1}\}$. Let k_0 be such that $m_{p_k+j,0} \leqslant m_{0,1}$ for $k \geqslant k_0$. For $k \geqslant k_0$, let $\gamma_k : K_k^1 \to K_k^2$ be an injection. Note that for all $\lambda \in K_k^1$, we have $\gamma_k(\lambda) = \exp(-\alpha_k)\cdot\lambda$. Now

$$(\# K_k^1)\,\lambda_{p_k+j,0} = \mu_k(p_k+j)\,\mu_{k+1}(0) \geqslant \frac{t^{p_k+j}}{2(p_k+j)!\,e^t}\,e^{-t}.$$

Hence

$$\begin{aligned}
\sum_{k\geqslant 1} (\# K_{2k}^1)\,\lambda_{p_{2k}+j,0} &\geqslant \sum_{k\geqslant 1} \frac{t^{p_{2k}+j}}{2(p_{2k}+j)!}\,e^{-2t} \\
&\geqslant \sum_{p\geqslant 2}\ \sum_{k=p!/2}^{(p+1)!/2-1} \frac{t^{p+j}e^{-2t}}{2(p+j)!} \\
&= \sum_{p\geqslant 2} \frac{(p+1)!-p!}{4(p+j)!}\,t^{p+j}e^{-2t} \\
&\geqslant \frac{e^{-2t}}{4}\sum_{p\geqslant 2}\frac{pp!}{(p+j)!}\,t^{p+j}.
\end{aligned}$$

If $j=2$ and $t\geqslant 1$, we have

$$\sum_{k\geqslant 1}(\# K_{2k}^1)\,\lambda_{p2k+j,0}\geqslant\frac{e^{-2t}}{4}\sum_{p\geqslant 2}\frac{p}{(p+1)(p+2)}\,t^{p+2}=+\infty.$$

If $j=3$ and $t>1$, this is also the case. Therefore, $e^{-\alpha}\in S(M_{t,j})$ (by [1, Definition 3.2]). The same computation shows that $e^{-\beta}\in S(M_{t,j})$. ∎

In particular $M_{3,1}$ is of type III_0, but $M_{3,2}\cong M_{3,1}\otimes M_{3,1}$ is of type III_1 if α and β are rationally independent and of type $\mathrm{III}_{e^{-\alpha}}$ if $\alpha=\beta$.

One checks also that if $t\neq s$, $t, s<1$, then $M_{2,t}\not\cong M_{2,s}$ and $M_{3,t}\not\cong M_{3,s}$. To see this assume $t<s$ and let $u=1-(t+s)/2$. Then $M_{j,t}\otimes M_{j,u}$ is of type III_0 though $M_{j,s}\otimes M_{j,u}$ is of type III_λ, $\lambda\neq 0$.

II. Krieger Factors Isomorphic to Their Tensor Square

If M is a Krieger factor of type III, whose flow of weights (is finite measure preserving and) has pure point spectrum, then M is isomorphic to $M\otimes M$ [15, Theorem 6] and the Sakai flip on $M\otimes M$ is approximately inner (using [7, Remark on p. 230]). Theorem 2.1 provides the converse of this assertion. The idea of the proof of 2.1 comes directly from commutative harmonic analysis (cf. [2, 20, 13, Theorem 6.1.8]).

Then we apply Theorem 2.1 to the case of an ITPFI_2 factor (Corollary 2.6).

We may note that 2.1 is part of the bilingual dictionary between properties of the factor and its flow of weights (cf. [22, Sect. 6]).

An other characterization of a Krieger factor M, whose flow of weights is pure point spectrum (namely, $\mathrm{Aut}(M)/\overline{\mathrm{Int}}(M)$ is compact), was obtained by Hamachi [14].

Notation. $U(1)$ denotes the multiplicative group of complex numbers of modulus 1.

2.1. Theorem. *Let M be a Krieger factor of type III_0 such that $M\cong M\otimes M$ and its Sakai flip σ is approximately inner.*

Then the flow of weights of M (is finite measure preserving and) has pure point spectrum.

Proof. We just have to prove that the flow of weights of M admits a finite invariant measure: indeed by Lemma 1 of [15] the flow of weights of $M\otimes M$ is pure point spectrum.

If (Ω, P, F_t) denotes the flow of weights of M, then let (Ω', P') be the ergodic decomposition of the measure space $(\Omega\times\Omega, P\times P)$, under the action of $\mathbb{R}$, given by $F_t \times F_{-t}$, $t \in \mathbb{R}$. The flow of weights of $M \otimes M$ is (Ω', P', F'_t) where F'_t is induced by $F_t \times \text{id}$, acting on $(\Omega \times \Omega, P \times P)$ [6, Corollary II.6.8].

Let $p: \Omega \times \Omega \to \Omega'$ be the projection. As M and $M \otimes M$ are isomorphic, there exists a measure-class preserving isomorphism $i: (\Omega', P') \to (\Omega, P)$, intertwining the two $\mathbb{R}$-actions F' and F. Let $a = i \circ p: \Omega \times \Omega \to \Omega$. We have

(i) $a(P \times P)$ is equivalent to P.

(ii) $a(F_t x, F_{-t} y) = a(x, y)$ for almost all $(x, y) \in \Omega \times \Omega$.

(iii) $a(F_t x, y) = F_t a(x, y)$, for all $t \in \mathbb{R}$ and almost all $(x, y) \in \Omega \times \Omega$.

As the Sakai flip is approximately inner, we have $p(x, y) = p(y, x)$, for almost all $x, y \in \Omega$ [6, Theorem IV.1.9], which gives

(iv) $a(x, y) = a(y, x)$, for almost all $(x, y) \in \Omega \times \Omega$.

If $p': (\Omega, P)^3 \to (\Omega'', P'')$ is the ergodic decomposition under the action of $\mathbb{R}^2$, given by $F_t \times F_s \times F_{-t-s}$, we have $p'(x, y, z) = p'(y, z, x)$, for almost all $x, y, z \in \Omega$. As $(x, y, z) \mapsto a(a(x, y), z)$ is $F_t \times F_s \times F_{-t-s}$-invariant, we get $a(a(x, y), z) = a(a(y, z), x)$, which gives

(v) $a(a(x, y), z) = a(x, a(y, z))$, for almost all $x, y, z \in \Omega$.

Theorem 2.1 follows from

2.2. LEMMA. *Let (Ω, P, F_t) be a non-transitive, ergodic flow and let $a: \Omega \times \Omega \to \Omega$ satisfy the conditions* (i) *to* (v) *above. Then (Ω, P, F_t) admits a finite invariant measure.*

Proof. For $\mu, \nu \in L^1(\Omega, P)$, let $\mu * \nu \in L^1(\Omega, P)$ be defined by $\mu * \nu = a(\mu \times \nu)$. ($L^1(\Omega, P)$ is considered as a space of measures on Ω.)

Conditions (iv) and (v) show that $(L^1(\Omega, P), *)$ is an abelian Banach algebra.

Let Δ denote the (locally compact, separable) space of characters of $(L^1(\Omega, P), *)$. It is included in the dual space of $L^1(\Omega, P)$, namely, $L^\infty(\Omega, P)$. Let $\chi \in L^\infty(\Omega, P)$. We have

$$\int \chi(x)\, d\mu * \nu(x) = \iint \chi(a(x, y))\, d\mu(x)\, d\nu(y)$$

and

$$\left(\int \chi(x)\, d\mu(x)\right) \cdot \left(\int \chi(x)\, d\nu(x)\right) = \iint \chi(x)\, \chi(y)\, d\mu(x)\, d\nu(y).$$

Therefore, χ is a character iff

$$\chi(a(x, y)) = \chi(x)\, \chi(y) \qquad P \times P\text{-a.e.} \tag{1}$$

If χ is a character, using condition (ii) we get $\chi(F_t x)\,\chi(y)=\chi(x)\,\chi(F_t\,y)$ for almost all (x, y). This shows that $\chi(F_t x)\,\chi(x)^{-1}$ is essentially constant so that χ is also an eigenfunction for F_t. Therefore the modulus of χ is constant and by (1) it is equal to 1. This shows that Δ equipped with the multiplication in $L^\infty(\Omega, P)$ is a (separable, locally compact) abelian subgroup of the unitary group of $L^\infty(\Omega, P)$ (using condition (1)).

Moreover, the map $\Phi\colon \Delta\times\mathbb{R}\to L^\infty(\Omega, P)$ given by $\Phi(\chi, t)=\chi\circ F_t\cdot\bar{\chi}$ is weakly continuous. As Φ takes its values in the constant functions of modulus one, it determines a continuous group homomorphism $\Pi\colon \Delta\to\hat{\mathbb{R}}$. As the image of Π is included in the point spectrum of the flow, Π is not surjective [3, Théorème 1.3.4]. If $\chi\in\ker\Pi$, then (by ergodicity of F_t) χ is constant, and using (1), $\chi=1$. Hence Π is injective. By [17, Lemma 2.1] we deduce that Δ is discrete.

By Silov's idempotent theorem (cf., for instance, [11, p. 88], there exists $\mu\in L^1(\Omega, P)$ such that $\mu*\mu=\mu$ and

$$\mu(\chi)=1 \qquad \text{for} \quad \chi=1\ (\in L^\infty(\Omega, P))$$
$$=0 \qquad \text{for} \quad \chi\in\Delta, \chi\neq 1.$$

Let $A=\mu*L^1(\Omega, P)$. It is a Banach subalgebra of $L^1(\Omega, P)$ with unit μ and only character given by $\chi=1$.

For $t\in\mathbb{R}$, $F_t(\mu)=\mu*F_t(\mu)\in A$. Therefore A is closed under the action of F_t. Moreover $F_t(\mu)*F_s(\mu)=F_{t+s}(\mu)$ and $\|F_t(\mu)\|=\|\mu\|$. Let $u_t\in B(A)$ be the restriction of F_t to A. It coincides with the multiplication by $F_t(\mu)$ and is an isometry of A.

As $(F_t(\mu)-\lambda\mu)(\chi)=1-\lambda$, for $\lambda\in\mathbb{C}$, the spectrum of u_t is reduced to 1 ($u_t-\lambda 1$ is equal to the multiplication by $F_t(\mu)-\lambda(\mu)$. By Corollaire 6.4 of [23], $u_t=1$ and $F_t(\mu)=\mu$. ∎

2.3. *Remark.* (a) The converse of Theorem 2.1 is true (cf. [15, Lemma 1]).

(b) We do not know whether the condition $M\otimes M\cong M$ implies the condition on the Sakai flip.

(c) Using Lemma 2.2, we see that Theorem 2.1 generalizes to the case where M is not necessarily a Krieger factor. In this case, one should state it in the following form:

> If $M\otimes M\cong M$ and $\mathrm{mod}(\sigma)=\mathrm{id}$ [6, Definition IV.1.1], then the flow of weights of M has pure point spectrum. (By Theorem 5.1 of [5], we know that if the Sakai flip is approximately inner, then M is injective. Moreover, for a Krieger factor, $\ker(\mathrm{mod})=\overline{\mathrm{Int}}(M)$.)

For every countable subgroup T of $\mathbb{R}$, let R_T denote the Krieger factor whose flow of weights has pure point spectrum, with point spectrum T (cf. [15, 16]).

2.4. *Remark.* (a) With the notation of Definition 5 of [15], R_T is the Krieger factor of type III^T. In particular,

$$\text{if } T = -\frac{2\pi}{\operatorname{Log} \lambda}\mathbb{Z},\ \lambda \in (0, 1), \text{ then } R_T = R_\lambda;$$

$$\text{if } T = \{0\},\ R_T = R_\infty.$$ [3]

(b) To a subgroup T of $\mathbb{R}$ and a factor P of type III_1, A. Connes associated in [3, Corollaire 1.5.8] a factor $M(T, P) = W^*(P, T, \sigma^\phi)$, with $T(M) = T$.

If T is countable and $P = R_\infty$, one can show that $M(T, P) = R_{T(M)}$.

2.5. Corollary. *Let M be a Krieger factor of type III such that the Sakai flip on $M \otimes M$ is approximately inner. Then, either $M^{\otimes p}$, $p \geqslant 1$, are pairwise non-isomorphic, or there exists $q \geqslant 1$, satisfying: $M^{\otimes k}$, $1 \leqslant k < q$, are pairwise non-isomorphic and $M^{\otimes p}$, $p \geqslant q$, is isomorphic to $R_{T(M)}$.*
In particular, if $T(M)$ is uncountable, $M^{\otimes p}$ are pairwise non-isomorphic.

Proof. If N is a Krieger factor, $N \otimes R_T \cong R_{T \cap T(N)}$ (cf. [15, Theorem 6]). Therefore, there exists $q_M \geqslant 1$, possibly infinite, such that $M^{\otimes p} \cong R_{T(M)}$ iff $p \geqslant q_M$. Hence, it is enough to show that if $M^{\otimes n} \cong M^{\otimes m}$, $n \neq m$, then $M^{\otimes n} \cong R_{T(M)}$, i.e., $n \geqslant q_M$.
If $M^{\otimes n} \cong M^{\otimes m}$, for $1 \leqslant n < m$, we have $M^{\otimes m} \cong M^{\otimes n} \otimes M^{\otimes (m-n)}$. So that $M^{\otimes n} \cong M^{\otimes (n + k(m-n))}$, for $k \geqslant 1$. Choose k such that $p = n + k(m-n) > 2n$. As $M^{\otimes p} \cong M^{\otimes n}$, we have $M^{\otimes p} \otimes M^{\otimes (p-2n)} \cong M^{\otimes n} \otimes M^{(p-2n)}$. Therefore, $M^{\otimes (p-n)} \cong M^{\otimes (p-n)} \otimes M^{\otimes (p-n)}$. By Theorem 2.1, we get $M^{\otimes (p-n)} \cong R_{T(M)}$. Therefore, $p > p - n \geqslant q_M$; so that $M^{\otimes n} \cong M^{\otimes p} \cong R_{T(M)}$. ∎

Let $(L_k)_{k \geqslant 1}$ be a sequence of positive integers and $(\lambda_k)_{k \geqslant 1}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{k \geqslant 1} \lambda_k < \infty$ and $\sum_{k \geqslant 1} L_k \lambda_k = \infty$. Let M be the corresponding ITPFI_2 factor. For $t \in \mathbb{R}$, $t > 0$, put $L_{k,t} = [tL_k]$ and let M_t be the ITPFI_2 factor, associated with $(L_{k,t}, \lambda_k)_{k \geqslant 1}$. We have $M_t \otimes M_s \cong M_{t+s}$. Replacing integers by reals in the proof of Corollary 2.5, we get:

2.6. Corollary. *Exactly one of the following holds:*

(i) *M_t, $t > 0$, are pairwise non-isomorphic.*

(ii) *There exists $t_0 > 0$ such that the M_t, $t < t_0$, are pairwise non-isomorphic and $M_t \cong R_{T(M)}$, for $t \geqslant t_0$.*

[3] Here, R_λ denotes the Powers factor of type III_λ, for $\lambda \in (0, 1)$, and R_∞ the Araki–Woods III_1 factor.

(iii) *There exists* $t_0 \geqslant 0$ *such that the* $M_t, t \leqslant t_0$, *are pairwise non-isomorphic and* $M_t \cong R_{T(M)}$ *for* $t > t_0$.

In particular, if $T(M)$ *is uncountable, the* M_t *are pairwise non-isomorphic. Therefore, if* T *is an uncountable subgroup of* $\mathbb{R}$, $T \neq R$, *the cardinality of isomorphism classes of* $ITPFI_2$ *factors* M *with* $T(M) = T$ *is either zero or the continuum.*

Example 1.8 provides sequences (L_k, λ_k) proving that cases (ii) and (iii) of Corollary 2.6 are non-empty with $T(M) = \{0\}$ or $T(M) = \theta Z, \theta \neq 0$. The example of Theorem 1.4 gives the case (i).

It is also very easy to construct a sequence (L_k, λ_k) such that (iii) is true with $t_0 = 0$ and $T(M) = \{0\}$ or $-(2\pi/\text{Log}\,\lambda)\,\mathbb{Z}, 0 < \lambda < 1$ (e.g., take $\lambda_1 = \lambda$ and $L_1 = +\infty$).

III. Examples of ITPFI_2 and Hierarchy of Krieger

In [18, Sect. 7], W. Krieger considers the following construction. Let M be a Krieger factor of type III_0 and let (X_M, μ_M, F^M) be its flow of weights. Let $\Phi(M) = W^*(L^\infty(X_M, \mu_M), \mathbb{R}, F^M)$ be the corresponding crossed product factor.

To the factor M, Krieger associates the first number $n = n(M)$ $(1 \leqslant n \leqslant \infty)$ such that $\Phi^n(M)$ fails to be of type III_0.

In [4], A. Connes constructed a type III_0 ITPFI factor M such that $\Phi(M)$ is isomorphic to M. In this case, the number $n(M)$ is infinite.

We give here examples of ITPFI_2 factors with analogous properties: given an integer N, we construct an ITPFI_2 factor M, with $\Phi^k(M) \cong M$ if and only if k is a multiple of N. Finally, we build an ITPFI_2 factor M, with $\Phi^k(M) \not\cong \Phi^l(M)$ for all k and l $(k \neq l)$ (Proposition 3.6).

3.1. *Remark.* If (X_M, μ_M, F^M) is the flow constructed over a base transformation (A, ν, T) under a positive ceiling function, then $\Phi(M)$ is isomorphic to the tensor product of $W^*(L^\infty(A, \nu), \mathbb{Z}, T)$ with the factor of type I_∞ ([19, Theorem 5.1]; see also [10, Theorem 9.3]).

To construct the examples of this section, we need the two following lemmas.

3.2. Lemma. *Let* $N \in \mathbb{N} \cup \{+\infty\}$, $N \geqslant 1$. *For* $k \geqslant 1$, $0 \leqslant j < N$, *let* $\lambda_{k,j} = 2^{-3^k - j}$.

If $(a_j)_{0 \leqslant j < N}$ *is a sequence of integers* $\geqslant 1$, *then there exist two sequences of positive integers* $\{L_{k,j}; k \geqslant 1, 0 \leqslant j < N\}$ *and* $\{n(k, j); k \geqslant 2, 0 \leqslant j < N\}$ *such that*

(a) $L_{1,j} = a_j$ *for* $0 \leqslant j < N$;

(b) $L_{k,j}\lambda_{k,j} = \lambda_{n(k,j)),j+1}$ *for* $k \geqslant 2$ (*if* $N \neq \infty$, *the indices j are written modulo* N);

(c) $1 + \sum_{p=1}^{n(k,j)-1} L_{p,j+1} < k \leqslant 1 + \sum_{p=1}^{n(k,j)} L_{p,j+1}$;

(d) $L_{k,j} \geqslant 1$ *and* $n(k, j) < k$ *for all* k, j.

Proof. Note that $n(2, j) = 1$ and $L_{2,j} = \lambda_{1,j+1}/\lambda_{2,j} \geqslant 1$. Assume that $L_{k,j}$ and $n(k, j)$ $(0 \leqslant j < N)$ are constructed up to $k = m \geqslant 2$ and determine $L_{m+1,j}$ and $n(m+1, j)$.

As $m+1 \leqslant 1 + \sum_{p=1}^{m} L_{p,j}$, there exists $n \leqslant m$ such that $1 + \sum_{p=1}^{n-1} L_{p,j+1} < m+1 \leqslant 1 + \sum_{p=1}^{n} L_{p,j+1}$. Put $n(m+1, j) = n$. As $n(m+1, j) < m+1$, one has $\lambda_{n(m+1,j)),j+1}\lambda_{m+1,j}^{-1} \geqslant 2^{3-1} \geqslant 1$. Hence the lemma is proved. ∎

3.3. *Remark.* For all $j, 0 \leqslant j < N$, and $n \geqslant 1$, we have $\#\{k \geqslant 2; L_{k,j}\lambda_{k,j} = \lambda_{n,j+1}\} = L_{n,j+1}$, by construction of the $L_{n,j}$'s (if $N \neq \infty$, the indices j are written modulo N).

3.4. Lemma. *Let* $N, \lambda_{k,j}, L_{k,j}$, *and* $n(k, j)$ *be as above. If* $N = \infty$, *let* $(a_j)_{j \geqslant 0}$ *be such that* $\sum_{j \geqslant 0} a_j 2^{-j} = \infty$. *Then, we have* $\sum_{k \geqslant 1} L_{k,j}\lambda_{k,j} = \infty$ *but* $\sum_{k \geqslant 1} (L_{k,j}\lambda_{k,j})^2 < \infty$ *for all* $j, 0 \leqslant j < N$.

Proof. Let $A_j = \sum_{k \geqslant 1} L_{k,j}\lambda_{k,j}$. By Remark 3.3, we get

$$A_j = a_j\lambda_{1,j} + \sum_{k \geqslant 2} L_{k,j}\lambda_{k,j}$$

$$= a_j\lambda_{1,j} + \sum_{n \geqslant 1} L_{n,j+1}\lambda_{n,j+1} = a_j\lambda_{1,j} + A_{j+1}.$$

If N is finite, then $A_j = \sum_{i=0}^{N-1} a_i\lambda_{1,i} + A_j$. Hence, $A_j = \infty$ for $0 \leqslant j < N$. If $N = \infty$, we have for all $j \geqslant 0$, $A_j \geqslant \sum_{i \geqslant j} a_i 2^{-3-i} = \infty$.

By 3.3, we obtain also

$$\sum_{k \geqslant 2} (L_{k,j}\lambda_{k,j})^2 = a_j\lambda_{1,j}^2 + \sum_{n \geqslant 1} L_{n,j+1}\lambda_{n,j+1}^2.$$

As $L_{n,j}\lambda_{n,j} < 1$ for $n \geqslant 2$ and as $\sum_{n \geqslant 1} \lambda_{n,j} < \infty$, we get $\sum_k (L_{k,j}\lambda_{k,j})^2 < \infty$. ∎

Let N and $\lambda_{k,j}$ be as in Lemma 3.2. Take, if $N \neq \infty$, $a_j = 1$ for all j, $0 \leqslant j < N$, and if $N = \infty$, let $(a_j)_{j \geqslant 0}$ be such that $\sum_{j \geqslant 0} a_j 2^{-j} = \infty$.

Let $L_{k,j}$ and $n(k, j)$ be the corresponding sequences obtained from Lemma 3.2 for $0 \leqslant j < N$, let $M_j = M(L_{k,j}, \lambda_{k,j})$.

3.5. Lemma. *Let* $N \in \mathbb{N} \cup \{+\infty\}$, $N \geqslant 1$, *and* $(a_j)_{j \geqslant 1}$ *be as in Lemma* 3.4. *Let* $p \geqslant 0$ *be an integer and* $0 \leqslant j < N$.

Then, $2\pi/(3^p \operatorname{Log} 2) \in T(M_j)$ *iff* j *is a multiple of* 3^p.

Proof. We have

$$\frac{2\pi}{3^p \operatorname{Log} 2} \in T(M_j) \Leftrightarrow \sum_{k \geqslant 1} L_{k,j}\lambda_{k,j}\, d\left(\frac{3^k + j}{3^p}, \mathbb{Z}\right)^2 < +\infty$$

$$\Leftrightarrow \sum_{k \geqslant p} L_{k,j}\lambda_{k,j}\, d\left(\frac{j}{3^p}, \mathbb{Z}\right)^2 < +\infty$$

ans as $\sum_{k \geqslant p} L_{k,j}\lambda_{k,j} = +\infty$, this is equivalent to $d(j/3^p, \mathbb{Z}) = 0$. ∎

3.6. PROPOSITION. *For $0 \leqslant j < N$, the M_j's are type III_0 factors, which are pairwise non-isomorphic. Moreover, $\Phi(M_j) \cong M_{j+1}$ (where $j \in \mathbb{Z}/N\mathbb{Z}$ if N is finite).*

Proof. As by Lemma 3.4, $\sum_{k \geqslant 1} L_{k,j}\lambda_{k,j} = +\infty$ for $0 \leqslant j < N$, the M_j's are of type III. Let us compute their flow of weights.

Let (Ω_j, μ_j) be the product measure space $\prod_{k \geqslant 1} (\{0, 1,..., L_k\}, \mu_{j,k}$ is the measure given by

$$\mu_{j,k}(p) = \frac{L_{k,j}!}{(L_{k,j} - p)!\, p!} \cdot \frac{\lambda_{k,j}^p}{(1 + \lambda_{k,j})^{L_{k,j}}}.$$

Let $A_j = \{\omega \in \Omega_j;\, \omega_k = 0 \text{ or } 1 \text{ for all } k \geqslant 1\}$. We have

$$\mu_j(A_j) = \prod_{k \geqslant 1} \frac{1 + L_{k,j}\lambda_{k,j}}{(1 + \lambda_{k,j})^{L_{k,j}}} \geqslant \prod_{k \geqslant 1} (1 + L_{k,j}\lambda_{k,j})\, e^{-L_{k,j}\lambda_{k,j}},$$

so

$$\operatorname{Log} \mu_j(A_j) \geqslant \sum_{k \geqslant 1} (\operatorname{Log}(1 + L_{k,j}\lambda_{k,j}) - L_{k,j}\lambda_{k,j})$$

$$\geqslant \sum_{k \geqslant 1} \left(\frac{L_{k,j}\lambda_{k,j}}{1 + L_{k,j}\lambda_{k,j}} - L_{k,j}\lambda_{k,j}\right) \geqslant - \sum_{k \geqslant 1} \frac{(L_{k,j}\lambda_{k,j})^2}{1 + L_{k,j}\lambda_{k,j}}.$$

Hence by Lemma 3.4, $\mu_j(A_j) > 0$.

Therefore the flow of weights of M_j is given by the action of $\mathbb{R}$ by translation on $(A_j \times \mathbb{R}, \mu_j | A_j \times dx)/\mathscr{R}$, where $\mathscr{R}$ is the equivalence relation of $A_j \times \mathbb{R}$, defined by $(x, t)\, \mathscr{R}(y, s)$ iff there exists K such that $x_k = y_k$, for $k > K$ and

$$- \sum_{i=1}^{K} x_i \operatorname{Log} \lambda_{k,i} + t - \sum_{i=1}^{K} y_i \operatorname{Log} \lambda_{k,i} + s.$$

(See Appendix of [12].)

If $\nu_j = \prod_{k \geqslant 1} \nu_{j,k}$ denotes the product measure on A_j, given by $\nu_{j,k}(0) =$

$1/(1+L_{k,j}\lambda_{k,j})$ and $v_{j,k}(1)=L_{k,j}\lambda_{k,j}/(1+L_{k,j}\lambda_{k,j})$ and if T is the odometer on (A_j, v_j), note that the flow of weights of M_j can be written as the flow over the base transformation (A_j, v_j, T) and under the ceiling function $\xi_j(x) = -\sum_{k\geq 1} ((Tx)_k - x_k) \operatorname{Log} \lambda_{k,j}, x \in A_j$. Hence by Remark 3.1, $\Phi(M_j) \cong W^*(L^\infty(A_j, v_j), T)$. Therefore $\Phi(M_j)$ is an ITPFI_2 factor; more precisely $\Phi(M_j) \cong \bigotimes_{k\geq 1} (M_2(C), \phi_{L_{k,j}\lambda_{k,j}})$, where $\phi_{L_{k,j}\lambda_{k,j}}$ is a state on $M_2(C)$, with eigenvalues $1/(1+L_{k,j}\lambda_{k,j})$ and $L_{k,j}\lambda_{k,j}/(1+L_{k,j}\lambda_{k,j})$. But by construction of the $L_{k,j}$ and $n(k, j)$ (Lemma 3.1 and Remark 3.2), we have $\Phi(M_j) \cong M(L_{n,j+1}, \lambda_{n,j+1}) = M_{j+1}$ and therefore M_j is of type III_0.

Finally, take $0 \leq i < j < N$. If $i=0$, then $3^{-p}2\pi/\operatorname{Log} 2$ belongs to $T(M_i)$ for all p, but does not belong to $T(M_j)$ for p large enough (Lemma 3.5). If $N < \infty$ and $i \neq 0$, then by the above discussion $\Phi^{N-j}(M_j) \cong M_0 \not\cong M_{N-j+i} \cong \Phi^{N-j}(M_i)$ and consequently $M_i \not\cong M_j$. If $N=\infty$, let us write $j-i=l.\,3^{p_0}$, where $(l, 3)=1$, and choose $3^p \geq j, p > p_0$. Then, $\Phi^{3^p-j}(M_j) \cong M_{3^p}$ and $\Phi^{3^p-j}(M_i) \cong M_{3^p+i-j}$. Hence, $3^{-p}2\pi/\operatorname{Log} 2$ belongs to $T(\Phi^{3^p-j}(M_j))$, but not to $T(\Phi^{3^p-j}(M_i))$ (Lemma 3.5). Therefore, $M_j \not\cong M_i$. ∎

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