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EXACT SEQUENCES FOR THE KASPAROV GROUPS OF GRADED ALGEBRAS

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Introduction. In [**11**] G. G. Kasparov defined the "operator K-functor" $KK(A, B)$ associated with the graded C^*-algebras A and B. If the algebras A and B are trivially graded and A is nuclear he proves six term exact sequence theorems. He asks whether this extends to the graded case.

Here we prove such "six-term exact sequence" results in the graded case. Our proof does not use nuclearity of the algebra A. This condition is replaced by a completely positive lifting condition (Theorem 1.1).

Using our result we may extend the results by M. Pimsner and D. Voiculescu on the K groups of crossed products by free groups to KK groups [**15**]. We give however a different way of computing these groups using the equivariant KK-theory developed by G. G. Kasparov in [**12**]. This method also allows us to compute the KK groups of crossed products by $PSL_2(\mathbf{Z})$.

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Notations. We use here essentially the definitions of the KK theory and notations of [**17**] which are slightly different from the original ones [**11**].

If A is a C^*-algebra and X is a locally compact space we use the notation $A(X)$ (rather than $A \otimes C_0(X)$) to denote the continuous functions vanishing at ∞ on X with values in A.

A shorter and more conceptual proof of Theorem 1.1 is in preparation jointly with J. Cuntz.

1. Statement of the theorem. In this section we state our main theorem and make some reductions. We then compute the connecting maps appearing in the theorem.

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Let

$$0 \to I \xrightarrow{i} A \xrightarrow{p} \frac{A}{I} \to 0 \quad \text{and} \quad 0 \to J \xrightarrow{j} B \xrightarrow{q} \frac{B}{J} \to 0$$

be the grading preserving exact sequences associated with the graded ideals $I \subseteq A$ and $J \subseteq B$.

1.1. THEOREM. a) *Assume that A is separable and that the quotient map p admits a completely positive, grading preserving, cross-section of norm 1. Then we have the six-term exact sequence*

$$\begin{array}{ccccc} KK\left(\frac{A}{I}, B\right) & \xrightarrow{p^*} & KK(A, B) & \xrightarrow{i^*} & KK(I, B) \\ \delta \uparrow & & & & \downarrow \delta \\ KK^1(I, B) & \xleftarrow[i^*]{} & KK^1(A, B) & \xleftarrow[p^*]{} & KK^1\left(\frac{A}{I}, B\right) \end{array}$$

b) *Assume that A is separable and q admits a completely positive, grading preserving, cross-section of norm 1. Then we have the six-term exact sequence*

$$\begin{array}{ccccc} KK(A, J) & \xrightarrow{j_*} & KK(A, B) & \xrightarrow{q_*} & KK\left(A, \frac{B}{J}\right) \\ \delta' \uparrow & & & & \downarrow \delta' \\ KK^1\left(A, \frac{B}{J}\right) & \xleftarrow[q_*]{} & KK^1(A, B) & \xleftarrow[j_*]{} & KK^1(A, J) \end{array}$$

1.2. *Remarks.* a) One can weaken the hypothesis in part b) of Theorem 1.1 (see remark 3.6).

b) (cf [**11**] p. 569). Let $s_0 : \frac{A}{I} \to A \left(\text{or } \frac{B}{J} \to B\right)$ be a completely positive cross-section, which is not necessarily grading preserving. Put

$$s(x) = s_0(x^{(0)})^{(0)} + s_0(x^{(1)})^{(1)}.$$

It is still completely positive and now it is grading preserving. If s_0 is of norm 1, so is s.

We may note that the proof of Lemmas 4 and 5 of Section 7 of [**11**] needs no change to get:

1.3. LEMMA. *Let L be a homotopy invariant functor (covariant or contravariant) from the category of graded C^*-algebras (or separable graded*

C^*-algebras) to the category of abelian groups. Assume that for each short exact sequence

$$0 \to I \xrightarrow{i} A \xrightarrow{p} \frac{A}{I} \to 0$$

such that p admits a completely positive lifting of norm 1 *the sequence*

$$L(I) \xrightarrow{L(i)} L(A) \xrightarrow{L(p)} L\left(\frac{A}{I}\right)$$

$$\left(or\ L\left(\frac{A}{I}\right) \xrightarrow{L(p)} L(A) \xrightarrow{L(i)} L(I)\right)$$

is exact. Then every such exact sequence gives rise to a long exact sequence:

$$\dots \xrightarrow{L(i)} L(A(\mathbf{R}^n)) \xrightarrow{L(p)} L\left(\frac{A}{I}(\mathbf{R}^n)\right) \xrightarrow{\delta} L(I(\mathbf{R}^{n-1}))$$

$$\xrightarrow{L(i)} L(A(\mathbf{R}^{n-1})) \xrightarrow{L(p)} \dots$$

(*the other way for L contravariant*).

Hence to prove Theorem 1.1 we just have to prove exactness of the sequences (using Bott periodicity [**11**] Section 5)

$$KK\left(\frac{A}{I}, B\right) \xrightarrow{p^*} KK(A, B) \xrightarrow{i^*} KK(I, B)$$

$$KK(A, J) \xrightarrow{j_*} KK(A, B) \xrightarrow{q_*} KK\left(A, \frac{B}{J}\right).$$

This will be done in Sections 2 and 3. The end of the present section is devoted to computing the connecting maps.

We recall that the connecting maps in Lemma 1.3 (and thus also in Theorem 1.1) are constructed in the following way ([**11**], Section 7 Lemma 5):

Let

$$S_p = \left\{(x, f), x \in A, f \in \frac{A}{I}[0, 1), f(0) = p(x)\right\}$$

be the cone of p. Let

$$e: I \to S_p, \ (e(a) = (a, 0)) \quad \text{and}$$

$$j: \frac{A}{I}(0, 1) \to S_p \quad (j(f) = (0, f))$$

be the inclusions.

The map $L(e):L(I) \to L(S_p)$ is an isomorphism. One then puts

$$\delta = L(e)^{-1} \circ L(j)$$

($L(j) \circ L(e)^{-1}$ in the contravariant case). We may notice that if $x \in KK(I, B)$, $x = 1_I \otimes_I x$ where 1_I is the unit of $KK(I, I)$. Hence

$$\delta(x) = \delta(1_I) \otimes_I x$$

where

$$\delta(1_I) \in KK\left(\frac{A}{I}(\mathbf{R}), I\right).$$

In the same way, if $x \in KK\left(A, \frac{B}{J}(\mathbf{R})\right)$ and $\frac{B}{J}$ is separable then

$$x = x \otimes_{B/J(\mathbf{R})} 1_{B/J(\mathbf{R})}.$$

Hence

$$\delta'(x) = x \otimes_{B/J(\mathbf{R})} \delta(1_{B/J(\mathbf{R})}).$$

We now give an explicit form for $\delta(1_I)$ and $\delta'(1_{B/J(\mathbf{R})})$.

Let

$$0 \to B \xrightarrow{j} D \xrightarrow{p} A \to 0$$

be a short exact sequence of graded algebras with A separable. To a grading-preserving completely positive cross-section s of norm 1 of p, there corresponds, thanks to Kasparov's "generalized Stinespring theorem" ([**10**], Theorem 3), a graded countably generated $\widetilde{D}$ module $\mathscr{E}_0$ and a grading preserving unital *-homomorphism

$$\pi:\widetilde{A} \to \mathscr{L}(\widetilde{D} \oplus \mathscr{E}_0),$$

such that $s(x) = Q\pi(x)Q$ for all $x \in A$ (Q denotes the projection on the summand $\widetilde{D}$).

We may note that if we require that $\mathscr{E}_0$ be generated by $(1 - Q)\pi(x)Qy$, $x \in \widetilde{A}$, $y \in \widetilde{D}$, $\mathscr{E}_0$ and π are uniquely determined by s. Denote by $E = C^*(\widetilde{A}, Q) \subseteq \mathscr{L}(\widetilde{D} \oplus \mathscr{E}_0)$ the C^*-algebra generated by $\pi(\widetilde{A})$ and Q. Let $J \subseteq E$ be the ideal generated by $[\pi(\widetilde{A}), Q]$ in E. Let $\mathscr{E}_1 = J \cdot (\widetilde{D} \oplus \mathscr{E}_0)$ be the submodule of $\widetilde{D} \oplus \mathscr{E}_0$ generated by

$$\{x\xi | x \in J; \xi \in \widetilde{D} \oplus \mathscr{E}_0\}.$$

Let $\mathscr{C}_1 = \mathscr{C}_{1,0}$ be the first Clifford algebra and let $\epsilon \in \mathscr{C}_1$ be an orientation of $\mathscr{C}_1(\epsilon = \epsilon^*, \epsilon^2 = 1, \partial\epsilon = 1)$.

Put

$$\mathscr{E} = \mathscr{E}_1 \hat{\otimes} \mathscr{C}_1,$$

$$F = (2Q - 1) \hat{\otimes} \varepsilon \in \mathscr{L}_{\tilde{D} \hat{\otimes} \mathscr{C}_1}(\mathscr{E}).$$

1.4. LEMMA. a) *The module* $\mathscr{E}$ *is a* $B \hat{\otimes} \mathscr{C}_1$ *module; i.e.*

$$\forall \xi \in \mathscr{E}, \langle \xi, \xi \rangle \in B \hat{\otimes} \mathscr{C}_1 \subseteq \tilde{D} \hat{\otimes} \mathscr{C}_1.$$

b) *The pair* $(\mathscr{E}, F)$ *is a Kasparov* $(A, B \hat{\otimes} \mathscr{C}_1)$ *bimodule.*

c) *Let* $\mathscr{E}'_1 \subseteq \tilde{D} \otimes \mathscr{E}_0$ *contain* $\mathscr{E}_1$ *and be stable under the action of* $\tilde{A}$ *and* Q. *Suppose that* $\mathscr{E}'_1$ *is a countably generated B module. Then the pairs* $(\mathscr{E}, F)$ *and* $(\mathscr{E}'_1 \hat{\otimes} \mathscr{C}_1, (2Q - 1) \hat{\otimes} \mathscr{E})$ *define the same element of* $KK^1(A, B)$.

d) *The class* δ_p *of* $(\mathscr{E}, F)$ *in* $KK^1(A, B)$ *does not depend upon the cross-section s.*

Proof. a) One has:

$$\langle (1 - Q)\pi(x)Q\xi, (1 - Q)\pi(x)Q\xi \rangle$$
$$= \xi^*(s(x^*x) - s(x)^*s(x))\xi \in B$$

for all $x \in \tilde{A}$, $\xi \in \tilde{D}$. Hence $\mathscr{E}_1$ is a B module.

b) As $Q \in \mathscr{K}(\tilde{D} \oplus \mathscr{E}_0)$, $J \subset \mathscr{K}(\tilde{D} \oplus \mathscr{E}_0)$. Hence $\mathscr{E}_1$ is countably generated and $J \subset \mathscr{K}(\mathscr{E}_1)$ ([**18**], Lemma 1.11). Hence $\mathscr{E}$ is countably generated and

$$[a, F] = 2[\pi(a), Q] \hat{\otimes} \varepsilon \in \mathscr{K}(\mathscr{E}) \quad \text{for all } a \text{ in } A.$$

Moreover $F^2 = 1$ and $F = F^*$.

c) is a consequence of [**5**], Remark A.6.2.

d) The set of the completely positive grading preserving cross-sections of p which extend unitarily to $\tilde{A} \to \tilde{D}$ is convex.

Note that the proof of a) shows that $\mathscr{E}_0$ is a B module. If B has a countable approximate unit one may take instead of $\mathscr{E}$ $(B \oplus \mathscr{E}_0) \hat{\otimes} \mathscr{C}_1$ (thanks to c).

This shows that in the non graded case δ_p is the element of $KK^1(A, B)$ corresponding to the exact sequence

$$0 \to B \xrightarrow{j} D \xrightarrow{p} A \to 0$$

through the identification

$$\text{Ext}(A, B)^{-1} \cong KK^1(A, B)$$

([**11**], Section 7, Theorem 1).

1.5. LEMMA. ([**11**], Section 7, Remark 3). *Let*

$$0 \to B_1 \to D_1 \xrightarrow{p_1} A_1 \to 0$$

be another short exact sequence with A_1 *separable and* p_1 *admitting a completely positive cross-section of norm* 1.

Let $\phi: D_1 \to D$ *be a grading preserving homomorphism with* $\phi(B_1) \subseteq B$. *Call* $\phi': B_1 \to B$ *and* $\phi'': A_1 \to A$ *the induced maps. One has*

$$\phi''^*(\delta_p) = \phi'_*(\delta_{p_1}).$$

Proof. Put

$$E = D \oplus_A A_1 = \{ (x, y) \mid x \in D, y \in A_1, p(x) = \phi''(y) \}.$$

Let $q: E \to A_1$ be the projection $q(x, y) = y$. The kernel of q identifies with I. Let s_1 and s be suitable cross-sections of p_1 and p. Put

$$s'_1(y) = (\phi s_1(y), y), \; s'(y) = (s\phi''(y), y), \; (y \in A_1).$$

Using the cross-section s'_1 of q we get

$$\delta_q = \phi'_*(\delta_{p_1}).$$

Using s' we get

$$\delta_q = \phi''^*(\delta_p).$$

We may now use the proofs of Lemmas 6 and 8 of Section 7 of [**11**] to get:

1.6. PROPOSITION. a) *The connecting map*

$$\delta: KK(I, B) \to KK^1\left(\frac{A}{I}, B\right)$$

in Theorem 1.1 *is given by*

$$x \mapsto \delta_p \otimes_I x.$$

b) *If* $\frac{B}{J}$ *is separable the connecting map*

$$\delta': KK\left(A, \frac{B}{J}\right) \to KK^1(A, J)$$

is given by

$$x \mapsto x \otimes_{B/J} \delta_q.$$

1.7. *Remark.* In the language of [**11**], Section 7, p. 569 we have just constructed a map

$$\operatorname{Ext}(A, B)^{-1} \to KK^1(A, B)$$

(the map which to the exact sequence $0 \to B \hat{\otimes} \mathcal{K} \to D \xrightarrow{p} A \to 0$ associates δ_p). This map is an (obvious) extension to the graded case of the isomorphism constructed by G. G. Kasparov in the non graded case ([**11**], Section 7, Theorem 1). But in the graded case this map is no longer an isomorphism ([**11**], p. 569).

2. Exactness of $A \to KK(A, B)$. In this section we prove:

2.1. PROPOSITION. *Let*

$$0 \to I \xrightarrow{i} A \xrightarrow{p} \frac{A}{I} \to 0$$

be an exact sequence of separable graded C^**-algebras. Assume that p admits a completely positive, grading preserving cross-section of norm* 1. *Let B be a graded* C^**-algebra. Then the sequence*

$$KK\left(\frac{A}{I}, B\right) \xrightarrow{p^*} KK(A, B) \xrightarrow{i^*} KK(I, B)$$

is exact.

Obviously $i^*p^* = 0$.

To prove the inclusion $\operatorname{Ker} i^* \subseteq \operatorname{Im} p^*$ let us make the following observation:

2.2. LEMMA. ([**11**], Theorem 1, Section 6). *Let* $(\mathscr{E}, F) \in \mathfrak{E}(A, B)$ *be such that* $i^*(\mathscr{E}, F)$ *is the zero element of* $KK(I, B)$. *Then there exists* $(\mathscr{E}', F') \in \mathfrak{D}(A, B)$ *such that* $i^*(\mathscr{E} \oplus \mathscr{E}', F \oplus F')$ *is operatorially homotopic to an element of* $\mathfrak{D}(I, B)$.

Proof. Let $(\bar{\mathscr{E}}, \bar{F})$ be a homotopy

$$(\bar{\mathscr{E}}, \bar{F}) \in \mathfrak{E}(I, B[0, 1])$$

such that

$$(\bar{\mathscr{E}}_t, \bar{F}_t) = i^*(\mathscr{E}, F) \quad \text{for } t \in [0, 1/2]$$

and $\bar{\mathscr{E}}_1 = 0$ (cf [**17**], Remark 3a).

Set

$$I' = \{f \in \tilde{A}[0, 1] \,|\, f(t) \in I, t \geqq \tfrac{1}{2}\}.$$

Let I' act naturally in $\bar{\mathscr{E}}$ and put $\tilde{\mathscr{E}} = I'\bar{\mathscr{E}}$. $\bar{F}$ admits an I' connexion $\tilde{F}$. We can change $\tilde{F}_t$ in $[0, 1/2)$ by adding a continuous family of zero connexions, in order to obtain $\tilde{F}_0 = F$.

Replacing $(\bar{\mathscr{E}}, \bar{F})$ by $(\tilde{\mathscr{E}}, \tilde{F})$ we may thus assume that A acts on $\bar{\mathscr{E}}$.

Let H_0 and $\mathbf{H}_1$ be the zero graded $(\mathbf{C}[0, 1], \mathbf{C})$ bimodules $H_0 = H_1 = C$; the action of $\mathbf{C}[0, 1]$ in H_0 is given by $f\xi = f(0)\xi$ and in H_1 by $f\xi = f(1)\xi$. Let $(H_0', F_0'), (H_1', F_1') \in \mathfrak{D}(\mathbf{C}[0, 1], \mathbf{C})$ be such that $(H_0 \oplus H_0', 0 \oplus F_0')$ and $(H_1 \oplus H_1', 0 \oplus F_1')$ are operatorially homotopic ([**11**], Section 6, Theorem 1, see also [**1**], Theorem 2.14, [**9**], Section 2).

Then any Kasparov product of $(\bar{\mathscr{E}}, \bar{F})$ by $(H_0 \oplus H_0', 0 \oplus F_0')$ is operatorially homotopic in $KK(I, B)$ to any Kasparov product of $(\bar{\mathscr{E}}, \bar{F})$ by $(H_1 \oplus H_1', 0 \oplus F_1')$ ([**17**], Theorem 12).

Put then

$$\begin{aligned}\mathscr{E}'' &= \bar{\mathscr{E}} \hat{\otimes}_{B[0,1]} (B \hat{\otimes}_C (H_0 \oplus H_0')) \\ &= \bar{\mathscr{E}}_0 \oplus \bar{\mathscr{E}} \hat{\otimes}_{B[0,1]} (B \hat{\otimes}_C H_0') \\ &= \mathscr{E} \oplus \mathscr{E}'.\end{aligned}$$

Put also

$$F' = 1 \hat{\otimes}_{B[0,1]} (1 \hat{\otimes} F_0') \in \mathscr{L}(\mathscr{E}').$$

Then $(\mathscr{E} \oplus \mathscr{E}', F \oplus F')$ is operatorially homotopic to a Kasparov product of $(\bar{\mathscr{E}}, \bar{F})$ by $(H_0 \oplus H_0', 0 \oplus F_0')$ (cf proof of [**17**], Theorem 12). Moreover

$$(\mathscr{E}', F') \in \mathfrak{D}(\mathscr{L}(\bar{\mathscr{E}}), B).$$

We may hence consider it as an element of $\mathfrak{D}(A, B)$.

Also

$$\bar{\mathscr{E}} \otimes_{B[0,1]} (B \hat{\otimes} H_1) = \bar{\mathscr{E}}_1 = 0.$$

So that

$$(\bar{\mathscr{E}} \hat{\otimes}_{B[0,1]} (B \hat{\otimes} H_1'), 1 \hat{\otimes} (1 \hat{\otimes} F_1')) \in \mathfrak{D}(I, B)$$

is operatorially homotopic to a Kasparov product of $(\bar{\mathscr{E}}, \bar{F})$ by $(H_1 \oplus H_1', 0 \oplus F_1')$.

As the class of $(\mathscr{E}, F)$ in $KK(A, B)$ does not change when adding $(\mathscr{E}', F')$ we may suppose that there exists a norm continuous path $(F_t)_{t \in [0,1]}$, $F_t \in \mathscr{L}(\mathscr{E})$ such that:

$$F = F_0, (\mathscr{E}, F_t) \in \mathfrak{E}(I, B), (\mathscr{E}, F_1) \in \mathfrak{D}(I, B).$$

We next construct an operatorial homotopy G_t, $G_t \in \mathscr{L}(\mathscr{E})$ such that

$$(\mathscr{E}, G_t) \in \mathfrak{E}(A, B) \quad \text{and} \quad (G_t - F_t)I \in \mathscr{K}(\mathscr{E}).$$

To do so we need the following lemma.

2.3. Lemma. *Let*

$$\mathscr{A} = \{x \in \mathscr{L}(\mathscr{E}) \mid [x, a] \in \mathscr{K}(\mathscr{E}), \forall a \in A],$$

$$\mathscr{A}' = [x \in \mathscr{L}(\mathscr{E}) \mid [x, a] \in \mathscr{K}(\mathscr{E}), \forall a \in I\},$$

$$\mathscr{J}' = \{x \in \mathscr{A}' | xa \in \mathscr{K}(\mathscr{E}), \forall a \in I\}.$$

Then one has $\mathscr{A} + \mathscr{J}' = \mathscr{A}'$.

Proof. Let $x \in \mathscr{A}'$ be homogeneous for the grading.

Put $E = \mathcal{K}(\mathcal{E})$, $E_1 = I + \mathcal{K}(\mathcal{E}) \subseteq \mathcal{L}(\mathcal{E})$.

Let E_2 be the C^*-subalgebra of $\mathcal{L}(\mathcal{E})$ generated by $[x, a]$ $(a \in A)$, and let $\mathfrak{F}$ be the vector space generated by x and A.

We see that $E \subseteq E_1$, $[\mathfrak{F}, E] \subseteq E$, $[\mathfrak{F}, E_1] \subseteq E_1$. Also if $a \in A$, $a' \in I$ are homogeneous we get

$$a'[a, x] = [a'a, x] - (-1)^{\partial x \partial a} [a', x]a \in \mathcal{K}(\mathcal{E}).$$

Hence $E_1 \cdot E_2 \subseteq E$.

Applying Theorem 4 of Section 3 of [**11**] we get M and N with $M + N = 1$, $M \geqq 0$, $N \geqq 0$, $[M, \mathfrak{F}] \subseteq \mathcal{K}(\mathcal{E})$, $M \cdot E_1 \subseteq \mathcal{K}(\mathcal{E})$, $N \cdot E_2 \subseteq \mathcal{K}(\mathcal{E})$.

Hence $Mx \in \mathcal{J}'$ and $Nx \in \mathcal{A}$.

2.4. LEMMA. *Assume that* $(\mathcal{E}, F) \in \mathfrak{E}(A, B)$ *and that* $(F_t)_{t \in [0,1]}$ *is an operatorial homotopy* $(\mathcal{E}, F_t) \in \mathfrak{E}(I, B)$ *such that* $F_0 = F$.

Then there exists an operatorial homotopy G_t *such that*

$$(\mathcal{E}, G_t) \in \mathfrak{E}(A, B), \ G_0 = F \quad \text{and}$$

$$(G_t - F_t)a \in \mathcal{K}(\mathcal{E}), \ \forall a \in I, \ \forall t \in [0, 1].$$

Proof. Define

$$\mathcal{J} = \{x \in \mathcal{A} | xa \in \mathcal{K}(\mathcal{E}), \ \forall a \in A\}.$$

Put

$$\mathcal{D} = \frac{\mathcal{A}}{\mathcal{J}}, \ \mathcal{D}' = \frac{\mathcal{A}'}{\mathcal{J}'}.$$

Lemma 2.3 asserts that the map

$$\mathcal{D} \xrightarrow{q} \mathcal{D}'$$

(induced by the inclusions $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{J} \subseteq \mathcal{J}'$) is onto.

Call f the image of $F (\in \mathcal{A})$ in $\mathcal{D}$ and f_t the images of $F_t (\in \mathcal{A}')$ in $\mathcal{D}'$. We have $f^2 = 1$, $f = f^*$, $\partial f = 1$ and $f_t^2 = 1$, $f_t = f_t^*$, $\partial f_t = 1$.

Choose a sequence $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n = 1$ such that

$$\|f_t - f_{\alpha_k}\| \leqq \frac{1}{2} \text{ for } t \in [\alpha_k, \alpha_{k+1}] \quad (k = 0, 1, \ldots, n - 1).$$

Assume constructed the continuous path $(g_t)_{t \in [0, \alpha_k]}$ $(k = 0, 1, \ldots, n - 1)$, $g_t \in \mathcal{D}$ such that:

$$q(g_t) = f_t, \ g_t^2 = 1, \ g_t = g_t^*, \ \partial g_t = 1, \ g_0 = f.$$

Choose then a continuous path $(g_t')_{t \in [\alpha_k, \alpha_{k+1}]}$ such that

$$q(g_t') = f_t, \ g_t' = g_t'^*, \ \partial g_t' = 1, \ g_{\alpha_k}' = g_{\alpha_k} \quad \text{and}$$

$$\|g'_t - g_{\alpha_k}\| \leqq \frac{1}{2}.$$

Then

$$\| \, |g'_t| - 1\| \leqq \frac{1}{2} \quad (\, |g_{\alpha_k}| = 1).$$

Put then $g_t = g'_t|g'_t|^{-1}$.

By induction we thus construct $(g_t)_{t \in [0, 1]}$ with

$$q(g_t) = f_t, \, g_t^2 = 1, \, \partial g_t = 1, \, g_t = g_t^*, \, g_0 = f.$$

Let then $(G_t)_{t \in [0,1]}$ be a continuous lifting of degree 1 of g_t in $\mathscr{A}$ such that $G_0 = F$.

Now $(\mathscr{E}, G_1)$ is operatorially homotopic to $(\mathscr{E}, F)$ and

$$I(G_1 - F_1) \in \mathscr{K}(\mathscr{E}).$$

The proof of Proposition 2.1 will be complete after the following:

2.5. Lemma. *Let* $(\mathscr{E}, G) \in \mathfrak{E}(A, B)$ *and* $F \in \mathscr{L}(\mathscr{E})$ *be such that*

$$(\mathscr{E}, F) \in \mathfrak{D}(I, B) \text{ and } I \cdot (F - G) \subseteq \mathscr{K}(\mathscr{E}).$$

Then the class of $(\mathscr{E}, F)$ *in* $KK(A, B)$ *is in the image of* p^*.

Proof. Let

$$s: \frac{A}{I} \to A$$

be a grading preserving, completely positive norm decreasing cross-section of the map p. Extend

$$s: \frac{\tilde{A}}{I} \to \tilde{A}$$

putting $s(1) = 1$.

Thanks to Kasparov's "generalized-Stinespring theorem" ([**10**], Theorem 3), there exist a separable graded Hilbert $\tilde{A}$ module $\mathscr{E}_0$ and a grading preserving *-homomorphism

$$\pi: \frac{\tilde{A}}{I} \to \mathscr{L}(\tilde{A} \oplus \mathscr{E}_0)$$

such that

$$Q\pi(a)Q = s(a), \quad \text{for all } a \in \frac{\tilde{A}}{I},$$

where $Q \in \mathcal{L}(\tilde{A} \oplus \mathcal{E}_0)$ is the projection in the summand $\tilde{A}$.

Replace $\mathcal{E}_0$ by the Hilbert $\tilde{A}$ module generated by

$$(1 - Q)\pi(a)\xi \quad \left(a \in \frac{\tilde{A}}{I}, \xi \in \tilde{A}\right).$$

Note that

$$\langle (1 - Q)\pi(a)\xi, (1 - Q)\pi(a)\xi\rangle = \xi^*(s(a^*a) - s(a^*)s(a))\xi \in I.$$

Hence $\mathcal{E}_0$ is an I module.

Let now $\mathcal{E}$, F, G be as in the statement. Let $\tilde{\mathcal{E}}$ be the $\left(\frac{A}{I}, B\right)$ bi-module

$$\tilde{\mathcal{E}} = (\tilde{A} \oplus \mathcal{E}_0) \hat{\otimes}_{\tilde{A}} \mathcal{E} = \mathcal{E} \oplus (\mathcal{E}_0 \hat{\otimes}_I \mathcal{E}).$$

As $(\mathcal{E}, F) \in \mathfrak{D}(I, B)$, $1 \hat{\otimes} F$ makes sense in $\mathcal{L}(\mathcal{E}_0 \hat{\otimes}_I \mathcal{E})$. Put

$$\tilde{F} = G \oplus (1 \hat{\otimes} F) \in \mathcal{L}(\mathcal{E} \oplus \mathcal{E}_0 \hat{\otimes}_I \mathcal{E}) = \mathcal{L}(\tilde{\mathcal{E}}).$$

$\tilde{F}$ is an $\tilde{A} \oplus \mathcal{E}_0$ connexion for G.

Moreover

$$(\tilde{\mathcal{E}}, \tilde{F}) \in \mathfrak{E}(\mathcal{L}(0 \oplus \mathcal{E}_0) \oplus \mathcal{K}(A \oplus \mathcal{E}_0), B).$$

But as

$$Q\pi\left(\frac{A}{I}\right)Q = s\left(\frac{A}{I}\right) \subseteq A = \mathcal{K}(A),$$

$$\pi\left(\frac{A}{I}\right) \subseteq \mathcal{L}(0 \oplus \mathcal{E}_0) \oplus \mathcal{K}(A \oplus \mathcal{E}_0),$$

so that $(\tilde{\mathcal{E}}, \tilde{F})$ defines an element of $\mathfrak{E}\left(\frac{A}{I}, B\right)$. Let us prove that $p^*(\tilde{\mathcal{E}}, \tilde{F})$ and $(\mathcal{E}, G)$ define the same element of $KK(A, B)$. Note that given an algebra D and a completely positive $\psi:D \to A$ (ψ of norm 1) such that

$$p \circ \psi : D \to \frac{A}{I}$$

is a $*$ homomorphism we thus get an element

$$\psi^*(\mathcal{E}, G, F) \in \mathfrak{E}(D, B).$$

Let then $\psi:A \to A[0, 1]$ be defined by

$$\psi(a)(t) = (1 - t)a + ts \circ p(a) \quad (a \in A, t \in [0, 1]).$$

Now

$$\psi^*(\mathscr{E}[0, 1], G \otimes 1, F \otimes 1) \in \mathfrak{E}(A, B[0, 1])$$

is a homotopy between $(\mathscr{E}, G)$ and $p^*(\widetilde{\mathscr{E}}, \widetilde{F})$.

3. Exactness of $B \to KK(A, B)$. In this section we prove:

3.1. PROPOSITION. *Let*

$$0 \to J \xrightarrow{j} B \xrightarrow{q} \frac{B}{J} \to 0$$

be an exact sequence of graded C^*-algebras. Assume that q admits a completely positive, grading preserving, cross-section of norm 1. Let A be a separable graded C^*-algebra. Then the sequence

$$KK(A, J) \xrightarrow{j_*} KK(A, B) \xrightarrow{q_*} KK\left(A, \frac{B}{J}\right)$$

is exact.

Obviously $q_* j_* = 0$.

We first prove the inclusion $\operatorname{Ker} q_* \subseteq \operatorname{Im} j_*$ assuming B separable.

3.2. LEMMA. *Let $(\mathscr{E}, F) \in \mathfrak{E}(A, B)$ be such that*

$$q_*(\mathscr{E}, F) \in \mathscr{D}\left(A, \frac{B}{J}\right).$$

Then the class of $(\mathscr{E}, F)$ in $KK(A, B)$ is in the image of j_*.

Proof. Let

$$\mathscr{E}' \subseteq \mathscr{E}, \mathscr{E}' = \{\xi \in \mathscr{E}, \langle \xi, \xi \rangle \in J\}.$$

Put

$$\widetilde{\mathscr{E}} = \{\xi \in \mathscr{E}[0, 1], \xi(1) \in \mathscr{E}'\}.$$

Notice that as

$$q_*(\mathscr{E}, F) \in \mathfrak{D}\left(A, \frac{B}{J}\right), \quad \text{for all } a \in A$$

$[a, F]$, $a(F^2 - 1)$, $a(F - F^*)$ are elements of $\mathscr{K}(\mathscr{E}') \subseteq \mathscr{K}(\mathscr{E})$. Hence

$$(\widetilde{\mathscr{E}}, F \otimes 1) \in \mathfrak{E}(A, B[0, 1])$$

(as $\mathscr{K}(\mathscr{E}') \otimes 1 \subseteq \mathscr{K}(\widetilde{\mathscr{E}})$). But

$$(\widetilde{\mathscr{E}}_0, (F \otimes 1)_0) = (\mathscr{E}, F) \quad \text{and}$$

$$(\widetilde{\mathscr{E}}_1, (F \otimes 1)_1) = (\mathscr{E}', F) \in j_*(\mathfrak{E}(A, J)).$$

3.3. LEMMA. *Assume that* $(\mathscr{E}, F) \in \mathfrak{E}(A, B)$ *and that* $(F_t)_{t\in[0,1]}$ *is an operatorial homotopy*

$$F_t \in \mathscr{L}\left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}\right), \quad \left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}, F_t\right) \in \mathfrak{E}\left(A, \frac{B}{J}\right)$$

such that $F \hat{\otimes} 1 = F_0$.

Then there exists an operatorial homotopy G_t *such that*

$$(\mathscr{E}, G_t) \in \mathfrak{E}(A, B), G_0 = F \text{ and } G_t \hat{\otimes} 1 = F_t.$$

Proof. Define

$$\mathscr{A} = \{x \in \mathscr{L}(\mathscr{E}) | [x, a] \in \mathscr{K}(\mathscr{E}), \forall a \in A\}$$

$$\mathscr{A}' = \left\{x \in \mathscr{L}\left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}\right) \middle| [x, a] \in \mathscr{K}\left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}\right), \forall a \in A\right\}$$

$$\mathscr{J} = \{x \in \mathscr{A} | xa \in \mathscr{K}(\mathscr{E}), \forall a \in A\}$$

$$\mathscr{J}' = \left\{x \in \mathscr{A}' | xa \in \mathscr{K}\left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}\right), \forall a \in A\right\}$$

$$\mathscr{D} = \frac{\mathscr{A}}{\mathscr{J}}, \mathscr{D}' = \frac{\mathscr{A}'}{\mathscr{J}'}.$$

In view of the proof of Lemma 2.4 it is enough to show that the map $\mathscr{D} \to \mathscr{D}'$ is onto. This follows from:

3.4. LEMMA. *The map* $q_*: \mathscr{A} \to \mathscr{A}'$ *is onto* ($q_*(x) = x \hat{\otimes} 1$).

Proof. The map

$$\mathscr{K}(\mathscr{E}) \xrightarrow{q_*} \mathscr{K}\left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}\right)$$

is onto. (In fact with the notations of Lemma 3.2

$$\mathscr{K}\left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}\right) = \mathscr{K}(\mathscr{E})/\mathscr{K}(\mathscr{E}')\text{)}.$$

As $\mathscr{K}(\mathscr{E})$ is separable it follows from [**13**], Proposition 3.12.10 (using [**10**], Theorem 1) that the map

$$\mathscr{L}(\mathscr{E}) \xrightarrow{q_*} \mathscr{L}\left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}\right)$$

is onto.

Let $x \in \mathscr{A}'$ and $y \in \mathscr{L}(\mathscr{E})$ such that $q_*(y) = x$. Put

$$E = \mathscr{K}(\mathscr{E}'), E_1 = \mathscr{K}(\mathscr{E}), \mathfrak{F} = A \oplus Cy^{(0)} \oplus Cy^{(1)}$$

($y = y^{(0)} + y^{(1)}$ is the decomposition of y for the grading).

Let also E_2' be the C^*-subalgebra of $\mathscr{L}(\mathscr{E})$ generated by $[y^{(0)}, a]$, $[y^{(1)}, a]$ $(a \in A)$. Put

$$E_2 = (\mathscr{K}(\mathscr{E}) + E_2') \cap \operatorname{Ker} q_*.$$

Note that

$$\operatorname{Ker} q_* = \{z \in \mathscr{L}(\mathscr{E}) \mid z \cdot E_1 \subseteq E, E_1 \cdot z \subseteq E\}.$$

We may hence apply Theorem 4 of Section 3 of **[11]** to get $M, N \geqq 0$, $\partial N = 0$, $M + N = 1$, $ME_1 \subseteq E$, $(M \in \operatorname{Ker} q_*)$, $N \cdot E_2 \subseteq E$, $[N, \mathfrak{F}] \subseteq E$.

Then $q_*(Ny) = x$. Moreover for all $a \in A$

$$[a, Ny] = [a, N]y + N[a, y].$$

But

$$q_*[a, y] \in \mathscr{K}\left(\mathscr{E} \hat{\otimes}_B \frac{B}{J}\right).$$

So that there exists $z \in \mathscr{K}(\mathscr{E})$ with

$$q_*[a, y] = q_*(z).$$

Then $([a, y] - z) \in E_2$. We thus get

$$[a, Ny] \in \mathscr{K}(\mathscr{E}) \quad \text{and} \quad Ny \in \mathscr{A}.$$

Following the same procedure as in Section 2 the proof of Proposition 3.1 will be complete for separable B after the following:

3.5. Lemma. *For all* $(\mathscr{E}, F) \in \mathfrak{D}\left(A, \frac{B}{J}\right)$ *there exists* $(\mathscr{E}', F') \in \mathfrak{D}(A, B)$ *with*

$$q_*(\mathscr{E}', F') = \mathscr{E}, F).$$

Proof. As in Lemma 2.5, we use Theorem 3 of **[10]** to construct a separable graded Hilbert J module $\mathscr{E}_0$ and a grading preserving * homomorphism

$$\frac{B}{J} \xrightarrow{\pi} \mathscr{L}_B(B \oplus \mathscr{E}_0)$$

such that

$$Q\pi(b)Q = s(b) \quad \text{for all } b \in \frac{B}{J},$$

where $Q \in \mathscr{L}(B \oplus \mathscr{E}_0)$ is the projection in the summand B.

Put then

$$\mathscr{E}' = \mathscr{E} \hat{\otimes}_{B/J} (B \oplus \mathscr{E}_0) \text{ and } F' = F \hat{\otimes} 1.$$

Then obviously $(\mathscr{E}, F) \in \mathfrak{D}(A, B)$. Note that

$$\mathscr{E}' \hat{\otimes}_B \frac{B}{J} = \mathscr{E} \hat{\otimes}_{B/J} \left[\left(B \oplus \mathscr{E}_0) \hat{\otimes}_B \frac{B}{J} \right) \right] = \mathscr{E} \hat{\otimes}_{B/J} \frac{B}{J} = \mathscr{E}.$$

Also, for all $T \in \mathscr{L}(\mathscr{E})$,

$$(T \hat{\otimes}_{B/J} 1_{B \oplus \mathscr{E}_0}) \hat{\otimes}_B 1_{B/J} = T.$$

Hence $q_*(\mathscr{E}', F') = (\mathscr{E}, F)$.

End of the proof of Proposition 3.1. Let $(\mathscr{E}, F) \in \mathfrak{E}(A, B)$ be such that

$$q_*(\mathscr{E}, F) = 0 \quad \text{in } KK\left(A, \frac{B}{J}\right).$$

Let

$$(\overline{\mathscr{E}}, \overline{F}) \in \mathfrak{E}\left(A, \frac{B}{J}[0, 1]\right)$$

be a homotopy between $q_*(\mathscr{E}, F)$ and $(0, 0)$. Then the pair $((\mathscr{E}, F), (\overline{\mathscr{E}}, \overline{F}))$ defines an element $x \in KK(A, S_q)$ where

$$S_q = \left\{ (b, f) | b \in B, f \in \frac{B}{J}[0, 1), f(0) = q(b) \right\}$$

is the cone of the map q.

Let $D \subseteq S_q$ be a separable subalgebra and $y \in KK(A, D)$ be such that $i_*(y) = x$ where $i:D \to S_q$ is the inclusion ([**18**], Remark 3.2).

Each element d of D is given by an element

$$\phi(d) \in B \quad \text{and} \quad \psi(d) \in \frac{B}{J}[0, 1).$$

Let B_1 be the (separable) subalgebra of B generated by

$$\{\phi(d), s(\psi(d)(t)) \, | d \in Dt \in [0, 1) \}.$$

Set $J_1 = B_1 \cap J$. Let

$$q_1 : B_1 \to \frac{B_1}{J_1}$$

be the quotient map. Note that it admits the completely positive lifting $s_{|_{B_1/J_1}}$. Let

$$j_1 : J_1 \to B_1, \; l : J_1 \to J, \; l' : B_1 \to B, \; i_1 : D \to S_{q_1}$$

be the inclusions and

$$p:S_q \to B, p_1:S_{q_1} \to B_1$$

be the projections. Put

$$x_1 = p_{1_*}i_{1_*}(y), x_1 \in KK(A, B).$$

As $x_1 \in p_{1_*}(KK(A, S_{q_1}))$, $q_{1_*}(x_1) = 0$. Hence applying Proposition 3.1 to the separable algebra B_1 we get $x'_1 \in J_1$ with $j_{1_*}(x'_1) = x_1$. But $l'_* x_1 = x$ and $l' \circ j_1 = j \circ l$.

Hence $x = j_*(l_* x'_1)$.

3.6. *Remark.* It is of course obvious that Proposition 3.1 (and hence Theorem 1.1) holds if we just assume the existence of a completely positive, grading preserving cross-section of norm 1 for any separable subalgebra B_1 of $\frac{B}{J}$.

In fact we only used this assumption in Lemma 3.5. Let $\mathscr{C}_1$ be the first Clifford algebra. An element of $\mathfrak{D}\left(A, \frac{B}{J}\right)$ is a * homomorphism

$$A \hat{\otimes} \mathscr{C}_1 \xrightarrow{\pi} \mathscr{L}(\mathscr{H}_{B/J}).$$

We want to ensure that such a π admits a completely positive lifting

$$A \hat{\otimes} \mathscr{C}_1 \to \mathscr{L}(\mathscr{H}_B).$$

This condition is also satisfied if the algebra A is nuclear [**2**].

4. Crossed products. Let us first note that the "mapping torus exact sequence" ([**3**], Section 5, Corollary 6) has the completely positive lifting property. Hence no nuclearity assumptions are needed in the exact sequence of [**8**] Theorem 3.

We may next notice that in M. Pimsner and D. Voiculescu's work on crossed-products by free groups [**15**] the exact sequence

$$0 \to \mathscr{K} \otimes A \to \mathfrak{C}_n \to A \rtimes_{\alpha,r} \mathbf{F}_n \to 0$$

([**15**], Lemma 1.1]) has also this property. Observe then that the element Δ that they construct ([**15**], Lemma 2.1) is actually an element of

$$KK(\mathfrak{C}_n, A \rtimes_{\alpha,r} \mathbf{F}_{n-1})$$

inverse to the element

$$d^*(1_{\mathfrak{C}_n}) \in KK(A \rtimes_{\alpha,r} \mathbf{F}_{n-1}, \mathfrak{C}_n).$$

Hence their results for reduced crossed-products extend to the KK groups. Thanks to J. Cuntz's results on "K amenability" ([7], Theorems 2.1 and 2.4) this is also true for the full crossed-products.

We now give another proof of this result. We also give exact sequences allowing to compute the KK groups of the crossed-products by $PSL_2(\mathbf{Z})$.

To do so we use the equivariant KK theory defined by Kasparov in [**12**].

Let $G = PSL_2(\mathbf{R})$ and let K be the maximal compact subgroup of G.

Let

$$\alpha \in KK_G\left(\mathbf{C}\left(\frac{G}{K}\right), \mathbf{C}\right) \quad \text{and} \quad \beta \in KK_G\left(\mathbf{C}, \mathbf{C}\left(\frac{G}{K}\right)\right)$$

be the elements defined by Kasparov in [**12**], Section 5, definition 3, 4 (see also [**4**], Section 12). One has

$$\alpha \otimes_{\mathbf{C}} \beta = 1_{\mathbf{C}(G/K)} \text{ and } r_{G,K}(\beta \otimes_{\mathbf{C}(G/K)} \alpha) = 1$$

([**12**], Section 5, Theorem 2). Here

$$r_{G,K}: KK_G(\mathbf{C}, \mathbf{C}) \to KK_K(\mathbf{C}, \mathbf{C})$$

is the restriction map.

4.1 LEMMA. *Set* $\Gamma = PSL_2(\mathbf{Z}) \subseteq PSL_2(\mathbf{R}) = G$. *Then*

$$r_{G,\Gamma}(\beta \otimes_{\mathbf{C}(G/K)} \alpha) = 1.$$

Proof. Let $h_t:\Gamma \to G(t \in [0, 1])$ be a continuous family of continuous group homomorphisms where Γ and G are second countable locally compact groups.

Let

$$h_t^*: KK_G(\mathbf{C}, \mathbf{C}) \to KK_\Gamma(\mathbf{C}, \mathbf{C})$$

be the restriction map associated to the homomorphism h_t. Then obviously $h_0^* = h_1^*$.

Put

$$a_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$b_t = \begin{pmatrix} \frac{1-t}{2} & -\sqrt{\frac{3+t^2}{4}} \\ \sqrt{\frac{3+t^2}{4}} & \frac{1+t}{2} \end{pmatrix} \in G.$$

Let $a = a_1$, $b = b_1$ be the generators of

$$\Gamma = \frac{\mathbf{Z}}{2} * \frac{\mathbf{Z}}{3}.$$

The families a_t, $b_t t \in [0, 1]$ determine a continuous family h_t of homomorphisms $\Gamma \to G$. Hence

$$r_{G,\Gamma}(\beta \otimes \alpha) = h_1^*(\beta \otimes \alpha) = h_0^*(\beta \otimes \alpha) = h_0'^* \, r_{G,K}(\beta \otimes \alpha) = 1$$

where h_0' is the homomorphism $h_0:\Gamma \to G$ seen as a homomorphism from Γ to K (as $h_0(\Gamma) \subseteq K$).

4.2. COROLLARY. *Let $\mathbf{F}_n$ be the free group with n generators embedded in $PSL_2(\mathbf{R})$ in the usual way. One has*

$$r_{G,\mathbf{F}_n}(\beta \otimes_{\mathbf{C}(G/K)} \alpha) = 1.$$

Proof. One has $\mathbf{F}_n \subseteq PSL_2(\mathbf{Z})$.

We may also note that the usual embedding of $\mathbf{F}_n$ in $PSL_2(\mathbf{R})$ is homotopic to the trivial map as $PSL_2(\mathbf{R})$ is connected.

Let us also notice that one could deduce this result from J. Cuntz's results in **[6]**. From this result we know that $KK(C^*(\mathbf{F}_n), \mathbf{C})$ $(\cong KK_{\mathbf{F}_n}(\mathbf{C}, \mathbf{C}))$ is isomorphic to $\mathbf{Z}$ (as a group) and that the map

$$KK(C^*(\mathbf{F}_n), \mathbf{C}) \xrightarrow{i^*} KK(\mathbf{C}, \mathbf{C})$$

induced by the inclusion

$$\mathbf{C} \xrightarrow{i} C^*(\mathbf{F}_n)$$

is an isomorphism. Note then that

$$i^*(\beta \otimes_{\mathbf{C}(G/K)} \alpha) = 1 \in KK(\mathbf{C}, \mathbf{C}).$$

4.3. COROLLARY. *Let Γ be $PSL_2(\mathbf{Z})$ or $\mathbf{F}_n$ embedded in $PSL_2(\mathbf{R})$. Let ρ be a grading preserving Γ action on the graded algebra A. Let B be a graded algebra. One has*:

$$\text{a) } KK(A \rtimes_{\rho} \Gamma, B) = KK(A \rtimes_{\rho,r} \Gamma, B) = KK\left(A\left(\frac{G}{K}\right) \rtimes_{\rho'} \Gamma, B\right)$$

(*A separable*)

$$\text{b) } KK(B, A \rtimes_{\rho} \Gamma) = KK(B, A \rtimes_{\rho,r} \Gamma) = KK\left(B, A\left(\frac{G}{K}\right) \rtimes_{\rho'} \Gamma\right)$$

(*B separable*).

(The action ρ' is defined by

$$(\rho'(g)f)(x) = \rho(g)f(g^{-1}x)$$

for all $g \in \Gamma, f \in A\left(\frac{G}{K}\right)$ and $x \in \frac{G}{K}\Big)$.

Proof. Assume A is separable. Put

$$\tilde{\alpha} = j_\Gamma(\tau_A(r_{G,\Gamma}(\alpha))) \in KK\left(A\left(\frac{G}{K}\right) \underset{\rho'}{\rtimes} \Gamma, A \underset{\rho}{\rtimes} \Gamma\right)$$

([**12**], Theorem 1, Section 6)

$$\tilde{\beta} = j_\Gamma(\tau_A(r_{G,\Gamma}(\beta))) \in KK\left(A \underset{\rho,r}{\rtimes} \Gamma, A\left(\frac{G}{K}\right) \underset{\rho'}{\rtimes} \Gamma\right)$$

([**12**], Theorem 1, Section 6 and remark 2, Section 6).
Let

$$p:A \underset{\rho}{\rtimes} \Gamma \to A \underset{\rho,r}{\rtimes} \Gamma$$

be the quotient map. We have

$$p^*(\tilde{\beta} \otimes_{A(G/K)\rtimes_{\rho'}\Gamma} \tilde{\alpha}) = 1_{A\rtimes_\rho\Gamma}, p_*(\tilde{\beta} \otimes \tilde{\alpha}) = 1_{A\rtimes_{\rho,r}\Gamma}$$

and

$$\tilde{\alpha}\, x_{A\rtimes_\rho\Gamma}\, p_*(\tilde{\beta}) = 1_{A(G/K)\rtimes_{\rho'}\Gamma}$$

([**12**], Theorems 4 and 5, Section 4 and 1, Section 6). The result follows then from [**11**], Theorem 6, Section 4 for A separable. To prove b) for non separable A argue as in [**18**], 3.3.

4.4. *Remark.* The isomorphisms

$$KK(A \underset{\rho}{\rtimes} \Gamma, B) \cong KK(A \underset{\rho,r}{\rtimes} \Gamma, B) \text{ and}$$

$$KK(B, A \underset{\rho}{\rtimes} \Gamma) \cong KK(B, A \underset{\rho,r}{\rtimes} \Gamma)$$

were proved by J. Cuntz in [**7**], Theorem 2.1.
Let us now show how one may compute

$$KK\left(A\left(\frac{G}{K}\right) \underset{\rho'}{\rtimes} \Gamma, B\right) \quad \text{and} \quad KK\left(B, A\left(\frac{G}{K}\right) \underset{\rho'}{\rtimes} \Gamma\right).$$

Let first Γ be $\mathbf{F}_2$.
Taking fundamental domains one gets that

$$A\left(\frac{G}{K}\right) \underset{\rho'}{\rtimes} \Gamma$$

is Morita equivalent to the subalgebra A' of $A([0,1] \times [0,1])$:

$$A' = \{f | f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0,$$
$$f(t,0) = \rho(a)f(0,t),$$
$$f(1,t) = \rho(b)f(t,1)\}$$

(where a and b are the generators of $\mathbf{F}_2$).

One thus gets the following exact sequence:

$$0 \to A((0,1) \times (0,1)) \to A' \xrightarrow{q} A(0,1) \oplus A(0,1) \to 0$$

where

$$q(f) = f(\cdot, 0) + f(1, \cdot).$$

This exact sequence obviously satisfies the completely positive lifting property.

We may compute the corresponding element

$$\delta_q \in KK^1(A(0,1) \oplus A(0,1), A((0,1) \times (0,1)))$$

assuming that A is separable. This is done (using Lemma 1.5) as in [**8**], Lemma p. 12. We thus get:

$$\delta_q = (\mathrm{id}^* - \rho(a)^*)\tau_{A(0,1)}(\beta_1) + (\mathrm{id}^* - \rho(b)^*)\tau_{A(0,1)}(\beta_1)$$

where

$$\beta_1 \in KK^1(C, C(\mathbf{R}))$$

is the Bott element ([**11**], p. 546). (This decomposition of δ_q corresponds to the equality

$$KK^1(A(0,1) \oplus A(0,1), A((0,1) \times (0,1)))$$
$$= KK^1(A(0,1), A((0,1) \times (0,1)))$$
$$\oplus KK^1(A(0,1), A((0,1) \times (0,1)))$$

[**11**], Corollary 1, Section 4). We thus get:

4.5. THEOREM. ([**15**], Theorem 3.5). *If* $(A, \mathbf{F}_2, \rho)$ *is a* (*graded*) C^* *dynamical system and B is a* (*graded*) C^**-algebra we have the following exact sequences*:

$$\begin{array}{ccccc} KK(A \rtimes_\rho \mathbf{F}_2, B) & \to KK(A, B) & \xrightarrow{(\mathrm{id}^* - \rho(a)^*, \mathrm{id}^* - \rho(b)^*} & KK(A, B) \oplus KK(A, B) \\ \uparrow & & & \downarrow \\ KK^1(A, B) \oplus KK^1(A, B) & \leftarrow KK^1(A, B) & \leftarrow & KK^1(A \rtimes_\rho \mathbf{F}_2, B) \end{array}$$

if A is separable and

$$\begin{array}{ccc} KK(B, A) \oplus KK(B, A) \to KK(B, A) \to KK(B, A \rtimes_\rho \mathbf{F}_2) \\ \uparrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ KK(B, A \rtimes_\rho \mathbf{F}_2) \leftarrow KK^1(B, A) \leftarrow KK^1(B, A) \oplus KK^1(B, A) \end{array}$$

if B is separable. The same is true for the reduced crossed-products.

To get the same results for the free group with n generators, one may proceed similarly. Let X be a polygon with $2n$ sides. Call $(S_k)_{k=1\dots 2n}$ the sides. For $j = 1 \dots n$, α_j is an identification of S_{2j-1} with S_{2j} which preserves the intersection vertex. Let $X' = X\backslash\{\text{vertices}\}$. Then

$$A\left(\frac{G}{K}\right) \rtimes_\rho \mathbf{F}_n$$

is Morita equivalent to the subalgebra A' of $A(X')$

$$A' = \{f | f(\alpha_j(t)) = \rho(a_j) \cdot f(t), \ t \in S_{2j-1}, j = 1 \dots n\}$$

($a_1 \dots a_n$ are the generators of $\mathbf{F}_n$).

One has the exact sequence

$$0 \to A(X'') \to A' \xrightarrow{p} A(0, 1) \xrightarrow{n} 0$$

(X'' = interior of X' is homeomorphic to $\mathbf{R}^2$). We get

$$\delta_p \in KK^1(A(0, 1)^n, A(\mathbf{R}^2)) \cong KK(A, A)^n$$

$$\delta_p = ((-1)^j(1 - (a_j)^{*1}))_{j=1\dots n}.$$

Thus we get the generalization of the Pimsner-Voiculescu exact sequence (to the KK groups).

4.6. *Remark.* Another way of getting this result is to embed $\mathbf{F}_n$ in $\mathbf{F}_2$ mapping a_k to $b^{k-1}ab^{1-k}$ $(k = 1, \dots, n - 1)$ and a_n to b^{n-1}.

If $(A, \mathbf{F}_n, \rho)$ is a C^*-dynamical system, we induce ρ to an action of $\mathbf{F}_2$. We then obtain the action $\bar{\rho}$ of $\mathbf{F}_2$ in A^{n-1} given by:

$$\bar{\rho}(a)(x_1, \dots, x_{n-1}) = (\rho(a_1)x_1, \dots, \rho(a_{n-1})x_{n-1})$$

$$\rho(b)(x_1, \dots, x_{n-1}) = (x_2, \dots, x_{n-1}, \rho(a_n)x_1).$$

As $A \rtimes_\rho \mathbf{F}_n$ is Morita equivalent to

$$A^{n-1} \rtimes_{\bar{\rho}} \mathbf{F}_2$$

we now just have to apply Theorem 4.5 in order to compute the "*KK* groups" of $A \rtimes_\rho \mathbf{F}_n$.

Let us now come to the case $\Gamma = PSL_2(\mathbf{Z})$.

Let

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, c = ab = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Taking first in account the points of G/K which have non trivial stabilizer we get the exact sequence:

$$0 \to A_1 \otimes \mathscr{K} \to A\left(\frac{G}{K}\right) \rtimes_{\rho'} \Gamma \to \left(A \rtimes_{\rho(a)} \frac{\mathbf{Z}}{2\mathbf{Z}} \oplus A \rtimes_{\rho(b)} \frac{\mathbf{Z}}{3\mathbf{Z}}\right) \otimes \mathscr{K} \to 0$$

where $\mathscr{K}$ is the algebra of the compacts and A_1 is the subalgebra of $A([0, 1] \times [0, 1])$

$$A_1 = \{f | f(0, 0) = f(1, 1) = f(0, 1) = f(1, 0) = 0;$$

$$f(t, 0) = \rho(a) f(0, t); f(1, t) = \rho(c) f(t, 1)\}.$$

We then get the exact sequence:

$$0 \to A((0, 1) \times (0, 1)) \to A_1 \xrightarrow{q} A(0, 1) \oplus A(0, 1) \to 0.$$

Both these exact sequences satisfy the completely positive lifting property. The element

$$\delta_q \in KK^1(A(0, 1) \oplus A(0, 1), A((0, 1) \times (0, 1)))$$

$$\cong KK(A, A) \oplus KK(A, A)$$

is easy to compute. We get

$$\delta_q = (1 - \rho(a)^* 1, 1 - \rho(c)^* 1).$$

4.7. THEOREM. *If $(A, PSL_2(\mathbf{Z}), \rho)$ is a (graded) C^*-dynamical system and B is a (graded) C^*-algebra we have the following exact sequences*:

a)

$$\begin{array}{ccc} KK(A \rtimes_\rho PSL_2(\mathbf{Z}), B) \to KK(A_1, B) \to KK^1(A \rtimes \mathbf{Z}_2 \oplus A \rtimes \mathbf{Z}_3, B) \\ \uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ KK^1(A \rtimes \mathbf{Z}_2 \oplus A \rtimes \mathbf{Z}_3, B) \leftarrow KK^1(A_1, B) \leftarrow KK^1(A \rtimes_\rho PSL_2(\mathbf{Z}), B) \end{array}$$

and

$$\begin{array}{ccccc} KK(A_1, B) & \rightarrow & KK(A, B) & \xrightarrow{\delta_q \otimes} & KK(A \oplus A, B) \\ \uparrow & & & & \downarrow \\ KK^1(A \oplus A, B) & \xleftarrow{\delta_q \otimes} & KK^1(A, B) & \leftarrow & KK^1(A_1, B) \end{array}$$

for A separable.

b)

$$\begin{array}{ccccc} KK(B, A \rtimes \mathbf{Z}_2 \oplus A \rtimes \mathbf{Z}_3) & \rightarrow & KK(B, A_1) & \rightarrow & KK(B, A \rtimes_\rho PSL_2(\mathbf{Z})) \\ \uparrow & & & & \downarrow \\ KK^1(B, A \rtimes_\rho PSL_2(\mathbf{Z})) & \leftarrow & KK^1(B, A_1) & \leftarrow & KK^1(B, A \rtimes \mathbf{Z}_2 \oplus A \rtimes \mathbf{Z}_3) \end{array}$$

and

$$\begin{array}{ccccc} KK(B, A \oplus A) & \xrightarrow{\otimes \delta_q} & KK(B, A) & \rightarrow & KK(B, A_1) \\ \uparrow & & & & \downarrow \\ KK^1(B, A_1) & \leftarrow & KK^1(B, A) & \leftarrow & KK^1(B, A \oplus A) \end{array}$$

(*B separable*).

4.8. *Remark*. a) If the actions $\rho(a)$ and $\rho(b)$ commute with each other, these results are obtained by J. Cuntz, in **[6]** (see remark at the end of [7]).

b) One would like to extend this method to other subgroups of $PSL_2(\mathbf{R})$. However this method fails if we take for instance $\Gamma \subseteq PSL_2(\mathbf{R})$ to be the fundamental group of a Riemann surface of genus $\geqq 2$. In that case the usual map $\Gamma \hookrightarrow PSL_2(\mathbf{R})$ is not homotopic to the trivial map $\Gamma \rightarrow 1 \subseteq PSL_2(\mathbf{R})$. However this method works to give an easy proof of **[12]**, Lemma 4, Section 5: One notices that the identity map of the proper motion group is homotopic to the map

$$G \xrightarrow{T} S0_n \hookrightarrow G$$

where $T(f)$ is the tangent linear map associated with the affine transformation T.

c) Our results allow one to extend the "universal coefficient formula" of [**16**] to a larger class of algebras which are not nuclear.

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