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Powers' Property and Simple C^*-Algebras

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Let A be a C^*-algebra with unit, let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action of some (discrete) group Γ on A and let $B = A \underset{\alpha, r}{\rtimes} \Gamma$ be the corresponding reduced crossed product. It would be desirable to understand the ideal structure of B in terms of the data; less ambitiously, we are interested here in criteria for B to be simple. This is a popular problem: it is for example discussed for Γ abelian at the end of [13], or for α outer in the introduction of [5].

The present work concentrates on cases where Γ is a so-called Powers' group (we recall the definition after Proposition 2 below), but where α is arbitrary.

Recall from [8] that the following are Powers' groups:

1) non abelian free groups, and more generally free products which are neither trivial nor $(\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/2\mathbf{Z})$;

2) non amenable subgroups of $PSL(2, \mathbf{C})$;

3) $SL(3, \mathbf{Z})$, and more generally subgroups of $SL(3, \mathbf{C})$ containing $SL(3, \mathbf{Z})$.

For the two next theorems, α is an action of a Powers' group Γ on a C^*-algebra A with 1 and

$$e : B = A \underset{\alpha, r}{\rtimes} \Gamma \to A$$

is the canonical conditional expectation.

Theorem I. *If the only Γ-invariant ideals in* A *are trivial, then* B *is simple. In particular if* A *is simple, then* B *is simple for any possible action α.*

Theorem II. *Any trace τ on* B *can be written as $\sigma \circ$* e *where σ is a Γ-invariant trace on* A. *In particular, if* A *has a unique trace, then* B *has a unique trace for any possible action α.*

The example with $A = \mathbf{C}$ and Γ free non abelian is that of Powers' original paper [14]. Examples show that Theorem I would not hold if A was without unit.

In the first section, we introduce a property of a pair (A, τ) where A is a C^*-algebra with unit and where τ is a faithful trace on A. This property implies that A is simple and that τ is the unique trace on A. Examples are provided by reduced

C^*-algebras of Powers' groups. We check also that this property is not shared by nuclear algebras, despite the fact that many of them are simple and have a unique trace.

In the second section, we prove the theorems above and we discuss the particular case given by A abelian.

Theorem I suggests the following

Problem. Let A be a simple C^*-algebra with unit, let Γ be a group with reduced C^*-algebra $C_r^*(\Gamma)$ simple, and let α be an action of Γ on A. Is it true that the reduced crossed product $A \underset{\alpha, r}{\rtimes} \Gamma$ is simple?

The answer is known to be yes in several cases, including that of a trivial action α [16].

1. Powers' Property Revisited

We consider a C^*-algebra A with unit and a faithful trace τ on A. By a trace, we mean here a positive linear functional normalized by $\tau(1)=1$, such that $\tau(xy) =\tau(yx)$ for $x, y \in A$. We denote by $L^2(A,\tau)$ or simply by H the Hilbert space obtained by completion of A for the scalar product $(x|y)=\tau(y^*x)$. Left multiplication by x in A extends to a bounded operator on H, denoted by x again. Thus, we identify A to a subalgebra of the algebra $L(H)$ of all bounded operators on H. Also τ extends to a linear form on H, denoted by τ again. For any pair (A,τ) as above and for any integer $n \geqq 2$, we define

Property P_n. *For any* $x \in A$ *and for any* $\varepsilon > 0$, *there exist a projection* $p \in L(H)$ *and unitaries* $(u_k)_{1 \leqq k \leqq n}$ *in* A *such that*

(1) $\|p(x-\tau(x))p\| \leqq \varepsilon$

(2) *the projections* $u_k(1-p)u_k^*$ *are pairwise orthogonal for* $k=1, \dots, n$.

Given a dense subset $\mathscr{A}$ of A, *observe one does not change P_n upon replacing "for any* $x \in A$" *by "for any* $x \in \mathscr{A}$".

Lemma 1. *Let* (A,τ) *be a pair which satisfies* P_3 *and let* c *be a real number with*

$$0.9714 \approx \frac{3+2\sqrt{2}}{6} < c < 1.$$

Let $x \in A$ *with* $x^* = x$ *and* $\tau(x)=0$. *Then there exist unitaries* $u_1, u_2, u_3 \in A$ *with*

$$\left\| \frac{1}{3} \sum_{k=1}^{3} u_k x u_k^* \right\| \leqq \frac{2+c}{3} \|x\|.$$

Proof. We may assume $\|x\|=1$. Choose ε such that

$$1-\frac{1}{2}\left(\frac{\sqrt{2}-1}{\sqrt{3}}-\varepsilon\right)^2 \leqq c$$

and let p, u_1, u_2, u_3 be as in the definition of P_3. Define $y=\frac{1}{3}\sum_{k=1}^{3} u_k x u_k^*$ and choose $\xi \in H$ with $\|\xi\|=1$; we want to estimate $|(y\xi|\xi)|$.

Projections $u_k(1-p)u_k^*$ being pairwise orthogonal, there exists $j \in \{1,2,3\}$ with $\|u_j(1-p)u_j^*\xi\|^2 \leqq \frac{1}{3}$; we may assume $j=1$, and we set $\xi_1 = u_1^*\xi$. As $\|(1-p)\xi_1\|^2 \leqq \frac{1}{3}$ one has $\|p\xi_1\|^2 \geqq \frac{2}{3}$ and $\|px(1-p)\xi_1\|^2 \leqq \frac{1}{3}$. Consequently

$$\begin{aligned}\|x\xi_1 - \xi_1\| &\geqq \|p\xi_1 - px(1-p)\xi_1 - pxp\xi_1\| \\ &\geqq \|p\xi_1\| - \|px(1-p)\xi_1\| - \|pxp\xi_1\| \geqq \frac{\sqrt{2}-1}{\sqrt{3}} - \varepsilon = \delta\end{aligned}$$

where the last equality defines δ. As

$$\|x\xi_1 - \xi_1\|^2 = \|x\xi_1\|^2 + \|\xi_1\|^2 - 2(x\xi_1|\xi_1) \leqq 2(1-(x\xi_1|\xi_1))$$

it follows that

$$(x\xi_1|\xi_1) \leqq 1 - \tfrac{1}{2}\|x\xi_1 - \xi_1\|^2 \leqq 1 - \tfrac{1}{2}\delta^2 \leqq c\,.$$

Now

$$\begin{aligned}(y\xi|\xi) &\leqq \tfrac{1}{3}(x\xi_1|\xi_1) + \tfrac{1}{3}\|u_2xu_2^*\|\ \|\xi\|^2 + \tfrac{1}{3}\|u_3xu_3^*\|\ \|\xi\|^2 \\ &\leqq \frac{c+2}{3}\|x\|\end{aligned}$$

and the same argument with $-x$ shows that

$$|(y\xi|\xi)| \leqq \frac{c+2}{3}\|x\|\,.$$

As ξ may be any vector of norm 1, this ends the proof. □

It follows immediately from Lemma 1 that, if (A,τ) satisfies P_3, then A has the Dixmier property (see [1, 7]), so that the next proposition is completely standard. We repeat it for the reader's convenience.

Proposition 2. *Let* (A, τ) *be a pair which satisfies* P_3. *Then* A *is simple and* τ *is the unique trace on* A.

Proof. Let $\mathscr{I}$ be a non-zero two-sided ideal in A. Choose $y \neq 0$ in $\mathscr{I}$; upon changing from y to $\tau(y^*y)^{-1}y^*y$, we may assume that $y^* = y$ and that $\tau(y) = 1$. By Lemma 1 applied several times (firstly to $x = y-1$), there exist unitaries $u_1, \ldots, u_N \in A$ with

$$\left\| \frac{1}{N} \sum_{1 \leqq k \leqq N} u_k y u_k^* - 1 \right\| < 1 \tag{$*$}$$

Consequently $\mathscr{I}$ contains the invertible element $z = \frac{1}{N} \sum_{1 \leqq k \leqq N} u_k y u_k^*$, and $\mathscr{I} = A$. Hence A is simple.

Let σ be a trace on A. By increasing N, one may assume that the left-hand side of ($*$) is arbitrarily small, so that $\sigma(z-1) = 0$, and $\sigma(y) = \sigma(z) = 1 = \tau(y)$. As σ and τ agree on hermitian elements, $\sigma = \tau$. □

The proof that property P_n implies the simplicity of A and the unicity of τ may be found (with minor variations) for $n=20$ in [14], $n=6$ in [9] and $n=5$ in [8]. But P_2 is not sufficient (even if phrased with $\varepsilon = 0$), as the next example shows.

Let A be the C^*-algebra $\mathbf{C}\oplus\mathbf{C}$, multiplication being

$$(\lambda,\mu)(\lambda',\mu')=(\lambda\lambda',\mu\mu')$$

and let τ be the trace $(\lambda,\mu)\to\lambda+\mu$. This A is not simple because $\mu=0$ defines a two-sided ideal. But (A,τ) satisfies P_2. Indeed, any $x\in A$ with $\tau(x)=0$ is a scalar multiple of $(1,-1)$. Define a projection p by $p(\lambda,\mu)=(\frac{1}{2}(\lambda-\mu),\frac{1}{2}(\mu-\lambda))$, so that $pxp=0$. Set $u_1=(1,1)$ and $u_2=(1,-1)$. Then $u_1^*(1-p)u_1=1-p$ is orthogonal to $u_2^*(1-p)u_2=p$.

Let Γ be a group (without topology). We denote by $C_r^*(\Gamma)$ the reduced C^*-algebra of Γ and by τ the canonical trace, defined by $\tau(\sum\lambda_g u_g)=\lambda_1$. Recall from [8] the

Definition. The group Γ is a *Powers' group* if the following holds. Given any non empty finite subset $\mathrm{F}\subset\Gamma-\{1\}$ and any integer $\mathrm{n}\geqq 1$, there exist a partition $\Gamma=\mathrm{D}\coprod\mathrm{E}$ and elements $\mathrm{g}_1,\ldots,\mathrm{g}_n\in\Gamma$ such that

(1) $\mathrm{fD}\cap\mathrm{D}=\emptyset$ for any $\mathrm{f}\in\mathrm{F}$

(2) $\mathrm{g_jE}\cap\mathrm{g_kE}=\emptyset$ for $\mathrm{j,k}\in\{1,\ldots,\mathrm{n}\}$ with $\mathrm{j}\neq\mathrm{k}$.

It is easy to check [8] that a Powers' group Γ is not amenable, and that each conjugacy class in $\Gamma-\{1\}$ is infinite. With this terminology, part of Powers' original paper is the proof that non abelian free groups are Powers' groups. It is also easy to check that one obtains the same notion when, in the definition, "$F\subset\Gamma-\{1\}$" is replaced by "$F\subset\Gamma$" and (1) by

(1') $fD\cap f'D=\emptyset$ for any $f,f'\in F$ with $f\neq f'$.

Proposition 3. *Let Γ be a Powers' group. Then* $(\mathrm{C}_r^*(\Gamma),\tau)$ *satisfies property P_n for any* $\mathrm{n}\geqq 2$.

Proof. The space $H=L^2(C_r^*(\Gamma),\tau)$ is simply $l^2(\Gamma)$. Let $\mathscr{A}$ be the dense subalgebra of $C_r^*(\Gamma)$ consisting of finite sums $\sum\lambda_g u_g$. Given $x\in\mathscr{A}$, we are going to find a projection $p\in L(H)$ and unitaries $u_1,\ldots,u_n$ in A such that (1) and (2) in the definition of P_n hold (with $\varepsilon=0$).

Let F be the finite subset of Γ consisting of those g such that $\lambda_g\neq 0$, where $x-\tau(x)1=\sum\lambda_g u_g$. Let $\Gamma=D\coprod E$ and let $g_1,\ldots,g_n$ be as in the definition of Γ being a Powers' group. Let p be the projection of H onto $l^2(D)$. Then $fD\cap D=\emptyset$ for $f\in F$ implies $p(x-\tau(x))p=0$. Let $u_k=u_{g_k}$ for $k=1,\ldots,n$. Then $u_k(1-p)u_k^*$ is the projection of H onto $l^2(g_kE)$, and these are pairwise orthogonal because the g_kE's are pairwise disjoint. □

One may also find a simple C^*-algebra A with unit having a unique trace τ (which is faithful) such that (A,τ) does not satisfy P_3. Before giving examples, we recall some definitions.

Let A be a C^*-algebra with unit, let τ be a faithful trace on A, and identify as above A with a subalgebra of $L(H)$, where $H=L^2(A,\tau)$. Denote by $W^*(A,\tau)$ the weak closure of A in $L(H)$. The trace τ is *factorial* if $W^*(A,\tau)$ is a factor; for example, if τ is the unique trace on A, then τ is factorial (because τ is obviously extremal in the appropriate sense – see [4, Theorem 6.7.3 and No. 6.8]). The trace τ is a *hypertrace* if there exists a positive linear form $T: L(H)\to\mathbf{C}$ extending τ such that $T(xY)=T(Yx)$ for any $x\in A$ and for any $Y\in L(H)$. The main theorem in

Connes' classification of injective factors states that $W^*(A,\tau)$ is injective if and only if τ is a hypertrace (see [3, Theorem 5.1, and also Remark 5.34]). The algebra A is said to be *nuclear* if its enveloping von Neumann algebra A^{**} is injective; if this holds, then $W^*(A,\tau)$ is also injective because it is an epimorphic image of A^{**}; for a review of nuclearity, see [10].

The next proposition generalizes the fact that an amenable group is not a Powers' group.

Proposition 4. *Let* A *be a nuclear C^*-algebra with unit, not isomorphic to* $\mathbf{C}$, *and let* τ *be a faithful factorial trace on* A. *Then* (A,τ) *does not satisfy property* P_3.

Proof. As A is nuclear, τ is a hypertrace and there exists $T: L(H)\to\mathbf{C}$ as above. Assuming that (A,τ) satisfies P_3, we shall deduce a contradiction.

Let $\varepsilon>0$. We claim first that there exists a unitary $x\in A$ with $|\tau(x)|\leqq\frac{\varepsilon}{2}$. Indeed, there exists a unitary $y\in W^*(A,\tau)$ with $\tau(y)=0$ because $\dim_{\mathbf{C}}(W^*(A,\tau))>1$, and the unitary group of A is strongly dense in that of $W^*(A,\tau)$ by Kaplansky's density theorem (Theorem 2.3.3 in [12]).

Let p, u_1, u_2, u_3 be as in the definition of P_3, satisfying

(1) $\|p(x-\tau(x))p\|\leqq\frac{\varepsilon}{2}$, so that $\|pxp\|\leqq\varepsilon$

(2) the $u_k(1-p)u_k^*$ are pairwise orthogonal $(k=1,2,3)$.

By (2) one has $T(1-p)\leqq\frac{1}{3}$ and thus $T(p)\geqq\frac{2}{3}$. By (1) one has $\|x^*pxp\|\leqq\varepsilon$; as

$$T(p)(T(p)-2\varepsilon)\leqq(T(p)-\varepsilon)^2\leqq|T((1-x^*px)p)|^2$$

it follows from Cauchy-Schwarz inequality (Theorem 3.1.3 in [12]) that

$$T(p)(T(p)-2\varepsilon)\leqq T(1-x^*px)T(p)=(1-T(p))T(p)$$

and finally $T(p)\leqq\frac{1}{2}+\varepsilon$. This is absurd when $6\varepsilon<1$. □

It is well-known that there is an abundance of nuclear C^*-algebras with unit which are simple with a unique trace.

In the spirit of Proposition 3, one may prove the following: if Γ is a Powers' group and if U is a UHF-algebra, then $A=C_r^*(\Gamma)\otimes U$ has property P_n for all $n\geqq 2$. If τ_A is the (unique) trace on A, this shows an example of a pair (A,τ_A) such that the factor $W^*(A,\tau_A)$ is McDuff. On the other hand, the factor associated to $(C_r^*(\Gamma),\tau)$ is full when Γ is free non abelian (or more generally when Γ is any of the groups in Sect. 3 of [2]).

It is also possible to construct a Powers' group Γ with associated factor $W^*(\Gamma)$ having "property gamma" of Murray and von Neumann. For example, define inductively a group Γ_n by $\Gamma_1=C_2$ (the group with 2 elements denoted by 1 and j) and $\Gamma_{n+1}=(\Gamma_n\times C_2)*C_2$ (where $\times$ indicates a direct product and $*$ a free product), and set $\Gamma=\varinjlim\Gamma_n$. Set $\gamma_n=(1,j)\in\Gamma_n\times C_2\subset\Gamma_{n+1}$. For any $\gamma\in\Gamma$ one has $\gamma\gamma_n=\gamma_n\gamma$ for n large enough, so that $W^*(\Gamma)$ has property gamma (Lemma 6.1.1 in [11]); in other words, $W^*(\Gamma)$ is not full [3]. As Γ is locally a non trivial free product, Γ is a Powers' group.

2. Crossed Products with Powers' Groups

Let $\alpha: \Gamma \to \mathrm{Aut}(A)$ and let $B = A \rtimes_{\alpha,r} \Gamma$ be as in the introduction. When appropriate, we may assume that A acts faithfully on a Hilbert space H; then B acts faithfully on the Hilbert space K of square summable functions from Γ to H according to the usual formulas (Theorem 7.7.5 in [12])

$$\begin{array}{ll} (x\xi)(g) = \alpha_{g^{-1}}(x)\xi(g) & x \in A \\ (u_h\xi)(g) = \xi(h^{-1}g) & h \in \Gamma \end{array} \quad \xi \in K \quad g \in \Gamma$$

and any $x \in B$ can be written in a unique way as

$$x = \sum_{g \in \Gamma} x_g u_g$$

where $x_g \in A$ for all $g \in \Gamma$. We identify A and $\{x \in B | x = x_1 u_1\}$; the canonical conditional expectation $e: B \to A$ is given by $e(\sum x_g u_g) = x_1$.

We want to show firstly examples where B has a unique trace, and secondly examples where B is simple. In both cases, we prove a general proposition, and specialize then to abelian A's.

Lemma 5. *Assume Γ to be a Powers' group. Consider a finite subset F of $\Gamma - \{1\}$ and an element $x \in B$ of the form*

$$x = \sum_{f \in F} x_f u_f \qquad x^* = x$$

Then there exist $g_1, g_2, g_3 \in \Gamma$ such that (with $u_k = u_{g_k}$)

$$\left\| \frac{1}{3} \sum_{k=1}^{3} u_k x u_k^* \right\| \leq \frac{1}{3} \left(\frac{3 + 2\sqrt{2}}{6} + 2 \right) \|x\| .$$

Proof. As in the proof of Lemma 1, we may assume $\|x\| = 1$. As Γ is a Powers' group, there exist a partition $\Gamma = D \coprod E$ and elements $g_1, g_2, g_3 \in \Gamma$ with

(1) $fD \cap D = \emptyset$ for any $f \in F$

(2) $g_j E \cap g_k E = \emptyset$ for $j, k \in \{1, 2, 3\}$ with $j \neq k$.

Let $K = l^2(\Gamma, H)$ be as above. The orthogonal sum

$$K = l^2(D, H) \oplus l^2(E, H)$$

is invariant by A. Let p be the orthogonal projection from K onto $l^2(D, H)$. One has $pxp = 0$ by (1), and the projections $1 - u_k p u_k^*$ are pairwise orthogonal for $k = 1, 2, 3$ by (2). The inequality to be proved follows as in Lemma 1. □

Proposition 6. *Assume Γ to be a Powers' group. For any trace τ on B, there exists a Γ-invariant trace σ on A with $\tau = \sigma e$.*

Proof. Let $x = \sum x_g u_g \in B$. As $3 + 2\sqrt{2} < 6$, Lemma 5 implies that the closed convex hull of

$$\{y \in B | y = u_g(x - x_1)u_g^* \text{ for some } g \in \Gamma\}$$

contains 0. Consequently $\tau(x - x_1) = 0$, and the proposition follows. □

Corollary. 7. *If Γ is a Powers' group and if there exists a unique Γ-invariant trace on* A, *then there exists a unique trace on* B.

Let us particularize to the case where A is abelian. By Gelfand's theory $A=\mathscr{C}(X)$ for a compact space X (the spectrum of A) and α may be viewed as an action of Γ on X by homeomorphisms. In this case, the existence of a unique Γ-invariant trace on A means that the action of Γ on X is uniquely ergodic (for some probability Radon measure). Corollary 7 becomes:

Corollary 8. *If a Powers' group Γ acts in a uniquely ergodic way by homeomorphisms on a compact space X, then the reduced crossed product* $\mathrm{B}=\mathscr{C}(\mathrm{X}) \rtimes_{\alpha,\mathrm{r}} \Gamma$ *has a unique trace.*

Corollary 8 does not hold for an arbitrary group: think of $\Gamma=\mathbf{Z}$ acting on X a point, so that $B=C^*(\mathbf{Z})=\mathscr{C}(\mathbf{S}^1)$.

Digression. Let Γ be a countable group (not necessarily a Powers' group) acting by homeomorphisms on a compact space X in a uniquely ergodic way, for some probability Radon measure μ. Let

$$X_1=\{w\in X|g\in\Gamma \text{ with } gw=w \text{ implies } g=1\}$$

be the subspace of points without isotropy. As Γ is countable, X_1 is Borel; as X_1 is Γ-invariant, $\mu(X_1)\in\{0,1\}$. Claim: if $\mu(X_1)=1$, then $B=\mathscr{C}(X) \rtimes_r \Gamma$ has a unique trace.

Indeed, let τ be a trace on B. By unique ergodicity one may identify $\tau|\mathscr{C}(X)$ with μ. It is enough to show that $\tau(xu_g)=0$ for any $x\in\mathscr{C}(X)$ and for any $g\in\Gamma-\{1\}$. Denote by X^g the fixed point set $\{w\in X|gw=w\}$; as $X^g\cap X_1=\emptyset$ one has $\mu(X^g)=0$. Suppose first that supports are such that $\mathrm{Supp}(x)\cap g\,\mathrm{Supp}(x)=\emptyset$; as one may write $x=x_1x_2$ for $x_j\in\mathscr{C}(X)$ with $\mathrm{Supp}(x_j)=\mathrm{Supp}(x)$ for $j=1,2$, one has

$$\tau(xu_g)=\tau(x_2u_gx_1u_g^*u_g)=\tau(x_2\alpha_g(x_1)u_g)=0$$

because $x_2\alpha_g(x_1)=0$. Suppose more generally that $\mathrm{Supp}(x)\cap X^g=\emptyset$; as one may write $x=\sum x_j$ (finite sum) with $\mathrm{Supp}(x_j)\cap g\,\mathrm{Supp}(x_j)=\emptyset$, one has again $\tau(xu_g)=0$. Let finally x be arbitrary; for any $\varepsilon>0$ there exists $x'\in\mathscr{C}(X)$ with $\mathrm{Supp}(x')\cap X^g=\emptyset$ and $\int_X |x-x'|d\mu<\varepsilon$ (because X^g has measure 0); then $\tau(x'u_g)=0$ as above and

$$|\tau((x-x')u_g)|\leqq\tau(|x-x'|)=\int_X |x-x'|d\mu<\varepsilon$$

by Cauchy-Schwarz; consequently $\tau(xu_g)=0$. The claim is proved.

Let again A be an arbitrary C^*-algebra with unit and let $\alpha:\Gamma\to\mathrm{Aut}(A)$ be an action of a group Γ on A. Recall that A is *Γ-simple* if any $\alpha(\Gamma)$-invariant closed two-sided ideal in A is either $\{0\}$ or A. (As A has a unit, one could suppress "closed" without changing the notion.)

Lemma 9. *Assume that* A *is Γ-simple. Let* $\mathrm{x}\in\mathrm{A}$ *with* $\mathrm{x}\geqq 0$ *and* $\mathrm{x}\neq 0$. *There exist* $\mathrm{g}_1,\ldots,\mathrm{g}_\mathrm{n}\in\Gamma$ *and* $\mathrm{z}_1,\ldots,\mathrm{z}_\mathrm{n}\in\mathrm{A}$ *such that*

$$\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{z}_\mathrm{j}\alpha_\mathrm{j}(\mathrm{x})\mathrm{z}_\mathrm{j}^*\geqq 1 \quad \textit{where} \quad \alpha_\mathrm{j}=\alpha_{\mathrm{g}_\mathrm{j}}.$$

Proof. As the two-sided ideal generated by $(\alpha_g(x))_{g\in\Gamma}$ is A, there exist $g_1, \dots, g_n \in \Gamma$ and $x_1, y_1, \dots, x_n, y_n \in A$ with

$$\sum_{j=1}^{n} x_j\alpha_j(x)y_j^* = \frac{1}{2}.$$

Set $z_j = x_j + y_j$ for $j = 1, \dots, n$. Then

$$\sum_{j=1}^{n} (z_j\alpha_j(x)z_j^* - x_j\alpha_j(x)y_j^* - y_j\alpha_j(x)x_j^*) = \sum_{j=1}^{n} (x_j\alpha_j(x)x_j^* + y_j\alpha_j(x)y_j^*) \geqq 0$$

and thus

$$\sum_{j=1}^{n} z_j\alpha_j(x)z_j^* \geqq \sum_{j=1}^{n} x_j\alpha_j(x)y_j^* + \left\{\sum_{j=1}^{n} x_j\alpha_j(x)y_j^*\right\}^* = 1$$

as claimed. □

Proposition 10. *Assume* Γ *to be a Powers' group. If the unital* C*-*algebra* A *is* Γ*-simple, then* $\mathrm{B} = \mathrm{A} \rtimes_{\alpha,r} \Gamma$ *is simple.*

Proof. Let $\mathscr{J}$ be a two-sided ideal in B and assume that $\mathscr{J}$ contains $x = \sum x_g u_g \neq 0$. Upon replacing x by x^*x, we may assume that $x \geqq 0$, $x_1 \geqq 0$, $x_1 \neq 0$. By Lemma 9, there exist $g_1, \dots, g_n \in \Gamma$ and $z_1, \dots, z_n \in A$ such that (with $\alpha_j = \alpha_{g_j}$)

$$\sum_{j=1}^{n} z_j\alpha_j(x_1)z_j^* \geqq 1.$$

Consequently, upon replacing x by $\sum_{j=1}^{n} z_j u_j x(z_j u_j)^*$, we may furthermore assume that $x_1 \geqq 1$.

By Lemma 5 applied several times, there exist $h_1, \dots, h_N \in \Gamma$ such that

$$\left\|\frac{1}{N}\sum_{k=1}^{N} v_j(x - x_1)v_j^*\right\| \leqq \frac{1}{2} \quad \text{with} \quad v_j = u_{h_j}.$$

If we set

$$r = \frac{1}{N}\sum_{k=1}^{N} v_k x v_k^* = \sum_{g\in\Gamma} r_g u_g \in \mathscr{J}$$

one has

$$r_1 = \frac{1}{N}\sum v_k x_1 v_k^* \geqq 1 \quad \text{and} \quad \|r - r_1\| \leqq \frac{1}{2}.$$

It follows that r is invertible, and thus that $\mathscr{J} = B$. □

Corollary 11. *If* Γ *is a Powers' group and if* A *is simple, then* B *is simple.*

Let us particularize to the case where A is abelian.

Lemma 12. *Let* Γ *be a group acting by homeomorphisms on a compact space* X, *and thus on the algebra* $\mathrm{A} = \mathscr{C}(\mathrm{X})$. *Then* Γ *is minimal on* X *if and only if* A *is* Γ*-simple.*

Proof. Closed ideals in A are in bijection with closed subset of X by Gelfand's theory; the lemma follows. □

Corollary 13. *If a Powers' group Γ acts minimally by homeomorphisms on a compact space* X, *then the reduced crossed product* $\mathbf{B} = \mathscr{C}(\mathrm{X}) \underset{\alpha, r}{\rtimes} \Gamma$ *is simple.*

Observe first that Corollary 13 does not hold for an arbitrary group. For example, if an amenable group Γ acts on a point, then $B = C^*(\Gamma)$ is not simple.

Observe also that Proposition 10 does not hold for A without unit, and in particular that Corollary 13 does not hold for a locally compact space X. Indeed, let Γ be a Powers' group, let H be an amenable subgroup of Γ (necessarily of infinite index in Γ), and consider the canonical action of Γ on $X = \Gamma/H$. Then $A = c_0(X)$ is the algebra of functions $X \to \mathbf{C}$ vanishing at infinity, and $A \underset{\alpha, r}{\rtimes} \Gamma$ is known to be Morita equivalent to $C^*(H)$; see [15, Example 1]. It follows that $A \underset{\alpha, r}{\rtimes} \Gamma$ is not simple. (It is known that $A \underset{\alpha, r}{\rtimes} \Gamma$ is isomorphic to the tensor product of $C^*(H)$ by the compact operators on $l^2(\Gamma/H)$; see Theorem 4.1 in [6].)

In this last example, let $\tilde{A}$ be the algebra obtained by adding a unit to A. Then $\tilde{A}$ has a unique non trivial Γ-invariant ideal, which is A; but $\tilde{A} \underset{\alpha, r}{\rtimes} \Gamma$ may have a lot of non trivial ideals, and indeed the ideal $A \underset{\alpha, r}{\rtimes} \Gamma$ which is Morita equivalent to $\mathscr{C}(\mathrm{S}^1)$ in case $H \approx \mathbf{Z}$. This shows that the elementary methods of the present work cannot solve the problem alluded to in the introduction, of understanding in general the ideal structure of reduced crossed products by Powers' groups.

Consider finally the situation of a previous digression: Γ is an arbitrary countable group acting by homeomorphisms on a compact space X, now in a minimal way. If

$$X_1 = \{w \in X \mid g \in \Gamma \text{ with } gw = w \text{ implies } g = 1\}$$

is not empty, then it can be shown that $\mathscr{C}(X) \underset{r}{\rtimes} \Gamma$ is simple. We refer to [5].

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