KASPAROV'S BIVARIANT K-THEORY AND APPLICATIONS

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Introduction

The study of operator K-theory started only a few years ago but its evolution was extremely spectacular. A large part of the recent work performed on operator algebras is concerned with K-theory and related topics. Also K-theory provided operator algebras with many important applications in other fields of mathematics.

Many new methods for dealing with operator K-theory have appeared this last decade. Most important is Kasparov's bivariant K-theory: being a generalization of K-theory, it gives new invariants for the classification of C*-algebras; as these invariants appear naturally in many concrete situations, they are very useful in many applications of K-theory; moreover Kasparov's theory helped with a better understanding of the K-theory itself and was used to compute the K-groups of many C*-algebras.

This paper is just an attempt to give some explanations concerning Kasparov's K-theory and some of its applications. It contains no new material at all and provides no complete proofs. Also it only gives a partial view of the theory based on my personal centers of interest. Many important developements of C*-algebraic K-theory have been knowingly left untouched. In particular I chose not to talk about Connes' cyclic cohomology: although this theory gives important methods for dealing with the same kind of problems, it seamed to me too difficult to treat in a reasonably short text, both Connes' and Kasparov's approach. We will also have to ommit some interesting applications of K-theory and Kasparov's theory: we give no information on the classification of AF algebras (although it witnessed the first appearence of K-theory in connection with C*-algebras; see the survey paper [41]); also we will not give the application of Kasparov's theory to Riemannian manifolds with positive scalar curvature (mainly Rosenberg's work cf. [93,94,95]). But it is certainly shorter to list the topics that we will see:

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We gave a quite long list of references; however, it is certainly not exhaustive. A more complete one can be found in Blackadar's book ([18]). This book contains lots of information on C*-algebraic K-theory and the reader will find there proofs (together with more precise statements) concerning most of the topics encountered here. Several survey papers, with quite different points of view, have been writen on C*-algebraic K-theory and related topics (cf. [35,43,71,92]) and the interested reader is encouraged to consult them. Let me recommend particularly Kasparov's paper [71].

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1 Definition and some properties of C*-algebras

In this section we give the definition, some examples and some properties of C*-algebras.

1.1 Definition: *A C*-algebra is a complex Banach algebra* A *endowed with an involution* $x \to x^*$ *such that for all* x *in* A $\|x^*x\| = \|x\|^2$.

Recall that an involution in a complex Banach algebra A is a map $x \to x^*$ such that $(x^*)^*=x$, $(x+y)^*=x^*+y^*$, $(\lambda x)^*= \bar{\lambda} x^*$ and $(xy)^*=y^*x^*$ $(x,y \in A$, $\lambda \in \mathbb{C})$.

There are many results in the theory of C*-algebras which require hard analysis. However, as the norm is given in purely algebraic terms ($\|x\|^2$ is the spectral radius of x^*x), for many purposes the algebraic behaviour takes the analysis into account. Let us give some examples illustrating this device:

Properties and examples

1.2. Let A and B be two C*-algebras and $f:A \to B$ a *-homomorphism (i.e. a continuous homomorphism such that $f(x^*)=f(x)^*$ for all x in A). Then the image of f is closed in B , hence it is a C*-algebra and the induced homomorphism $A/\mathrm{Ker}(f) \to f(A)$ is an isometry. In particular every injective *-homomorphism is isometric.

1.3. Let X be a compact space. Denote by C(X) the Banach algebra of continuous complex valued functions on X with the supremum norm . When endowed with the involution $f \to \bar{f}$, C(X) is a C*-algebra. If X is locally compact, $C_0(X)$ denotes the C*-algebra of continuous functions vanishing at ∞ on the one point compactification of X .
Conversely every **commutative** C*-algebra A is isomorphic (hence isometric) to some C(X) if it has a unit, or to some $C_0(X)$ if it has no unit. More precisely, every character χ of A (i.e. a continuous homomorphism of algebras $\chi:A \to \mathbb{C}$) is self adjoint (i.e. $\chi(x^*)=\overline{\chi(x)}$ for all x in A). Let X denote the spectrum of A (i.e. the locally compact space of characters of A). The **Gelfand transform** $\mathcal{F}:A \to C_0(X)$ given by $\mathcal{F}(x)(\chi)=\chi(x)$ is the desired *-isomorphism.

1.4. Because of 1.3, the non commutative C*-algebras can be thought of as being the algebras of continuous functions on a "non commutative locally compact space". Here are some examples of such spaces and the corresponding C*-algebras:

a) Let G be a locally compact group. To the "dual space" of G corresponds a C*-algebra called the **group C*-algebra** of G and denoted by $C^*(G)$; it is a completion of the convolution algebra of G . If G is commutative $C^*(G)$ is isomorphic, via the Fourier transform, to the algebra $C_0(\hat{G})$ where $\hat{G}$ is the dual group of G (the locally compact group of continuous group homomorphisms from G to $U(1) = \{ z \in \mathbb{C} \ / \ |z|=1 \}$). This example will be discussed in somewhat greater detail in §9 .

b) Let G be a locally compact group acting by homeomorphisms on the locally compact space X . The topology of the space of orbits X/G is not in general Hausdorff. To this "bad space" corresponds the **crossed product** C*-algebra $C_0(X)\times G$ (see §9 for a description of this C*-algebra). If G acts freely and X/G is Hausdorff (more precisely if the action of G is free and proper), $C_0(X)\times G$ is (Morita) equivalent to $C_0(X/G)$.

c) Let (X,F) be a foliation. To the space of leaves X/F , which is again a "bad space", corresponds the C*-algebra $C^*(X,F)$. If X/F is Hausdorff (more precisely if the foliaton is a fibration) then $C^*(X,F)$ is (Morita) equivalent to $C_0(X/F)$. We will discuss this example in §11 .

1.5. Let H be a Hilbert space. The algebra $\mathcal{L}(H)$ of all continuous linear transformations of H is a C*-algebra. Recall that the involution of $\mathcal{L}(H)$ is given by the formula $\langle x^*\xi,\eta\rangle=\langle\xi,x\eta\rangle$ $(\xi,\eta\in H)$. The algebra $\mathcal{L}(H)$ is not separable (unless H has finite dimension). However $\mathcal{L}(H)$ is a fundamental C*-algebra for the following reasons:

a) If dim(H)=n is finite, $\mathcal{L}(H)$ is the algebra $M_n(\mathbb{C})$ of n by n matrices over $\mathbb{C}$ and plays an important role. In the infinite dimensional case the role of $M_n(\mathbb{C})$ is played by the algebra $\mathcal{K}=\mathcal{K}(H)$ of compact operators of H which is an ideal in $\mathcal{L}(H)$.

b) Every C*-algebra A admits a faithful *-representation, i.e. an injective *-homomorphism $\pi: A \to \mathcal{L}(H)$, where H is a Hilbert space. By 1.2, π being injective it is isometric. Therefore every C*-algebra has an isomorphic image sitting as a closed *-subalgebra in some $\mathcal{L}(H)$. When A is separable, H can be taken to be the (unique) separable Hilbert space. This property gives a way of constructing many examples of C*-algebras:

Given a complex *-algebra $\mathcal{A}$ and a bounded family π_α of *-representations $\pi_\alpha:\mathcal{A}\to\mathcal{L}(H_\alpha)$, the Hausdorff completion of $\mathcal{A}$ for the seminorm $\|x\|=\sup_\alpha(\|\pi_\alpha(x)\|)$ is a

C*–algebra. The examples in 1.4. are constructed using this scheme. Taking $\mathcal{A}$ to be a Banach *–algebra and π_α the family of all continuous *–representations we obtain the so-called **envelopping C*-algebra** of $\mathcal{A}$.

1.6. Functional calculus on C*-algebras:

If A is a C*–algebra and $x \in A$ is **self-adjoint** (i.e. $x = x^*$), its spectrum is included in $\mathbb{R}$. The equality between the spectral radius and the norm tells us that for any polynomial $P \in \mathbb{C}[X]$, $\|P(x)\| = \sup\{P(t)/t \in Spec(x)\}$. Therefore, using Weierstrass' theorem, we may define $f(x) \in A$ for every continuous function $f: Spec(x) \to \mathbb{C}$ setting $f(x) = \lim P_n(x)$ where P_n is a sequence of polynomials converging uniformly to f on Spec(x) .

An element $x \in A$ is said to be **normal** if $x^*x = xx^*$. Using polynomials $P(x,x^*)$, we may perform the continuous functional calculus for normal elements as well. Finally an element $u \in A$ is said to be **unitary** if $u^*u = uu^* = 1$. It is normal and its spectrum is included in U(1) .

1.7. Positivity: This is a very important notion in the theory of C*–algebras. In particular it allows to construct faithful *–representations (1.5.b). The basic result is the following:

Theorem: *For* $x \in A$ *the following are equivalent:*

(i) $x = x^*$ *and* $Spec(x) \subset \mathbb{R}_+$.

(ii) There exists $y \in A$ *with* $x = y^*y$.

(iii) There exists $y \in A$ *with* $y = y^*$ *and* $x = y^2$.

The set of such elements is a convex cone in A .

An element x satisfying these conditions is said to be **positive**. We write $x \geqslant 0$. If $x \geqslant 0$ and $-x \geqslant 0$ then $x = 0$. Positivity defines a partial ordering in A : we write $x \leqslant y$ or $y \geqslant x$ if $y - x \geqslant 0$. The cone of positive elements of A is noted A_+ .

The reader is refered to [39,79,18] for all important properties of C*–algebras, examples and proofs.

2 . K-theory of C*-algebras

In this section we recall briefly the definitions and some fundamental properties of the K-theory of C*-algebras. All statements in this section hold for Banach algebras. See [2,18,65].

2.1. Definition: *Let* A *be a C*-algebra (or just a ring)* **with unit.** *Let* $\mathcal{K}_0(A)$ *denote the set of isomorphism classes of finitely generated projective (right) A-modules. The direct sum of modules defines a (commutative and associative) addition in* $\mathcal{K}_0(A)$. *The group of formal differences of elements of* $\mathcal{K}_0(A)$ *is denoted by* $K_0(A)$.

2.2. Remarks: a) Recall that the right A-module $\mathcal{E}$ is **projective** if for every diagram $\mathcal{E} \xrightarrow{g_2} \mathcal{E}_2 \xleftarrow{f} \mathcal{E}_1$ of A-module homomorphisms with f surjective there exists an A-module homomorphism $g_1:\mathcal{E}\to\mathcal{E}_1$ such that $g_2=fg_1$. If moreover $\mathcal{E}$ is finitely generated, there exists a surjective A-module homomorphism f from the free module A^n to $\mathcal{E}$. Lifting the identity $g_2:\mathcal{E}\to\mathcal{E}$, we find an isomorphism $g_1:\mathcal{E}\to pA^n$ where $p=g_1f$ is an **idempotent** of the algebra of A-module endomorphisms of the free module A^n, which is isomorphic to the C*-algebra $M_n(A)$ of n by n matrices over A. This idempotent may be chosen to be self-adjoint (a self-adjoint idempotent of a C*-algebra is called a **projection**). Conversely if $p\in M_n(A)$ is an idempotent the right A-module pA^n is finitely generated and projective.

b) Let X be a compact space and A the commutative C*-algebra C(X) of continuous complex valued functions on X. Let E be a complex vector bundle over X. The vector space of continuous sections of E is a finitely generated projective A-module $\mathcal{E}$. The correspondence $E\to\mathcal{E}$ identifies $\mathcal{K}_0(A)$ with the set Vect(X) of isomorphism classes of complex vector bundles over X and the group $K_0(A)$ with the group $K^0(X)$ of formal differences of complex vector bundles (Serre-Swan's theorem).

We next come to the higher K_n groups:

2.3. Definition: *Let* A *be a C*-algebra* **with unit.** *Denote by* $Gl_k(A)$ *the group of invertible elements of* $M_k(A)$ *and by* $Gl_\infty(A)$ *the union of the* $Gl_k(A)$ *where* $Gl_k(A)$ *embeds in* $Gl_{k+1}(A)$ *by the map* $a\to\begin{bmatrix} a & 0 \\ 0 & 1\end{bmatrix}$. *For* $n\geqslant 1$, *we set:* $K_n(A)=\pi_{n-1}(Gl_\infty(A))$.

2.4. Remarks: a) Denote by $U_k(A)$ the group of unitaries (cf. 1.6) of $M_k(A)$. The inclusion $U_k(A)\to Gl_k(A)$ is a homotopy equivalence, hence $K_n(A)=\pi_{n-1}(U_\infty(A))=\lim\limits_{\to}\pi_{n-1}(U_k(A))$.

b) The homotopy group π_n $(n\geq 2)$ of any topological space is abelian and π_1 of any topological group is abelian. Thus $K_n(A)$ is abelian for $n\geq 2$. To see that $K_1(A)$ is also abelian note that for every $x\in Gl_n(A)$, $\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$ are in the same connected component of $Gl_{2n}(A)$. Hence

$\begin{bmatrix} xyx^{-1}y^{-1} & 0 \\ 0 & 1 \end{bmatrix}$ is in the same component as $\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} x^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & y^{-1} \end{bmatrix}=1$

2.5. K_n as a homotopy functor:

Let A and B be two C^*-algebras and $f:A\to B$ a $*$-homomorphism with $f(1)=1$. If $\mathcal{E}$ is a finitely generated projective A-module then $\mathcal{E}\otimes_A B$ is a finitely generated projective B-module (if $\mathcal{E}=pA^n$ then $\mathcal{E}\otimes_A B=f(p)B^n$ where $p\in M_n(A)$ is a projection and f is extended to n by n matrices setting $(f(x))_{i,j}=f(x_{i,j})$). Hence f defines a group homomorphism $f_*:K_0(A)\to K_0(B)$. Also f defines a continuous group homomorphism $f:Gl_\infty(A)\to Gl_\infty(B)$ hence a group homomorphism $f_*: K_n(A)\to K_n(B)$ $(n\geq 1)$.

Proposition: *(homotopy invariance). Let A and B be two C^*-algebras and $f,g:A\to B$ two homotopic unital $*$-homomorphisms. Then for all $n\geq 0$, $f_*=g_*: K_n(A)\to K_n(B)$.*

Recall that the $*$-homomorphisms f and g are said to be homotopic if there exists a family $(f_t)_{t\in[0,1]}$ of $*$-homomorphisms with $f_0=f$, $f_1=g$ such that for all $x\in A$, $t\to f_t(x)$ is norm continuous. This proposition is obvious for $n\geq 1$; for $n=0$ it is a cosequence of the fact that two near by projections are equivalent [(1)], hence the corresponding modules are isomorphic.

[(1)] The projections p and q of the C^*-algebra B are said to be equivalent if there exists $u\in B$ with $u^*u=p$ and $uu^*=q$. Then $u:pB\to qB$ is a B-module isomorphism.

The next result is fundamental:

2.6. Theorem: *(Bott periodicity). For every* $n \geq 0$ *there is a* ***natural isomorphism***

$$\beta: K_n(A) \to K_{n+2}(A).$$

"Natural" means here that for any unital *-homomorphism $f: A \to B$ the diagram

$$\begin{array}{ccc} K_n(A) & \xrightarrow{f_*} & K_n(B) \\ \beta\downarrow & & \downarrow\beta \\ K_{n+2}(A) & \xrightarrow{f_*} & K_{n+2}(B) \end{array}$$

commutes.

Let me just indicate how $\beta: K_0(A) \to K_2(A)$ is defined: let $p \in M_n(A)$ be a projection. The image through β of the class of the module pA^n is the class of the loop $f: U(1) \to U_n(A)$, $f(z) = zp + 1 - p$.

2.7. Computation of $K_n(\mathbb{C})$:

A finitely generated projective $\mathbb{C}$-module is just a finite dimensional complex vector space. These are classified by their dimension; hence $\mathcal{K}_0(\mathbb{C}) = \mathbb{N}$ and $K_0(\mathbb{C}) = \mathbb{Z}$. Notice that for all $k \geq 1$, $Gl_k(\mathbb{C})$ is connected. Therefore $K_1(\mathbb{C}) = 0$. Using Bott periodicity we obtain: $K_{2n}(\mathbb{C}) = \mathbb{Z}$ and $K_{2n+1}(\mathbb{C}) = 0$.

K-theory of non unital C*-algebras: Let A be a C*-algebra without unit. We denote by $\tilde{A}$ the C*-algebra with an adjoined unit: as a vector space $\tilde{A} = A \oplus \mathbb{C}1$; the product rule is $(a + \lambda 1)(b + \mu 1) = (ab + \lambda b + \mu a) + \lambda\mu 1 \in A \oplus \mathbb{C}1$; also $(a + \lambda 1)^* = a^* + \bar{\lambda}$; the norm is given by $\|a + \lambda 1\| = \sup\{\|ab + \lambda b\|, b \in A, \|b\| \leq 1\}$; one checks easily that this norm is a C*-norm.

Moreover the map $a + \lambda 1 \to \lambda$ is a *-homomorphism noted $\varepsilon: \tilde{A} \to \mathbb{C}$. Its kernel can be identified with A. Finally ε admits a cross-section $\lambda \to 0 + \lambda 1$ which is a *-homomorphism. In other terms we have a **split exact sequence of C*-algebras**: $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$.

2.8. Definition : *For a non unital C*-algebra* A *and* $n \geq 0$ $K_n(A)$ *is defined to be the kernel of the map* $\varepsilon_*: K_n(\tilde{A}) \to K_n(\mathbb{C})$.

Using the splitting of ε we see that the map ε_* is onto and splits. Therefore $K_n(\tilde{A})\cong K_n(A)\oplus K_n(\mathbb{C})$.

Using naturality, Bott periodicity extends to the non unital case.

If A is a non unital C*-algebra , $Gl_k(A)$ denotes the subgroup of $Gl_k(\tilde{A})$ of matrices of the form $1+x$, $x\in M_k(A)$; in other words $Gl_k(A)$ is the kernel of the group homomorphism $\varepsilon: Gl_k(\tilde{A}) \to Gl_k(\mathbb{C})$. Obviously for all $n\geqslant 1$ we have $K_n(A)=\pi_{n-1}(Gl_\infty(A))= \lim_{\to} \pi_{n-1}(Gl_k(A))$. Using this fact, the exact sequence of the homotopy groups for a fibration and Bott periodicity, we get the following fundamental theorem:

2.9. Theorem: *Let* $0 \to J \xrightarrow{i} A \xrightarrow{p} B \to 0$ *be an exact sequence of C*-algebras. Then we have a periodic exact sequence of K-groups:*

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{i*} & K_0(A) & \xrightarrow{p*} & K_0(B) \\ \delta\uparrow & & & & \downarrow\delta \\ K_1(B) & \xleftarrow{p*} & K_1(A) & \xleftarrow{i*} & K_1(J) \end{array}$$

Rephrasing Bott periodicity: Without the use of Bott periodicity, theorem 2.9 gives long exact sequences:

$$\dots \xrightarrow{i*} K_{n+1}(A) \xrightarrow{p*} K_{n+1}(B) \xrightarrow{\delta} K_n(J) \xrightarrow{i*} K_n(A) \xrightarrow{p*} \dots$$

$(n\geqslant 0)$.

Let X be a locally compact space and B a C*-algebra . The C*-algebra of continuous functions on X with limit 0 at ∞ is denoted by $B(X)$.

The C*- algebra $B((0,1])$ is contractible i.e. there exists a continuous family $(\varphi_t)_{t\in[0,1]}$ of *-endomorphisms of $B((0,1])$ with φ_0=identity and $\varphi_1=0$ (set $\varphi_t(f)(s)=f(s-t)$ if $0<s<t\leqslant 1$, $\varphi_t(f)(s)=0$ if $0<s\leqslant t\leqslant 1$). Hence $K_n(B((0,1])=0$ for all $n\geqslant 0$ (use proposition 2.5). Applying the K-theory long exact sequence to the exact sequence: $0\to B((0,1))\to B((0,1])\to B\to 0$ we get an isomorphism $K_n(B(0,1))\cong K_{n+1}(B)$. Using a homeomorphism of $(0,1)$ with $\mathbb{R}$ we get $K_n(B(\mathbb{R}))=K_{n+1}(B)$ and by induction , $K_n(B)=K_0(B(\mathbb{R}^n))$. Bott periodicity then reads:

2.10. Theorem: *(Bott periodicity) For every C*-algebra* B , $K_0(B(\mathbb{R}^2)) \cong K_0(B)$.

Theorems 2.9 and 2.10 allow, in principle, to compute the K-theory of the C*-algebra C(X) where X is a finite CW complex.

To compute the K-theory of more complicated compact spaces one uses the good behaviour of the K-groups with respect to inductive limits:

let $A_n \xrightarrow{\varphi_n} A_{n+1}$ be an inductive system of C*-algebras. The algebraic inductive limit (the union) of the inductive system (A_n, φ_n) is a normed *-algebra. Its completion is a C*-algebra A . We write $A = \varinjlim (A_n, \varphi_n)$.

It is quite easy to get:

2.11. Proposition : *For every inductive system* (A_k, φ_k) *of C*-algebras we have:*

$$K_n(\varinjlim (A_k, \varphi_k)) = \varinjlim (K_n(A_k), \varphi_{k*}) \quad (n=0,1) .$$

In particular for a C*-algebra A let $\varphi_k: M_k(A) \to M_{k+1}(A)$ be the inclusion given by $x \to \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$. The inductive limit is the C*-tensor product (2) $A \otimes \mathcal{K}$ where $\mathcal{K}$ denotes the C*-algebra of compact operators in a separable Hilbert space. All the $(\varphi_k)_*$ being isomorphisms we deduce:

2.12. Proposition : *The inclusion* $A \to A \otimes \mathcal{K}$ *given by* $a \to a \otimes p$ *where* p *is a rank one projection induces an isomorphism* $K_n(A) \xrightarrow{\sim} K_n(A \otimes \mathcal{K})$ (n=0,1) .

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(2) There are in general several ways of defining a C*-tensor product $A \otimes B$ (cf[42]; see § 8 for more details). But the algbra $\mathcal{K}$ is nuclear (in the C*-algebraic sense) which means that all these constructions yield the same result.

3. Atiyah's Ell

In [4] Atiyah made the following observations: consider the functor K^0 which to a compact space X associates the abelian group $K^0(X)$ ($=K_0(C(X))$ by definition). This functor satisfies the abstract properties of a cohomological functor, so that there must exist a dual homological functor. This dual functor the **K-homology** admits a topological definition at least for "nice" spaces : $K_n(X)=K^n(DX)$ where DX is a (Spanier- Whitehead) dual of X. Motivated by the case of elliptic operators on compact manifolds Atiyah gave a definition of cycles for an analytical K-homology. This definition extends to the non commutative case : to a C*-algebra A one associates a set Ell(A). In order to define the "K-homology" of A, one needs to give the right equivalence relation on Ell(A). The answer to this question came later and will be discussed in the forthcoming sections.

In the present section we will consider the two following points:

a) The abstract properties that Atiyah extracted from the elliptic operators on compact manifolds.

b) How an operator satisfying these properties defines a homomorphism from the K-theory to $\mathbb{Z}$.

Point b) describes the pairing between K-theory and K-homology.

Let X be a smooth compact manifold. Without getting into the technical problems of the pseudodifferential calculus let us just recall the following basic facts.

A **pseudodifferential operator** P defines a continuous linear mapping $P:C^\infty(X;E^{(0)})\to C^\infty(X;E^{(1)})$ where $E^{(i)}$ is a smooth vector bundle over X and $C^\infty(X;E^{(i)})$ denotes the space of smooth sections of $E^{(i)}$ $(i=0,1)$.

Denote by $L^2(X;E^{(i)})$ the Hilbert space of L^2 sections of the bundle $E^{(i)}$ $(i=0,1)$. If P has **order zero**, it extends to a continuous linear operator, still noted $P:L^2(X;E^{(0)})\to L^2(X;E^{(1)})$.

A pseudodifferential operator P has a **symbol** σ_P which is a smooth section of the bundle $\mathcal{L}(E^{(0)},E^{(1)})$ over T^*X the total space of the cotangent bundle of X (i.e. $\sigma_P(x,\xi)$ is a linear map from $E^{(0)}_x$ to $E^{(1)}_x$ for $x\in X$, $\xi\in T^*_x(X)$). When P has order zero its symbol is bounded on T^*X.

When σ_P has a bounded inverse outside some compact subset of T^*X, the order zero pseudodifferential operator P is said to be **elliptic**. If P is elliptic it defines a **Fredholm**

operator which means that it has a **quasi-inverse** Q i.e. a pseudodifferential operator $Q:L^2(X;E^{(1)})\to L^2(X;E^{(0)})$ such that $PQ-1$ and $QP-1$ are compact operators.

Observe next that $C(X)$ is represented by multiplication operators $f\to M^{(i)}(f)$ on $L^2(X;E^{(i)})$ (where $M^{(i)}(f)\xi=f\xi$, $\xi\in L^2(X;E^{(i)})$). When $f\in C^\infty(X)$, $M^{(i)}(f)$ is a differential hence pseudodifferential operator of order zero. Therefore $PM^{(0)}(f)-M^{(1)}(f)P$ is a pseudodifferential operator of order 0 . Its symbol has negative order hence it defines a compact operator in L^2 . The map $f\to PM^{(0)}(f)-M^{(1)}(f)P$ being continuous we deduce that $PM^{(0)}(f)-M^{(1)}(f)P$ is compact for every $f\in C(X)$. We say that P **almost intertwines** the representations $M^{(0)}$ and $M^{(1)}$.

To simplify many formulae it is convenient to use the **$\mathbb{Z}/2\mathbb{Z}$-graded formalism** ([68]): instead of dealing with the pair of Hilbert spaces $L^2(X;E^{(i)})$ we consider the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $H=L^2(X;E^{(0)})\oplus L^2(X;E^{(1)})$. An element $\xi\in H$ is said to be **homogeneous of degree** $i\in\mathbb{Z}/2\mathbb{Z}$ (we write $\partial\xi=i$)[3] if $\xi\in H^{(i)}=L^2(X;E^{(i)})\subset H$. An operator $T\in\mathcal{L}(H)$ is said to be **homogeneous of degree** $i\in\mathbb{Z}/2\mathbb{Z}$ (write $\partial T=i$) if for every $\xi\in H$ homogeneous, $T\xi$ is homogeneous and $\partial T\xi=\partial\xi+i$. Denote by $\mathcal{L}(H)^{(i)}$ the space of homogeneous operators of degree i : in the decomposition $H=H^{(0)}\oplus H^{(1)}$, $\mathcal{L}(H)^{(0)}$ is the space of diagonal and $\mathcal{L}(H)^{(1)}$ the space of antidiagonal matrices. The action of $C(X)$ on H is $f\to M(f)$, $M(f)=M^{(0)}(f)\oplus M^{(1)}(f)\in\mathcal{L}(H)^{(0)}$.

Consider the operator $F=\begin{bmatrix}0 & Q\\ P & 0\end{bmatrix}\in\mathcal{L}(H)^{(1)}$ where Q is a quasi-inverse of P . We have:

$F^2-1\in\mathcal{K}(H)$ and for all $f\in C(X)$ $[F,M(f)]=FM(f)-M(f)F\in\mathcal{K}(H)$.

Atiyah's definition of Ell can be expressed in the following way:

3.1. Definition: *Let* A *be a unital C*-algebra. Denote by* $Ell(A)$ *the set of triples* (H,π,F) *where* $\pi:A\to\mathcal{L}(H)$ *is a unital *-representation of* A *by operators of degree zero in the graded Hilbert space* H *and* $F\in\mathcal{L}(H)^{(1)}$ *satisfies* $F^2-1\in\mathcal{K}(H)$ *and for all* $a\in A$ $[F,\pi(a)]\in\mathcal{K}(H)$.

By the above discussion an elliptic pseudodifferential operator of order zero on the smooth manifold X defines an element of $Ell(C(X))$.

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[3] This writing is a little improper as $\partial 0=0$ and $\partial 0=1$.

3.2. Pairing Ell(A) with $K_0(A)$:

Associated with a triple $(H,\pi,F)\in Ell(A)$ is a triple $(H^n,\pi_n,F_n)\in Ell(M_n(A))$, where $H^n=H\otimes\mathbb{C}^n$, π_n: $M_n(A)\to\mathcal{L}(H^n)=M_n(\mathcal{L}(H))$ is the natural extension of π to n by n matrices and $F_n=F\otimes id_{\mathbb{C}^n}\in\mathcal{L}(H^n)$; the grading of $H^n=H\otimes\mathbb{C}^n$ is given by $(H\otimes\mathbb{C}^n)^{(i)}=H^{(i)}\otimes\mathbb{C}^n$.

The triple (H,π,F) defines a homomorphism f from $K_0(A)$ to $\mathbb{Z}$ in the following way: let $p\in M_n(A)$ be a projection. Define $F_p\in\mathcal{L}(\pi_n(p)H^n)$ to be the restriction to $\pi_n(p)H^n$ of $\pi_n(p)F_n$. As $\partial F_p=1$, we may write (in the decomposition $\pi_n(p)H^n=(\pi_n(p)H^n)^{(0)}\oplus(\pi_n(p)H^n)^{(1)}$)

$F_p=\begin{bmatrix} 0 & F_p^- \\ F_p^+ & 0 \end{bmatrix}$. One checks that F_p^+ is a Fredholm operator (with quasi-inverse F_p^-). We then set $f([p])=\mathrm{index}(F_p^+)$[4] where [p] denotes the class of p in $K_0(A)$.

4. Hilbert C*-modules

We are now coming to Kasparov's theory. Actually to pass from Atiyah's "Ell" to Kasparov's bivariant theory, we just need a suitable generalization of the Hilbert spaces and of the notions of bounded and compact operators. This is provided by the Hilbert C*-modules which are Hilbert spaces with the scalars replaced by the elements of a C*-algebra.

The work on Hilbert C*-modules was not initiated by Kasparov (cf. [78,89]) but Kasparov's contribution is fundamental ([67,68]).

4.1. Definition: *A* ***prehilbert C*-module*** *over a C*-algebra B is a right B-module $\mathcal{E}$ equipped with a B-valued scalar product <,> such that:*

a) $\langle x,\lambda y\rangle=\lambda\langle x,y\rangle$, $\langle x,yb\rangle=\langle x,y\rangle b$, $(\forall x,y\in\mathcal{E}$, $\lambda\in\mathbb{C}$, $b\in B)$.

b) $\langle y,x\rangle=\langle x,y\rangle^*$, $(\forall x,y\in\mathcal{E})$.

c) $\langle x,x\rangle\geq 0$, $(\forall x\in\mathcal{E})$.

Setting then $\|x\|=\|\langle x,x\rangle\|^{1/2}$ *we get a semi norm on* $\mathcal{E}$ *. If* $\mathcal{E}$ *is Hausdorff and complete it is called a* ***Hilbert C*-module*** *over B or a* ***Hilbert B-module.***

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(4) If T is a Fredholm operator its kernel and cokernel have finite dimension. Its index is the difference $\mathrm{index}T=\dim(\ker T)-\dim(\mathrm{coker}T)$ (recall that $\mathrm{coker}T=\mathrm{im}T^\perp=\ker(T^*)$.

By the first two properties we get: $<\lambda x,y>=\bar{\lambda}<x,y>$ and $<xb,y>=b^*<x,y>$. Let $\mathcal{E}$ be a prehilbert C*-module. Note that $\mathcal{E}$ is Hausdorff if and only if $<x,x>=0 \Rightarrow x=0$. Moreover we have $\|xb\|\leqslant\|x\|.\|b\|$, $\|<x,y>\|\leqslant\|x\|.\|y\|$. We deduce that the set $N=\{x\in\mathcal{E} \ / <x,x>=0\}$ is a submodule of $\mathcal{E}$ and that the Hausdorff completion of $\mathcal{E}$ (i.e. the completion of $\mathcal{E}/N$) is a Hilbert C*-module.

4.2. *Examples:* a) The C*-algebra B itself is a Hilbert B-module with scalar product $<x,y>=x^*y$. Note that the fundamental property of the C*-algebras: $\|x^*x\|=\|x\|^2$, tells that the norm on the Hilbert B-module B coincides with the C*-algebra norm of B .

b) If n is an integer, B^n is a Hilbert B-module with scalar product $<(x_i),(y_i)>=\sum_{i=1}^{n} x_i^* y_i$.

More generally given a family $(\mathcal{E}_i)_{i\in I}$ of Hilbert B-modules, one defines the Hilbert B-module $\bigoplus_{i\in I}\mathcal{E}_i$ to be the completion of the algebraic direct sum which is a prehilbert B-module with scalar product $<(x_i),(y_i)>=\sum_{i\in I}<x_i,y_i>$.

c) The sum $\bigoplus_{i\in\mathbb{N}} B$ (in the sense of b above) is a fundamental Hilbert B-module noted $\mathcal{H}_B$.

d) Let X be a compact space. The Hilbert C(X)-modules are exactly the spaces of continuous sections of continuous fields of Hilbert spaces over X ([39,40]).

The Hilbert B-modules behave very much like Hilbert spaces. There are however some differences. The main one is probably the following: let $\mathcal{E}$ be a Hilbert B-module and $\mathcal{E}'\subset\mathcal{E}$ a closed submodule. Define the orthogonal of $\mathcal{E}'$ in $\mathcal{E}$ setting $\mathcal{E}'^{\perp}=\{x\in\mathcal{E} \ / <x,y>=0 \ ,\forall y\in\mathcal{E}'\}$. Contrarily to the case of Hilbert spaces (i.e. $B=\mathbb{C}$) we may have $\mathcal{E}'^{\perp}\oplus\mathcal{E}'\neq\mathcal{E}$ (e.g. take $B=C([0,1])=\mathcal{E}$, $\mathcal{E}'=\{f\in\mathcal{E} \ / \ f(0)=0\}$; then $\mathcal{E}'^{\perp}=0$, though $\mathcal{E}\neq\mathcal{E}'$). Because of this, a bounded linear operator on a Hilbert B-module does not have always an adjoint. Actually an operator $T:\mathcal{E}_1\to\mathcal{E}_2$ has an adjoint if its graph $G(T)\subset\mathcal{E}_1\oplus\mathcal{E}_2$ satisfies $G(T)^{\perp}\oplus G(T)=\mathcal{E}_1\oplus\mathcal{E}_2$. Then the graph of T* is given by $G(T^*)=\{(x_2,x_1)\in\mathcal{E}_2\oplus\mathcal{E}_1 \ / \ (x_1,-x_2)\in G(T)^{\perp}\}$. This leads to the following definition:

4.3. Definition: *Let $\mathcal{E}_1,\mathcal{E}_2$ be Hilbert B-modules. Then $\mathcal{L}(\mathcal{E}_1,\mathcal{E}_2)$ denotes the set of maps $T:\mathcal{E}_1\to\mathcal{E}_2$ which admit an adjoint $T^*:\mathcal{E}_2\to\mathcal{E}_1$ such that $<Tx_1,x_2>=<x_1,T^*x_2>$ $(\forall x_1\in\mathcal{E}_1, x_2\in\mathcal{E}_2)$. Put $\mathcal{L}(\mathcal{E})=\mathcal{L}(\mathcal{E},\mathcal{E})$.*

If $T\in\mathcal{L}(\mathcal{E}_1,\mathcal{E}_2)$ it is automatically linear, B-linear and bounded. With the norm of bounded operators $\mathcal{L}(\mathcal{E})$ is a C*-algebra.

We now construct rank one and compact operators:

4.4. Definition : *Let $\mathcal{E}_1,\mathcal{E}_2$ be two Hilbert B-modules and let $x_1\in\mathcal{E}_1$ and $x_2\in\mathcal{E}_2$. Define $\theta_{x_1,x_2}\in\mathcal{L}(\mathcal{E}_2,\mathcal{E}_1)$ by $\theta_{x_1,x_2}(x)=x_1<x_2,x>$. We have $\theta_{x_1,x_2}=\theta_{x_2,x_1}{}^*$. An operator $T\in\mathcal{L}(\mathcal{E}_1,\mathcal{E}_2)$ is said to be compact if it is in the norm closure of the linear span of rank one operators (i.e. operators of the form θ_{x_1,x_2}). Let $\mathcal{K}(\mathcal{E}_1,\mathcal{E}_2)$ denote the space of compact operators and put $\mathcal{K}(\mathcal{E})=\mathcal{K}(\mathcal{E},\mathcal{E})$. The equalities $T\theta_{x,y}=\theta_{Tx,y}$ and $\theta_{x,y}{}^*=\theta_{y,x}$ show that $\mathcal{K}(\mathcal{E})$ is a (closed two sided) ideal in $\mathcal{L}(\mathcal{E})$.*

Coming back to the examples 4.2 , one checks that $\mathcal{K}(B)=B$, $\mathcal{K}(B^n)=M_n(B)$, $\mathcal{K}(\mathcal{H}_B)=\mathcal{K}\otimes B$ where $\mathcal{K}$ is the algebra of compact operators in a separable Hilbert space .

***4.5.** Remark:* Assume that B has a unit and let $\mathcal{E}$ be a Hilbert B-module. The following are equivalent:

i) $\mathcal{E}$ as a B-module is finitely generated and projective.

ii) $\mathcal{K}(\mathcal{E})=\mathcal{L}(\mathcal{E})$.

iii)The identity of $\mathcal{E}$ is compact.

iv) The identity of $\mathcal{E}$ is a finite rank operator i.e. a finite combination of rank one operators.

We now come to a most important result on Hilbert C*-modules.

A Hilbert B-module $\mathcal{E}$ is said to be **countably generated** if there exists a countable subset $X\subset\mathcal{E}$ such that the smallest closed submodule of $\mathcal{E}$ containing X is $\mathcal{E}$.

4.6. Theorem: *(Kasparov's stabilization theorem [67]) For every countably generated*

Hilbert B-module $\mathcal{E}$, $\mathcal{H}_B \oplus \mathcal{E}$ is isomorphic to $\mathcal{H}_B$.

This means that there exists $U \in \mathcal{L}(\mathcal{H}_B \oplus \mathcal{E}, \mathcal{H}_B)$ with $U^*U = 1_{\mathcal{H}_B \oplus \mathcal{E}}$ and $UU^* = 1_{\mathcal{H}_B}$. This theorem implies in particular that for every countably generated Hilbert B-module there exists a projection $p \in \mathcal{L}(\mathcal{H}_B)$ with $\mathcal{E}$ isomorphic to $p\mathcal{H}_B$. To prove this theorem one has to construct (if B has a unit) an orthonormal B-basis of $\mathcal{H}_B \oplus \mathcal{E}$. This is done through a beautiful adaptation of the Gram-Schmidt orthonormalization method.

4.7. Tensor products of Hilbert C*-modules

There are in fact two quite different kinds of tensor products. Both play an essential role in the construction of the Kasparov product.

4.7. a) Outer tensor products:

For $i=1,2$ let B_i be a C*-algebra and $\mathcal{E}_i$ a Hilbert B_i-module. Let $B_1 \otimes B_2$ denote the C*-algebraic tensor product [(5)]. The Hilbert $B_1 \otimes B_2$-module $\mathcal{E}_1 \otimes \mathcal{E}_2$ is defined to be the completion of the algebraic tensor product $\mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$ for the norm $\|\xi\| = \|<\xi,\xi>\|^{1/2}$, $\xi \in \mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$, where $<,>$ denotes the $B_1 \otimes B_2$-valued scalar product given by: $<\xi_1 \otimes \xi_2, \zeta_1 \otimes \zeta_2> = <\xi_1, \zeta_1> \otimes <\xi_2, \zeta_2>$ $(\xi_i, \zeta_i \in \mathcal{E}_i)$.

4.7. b) Inner tensor products:

Let A and B be two C*-algebras. let $\mathcal{E}_1$ be a Hilbert A-module , $\mathcal{E}_2$ a Hilbert B-module and $\pi : A \to \mathcal{L}(\mathcal{E}_2)$ a *-homomorphism. The Hilbert B-module $\mathcal{E}_1 \otimes_\pi \mathcal{E}_2$ (we write sometimes $\mathcal{E}_1 \otimes_A \mathcal{E}_2$) is the Hausdorff completion of the algebraic tensor product $\mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$ with respect to the norm $\|\xi\| = \|<\xi,\xi>\|^{1/2}$, $\xi \in \mathcal{E}_1 \otimes_{alg} \mathcal{E}_2$, where $<,>$ denotes the B-valued scalar product given by: $<\xi_1 \otimes \xi_2, \zeta_1 \otimes \zeta_2> = <\xi_2, \pi(<\xi_1, \zeta_1>)\zeta_2>$ $(\xi_i, \zeta_i \in \mathcal{E}_i)$. The action of B is given by $(\xi_1 \otimes \xi_2)b = \xi_1 \otimes \xi_2 b$. Note that for $a \in A$, $\xi_1 \in \mathcal{E}_1$ and $\xi_2 \in \mathcal{E}_2$ we have $\xi_1 \otimes \pi(a)\xi_2 = \xi_1 a \otimes \xi_2$.

4.8. *Remark:* We will use $\mathbb{Z}/2\mathbb{Z}$-**graded Hilbert C*-modules**: a graded Hilbert B-module $\mathcal{E}$ admits a decomposition $\mathcal{E} = \mathcal{E}^{(0)} \oplus \mathcal{E}^{(1)}$ where $\mathcal{E}^{(i)}$ is the Hilbert B-submodule of homogeneous elements of degree i $(i \in \mathbb{Z}/2\mathbb{Z})$.

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(5) This is not unique in general see §8

5. Kasparov's bifunctor

We are now ready to give the definition of Kasparov's groups ([68]).

It is convenient to use the following notation:

5.1. Definition: *Let* A *and* B *be two C*-algebras. An* **A,B-bimodule** *is a pair* $(\mathcal{E},\pi)$ *where* $\mathcal{E}$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded Hilbert B-module acted upon by* A *through a *-homomorphism* $\pi:A\to\mathcal{L}(\mathcal{E})$ *such that for all* $a\in A$, $\pi(a)$ *is homogeneous of degree 0 (i.e.* $\pi(A)\subset\mathcal{L}(\mathcal{E})^{(0)}$*).*

5.2. Definition: *Let* A *and* B *be two C*-algebras. Let* $\mathbf{E}(A,B)$ *denote the set of triples* $(\mathcal{E},\pi,F)$ *where* $(\mathcal{E},\pi)$ *is an A,B-bimodule and* $F\in\mathcal{L}(\mathcal{E})$ *is homogeneous of degree 1 and satisfies:* $\forall a\in A$, $\pi(a)(F^2-1)\in\mathcal{K}(\mathcal{E})$ *and* $[\pi(a),F]\in\mathcal{K}(\mathcal{E})$. *The triple* $(\mathcal{E},\pi,F)$ *is said to be* **degenerate** *if* $\forall a\in A$, $\pi(a)(F^2-1)=0$ *and* $[\pi(a),F]=0$. *Denote by* $\mathbf{D}(A,B)$ *the set of degenerate triples.*

For $B=\mathbb{C}$ we recognise the definition of Atiyah's Ell : $\mathbf{E}(A,\mathbb{C})=\mathrm{Ell}(A)$. Let us discuss a little the case of a commutative C*-algebra $B=C(X)$.

5.3. *Example:* Let $\mathcal{E}$ be a Hilbert $C(X)$-module. It is the space of continuous sections of a continuous field of Hilbert spaces $(\mathcal{E}_x)_{x\in X}$. An element of $\mathcal{L}(\mathcal{E})$ is a ***-strongly continuous** family $(T_x)_{x\in X}$, $T_x\in\mathcal{L}(\mathcal{E}_x)$ in the sense that for every continuous section $(\xi_x)_{x\in X}$, the sections $(T_x\xi_x)_{x\in X}$ and $(T_x^*\xi_x)_{x\in X}$ are continuous. An element of $\mathcal{K}(\mathcal{E})$ is a **norm continuous** family $(T_x)_{x\in X}$, $T_x\in\mathcal{K}(\mathcal{E}_x)$ (to express the norm continuity we may assume using the stabilization theorem (4.6.) that $\mathcal{E}_x$ is a constant Hilbert space). In particular $\|T_x\|_{x\in X}$ is continuous for $T\in\mathcal{K}(\mathcal{E})$, whereas for $T\in\mathcal{L}(\mathcal{E})$ it needs not (it is semi-continuous being a supremum of continuous functions: $\|T_x\|=\sup\{\|T_x\xi_x\| , \xi\in\mathcal{E} , \|\xi\|=1\}$).

Up to this important distinction between the *-strong and the norm topologies, an element of $\mathbf{E}(A,C(X))$ is a continuous family $X\to\mathrm{Ell}(A)$. Examples of such a families are the families of elliptic pseudodifferential operators of Atiyah and Singer (cf. [12]).

More generally, an element of $\mathbf{E}(A,B(X))$ is (up to the same mixing of topologies) a continuous family $X\to\mathbf{E}(A,B)$.

To define the Kasparov group KK(A,B) we have to give an equivalence relation on **E**(A,B). This is provided by homotopy:

5.4. Definition: *A* ***homotopy*** *in* **E**(A,B) *is an element of* **E**(A,B[0,1]). *The addition in* **E**(A,B) *is given by* $(\mathcal{E},\pi,F)+(\mathcal{E}',\pi',F')=(\mathcal{E}\oplus\mathcal{E}',\pi\oplus\pi',F\oplus F')$. *Define* KK(A,B) *to be the set of homotopy classes of elements of* **E**(A,B). *It is an abelian group with respect to addition.*

Note that the bivariant theory provides Atiyah's Ell with the needed equivalence relation (cf. §3).

The homotopy used here is a quite weak kind of homotopy:

5.5. Examples: a) A triple $(\mathcal{E},\pi,F)\in$ **D**(A,B) is homotopic to the triple (0,0,0). Indeed define the triple $(\tilde{\mathcal{E}},\tilde{\pi},\tilde{F})\in$ **E**(A,B[0,1]) by $\tilde{\mathcal{E}}=\{\xi:[0,1]\to\mathcal{E}$, ξ norm continuous and $\xi(1)=0\}$. It is a Hilbert B[0,1]-module. The action $\tilde{\pi}$ is given by $(\tilde{\pi}(a)\xi)(t)=\pi(a)\xi(t)$. The element $\tilde{F}\in\mathcal{L}(\tilde{\mathcal{E}})$ is given by $(\tilde{F}\xi)(t)=F\xi(t)$. Being degenerate, $(\tilde{\mathcal{E}},\tilde{\pi},\tilde{F})\in$ **E**(A,B[0,1]). Its evaluation at $t=0$ is $(\mathcal{E},\pi,F)$ and at $t=1$ (0,0,0). Therefore every degenerate element defines the zero element of the group KK(A,B).

b) A homotopy in which the bimodule $(\mathcal{E},\pi)$ is fixed and the operator F varies in a **norm continuous** way is called an **operator homotopy**. It is a really much stronger kind of homotopy.

c) Let $(\mathcal{E},\pi,F)\in$ **E**(A,B). Let $-\mathcal{E}$ denote the same Hilbert B-module as $\mathcal{E}$ with the opposite grading $((-\mathcal{E})^{(i)}=\mathcal{E}^{(1-i)})$. Then $(-\mathcal{E},\pi,-F)$ defines the opposite element to $(\mathcal{E},\pi,F)$ in KK(A,B). Indeed let $F_t\in\mathcal{L}(\mathcal{E}\oplus(-\mathcal{E}))$ be given by $F_t=\cos(\pi t/2)(F\oplus(-F))+\sin(\pi t/2)U$ with $U=J+J^*$ where J is the identity from $\mathcal{E}$ to $-\mathcal{E}$ (note that $\partial(U)=\partial(J)=1$). It defines an operator homotopy between $(\mathcal{E},\pi,F)+(-\mathcal{E},\pi,-F)$ and the degenerate element $(\mathcal{E}\oplus(-\mathcal{E}),\pi\oplus\pi,U)$.

5.6. Functorial properties of KK(A,B)

a) Let $f:A_1\to A_2$ be a *-homomorphism and $(\mathcal{E},\pi,F)\in$ **E**(A_2,B). Then $(\mathcal{E},\pi\circ f,F)\in$ **E**(A_1,B). Let $f^*:KK(A_2,B)\to KK(A_1,B)$ be the group homomorphism which to the class of $(\mathcal{E},\pi,F)$ associates the class of $(\mathcal{E},\pi\circ f,F)$.

b) Let $g:B_1 \to B_2$ be a *-homomorphism and $(\mathcal{E},\pi,F)\in\mathbf{E}(A,B_1)$. Let $\pi\otimes 1$ denote the *-homomorphism $A\to\mathcal{L}(\mathcal{E}\otimes_g B_2)$ (cf 4.7.b) given by $(\pi\otimes 1)(a)(\xi\otimes b)=\pi(a)\xi\otimes b$. Then $(\mathcal{E}\otimes_g B_2,\pi\otimes 1,F\otimes 1)\in\mathbf{E}(A,B_2)$. Let $g_*:KK(A,B_1)\to KK(A,B_2)$ be the group homomorphism which to the class of $(\mathcal{E},\pi,F)$ associates the class of $(\mathcal{E}\otimes_g B_2,\pi\otimes 1,F\otimes 1)$.

From the definition of the equivalence relation defining KK(A,B) we immediately get:

5.7. Theorem: KK(A,B) *is a homotopy invariant bifunctor covariant in* B , *contravariant in* A

Finally K-theory is a particular case of KK-theory:

5.8. Theorem : *For every C*-algebra* B , $KK(\mathbb{C},B)=K_0(B)$.

More precisely assume that B has a unit. Then every finitely generated projective B-module $\mathcal{E}$ can be endowed with a (unique up to homotopy) structure of Hilbert B-module. Let $\mathcal{E}$ be graded by $\mathcal{E}=\mathcal{E}^{(0)}$, and $i:\mathbb{C}\to\mathcal{L}(\mathcal{E})$ be given by $i(\lambda)=\lambda 1$. By remark 4.5, $(\mathcal{E},i,0)\in\mathbf{E}(\mathbb{C},B)$. The theorem states that the map $\mathcal{E}\to(\mathcal{E},i,0)$ extends to a group isomorphism $K_0(B)\to KK(\mathbb{C},B)$.

***5.9.** Remark:* The higher K-groups $K_n(B)$ $n\geqslant 1$ can also be recognised as KK-groups. In particular $K_n(B)=K_0(B(\mathbb{R}^n))=KK(\mathbb{C},B(\mathbb{R}^n))$. In fact Kasparov gives a very natural definition of higher KK-groups $KK_n(A,B)$, $n\in\mathbb{Z}$, in terms of Clifford algebras and of $\mathbb{Z}/2\mathbb{Z}$-graded C*-algebras. He deduces from the periodicity of the Clifford algebras the "formal Bott periodicity": $KK_n(A,B)=KK_{n+2}(A,B)$. We have to refrain from giving this very elegant presentation here. One may set instead $KK_n(A,B)=KK(A,B(\mathbb{R}^n))$. The coincidence of these two definitions was proved by Kasparov using the Kasparov-product.

The group KK_1 admits the following defintion:

5.10. Definition: *Let* $\mathbf{E}_1(A,B)$ *denote the set of triples* $(\mathcal{E},\pi,F)$ *where* $(\mathcal{E},\pi)$ *is a trivially graded* A,B-*bimodule and* $F\in\mathcal{L}(\mathcal{E})$ *satisfies* $\pi(a)(F^2-1)\in\mathcal{K}(\mathcal{E})$ *and* $[\pi(a),F]\in\mathcal{K}(\mathcal{E})$ *for all* $a\in A$. *Then* $KK_1(A,B)$ *is the group of homotopy classes of elements of* $\mathbf{E}_1(A,B)$.

6. The Kasparov product

We now come to the heart of the theory (cf.[68]). The Kasparov product is what makes KK-theory computable and provides it with interesting applications.

One might expect that the product exists for some abstract reasons. This is not at all the case. Maybe the power of the theory comes from the fact that really deep analysis is needed to construct this product. In particular it is here that the properties of the C*-algebras are used in the deepest way. This is not the right place to explain the analysis involved. Let me just mention that the key notion is that of quasi-central approximate units (cf. [1]).

In this section we will just state the theorem of the existence of the Kasparov product and outline how some important corollaries can be drawn. In the next section we will see how this theorem can be reduced, thanks to Cuntz' work, to the homotopy equivalence of two C*-algebras.

Before we state the main theorem we will need a definition:

6.1. Definition : *a) Let* A *be a C*-algebra. The element* $1_A \in KK(A,A)$ *is the class of the triple* $(A, i_A, 0)$ *where* A *is graded by* $A^{(1)}=0$ *and* $i_A: A \to \mathcal{K}(A) \subset \mathcal{L}(A)$ *is given by* $i(a)b=ab$, $a,b \in A$.

b) Let A,B,D *be C*-algebras. We define* $\tau_D: KK(A,B) \to KK(A\otimes D, B\otimes D)$ *by* $\tau_D(x)=$ class of $(\mathcal{E}\otimes D, \pi\otimes i_D, F\otimes 1)$ *if* $x=$ class of $(\mathcal{E},\pi,F)$.

Note that $\tau_B(1_A)=1_{A\otimes B}$. (The tensor product $\mathcal{E}\otimes D$ is discussed in 4.7. a).

6.2. Theorem : *a) There is a well defined bilinear coupling* ***(the Kasparov product)*** $KK(A,D)\times KK(D,B) \to KK(A,B)$ *noted* $(x,y) \to x\otimes_D y$.

b) This coupling is covariant in B *, contravariant in* A *, and if* $f: D \to E$ *is a *-homomorphism,* $x \in KK(A,D)$ *and* $y \in KK(E,B)$ *then* $f_*(x)\otimes_E y = x\otimes_D f^*(y)$.

c) This product is associative: $\forall x \in KK(A,D)$, $y \in KK(D,E)$, $z \in KK(E,B)$ *we have:* $(x\otimes_D y)\otimes_E z = x\otimes_D (y\otimes_E z)$.

d) Moreover $\forall x \in KK(A,B)$, $x\otimes_B 1_B = 1_A \otimes x = x$ *and* $\forall y \in KK(B,D)$ *and every C*-algebra* E , $\tau_E(x\otimes_B y)=\tau_E(x)\otimes_{B\otimes E}\tau_E(y)$ $(\in KK(A\otimes E, D\otimes E))$.

For some technical reasons we have to assume that the C*–algebras appearing in this theorem are separable.

Some important simplifications of the proof of this basic theorem where given by Cuntz and Higson (cf. [34,35,36,55] see also [31,101]). We will go a little into the proof using Cuntz' approach in the next section. We now draw some consequences.

Thanks to the operation τ_D, the Kasparov product can immediately be extended to a more general situation:

6.3. Definition: *Let* A_1, A_2, B_1, B_2, D *be C*–algebras and let* $x \in KK(A_1, B_1 \otimes D)$, $y \in KK(D \otimes A_2, B_2)$. *The Kasparov product of* x *by* y, *noted* $x \otimes_D y$, *is defined to be equal to* $\tau_{A_2}(x) \otimes_{B_1 \otimes D \otimes A_2} \tau_{B_1}(y)$.

Note that $\tau_{A_2}(x) \in KK(A_1 \otimes A_2, B_1 \otimes D \otimes A_2)$ and $\tau_{B_1}(y) \in KK(B_1 \otimes D \otimes A_2, B_1 \otimes B_2)$ can be paired by theorem 6.2.

One deduces from theorem 6.2 that this generalized Kasparov product is covariant in B_i contravariant in A_i $(i=1,2)$, its behaviour with respect to the change of algebra D is functorial (in the sense of 6.2 b); this product is still associative, admits 1_D as a unit and behaves nicely with respect to operation τ_E.

Moreover, one deduces from the definition of the Kasparov product (that we didn't give here!):

6.4. Proposition: *The Kasparov product over the C*–algebra* $\mathbb{C}$ *is commutative.*

More precisely, for every $x \in KK(A_1, B_1)$, $y \in KK(A_2, B_2)$ we have $x \otimes_{\mathbb{C}} y = y \otimes_{\mathbb{C}} x$ (up to the obvious isomorphisms $A_1 \otimes A_2 \cong A_2 \otimes A_1$, $B_1 \otimes B_2 \cong B_2 \otimes B_1$).

From theorem 6.2 we immediately deduce:

6.5. Theorem: *(abstract periodicity). Let* D *and* E *be C*–algebras and assume that there exist elements* $\alpha \in KK(D,E)$, $\beta \in KK(E,D)$ *such that* $\alpha \otimes_E \beta = 1_D$ *and* $\beta \otimes_D \alpha = 1_E$.

Then for every pair A,B of C-algebras the group homomorphisms:*
$\cdot\otimes_D\alpha: KK(A,B\otimes D)\to KK(A,B\otimes E)$ *and* $\beta\otimes_D\cdot: KK(D\otimes A,B)\to KK(E\otimes A,B)$ *are isomorphisms with inverses respectively* $\cdot\otimes_E\beta$ *and* $\alpha\otimes_E\cdot$.

When such elements α and β exist the algebras D and E are said to be **K-equivalent.** This theorem states that K-equivalent C*-algebras cannot be distinguished by KK-theory.

From theorem 6.2 and proposition 6.4 we deduce:

6.6. Theorem: *(abstract duality) Let D and E be C*-algebras and assume that there exist elements* $\rho\in KK(D\otimes E,\mathbb{C})$, $\sigma\in KK(\mathbb{C},E\otimes D)$ *such that* $\sigma\otimes_D\rho=1_E$ *and* $\sigma\otimes_E\rho=1_D$.
Then for every pair A,B of C-algebras the group homomorphisms*
$\sigma\otimes_D\cdot: KK(D\otimes A,B)\to KK(A,E\otimes B)$ *and* $\sigma\otimes_E\cdot\ : KK(A\otimes E,B)\to KK(A,B\otimes D)$ *are isomorphisms (with inverses* $\cdot\otimes_E\rho$ *and* $\cdot\otimes_D\rho$ *).*

When such elements ρ and σ exist the C*-algebras D and E are said to be **K-dual** to each other.

We end this section with some examples of pairs of K-equivalent and K-dual C*-algebras. Each of these examples gives a powerful periodicity or duality theorem for the Kasparov groups.

6.7. Bott periodicity: Define the element $\beta\in KK(\mathbb{C},C_0(\mathbb{R}^2))$ to be the class of the triple $(\mathcal{E},\pi,F)$ where $\mathcal{E}^{(0)}=\mathcal{E}^{(1)}=C_0(\mathbb{R}^2)$, π is the unital *-homomorphism $\pi:\mathbb{C}\to\mathcal{L}(\mathcal{E})$ and F is defined by $F(\xi,\zeta)=(P^*\zeta,P\xi)$ where $P:C_0(\mathbb{R}^2)\to C_0(\mathbb{R}^2)$ is given by $P\xi(s,t)=(1+s^2+t^2)^{-1/2}(s+it)\xi(s,t)$, $(s,t)\in\mathbb{R}^2$. Note that F^2-1 is the multiplication by $(1+s^2+t^2)^{-1}\in C_0(\mathbb{R}^2)$ and thus $F^2-1\in\mathcal{K}(\mathcal{E})$. Therefore $(\mathcal{E},\pi,F)\in\mathbf{E}(\mathbb{C},C_0(\mathbb{R}^2))$.

Define also the element $\alpha\in KK((C_0(\mathbb{R}^2),\mathbb{C})$ to be the class of the triple (H,π,D) where $H^{(0)}=H^{(1)}=L^2(\mathbb{R}^2)$, π is the action by multiplication and D is defined by $D(\xi,\zeta)=(T\zeta,T^*\xi)$ where T is given by $(T\xi)^\wedge=P(\hat\xi)$ ($\xi\to\hat\xi$ is the Fourier transform). One checks that for $f\in C_0(\mathbb{R}^2)$, $[D,\pi(f)]\in\mathcal{K}(H)$ and $\pi(f)(D^2-1)\in\mathcal{K}(H)$. Therefore $(H,\pi,F)\in\mathbf{E}(C_0(\mathbb{R}^2),\mathbb{C})$. Computing the index of an operator one shows that $\beta\otimes_{C_0(\mathbb{R}^2)}\alpha=1_\mathbb{C}$.

One checks also that $\alpha\otimes_\mathbb{C}\beta=1_{C_0(\mathbb{R}^2)}$ (this can be deduced from the equality $\beta\otimes\alpha=1$ using proposition 6.4 and a trick coming from [3]).

Therefore the algebras $\mathbb{C}$ and $C_0(\mathbb{R}^2)$ are K-equivalent. We deduce:

Theorem: *(Bott periodicity). For every pair* A,B *of C*-algebras* $KK(A,B(\mathbb{R}^2))\cong KK(A,B)$ *and* $KK(A(\mathbb{R}^2),B)\cong KK(A,B)$.

Moreover we can write $C_0(\mathbb{R}^2)=D\otimes E$ where $D=E=C_0(\mathbb{R})$. Set then $\sigma=\beta\in KK(\mathbb{C},E\otimes D)$ and $\rho=\alpha\in KK(D\otimes E,\mathbb{C})$. Using the trick of [3], we can deduce from the equality $\alpha\otimes_{D\otimes E}\beta=1_{\mathbb{C}}$ the equalities $\sigma\otimes_D\rho=1_E$ and $\sigma\otimes_E\rho=1_D$. Hence $C_0(\mathbb{R})$ is K-dual to itself.

Theorem: *(Bott periodicity) For every pair* A,B *of C*-algebras and every pair of integers* $m,n\geqslant 0$ *we have:*

if $m+n$ *is even* $KK(A(\mathbb{R}^m),B(\mathbb{R}^n))=KK(A,B)$;

if $m+n$ *is odd* $KK(A(\mathbb{R}^m),B(\mathbb{R}^n))=KK(A,B(\mathbb{R}))=KK(A(\mathbb{R}),B)=KK_1(A,B)$.

Here is an important generalization of the Bott periodicity:

Theorem : *(Thom isomorphism) Let* X *be a locally compact space and let* E *denote the total space of a complex vector bundle over* X *. Then the C*-algebras* $C_0(X)$ *and* $C_0(E)$ *are K-equivalent.*

When X is just a point this theorem reduces to Bott periodicity.

In fact Thom isomorphism still holds for an even dimensional real vector bundle endowed with a $Spin^{\mathbb{C}}$ structure [8,53].

6.8. Morita equivalence:

Let A be a C*-algebra and let $\mathcal{E}$ be a Hilbert A-module. We say that $\mathcal{E}$ is **full** if the norm closure of the vector span of $\{\langle\xi,\zeta\rangle , \xi,\zeta\in\mathcal{E}\}$ is A (it is in general a two sided ideal because of the equalities $\langle\xi,\zeta\rangle^*=\langle\zeta,\xi\rangle$ and $\langle\xi,\zeta\rangle a=\langle\xi,\zeta a\rangle$).

Definition: *The C*-algebras* A *and* B *are said to be Morita equivalent if there exists a Hilbert A-module* $\mathcal{E}$ *which is full and an isomorphism of* B *with* $\mathcal{K}(\mathcal{E})$.

In the situation of this definition, identify B with $\mathcal{K}(\mathcal{E})$. Let $\mathcal{E}^*$ denote the A,B-bimodule obtained in the following way: as a set $\mathcal{E}^*$ is equal to $\mathcal{E}$; let $\xi\to\xi^*$ denote the identity from $\mathcal{E}$ to $\mathcal{E}^*$. The linear structure of $\mathcal{E}^*$ makes the map $\xi\to\xi^*$ antilinear (i.e.

$\lambda\xi^*=(\bar{\lambda}\xi)^*$). For $\xi,\zeta\in\mathcal{E}$, $a\in A$ and $b\in B$ we set $a\xi^*=(\xi a^*)^*$, $\xi^*b=(b^*\xi)^*$ and $<\xi^*,\zeta^*>=\theta_{\xi,\zeta}$. In particular $\mathcal{E}^*$ is a Hilbert B-module; it is full by definition of $\mathcal{K}(\mathcal{E})$ and the map $\theta_{\xi^*,\zeta^*}\to<\xi,\zeta>$ identifies $\mathcal{K}(\mathcal{E}^*)$ with A (hence B and A are Morita equivalent).

If A and B are Morita equivalent, $(\mathcal{E},i,0)\in\mathbf{E}(B,A)$, where $\mathcal{E}$ is trivially graded (i.e. $\mathcal{E}^{(1)}=0$) and $i:B\to\mathcal{K}(\mathcal{E})$ is the identity. Also $(\mathcal{E}^*,i',0)\in\mathbf{E}(A,B)$. Let α and β denote the classes of these elements in the KK-groups.

The map $\xi\otimes\zeta^*\to\theta_{\xi,\zeta}$ identifies the B,B-bimodule $\mathcal{E}\otimes_A\mathcal{E}^*$ with B .We deduce that $\alpha\otimes_A\beta=1_B$. In the same way $\beta\otimes_B\alpha=1_A$.

Theorem: *Morita equivalent C*-algebras are K-equivalent.*

In particular for every C*-algebra A , the C*-algebras A and $A\otimes\mathcal{K}$ are Morita equivalent hence K-equivalent.

6.9. Poincaré duality: Let M be a smooth compact manifold and let T*M denote the total space of the cotangent bundle of M . One can construct elements $\sigma_M\in KK(\mathbb{C},C_0(T^*M\times M))$ and $\rho_M\in KK(C_0(M\times T^*M),\mathbb{C})$ such that $\sigma_M\otimes_{C_0(T^*M)}\rho_M=1_{C(M)}$ and $\sigma_M\otimes_{C(M)}\rho_M=1_{C_0(T^*M)}$ ([69,31]).

Theorem: *(Poincaré duality) The C*-algebras* C(M) *and* $C_0(T^*M)$ *are K-dual to each other.*

Other examples of K-equivalent and K-dual C*-algebras will be considered in the following sections.

7. Cuntz' approach

In Cuntz' approach ([34,35,36]), KK(A,B) is shown to be the set of homotopy classes of *-homomorphisms from a C*-algebra qA naturally associated with A to the C*-algebra $B\otimes\mathcal{K}$. This is a first advantage of this picture as *-homomorphisms are very natural and familiar objects.

The main theorem on the existence of the Kasparov product is here expressed by the fact that the C*-algebras qA and q(qA) are (stably) homotopy equivalent. Once this still technically involved fact is established, the Kasparov product becomes just composition of

*-homomorphisms. Therefore its properties become easy to check. In particular its associativity which in Kasparov's work is really hard, reduces here to associativity of composition of *-homomorphisms.

We start with some elementary observations.

Let A and B be two C*-algebras and let $(\mathcal{E},\pi,F)\in\mathbf{E}(A,B)$ (definition 5.2).

Changing the bimodule $(\mathcal{E},\pi)$ without changing the class in KK(A,B) we may assume that $F^2=1$. Indeed let $-\mathcal{E}$ be the same Hilbert B-module as $\mathcal{E}$ with opposite grading ($-\mathcal{E}^{(0)}=\mathcal{E}^{(1)}$, $-\mathcal{E}^{(1)}=\mathcal{E}^{(0)}$). Let $\pi':A\to\mathcal{L}(\mathcal{E}\oplus(-\mathcal{E}))$ be defined by $\pi'(a)=\pi(a)\oplus 0$. Let $F'\in\mathcal{L}(\mathcal{E}\oplus(-\mathcal{E}))$ be given by $F'(\xi,\zeta)=(F\xi+U^*\zeta,U(1-F^2)\xi-UFU^*\zeta)$ where $U\in\mathcal{L}(\mathcal{E},-\mathcal{E})$ is the identification of $\mathcal{E}$ with $-\mathcal{E}$ (note that $\partial U=1$). Then $F'^2=1$, $(\mathcal{E}\oplus(-\mathcal{E}),\pi',F')\in\mathbf{E}(A,B)$ and its class in KK(A,B) is the same as the class of $(\mathcal{E},\pi,F)$.

Assume $(\mathcal{E},\pi,F)\in\mathbf{E}(A,B)$ and $F^2=1$. We may further assume that $F=F^*$. Indeed let $F'=F(FF^*)^{1/2}$. Note that $FF^*=(F^*F)^{-1}$, hence $(FF^*)^{1/2}=(F^*F)^{-1/2}$. In particular F' is invertible and as $F'F'^*=F(FF^*)F^*=1$, F' is a unitary. Moreover for every polynomial f we have $Ff(F^*F)=f(FF^*)F$. As $x\to x^{-1/2}$ is a limit of polynomials on the spectrum of FF^* (= spectrum of F^*F) we deduce $F'=F(F^*F)^{-1/2}=(FF^*)^{-1/2}F=F'^{-1}=F'^*$. Moreover $F_t=F(FF^*)^{t/2}$ ($t\in[0,1]$) is an operator homotopy between F and F'.

Let now $(\mathcal{E},\pi,F)\in\mathbf{E}(A,B)$ with $F=F^*=F^{-1}$. We may use F to identify $\mathcal{E}^{(0)}$ with $\mathcal{E}^{(1)}$. What is left is then a trivially graded Hilbert B-module (the module $\mathcal{E}^{(0)}$) and two actions of A, π^+ and π^- (π^+ is the restriction of π to $\mathcal{E}^{(0)}$, π^- is the restriction of π to $\mathcal{E}^{(1)}$ transported by F to $\mathcal{E}^{(0)}$ i.e. $\pi^-(a)$ is the restriction to $\mathcal{E}^{(0)}$ of $F\pi(a)F$). Using the stabilization theorem (4.6) we are led to the following definition:

7.1. Definition: *A* ***quasi-homomorphism*** *from A to B is a pair $\pi^\pm$ of *-homomorphisms from A to $\mathcal{L}(\mathcal{H}_B)$ such that for all $a\in A$, $\pi^+(a)-\pi^-(a)\in B\otimes\mathcal{K}=\mathcal{K}(\mathcal{H}_B)$.*

We find here Cuntz' first presentation of KK(A,B) : the set of homotopy classes of quasi-homomorphisms from A to B (cf. [34]).

We now contruct the C*-algebra qA. Its construction is based on the notion of free products of C*-algebras.

7.2. Free products of C*-algebras: Let A and B be two C*-algebras. Denote by $A*_{alg}B$ their algebraic free product. It is a *-algebra endowed with two *-homomorphisms $j_A:A\to A*_{alg}B$ and $j_B:B\to A*_{alg}B$. For every *-algebra D and a pair (φ,ψ) of *-homomorphisms $\varphi:A\to D$, $\psi:B\to D$ there exists a unique *-homomorphism $(\varphi,\psi):A*_{alg}B\to D$ such that $(\varphi,\psi)\circ j_A=\varphi$ and $(\varphi,\psi)\circ j_B=\psi$. On $A*_{alg}B$ define the norm $\|x\|=\sup\{\ \|(\varphi,\psi)(x)\|\ \}$ this supremum being taken over all pairs (φ,ψ) as above with D a C*-algebra. The completion $A*B$ of $A*_{alg}B$ for this norm is a C*-algebra called the C*-algebraic free product of A and B.

7.3. Definition: *Let* A *be a C*-algebra. Denote by* QA *the free product of* A *by itself and by* $i^{\pm}$ *the two natural inclusions of* A *in* QA. *Let* qA *be the ideal of* QA *generated by* $\{i^+(a)-i^-(a)\ /\ a\in A\}$. *It is the kernel of the *-homomorphism* $\Delta=(id_A,id_A):QA\to A$.

In other terms we have a short exact sequence $0\to qA\to QA\xrightarrow{\Delta}A\to 0$. Note that both i^+ and i^- split Δ.

We next remark that a quasi-homomorphism from A to B defines by the universal property of the free product a *-homomorphism $\pi=(\pi^+,\pi^-):QA\to\mathcal{L}(\mathcal{H}_B)$ such that $\pi(qA)\subset B\otimes\mathcal{K}$. Call $\varphi:qA\to B\otimes\mathcal{K}$ the restriction of π. Conversely, every *-homomorphism $\varphi:qA\to B\otimes\mathcal{K}$ determines (modulo a minor deformation using the stabilization theorem 4.6) a quasi-homomorphism from A to B. We therefore get:

7.4. Theorem: *The set of homotopy classes of *-homomorphisms from* qA *to* $B\otimes\mathcal{K}$ *is* $KK(A,B)$.

This is Cuntz' (second) definition of $KK(A,B)$.

We now give Cuntz' picture of the Kasparov product.

For every C*-algebra A, denote by $p_A:qA\to A$ the restriction to qA of the

*-homomorphism $(id_A,0):QA\to A$.

For every *-homomorphism $f:D\to E$ (D and E are C*-algebras) denote also by $f:M_2(D)\to M_2(E)$ its natural extension to 2 by 2 matrices (given by $(f(x))_{i,j}=f(x_{i,j})$, $i,j\in\{1,2\}$).
Denote by q^2A the C*-algebra $q(qA)$.

The construction of the Kasparov product is based on the following:

7.5. Theorem: *For every (separable) C*-algebra A there exists a *-homomorphism $\varphi_A:qA\to M_2(q^2A)$ such that the composition $\varphi_A\circ p_{qA}$ is homotopic (among *-homomorphisms) to the inclusion of q^2A in $M_2(q^2A)$ and $p_{qA}\circ\varphi_A$ is homotopic to the inclusion of qA in $M_2(qA)$.*

(The inclusions that we are refering to are: $x\to \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$).

From this theorem we deduce:

7.6. Corollary: *The map* $p_B:qB\to B$ *induces a group isomorphism* $\pi_B*:KK(A,qB)\to KK(A,B)$.

The inverse of π_B* is constructed as follows: let $\psi:qA\to B\otimes\mathcal{K}$ define an element of KK(A,B) . As the construction of qA is functorial we get a *-homomorphism $q\psi:q^2A\to q(B\otimes\mathcal{K})$. Although $q(B\otimes\mathcal{K})$ and $qB\otimes\mathcal{K}$ do not coincide, there is a natural *-homomorphism $\kappa:q(B\otimes\mathcal{K})\to qB\otimes\mathcal{K}$. Consider then the *-homomorphism $\chi:qA\to\mathcal{K}\otimes qB$ given by the composition:

$$qA \xrightarrow{\varphi_A} M_2(q^2A) \xrightarrow{q\psi} M_2(q(B\otimes\mathcal{K})) \xrightarrow{\kappa} M_2(qB\otimes\mathcal{K}) \cong qB\otimes\mathcal{K}.$$

It defines the desired element of KK(A,qB).

Now, as $\mathcal{K}\otimes\mathcal{K}$ and $\mathcal{K}$ are isomorphic, we deduce:

7.7. Corollary: KK(A,B) *is the set of homotopy classes of *-homomorphisms from* $qA\otimes\mathcal{K}$ *to* $qB\otimes\mathcal{K}$.

The Kasparov product becomes now just the composition of *-homomorphisms and its associativity is obvious.

***7.8.** Remark:* Cuntz' proof of theorem 7.5 is based on a theorem of Pedersen about lifting of derivations (cf. [79]).

Actually theorem 7.5 is in a way equivalent to the existence of the Kasparov product: the Kasparov product is a direct consequence of it; conversely, the identity of qA defines by theorem 7.4 an element $\ell_A \in KK(A,qA)$; then the Kasparov product $\ell_A \otimes_{qA} \ell_{qA} \in KK(A,q^2A)$ defines (using again theorem 7.4) the desired homomorphism $\varphi_A: qA \to q^2A \otimes \mathcal{K}$ (in fact the image of φ_A sits in $M_2(q^2(A)) = q^2A \otimes M_2(\mathbb{C}) \subset q^2A \otimes \mathcal{K}$ as stated in theorem 7.5).

8. Further properties of the Kasparov groups, computation in some cases

Besides the periodicity theorems based on the Kasparov product (i.e. on the abstract periodicity, theorem 6.5), there are some other important properties of the Kasparov groups. We indicate here the behaviour of these groups with respect to C*-algebraic extensions and to C*-algebraic inductive limits.

This enables to compute the KK-groups of a large class of C*-algebras in terms of their K-groups thanks to Rosenberg-Schochet's **universal coefficient theorem** .

8.1. C*-algebraic extensions: The Kasparov groups are closely related to C*-algebraic extensions as we will see in §10. Their behaviour with respect to extensions is very important and not yet completely settled.

Let $0 \to J \xrightarrow{i} A \xrightarrow{p} A/J \to 0$ be an exact sequence of C*-algebras (i.e. J is a closed two-sided ideal in A and p is the quotient map). The algebra A is then said to be an **extension of A/J by J** .

We need to introduce two notions in this discussion: the notion of **semi-split exact sequences** and that of **K-nuclear C*-algebras** .

For the first one, recall that a continuous linear map f from a C*-algebra A to a C*-algebra B is said to be **positive** if $x \geqslant 0 \Rightarrow f(x) \geqslant 0$. It is said to be **completely positive** if for every integer n the map $f \otimes id_{M_n(\mathbb{C})}: M_n(A) \to M_n(B)$ is positive. The exact sequence $0 \to J \to A \to A/J \to 0$ is said to be **semi-split** if the quotient map $p: A \to A/J$ admits a completely positive cross-section $s: A/J \to A$. Such a cross-section always exists when the C*-algebra A/J is **nuclear** (cf. [24]). This brings us to the second notion.

Given two C*-algebras A and B there are two extremal ways of completing their algebraic tensor product so to get a C*-algebra. These completions are called $A \otimes_{max} B$ and $A \otimes_{min} B$. Extending the identity of the algebraic tensor product, we get a surjective *-homomorphism $p: A \otimes_{max} B \to A \otimes_{min} B$. The algebra A is said to be **nuclear** if for every C*-algebra B this *-homomorphism p is an isomorphism. Loosely speaking, A is said to be **K-theoretically nuclear** or just **K-nuclear** if for every B the *-homomorphism p induces an isomorphism of the Kasparov groups (in fact the definition of K-nuclearity is a little more technical cf.[106]; it is inspired from a notion of K-theoretic amenability for groups due to Cuntz [33] and generalized by Julg and Valette [63]).

Theorem: *([68,37,104,106]). Let* $0 \to J \xrightarrow{i} A \xrightarrow{p} A/J \to 0$ *be an exact sequence of C*-algebras.*

a) If either this sequence is semi-split or the C-algebra* B *is K-nuclear we have a periodic exact sequence:*

$$\begin{array}{ccccc} KK(B,J) & \xrightarrow{i_*} & KK(B,A) & \xrightarrow{p_*} & KK(B,A/J) \\ \delta\uparrow & & & & \downarrow\delta \\ KK_1(B,A/J) & \xleftarrow{p_*} & KK_1(B,A) & \xleftarrow{i_*} & KK_1(B,J) \end{array}$$

b) If either this sequence is semi-split or the C-algebras* J , A *and* A/J *are K-nuclear, then for every C*-algebra* B *we have a periodic exact sequence:*

$$\begin{array}{ccccc} KK(A/J,B) & \xrightarrow{p^*} & KK(A,B) & \xrightarrow{i^*} & KK(J,B) \\ \delta\uparrow & & & & \downarrow\delta \\ KK_1(J,B) & \xleftarrow{i^*} & KK_1(A,B) & \xleftarrow{p^*} & KK_1(A/J,B) \end{array}$$

Whether these exact sequences hold in full generality is still, as far as I know, an open problem.

We now pass to the behaviour of the KK-groups with respect to inductive limits of C*-algebras.

We first consider the case of (countable) infinite sums: let $(A_n)_{n\in\mathbb{N}}$ be a sequence of C*-algebras and set $A=\underset{n}{\oplus} A_n$ (in the C*-algebraic sense i.e. $A=\{(x_n)\in\prod A_n \ / \ \lim \|x_n\|=0\}$).

Rosenberg proved [92]:

8.2. Proposition: *For every C*-algebra* B *,* $KK(A,B)=\prod_{n\in\mathbb{N}} KK(A_n,B)$.

Let (A_n,i_n) be an inductive system of C*-algebras and let $A=\underset{\rightarrow}{\lim}(A_n,i_n)$ (in the C*-algebraic sense).

Using proposition 8.2, theorem 8.1 and a construction due to L.G.Brown ([19]) one shows (cf. [92]):

8.3. Theorem: *For every C*-algebra* B *we have a Milnor* $\lim^1$ *exact sequence:*

$$0 \to \underset{\leftarrow}{\lim}{}^1 KK(A_n,B) \to KK(A,B) \to \underset{\leftarrow}{\lim}\, KK(A_n,B) \to 0$$

Remark: (cf. [18,19]) The corresponding result for the algebra B is not true in general: it is easy to construct algebras A and B_n such that $KK(A,\oplus B_n)\neq\oplus KK(A,B_n)$ (e.g. take $B_n=\mathbb{C}$, $A=\oplus B_n$).

We now come to the computation of the Kasparov groups in terms of the K-groups. If $\varphi:A\to B$ is a *-homomorphism its **mapping cone** is the C*-algebra:

$C_\varphi=\{ (x,f) \ / \ x\in A \ , f\in B[0,1] \, , f(0)=\varphi(x) \ , f(1)=0 \}$.

8.4. Theorem: *(Rosenberg – Schochet's universal coefficient theorem) Let* $\mathcal{N}$ *be the smallest class of separable C*-algebras, containing all the abelian C*-algebras, closed under K-equivalence, inductive limits and such that if* A *and* B *are in* $\mathcal{N}$ *and* $\varphi:A\to B$ *is* *a *–homomorphism* C_φ *is in* $\mathcal{N}$*. Then for every pair* A , B *of C*–algebras, with* A *in* $\mathcal{N}$*, we have a short exact sequence:*

$$0 \to \mathrm{Ext}(K_*(A),K_*(B)) \to KK_*(A,B) \to \mathrm{Hom}(K_*(A),K_*(B)) \to 0 .$$

This exact sequence is split (but not canonically).

(Here $K_*=K_0\oplus K_1$ and $KK_*=KK_0\oplus KK_1$, where $KK_0(A,B)=KK(A,B)$. Recall that $\mathrm{Ext}(K_*(A),K_*(B))$ is the group of extensions of abelian groups of the form $0\to K_*(B)\to G\to K_*(A)\to 0$).

To prove this theorem one first considers the case when $K_*(B)$ is a division group. The theorem reduces in that case to the equality: $KK_*(A,B)=\mathrm{Hom}(K_*(A),K_*(B))$. Call $\mathcal{N}_B$ the class of C*–algebras satisfying this equality. It is closed under all operations of the theorem. As it is closed under semi-split extensions (see remark 8.5 below) and inductive limits and contains $C_0(\mathbb{R}^n)$ for all n , it contains all abelian C*–algebras. Hence it contains $\mathcal{N}$. The case of general B is handled using an injective resolution (cf. [97]).

8.5. *Remarks:* a) It turns out (cf.[97]) that $\mathcal{N}$ is exactly the class of C*–algebras satisfying the universal coeficient theorem for all B. Moreover, it is the class of C*–algebras K–equivalent to an abelian one.

b) If $0\to J\to A\to A/J\to 0$ is a short exact sequence of C*–algebras, if A and A/J are in $\mathcal{N}$ and either the sequence is semi-split or J is K–nuclear then J is in $\mathcal{N}$. Indeed under this hypothesis J and C_p are K–equivalent where $p:A\to A/J$ is the quotient map. Similarly, one shows that if J and A (resp. J an A/J) are in $\mathcal{N}$ and if either the sequence is semi-split or A/J (resp. A) is K–nuclear then it is also in $\mathcal{N}$.

Also $\mathcal{N}$ contains all type I C*–algebras and is closed under many constructions: (min and max) tensor products, crossed products by $\mathbb{R}$, $\mathbb{Z}$ and free groups.

c) In fact $\mathcal{N}$ is contained in the class of K–nuclear C*–algebras. As there are examples of non K–nuclear C*–algebras ([106]), there are C*–algebras which are not in $\mathcal{N}$. In fact one may construct a C*–algebra A such that $K_*(A)=0$ but $KK(A,A)\neq 0$.

9. Kasparov's equivariant KK-theory

The basic interplay between the theory of C*-algebras and the theory of locally compact groups is expressed through the fundamental constructions of group C*-algebras and C*-algebraic crossed products.

The equivariant KK-theory ([69,72]) sets a perfect basis for the computation of the K-groups of group C*-algebras and crossed products. Kasparov defined an abelian group $KK_G(A,B)$, where G is a locally compact group and A, B are C*-algebras acted upon by G and showed that the Kasparov product is still defined in this context. Constructions such as the restriction to a subgroup, induction from a subgroup, and a homomorphism $j_G:KK_G(A,B)\to KK(A\times G,B\times G)$ are performed and shown to be compatible with the Kasparov product. When G is a connected Lie group, fundamental elements α_G, β_G and γ_G are constructed in order to relate the equivariant KK-theory of G to that of a maximal compact subgroup.

We begin by some explanations concerning the constructions of group C*-algebras and crossed products.

9.1. Crossed products: Let A be a C*-algebra and G a locally compact group. An **action α of G in A** is a (pointwise) continuous group homomorphism $g\to\alpha_g$ from G to the group of *-automorphisms of A. The continuity of α means that for every $x\in A$ the map $g\to\alpha_g(x)$ is continuous. A C*-algebra endowed with an action of G is called a **G-algebra.**

Let A be a G-algebra. The vector space $C_c(G;A)$ of continuous functions with compact support from G to A, is naturally a *-algebra: it is convenient to represent an element $g\to a_g\in C_c(G;A)$ by the formal integral $\int a_g u_g\, dg$, where dg is the left Haar measure of G, u_g is a formal letter satisfying $u_g u_{g'}=u_{gg'}$, $u_g^*=(u_g)^{-1}=u_{g^{-1}}$ and $u_g\, a\, u_g^*=\alpha_g(a)$ $(a\in A)$. The product and the adjoint in $C_c(G;A)$ are then given by:

$(\int a_g u_g dg)(\int b_g u_g dg)=\int c_g u_g dg$ where $c_g=\int a_h\alpha_h(b_{h^{-1}g})dh$ $(\in A)$.

$(\int a_g u_g dg)^*=\int u_g^* a_g^* dg=\int b_g u_g dg$ with $b_g=\Delta(g)\alpha_g(a_{g^{-1}})$ where $\Delta(g)$ is the modular function of G (recall that $dg^{-1}=\Delta(g)dg$).

There are two natural ways of completing $C_c(G;A)$. The corresponding completions are the

full crossed product noted $A\times_{\alpha}G$ and the **reduced crossed product** noted $A\times_{\alpha,r}G$. Both are C*–algebras. When G is **amenable** these two crossed products coincide.

When the algebra A is $\mathbb{C}$ (and the action of course trivial) the (full or reduced) crossed product is called the **(full or reduced) group C*-algebra of G** and is noted $C^*(G)$ or $C_r^*(G)$.

When G is abelian $C^*(G)$ ($=C_r^*(G)$ as abelian groups are amenable) is isomorphic via the Fourier transform to the algebra $C_0(\hat{G})$ of continuous functions vanishing at ∞ on the dual group $\hat{G}$.

When G acts by automorphisms on the group H , it acts on the C*–algebra $C^*(H)$ and the crossed product $C^*(H)\times G$ is the group C*–algebra of the semidirect product of G by H .

When G acts on the locally compact space X it acts on the abelian C*–algebra $C_0(X)$ and, if the action on X is **free and proper** , the quotient X/G is a "nice" locally compact space and the crossed-product $C_0(X)\times G$ is Morita equivalent to the abelian algebra $C_0(X/G)$.

We now come to the definition of the equivariant KK–theory.

Let G be a locally compact group and let A and B be G–algebras. Denote both actions of G by the same letter α .

9.2. Definition: *An* ***equivariant*** **A,B**-***bimodule*** *is an A,B bimodule* $(\mathcal{E},\pi)$ *(definition 5.1) endowed with a (pointwise) continuous action of G , still noted* α *such that for all* $a\in A$, $b\in B$, $\xi,\zeta\in\mathcal{E}$ *and* $g\in G$ *we have:* $\partial(\alpha_g(\xi))=\partial(\xi)$, $\alpha_g(\xi b)=\alpha_g(\xi)\alpha_g(b)$, $\alpha_g(\pi(a)\xi)=\pi(\alpha_g(a))\alpha_g(\xi)$, $\alpha_g(<\xi,\zeta>)=<\alpha_g(\xi),\alpha_g(\zeta)>$.

The continuity of the action means that for all $\xi\in\mathcal{E}$ the map $g\to\alpha_g(\xi)$ is continuous. Then G acts in $\mathcal{L}(\mathcal{E})$ and in $\mathcal{K}(\mathcal{E})$ in the following way: for $T\in\mathcal{L}(\mathcal{E})$ and $\xi\in\mathcal{E}$, $\alpha_g(T)\xi=\alpha_g(T\alpha_{g^{-1}}(\xi))$. However although the action of G in $\mathcal{K}(\mathcal{E})$ is continuous, its action in $\mathcal{L}(\mathcal{E})$ is in general not continuous (it is *–strong pointwise continuous but not norm pointwise continuous).

This leads to the following definition:

9.3. Definition: *The element* $T\in\mathcal{L}(\mathcal{E})$ *is said to be G–continuous if the map* $g\to\alpha_g(T)$ *is norm continuous.*

In particular if $T\in\mathcal{K}(\mathcal{E})$ or $T\in\pi(A)$ it is G–continuous.

9.4. Definition: *Let* $\mathbf{E}_G(A,B)$ *denote the set of triples* $(\mathcal{E},\pi,F)$ *where* $(\mathcal{E},\pi)$ *is an equivariant* A,B-*bimodule and* $F\in\mathcal{L}(\mathcal{E})^{(1)}$ *satisfies:*

a) $(\mathcal{E},\pi,F)\in\mathbf{E}(A,B)$

b) F *is G-continuous and* $\alpha_g(F)-F\in\mathcal{K}(\mathcal{E})$ *for all* $g\in G$.

A homotopy is defined as in §5.

9.5. Definition: *The group of homotopy classes of elements of* $\mathbf{E}_G(A,B)$ *is noted* $KK_G(A,B)$.

9.6. *Examples:* a) When G is compact we may average F over G without changing the class of $(\mathcal{E},\pi,F)$ in $KK_G(A,B)$. We may therefore change definition 9.4.b asking that $\alpha_g(F)=F$ for all $g\in G$, without modifying the group KK_G. Using this we see that $KK_G(\mathbb{C},\mathbb{C})$ is the representation ring of G. Also in this case $KK_G(\mathbb{C},B)\cong KK(\mathbb{C},B\times_\alpha G)$ (cf. [61]).

b) If G is discrete $KK_G(\mathbb{C},\mathbb{C})=KK(C^*(G),\mathbb{C})$. More generally $KK_G(A,\mathbb{C})=KK(A\times_\alpha G,\mathbb{C})$.

Kasparov proved that the Kasparov product still exists in the equivariant setting. In particular all the periodicity results discussed in §6 generalize to the equivariant case.

We now discuss some constructions which change the group G.

9.7. Definition: the restriction homomorphism. *Let* $r:G\to H$ *be a continuous group homomorphism. By composition, every H-algebra (resp. equivariant bimodule) becomes a G-algebra (resp. G-equivariant bimodule). In this way we obtain a homomorphism* $r^*:KK_H(A,B)\to KK_G(A,B)$ *called the restriction homomorphism for every pair* A,B *of H-algebras.*

In particular letting G to be the group with only one element we get the obvious forgetful map $KK_H(A,B)\to KK(A,B)$.

Kasparov defines also an induction homomorphism: if H is a closed subgroup of G and A is an H-algebra, there is a naturally associated G-algebra noted $C(G\times_H A)$. The induction homomorphism ind_H^G maps $KK_H(A,B)$ to $KK_G(C(G\times_H A),C(G\times_H B))$.

Both restriction and induction have a good behaviour with respect to the Kasparov product (cf. [69,72]).

We now come to the fundamental construction j_G .

To every equivariant A,B bimodule $(\mathcal{E},\pi)$ we associate an $A\times_\alpha G$, $B\times_\alpha G$ bimodule $(\mathcal{E}\times_\alpha G,\pi_\alpha)$: let B act on $B\times_\alpha G$ by left multiplication $(b\int b_g u_g dg = \int bb_g u_g dg)$. Set then $\mathcal{E}\times_\alpha G=\mathcal{E}\otimes_B(B\times_\alpha G)$ (cf. 4.7.b). For $g\in G$ let $U_g\in\mathcal{L}(\mathcal{E}\times_\alpha G)$ be given by the equality:

$$U_g(\xi\otimes(\int b_h u_h dh)) = \alpha_g(\xi)\otimes u_g\int b_h u_h dg = \alpha_g(\xi)\otimes\int\alpha_g(b_{g^{-1}h})u_h dh .$$

We have $U_gU_{g'}=U_{gg'}$, $(U_g)^{-1}=U_{g^{-1}}=(U_g)^*$ and $U_g(\pi(a)\otimes_B 1)U_g^*=\pi(\alpha_g(a))\otimes_B 1$. We get the *-representation π_α of $A\times_\alpha G$ in $\mathcal{E}\times_\alpha G$ by setting $\pi_\alpha(\int a_g u_g dg)=\int(\pi(a_g)\otimes_B 1)U_g dg \in\mathcal{L}(\mathcal{E}\times_\alpha G)$.

Kasparov proved ([69,72]):

9.8. Theorem: *a) If* $(\mathcal{E},\pi,F)\in\mathbf{E}_G(A,B)$ *then* $(\mathcal{E}\times_\alpha G,\pi_\alpha,F\otimes_B 1)\in\mathbf{E}(A\times_\alpha G,B\times_\alpha G)$.

b) The map $(\mathcal{E},\pi,F)\to(\mathcal{E}\times_\alpha G,\pi_\alpha,F\otimes_B 1)$ *defines a group homomorphism* $j_G:KK_G(A,B)\to KK(A\times_\alpha G,B\times_\alpha G)$.

c) This map behaves nicely with respect to the Kasparov product: $j_G(x\otimes_B y)=j_G(x)\otimes_{B\times_\alpha G}j_G(y)$ *and* $j_G(1_A)=1_{A\times_\alpha G}$.

Let us now consider the case of a connected Lie group G . Let K be a maximal compact subgroup of G . We assume to simplify the notation that G/K has even dimension and that it may be equipped with a G-invariant $\mathrm{Spin}^{\mathbb{C}}$ structure. Using the $\mathrm{Spin}^{\mathbb{C}}$ structure we may form the G-invariant Dirac operator D . It is a self-adjoint elliptic differential operator of order 1 and of degree 1 (it acts on the space of sections of the $\mathbb{Z}/2\mathbb{Z}$-graded bundle S of Spinors). Let F be the pseudodifferential elliptic operator of order zero: $F=D(1+D^2)^{-1/2}$ acting on the graded Hilbert space $H=L^2(G/K;S)$ of L^2 sections of the bundle S . Let $C_0(G/K)$ act on H by multiplication $f\to M(f)$. The triple (H,M,F) defines an element of $\mathbf{E}(C_0(G/K),\mathbb{C})$. As G acts on $C_0(G/K)$ and on H by left translation and as F is G-invariant we get an element of $\mathbf{E}_G(C_0(G/K),\mathbb{C})$.

9.9. Definition: *The fundamental element* $\alpha_G\in KK_G(C_0(G/K),\mathbb{C})$ *is the class of the triple* (H,M,F) .

9.10. Theorem: *([69,72]) There exists a unique element* $\beta_G \in KK_G(\mathbb{C}, C_0(G/K))$ *such that* $\alpha_G \otimes_{\mathbb{C}} \beta_G = 1_{C_0(G/K)}$.

The product $\beta_G \otimes_{C_0(G/K)} \alpha_G$ *is an idempotent* $\gamma_G \in KK_G(\mathbb{C},\mathbb{C})$. *If* G *is amenable* $\gamma_G = 1$. *If* $r: H \to G$ *is a proper homomorphism then* $r^*(\gamma_G) = \gamma_H$.

Let $i: K \to G$ *be the inclusion. Then for every pair* A,B *of G-algebras the homomorphism* $i^*: KK_G(A,B) \to KK_K(A,B)$ *is surjective. Its kernel is* $\ker(\gamma_G) = \{x \in KK_G(A,B) \,/\, \gamma_G \otimes_{\mathbb{C}} x = 0\}$. *In particular when* G *is amenable* i^* *is an isomorphism.*

(Note that $\gamma_G \otimes_{\mathbb{C}} x = x \otimes_{\mathbb{C}} \gamma_G$ – cf. proposition 6.4).

The element β_G is constructed in the following way: assume first that G/K is equipped with a metric of non positive sectional curvature. Let $\mathcal{E}$ be the Hilbert $C_0(G/K)$-module $\mathcal{E} = C_0(G/K;S^*)$ (the space of continuous sections vanishing at ∞ of the bundle S^* dual to S). Let $F \in \mathcal{L}(\mathcal{E})$ be defined by $(F\xi)(x) = t(x,x_0).\xi(x)$. Here $x_0 \in G/K$ is a base point; $t(x,x_0) \in T_x(G/K)$ is the unit vector along the geodesic pointing towards x_0 when the distance from x to x_0 is bigger than 1 and is extended in any continuous way for x close to x_0; the product $t(x,x_0).\xi(x)$ is the Clifford product.

As $(F^2\xi)(x) = \|t(x,x_0)\|^2 \xi(x)$, $(F^2 - 1) \in \mathcal{K}(\mathcal{E})$. Thus $(\mathcal{E},F) \in \mathbf{E}(\mathbb{C}, C_0(G/K))$. Moreover for $g \in G$, $(\alpha_g(F)\xi)(x) = t(x, g(x_0)).\xi(x)$. Now, because of the asumption on the sectional curvature for every pair x_0, x_1 of points in G/K the function $x \to \|t(x,x_0) - t(x,x_1)\|$ converges to 0 when $x \to \infty$; therefore $\alpha_g(F) - F \in \mathcal{K}(\mathcal{E})$. This convergence being uniform on $x_1 \in C$ for any compact $C \subset G/K$, we deduce that F is G-continuous and hence $(\mathcal{E},F) \in \mathbf{E}_G(\mathbb{C}, C_0(G/K))$. Its class is β_G.

The method discussed above contructs β_G for G semisimple or $G = \mathbb{R}^n$. The general case is then handled inductively (cf.[69,72]).

Let now A be a G-algebra. Then $(A \otimes C_0(G/K)) \times G$ is Morita equivalent to $A \times_\alpha K$ (where G acts on $A \otimes C_0(G/K)$ via the diagonnal action). We deduce:

9.11. Corollary: *If* $\gamma_G = 1$ *the algebras* $A \times_\alpha G$ *and* $A \times_\alpha K$ *are K-equivalent.*

Let Γ be a closed subgroup of G and let A be a Γ-algebra. Then the K-theory of $(A\otimes C_0(G/K))\times\Gamma$ can be considered as being computable in terms of the K-theory of A . In particular, if Γ is discrete and torsion free $(A\otimes C_0(G/K))\times\Gamma$ is Morita equivalent to the algebra of continuous sections of a (flat) bundle over $\Gamma\backslash G/K$ with fiber A . Its K-theory is computable (in principle) using a cell decomposition of $\Gamma\backslash G/K$. This explains the interest of the following result:

9.12. Corollary: *If* $\gamma_G=1$ *then* $A\times_\alpha\Gamma$ *and* $(A\otimes C_0(G/K))\times\Gamma$ *are K-equivalent.*

At the K-theory level, corollary 9.11 was established by A.Connes in [26] for G simply connected and solvable; corollary 9.12 for $\Gamma=\mathbb{Z}$ is Pimsner and Voiculescu's breakthrough cf.[84]. See [85,32] for the case of free groups. See also [46].

It is obviously of great interest to decide for which groups G , $\gamma_G=1$. Unfortunately this is not always the case:

9.13. Proposition: *(cf. [33,69]) If* G *has a closed non compact subgroup which satisfies Kazhdan's property* T *then* $\gamma_G\neq 1$.

In the language of C*-algebras, the group Γ has Kazhdan's property T ([73,38]) if there is a projection $p\in C^*(\Gamma)$ such that for every unitary representation π of Γ , $\pi(p)$ is the projection on the space of fixed vectors $\{\xi\in\mathcal{H}_\pi \,/\, \pi(g)\xi=\xi\ ,\ \forall g\in\Gamma\}$. The class [p] of p in $K_0(C^*(\Gamma))$ is not 0. One proves that $[p]\otimes j_\Gamma(r^*(\gamma_G))=0$. Hence $\gamma_G\neq 1$.

Recall that all simple Lie groups with real rank $\geqslant 2$ have Kazhdan's property T . Actually the only simple Lie groups which do not have property T are (locally isomorphic to) SO(n,1) and SU(n,1) , $n\geqslant 2$. Kasparov proved ([70]):

9.14. Theorem: *For* $G=SO(n,1)$, $\gamma_G=1$.

In order to decide for which Lie groups $\gamma_G=1$ we are left with the following (open) question.

9.15. Problem: *Is* $\gamma_{SU(n,1)}$ *equal to* 1 *(n⩾2)?*

There is some evidence for an affirmative answer coming from Fox and Haskell ([49]).

There are also some other interesting related questions. Most important is the "Connes–Kasparov–Rosenberg" conjecture that we now explain.

The proof of proposition 9.13 shows that if G has a closed non compact subgroup with property T, there exists a G–algebra A such that the element $j_G(\tau_A(\gamma_G)) \in KK(A\times_\alpha G, A\times_\alpha G)$ does not act as the identity in $K_0(A\times_\alpha G)$ (by Kasparov product). But it says nothing about the reduced crossed products.

9.16. Conjecture: *(Connes, Kasparov, Rosenberg)([71]). For every G–algebra* A *the element* $j_{G,r}(\tau_A(\gamma_G)) \in KK(A\times_{\alpha,r}G, A\times_{\alpha,r}G)$ *defines, by Kasparov product, the identity endomorphism of* $K_0(A\times_{\alpha,r}G)$.

If this conjecture is true the Dirac induction– i.e. the map from $K_0(A\times_\alpha K)$ to $K_0(A\times_{\alpha,r}G)$ corresponding to α_G is an isomorphism (see corollary 9.11).

One might hope to prove the equality $j_{G,r}(\tau_A(\gamma_G)) = 1_{A\times_{\alpha,r}G}$. Unfortunately this can be wrong: if Γ is a closed discrete subgroup with finite covolume in $G=Sp(n,1)$, $n\geqslant 2$ then for $A=C_0(G/\Gamma)$ this equality fails (cf. [106]). Therefore if true this conjecture cannot be proved only by means of KK–theory.

Finally, let us restrict to the case $A=\mathbb{C}$. Wassermann proved:

9.17. Theorem: *(cf.[115] see also [113]) For every reductive Lie group* G , $j_{G,r}(\gamma_G) = 1_{C_r^*(G)}$. *In particular the Dirac induction is an isomorphism between* $K_*(C^*(K))$ *and* $K_*(C^*(G))$.

10. C*-algebraic extensions

As many C*-algebras are formed out of simpler blocks through extensions (e.g. type I C*-algebras) it is important to give invariants that classify the different possible extensions of a pair of given C*-algebras.

The developement of the theory is quite recent. The first major achievement in this direction is Brown-Douglas-Fillmore's theory in which extensions of the type $0\to\mathcal{K}\to D\to C(X)\to 0$ are classified ([21]). These extensions are shown to form a group Ext(X) which is recognized to be the K-homology group $K^1(C(X))=K_1(X)$. Thus Ext is a beautiful answer to Atiyah's problem of finding an analytic definition of K-homology.

After this starting point, the subject became a center of interest of many C*-algebraists, evolved very rapidly and in many different directions (see the surveys in [92,18]). In this section we will mainly be concerned with Kasparov's fundamental contribution.

Let A and B be two C*-algebras. An extension of A by B is a short exact sequence of C*-algebras $0\to B\to D\to A\to 0$. When $A=C(X)$ and $B=C_0(Y)$ extensions of A by B correspond to the different ways of glueing the spectra X and Y through a continuous map from the boundary of Y (for some compactification) to X.

We will here be interested to the non commutative case. When B has a unit D is necessarily isomorphic to $A\oplus B$. Because of that and as we want to be able to add extensions we will assume that B is stable (i.e. $B\cong B\otimes\mathcal{K}$), or else consider extensions of A by $B\otimes\mathcal{K}$.

The addition of two extensions is defined in the following way:

10.1. Definition: *Given two extensions* $0\to B\otimes\mathcal{K}\xrightarrow{i_j} D_j\xrightarrow{p_j} A\to 0$ $(j=0,1)$ *of* A *by* $B\otimes\mathcal{K}$ *their sum is the extension* $0\to B\otimes\mathcal{K}\xrightarrow{i} D\xrightarrow{p} A\to 0$ *where* D *is the C*-algebra of 2 by 2 matrices*

$$D=\left\{\begin{bmatrix} x_1 & z \\ y & x_2\end{bmatrix},\ y,z\in B\otimes\mathcal{K},\ x_j\in D_j\ (j=1,2),\ p_1(x_1)=p_2(x_2)\right\}$$

(with the obvious rules for the product and the adjoint), $p:D\to A$ *is defined by*

$p\begin{bmatrix} x_1 & z \\ y & x_2 \end{bmatrix}=p_1(x_1)=p_2(x_2)$ *and* $i:B\otimes\mathcal{K}\to D$ *is the composition*

$B\otimes\mathcal{K} \xrightarrow{\sim} M_2(B\otimes\mathcal{K}) \xrightarrow{\ell} D$ *, where* ℓ *is given by* $\ell(\begin{bmatrix} x_1 & z \\ y & x_2 \end{bmatrix})=\begin{bmatrix} i_1(x_1) & z \\ y & i_2(x_2) \end{bmatrix}$.

The addition of extensions is well defined commutative and associative only up to unitary equivalence:

10.2. Definition: *The extensions* $0\to B\otimes\mathcal{K}\to D_1\to A\to 0$ *and* $0\to B\otimes\mathcal{K}\to D_2\to A\to 0$ *are said to be* ***unitarily equivalent*** *if there exists a unitary* u *in the multiplier algebra* [(6)] $M(B\otimes\mathcal{K})$ *and an isomorphism* $h:D_1\to D_2$ *such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 0 & \to & B\otimes\mathcal{K} & \to & D_1 & \to & A & \to & 0 \\ & & Ad(u)\downarrow & & h\downarrow & & \| & & \\ 0 & \to & B\otimes\mathcal{K} & \to & D_2 & \to & A & \to & 0 \end{array}$$

where $Ad(u)$ *is the inner automorphism defined by* u : $Ad(u)(x)=uxu^*$ $x\in B\otimes\mathcal{K}$.

10.3. Definition: *An extension* $0\to B\otimes\mathcal{K}\to D\to A\to 0$ *is said to be* ***trivial*** *if it splits i.e. if there is a *-homomorphism* $s:A\to D$ *which is a cross-section.*

We may now define Kasparov's semigroup Ext(A,B).

10.4. Definition: Ext(A,B) *is the semigroup of extensions of* A *by* $B\otimes\mathcal{K}$ *devided by the equivalence relation:* $x\sim x'$ *if there exist trivial extensions* σ *and* σ' *such that the extensions* $x+\sigma$ *and* $x'+\sigma'$ *are unitarily equivalent.*

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(6) The multiplier algebra M(J) of a C*-algebra J is the space of pairs (L,R) of bounded linear maps from J into J such that $\forall x,y\in J$, $L(x)y=L(xy)$, $xR(y)=R(xy)$, $xL(y)=R(x)y$. For $T=(L,R)\in M(J)$ and $x\in J$ we set $Tx=L(x)$, $xT=R(x)$, $T^*=(R^*,L^*)$, where $L^*(x)=L(x^*)^*$ and $R^*(x)=R(x^*)^*$. Finally the product in M(J) is given by: $(L,R)(L',R')=(L\circ L',R'\circ R)$. Kasparov proved that for every Hilbert C*-module $\mathcal{E}$, $M(\mathcal{K}(\mathcal{E}))=\mathcal{L}(\mathcal{E})$. In particular $M(B\otimes\mathcal{K})=\mathcal{L}(\mathcal{H}_B)$.

Remark: The fact that we have to mod out the trivial extensions can be explained in the following way: let $x: 0\to B\otimes\mathcal{K}\to D\to A\to 0$ be an extension and assume that there exists a splitting $s:A\to D$. Define D_∞ to be the *-subalgebra of $D\otimes\mathcal{K}$ generated by $B\otimes\mathcal{K}\otimes\mathcal{K}$ and $s(A)\otimes 1$. We have an extension $x_\infty: 0\to B\otimes\mathcal{K}\otimes\mathcal{K}\to D_\infty\to A\to 0$ and $x+x_\infty$ is unitarily equivalent to x_∞. So if we wish to end up with a group we have to set $x=0$.

Let $Ext(A,B)^{-1}$ be the group of invertible elements of $Ext(A,B)$, i.e. the group of classes of extensions x such that there exists an extension y such that $x+y$ is trivial.

10.5. *Remark:* Let $\tau_1\in Ext(A,B)^{-1}$ be the class of $0\to B\otimes\mathcal{K}\to D_1\to A\to 0$ and let $\tau_2: 0\to B\otimes\mathcal{K}\to D_2\to A\to 0$ be such that $\tau_1+\tau_2$ is trivial. Define $q:D\to D_1$ by

$$q\begin{bmatrix} x_1 & z \\ y & x_2 \end{bmatrix}=x_1$$

(where D is given by definition 10.1) and let $s:A\to D$ be a cross-section. Then q and hence $q\circ s$ is completely positive; therefore the extension τ_1 is semi-split (cf. 8.1). The converse is also true : generalising a theorem of Stinespring ([107]), Kasparov showed that if an extension is semi-split, its class sits in $Ext(A,B)^{-1}$ ([67]). A theorem of Choi-Effros ([24]) says that if A is a separable and nuclear C*-algebra all extensions of A by any algebra B are semi-split. Therefore if A is separable and nuclear $Ext(A,B)$ is a group.

Let us come back to the general case. Let $0\to B\otimes\mathcal{K}\to D_1\to A\to 0$ and $0\to B\otimes\mathcal{K}\to D_2\to A\to 0$ be two extensions whose sum $0\to M_2(B\otimes\mathcal{K})\to D\to A\to 0$ is trivial. Recall that $B\otimes\mathcal{K}=\mathcal{K}(\mathcal{H}_B)$; also $M_2(B\otimes\mathcal{K})=\mathcal{K}(\mathcal{H}_B\oplus\mathcal{H}_B)$. Let $P\in\mathcal{L}(\mathcal{H}_B\oplus\mathcal{H}_B)$ be the projection onto the first summand. The algebra D acts on $\mathcal{H}_B\oplus\mathcal{H}_B$: for $x\in D$, $\xi\in\mathcal{H}_B\oplus\mathcal{H}_B$ and $y\in\mathcal{K}(\mathcal{H}_B\oplus\mathcal{H}_B)$, we have $\pi(x)(y\xi)=xy.\xi$ (note that $xy\in\mathcal{K}(\mathcal{H}_B\oplus\mathcal{H}_B)$ as $\mathcal{K}(\mathcal{H}_B\oplus\mathcal{H}_B)$ is a two sided ideal in D). As the non diagonal terms of the elements of D sit in $B\otimes\mathcal{K}=\mathcal{K}(\mathcal{H}_B)$ we see that for all $x\in D$, the commutator $[\pi(x),P]$ is in $\mathcal{K}(\mathcal{H}_B\oplus\mathcal{H}_B)$.

Let $s:A\to D$ be a cross-section and put $\pi'=\pi\circ s:A\to\mathcal{L}(\mathcal{H}_B\oplus\mathcal{H}_B)$ and $F=2P-1$. Then the triple $(\mathcal{H}_B\oplus\mathcal{H}_B,\pi',F)$ is an element of $\mathbf{E}_1(A,B)$.

Kasparov proved:

10.6. Theorem: *([68]) When A is separable, the groups $\mathrm{Ext}(A,B)^{-1}$ and $KK_1(A,B)$ are isomorphic. Consequently $\mathrm{Ext}(A,B)^{-1}$ is homotopy invariant in both entries and satisfies all the periodicity results coming from the Kasparov product (e.g. Bott periodicity).*

The isomorphism between $\mathrm{Ext}(A,B)^{-1}$ and $KK_1(A,B)$ maps the class of the extension $0 \to B \otimes \mathcal{K} \to D_1 \to A \to 0$ to the class of the triple $(\mathcal{H}_B \oplus \mathcal{H}_B, \pi', F)$ constructed above. One has to prove that the equivalence relations of $\mathrm{Ext}(A,B)^{-1}$ and $KK_1(A,B)$ correspond to each other via this map. This is a quite difficult result based on the technique of the Kasparov product. The key lemma is the following:

10.7. Lemma: *Let $\check{K}K(A,B)$ be the group of classes of elements in $\mathbb{E}(A,B)$ for the equivalence relation: $x_1 \sim x_2$ if there exist degenerate elements $y_1, y_2 \in \mathbb{D}(A,B)$ such that $x_1 + y_1$ and $x_2 + y_2$ are* ***operator homotopic*** *(cf. 5.5.b). Then the Kasparov product may be defined at the level of $\check{K}K$.*

In other terms there is a Kasparov product $\check{K}K(A,D) \times \check{K}K(D,B) \to \check{K}K(A,B)$ which is compatible with the product at the level of KK (note that KK is a quotient of $\check{K}K$). One deduces from this lemma that $\check{K}K = KK$. Indeed one shows that $\check{K}K(C[0,1],\mathbb{C}) = \mathbb{Z} = KK(C[0,1],\mathbb{C})$. Hence $C[0,1]$ and $\mathbb{C}$ are $\check{K}K$-equivalent. Hence $\check{K}K(A,B[0,1]) = \check{K}K(A,B)$. But a homotopy defines an element of $\check{K}K(A,B[0,1])$. From these considerations the equality $KK = \check{K}K$ follows easily. But it is also rather easy to establish the equality $\mathrm{Ext}(A,B)^{-1} = \check{K}K_1(A,B)$.

***10.8.** Remark:* In fact one can prove along these lines that all "reasonable" equivalence relations on $\mathbb{E}(A,B)$ are the same. The weakest being homotopy and the strongest is called cobordism in [37]. In that sense the answer to Atiyah's question on finding the right equivalence relation on Ell is that all "reasonable" ones work.

Let us come back to the problem of classification of extensions. We already obtained the classification of semi-split extensions up to addition of trivial ones. But we can do more.

In the case $B = \mathbb{C}$ Voiculescu proved ([114]):

10.9. Theorem: *Let $0 \to \mathcal{K} \to D_1 \to A \to 0$ and $0 \to \mathcal{K} \to D_2 \to A \to 0$ be two extensions which* *define the same element of* Ext(A)=Ext(A,ℂ) *. If* D_1 *and* D_2 *are not unital and if* $\mathcal{K}$ *is an essential ideal in* D_1 *and in* D_2 *, these extensions are unitarily equivalent and the algebras* D_1 *and* D_2 *are isomorphic.*

(An ideal J is said to be essential in the algebra D if for every $x \in D$, $x \neq 0$, there exists $y \in J$ such that $xy \neq 0$).

This theorem says that if x is a nonunital essential extension and σ is trivial then $x+\sigma$ is unitarily equivalent to x. The problem of the unit will be discussed below (10.11).

Kasparov generalized Voiculescu's theorem in the following way:

10.10. Theorem: *([67]) Assume that* A *or* B *is nuclear. Then there exists a subsemigroup of extensions of* A *by* $B \otimes \mathcal{K}$ *whose elements are called* ***absorbing extensions,*** *such that in every class of* Ext(A,B) *there is exactly one absorbing extension (up to unitary equivalence).*

More precisely Kasparov proved the following: let $0 \to \mathcal{K} \to D_0 \to A \to 0$ be a trivial nonunital essential extension. Let $s: A \to D_0$ be a splitting and let $D \subset \tilde{B} \otimes D_0$ be the algebra generated by $B \otimes \mathcal{K}$ and $s(A) \otimes 1$. It defines a trivial extension σ: $0 \to B \otimes \mathcal{K} \to D \to A \to 0$. Then for every trivial extension σ', $\sigma + \sigma'$ and σ are unitarily equivalent. The absorbing extensions are the extensions of the form $x+\sigma$.

In the light of theorem 10.10, Voiculescu's theorem (10.9) says that, an extension of A by $\mathcal{K}$ is absorbing if and only if it is non unital and essential. The problem of characterising intinsiquely absorbing extensions in general seems difficult. Pimsner-Popa-Voiculescu ([81,82]) gave the answer in the case B=C(Y) where Y is a finite dimensional compact space.

10.11. *Remark:* When A is unital we may define a semigroup $\mathrm{Ext}_s(A,B)$ to classify unital extensions $0 \to B \otimes \mathcal{K} \to D \to A \to 0$ (i.e. with D unital): a unital extension is unitarily trivial if it admits a unital cross-section. Then $\mathrm{Ext}_s(A,B)$ is the semigroup of unital extensions divided by the equivalence relation $x \sim_s y$ if there exist unitarily trivial extensions σ and τ such that $x+\sigma$ and $y+\tau$ are unitarily equivalent.

One computes $\mathrm{Ext}_s(A,B)^{-1}$ thanks to a six term exact sequence (cf. [105]):

$$\begin{array}{ccccc} Ext_s(A,B)^{-1} & \rightarrow & Ext(A,B)^{-1} & \rightarrow & K_1(B) \\ \uparrow & & & & \downarrow \\ K_0(B) & \leftarrow & Ext(A,B(\mathbb{R}))^{-1} & \leftarrow & Ext_s(A,B(\mathbb{R}))^{-1} \end{array}$$

We also have $Ext_s(A,B)^{-1}=KK(C_i,B)$ where C_i is the mapping cone (cf.8.4) of the *-homomorphism $i:\mathbb{C}\rightarrow A$ given by $i(\lambda)=\lambda 1$.

Moreover $Ext(A,B)=Ext_s(\tilde{A},B)$ for every pair A , B of C*-algebras (recall that $\tilde{A}$ is obtained from B by adjoining a unit).

Voiculescu's theorem in fact says:

Two essential unital extensions which define the same element of $Ext_s(A)=Ext_s(A,\mathbb{C})$ *are unitarily equivalent*

Also Kasparov's generalization of Voiculescu's theorem holds for $Ext_s(A,B)$.

11. Index theory

Kasparov's theory is an ideal setting for the Atiyah-Singer index theorems and many generalizations.

The scheme of all these theorems is the following:

An elliptic pseudodifferential operator P defines an element of some Kasparov group. We want to describe this element and the map between K-groups that it defines in terms of the K-theory class of its symbol.

In this section we will present some situations where such theorems are obtained. It would be too long to give sketches of the proofs or even precise statements of these theorems. We will just indicate the Kasparov elements defined by an elliptic element and its symbol.

11.1. Atiyah-Singer index theorem (cf. [10]): Let M be a smooth compact manifold. A pseudodifferential operator of order 0 defines a bounded linear operator $P:L^2(M;E^{(0)})\rightarrow L^2(M;E^{(1)})$ where $L^2(M;E^{(i)})$ is the space of L^2 sections of the Hermitian vector bundle $E^{(i)}$ over M . The operator P has a symbol which is a bounded vector bundle homomorphism $\sigma_P:p^*E^{(0)}\rightarrow p^*E^{(1)}$ where $p:T^*M\rightarrow M$ is the projection.

Recall that P is said to be elliptic if there exists a bounded continuous vector bundle homomorphism $\sigma':p^*E^{(1)}\rightarrow p^*E^{(0)}$ such that $1-\sigma_P\sigma'$ and $1-\sigma'\sigma_P$ have compact supports.

When P is elliptic it defines an element $[P]\in KK(C(M),\mathbb{C})$ (cf. §3). Also the space $C_0(T^*M;p^*E)$ of continuous sections vanishing at ∞ of the Hermitian $\mathbb{Z}/2\mathbb{Z}$-graded bundle p^*E is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert $C_0(T^*M)$- module. The element $F\in\mathcal{L}(C_0(T^*M;p^*E)$ defined by $F(\xi^{(0)},\xi^{(1)})=(\sigma'\xi^{(1)},\sigma_P\xi^{(0)})$ $(\xi^{(i)}\in C_0(T^*M;p^*E^{(i)}))$ satisfies $F^2-1\in\mathcal{K}(C_0(T^*M;p^*E))$. Hence $(C_0(T^*M;p^*E),F)$ defines an element $[\sigma_P]\in KK(\mathbb{C},C_0(T^*M))=K_0(C_0(T^*M))=K^0(T^*M)$.

The index of P i.e. the homomorphism from $K_0(C(M))$ to $\mathbb{Z}=K_0(\mathbb{C})$ it defines is computed in terms of $[\sigma_P]$. Actually the correspondence between $[P]$ and $[\sigma_P]$ is an isomorphism (the Poincaré duality- cf. §6.9) of the K-homology of M with the K-theory of T^*M.

11.2. The Atiyah-Singer index theorem for families (cf.[12]): Let M be a smooth compact manifold and let $p:M\to B$ be a smooth fibration with fiber F. We are interested here with families of pseudodifferential operators $(P_y)_{y\in B}$ of order 0, defining bounded linear operators $P_y:L^2(F_y;E^{(0)})\to L^2(F_y;E^{(1)})$ where $F_y=p^{-1}(\{y\})$ is the fiber over y and E is a $\mathbb{Z}/2\mathbb{Z}$-graded Hermitian vector bundle over M. The continuity $y\to P_y$ is expressed by the fact $Pf\in C^\infty(M;E^{(1)})$ for every $f\in C^\infty(M;E^{(0)})$, where $(Pf)(x)=(P_yf_y)(x)$ with $y=p(x)$ and f_y the restriction of f to F_y. The collection of the $L^2(F_y;E)_{y\in B}$ is naturally endowed with a structure of a continuous field of Hilbert spaces over B, which means exactly that it admits a space of continuous sections and that this space is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert $C(B)$-module $\mathcal{E}$. By the continuity of $y\to P_y$ we get an operator $P\in\mathcal{L}(\mathcal{E}^{(0)},\mathcal{E}^{(1)})$.

The symbol of the family P is the collection of the symbols of the $(P_y)_{y\in B}$. Therefore σ_P is a smooth and bounded bundle homomorphism $\sigma_P:q^*E^{(0)}\to q^*E^{(1)}$ where $q:T_F^*M\to M$ is the projection, T_F^*M being the dual bundle to the subbundle T_FM of the tangent bundle TM of vectors tangent to the fibers of the fibration. The family P is said to be elliptic if there exists a bounded vector bundle homomorphism $\sigma':q^*E^{(1)}\to q^*E^{(0)}$ such that $\sigma_P\sigma'-1$ and $\sigma'\sigma_P-1$ have compact support in T_F^*M.

An elliptic pseudodifferential family P of order 0 defines an element $[P]\in KK(C(M),C(B))$. Also the symbol of P defines an element $[\sigma_P]\in K^0(T_F^*M)$ of the K-theory of T_F^*M.

The index of P is again computed in terms of the symbol class. Here the correspondence $[P]\leftrightarrow[\sigma_P]$ does not extend to an isomorphism in general. But $[P]$ is still obtained from $[\sigma_P]$ by a Kasparov product: there exists an element $\mathcal{P}\in KK(C_0(T_F^*M\times M),C(B))$ (which can be thought of as the family, indexed by B, of the Kasparov elements defining the Poincaré duality on F_y) such that $[P]=[\sigma_P]\otimes_{T_F^*M}\mathcal{P}$.

11.3. The longitudinal index theorem for foliations (cf. [25,27,31]): Let M be a compact manifold and let F be a smooth foliation of M, i.e. F is a subbundle of the tangent bundle which is **integrable** which means that for every pair ξ,ζ of smooth sections of F their bracket $[\xi,\zeta]$ lies in F. A fibration is a foliation (taking F to be the space of vectors tangent to the fibers). The local picture of a foliation is exactly the same as for a fibration. However the global picture may be far more complicated, for instance all the "fibers", called **leaves** can very well be dense in M. The topology of the space M/F of leaves can therefore be the coarse topology. Hence the space of continuous functions on M/F carries very little information about the foliation itself. This led Connes ([25, 27]) to construct a C*-algebra $C^*(M,F)$ to play the role of the algebra of continuous functions on M/F. The noncommutativity of $C^*(M,F)$ reflects the "distance of the foliation from a fibration". In the case of a fibration, the quotient space M/F is the base B and $C^*(M,F)$ is Morita equivalent to $C(B)$. At the other end $C^*(M,F)$ is simple if and only if the foliation is minimal (i.e. all leaves are dense cf.[45]).

The local picture being the same, one defines the longitudinal pseudodifferential operators of a foliation in the same way as for the families of pseudodifferential operators of a fibration (cf. [25]). When such an operator P is elliptic it defines an element $[P]\in KK(C(M),C^*(M,F))$. Its symbol σ_P defines an element $[\sigma_P]\in K^0(F^*)=KK(\mathbb{C},C_0(F^*))$ (where F^* is the total space of the dual bundle to F). In this situation too we have a formula of the type $[P]=[\sigma_P]\otimes_{F^*}\mathcal{P}$ where $\mathcal{P}\in KK(C_0(F^*\times M),C^*(M,F))$.

The longitudinal index theorem allows to construct a homomorphism $\mu:RK_{*,\tau}(BG)\to K_*(C^*(M,F))$ (cf. [27,15,31]) where BG is Haefliger's classifying space ([52]) of the holonomy groupoid G of the foliation (cf. [116]) and RK_* is the representable K-homology (cf. [69,99] see also definition 12.1 ; the subscript "τ" indicates some twisting by the transverse bundle which disappears when this bundle, over BG, is endowed with a $Spin^{\mathbb{C}}$ structure). This homomorphism μ is conjectured to be an isomorphism(7) by Baum and Connes (cf. [15], this conjecture is the analogue for foliations of the Connes-Kasparov-Rosenberg conjecture 9.16). Even if the Baum-Connes conjecture turns out to be wrong, it gives some good approximation of the K-theory of the C*-algebra of a foliation.

(8) If there is some torsion in the holonomy groups this statement has to be slightly modified.

11.4. There are several other index theorems along the same lines: e.g. Atiyah's index theorem for covering spaces (cf. [6]); this is a particular case of Mishchenko and Fomenko index theorem for flat A-bundles (i.e. flat bundles over a manifold M with fibers finitely generated projective modules over the C*-algebra A -cf. [77]). Another generalisation of the index theorem for covering spaces is Connes' index theorem for measured foliations (cf. [25] this theorem can be deduced from 11.3 using Connes' cyclic cohomology).

Also Atiyah's theory of operators transversally elliptic to a compact Lie group action (cf. [5]) can be expressed in terms of Kasparov theory (see Julg [62]). See also the index theorems in [66,101,17].

We end this section with two results which are related but not exactly of the same kind as the index theorems discussed above.

11.5. Signature operators on Lipschitz manifolds (Teleman [110,111], Hilsum [56]).

Let M be a compact topological manifold endowed with two different smooth structures. Associated to each of these structures are the Pontrjagin classes (in $H^*(M;\mathbb{Z})$) of the corresponding smooth manifolds. Novikov's theorem states that the rational Pontrjagin classes (i.e. their image in $H^*(M;\mathbb{Q})$) do not depend on the smooth structure. Teleman proved that the K-homology class of the signature operator of an oriented smooth compact manifold does not depend on the smooth structure. This shows that the L-genus of M (i.e. the homology class corresponding to the signature through the Chern character -see end of §12) is independent of the smooth structure, thus giving an analytic proof of Novikov's theorem (cf. [109] - it was a problem asked by Singer in [100] to find a proof along these lines).

In fact Teleman was able to perform a Hodge theory and to define the signature operator without using a smooth structure: one just needs a Lipschitz structure. But a result of Sullivan ([108]) states that on any topological manifold (of dimension different from 4) there is a Lipschitz structure unique up to isotopy. Therefore one can define the K-homology class of the signature operator for every oriented topological compact manifold.

In Teleman's work the signature operator does not define directly an element of $K_*(M)$ $(=KK_*(C(M),\mathbb{C}))$: it defines an element of a group called $K_*^{(1)}(M)$ which is then shown to be isomorphic to Kasparov's group $K_*(M)$. Because of that Teleman could not use directely Kasparov's theory and some of the proofs became quite tedious. However Hilsum ([56]), using

Baaj-Julg's unbounded Kasparov bimodules (cf. [14]), showed that the signature operator defines directly an element of $K_*(M)$.

Further applications of Kasparov's theory in connection with Lipschitz manifolds where obtained by Teleman ([112]), Hilsum ([57]) and Rosenberg-Weinberger ([98]). In particular Hilsum cnstructed families of signature operators for Lipschitz fibrations and longitudinal signature operators for Lipschitz foliations, generalizing the index theorems discussed in 11.2 and 11.3 to the Lipschitz case.

11.6. Transversally elliptic operators for foliations ([58]): Let (M,F) be a smooth foliated manifold. We want to construct operators which define elements of $KK_*(C^*(M,F),\mathbb{C})$.

The problem is that there are not in general enough symbols which are invariant under holonomy. Let us describe more precisely the situation in the (almost equivalent) case of a discrete group action on a smooth manifold [8].

Let M be a smooth manifold and let Γ be a countable group acting on M by diffeomorphisms. If there is no Riemannian structure on M invariant under the action of Γ, all the "classical" operators (e.g. Dirac, signature) cannot be defined in an invariant way, hence they will not define elements of $KK_*(C(M)\times\Gamma,\mathbb{C})$.

This difficulty can be bypassed, thanks to an idea of Connes ([29]), in the following way: let W be the fiber bundle over M whose fibers are the positive definite quadratic forms on TM. There are canonical elements $\beta \in KK(C_0(M)\times\Gamma, C_0(W)\times\Gamma)$ and $\alpha \in KK(C_0(W)\times\Gamma, C_0(M)\times\Gamma)$ associated in a natural way with Kasparov's fundamental elements α and β (see §9 – we have to make here some unimportant assumptions of orientability).

Now the manifold W is equipped with a Riemannian metric which is **quasi-invariant** under the action of Γ, which means that there is an invariant subbundle E of the tangent bundle TW (namely the bundle of vectors tangent to the fibers of the fibration $W \to M$) such that both E and TW/E have invariant metrics. It is then possible to construct almost invariant elliptic symbols of order 0 (and type ρ,δ). A pseudodifferential operator with such a symbol defines an

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(8) A foliation can always be treated, thanks to a transversal, as a discrete **pseudogroup** action on a smooth manifold. All the ideas explained here remain valid if the group is replaced by a pseudogroup.

element of $KK_*(C_0(W)\times\Gamma,\mathbb{C})$. Finally taking the Kasparov product with β yields the desiredelements of $KK_*(C_0(M)\times\Gamma,\mathbb{C})$.

Let us come back to the case of a foliation (M,F) . By the above procedure we obtain elements of $KK_*(C^*(M,F),\mathbb{C})$. Their Kasparov product with the elements of $KK(C(M),C^*(M,F))$ associated with longitudinal operators (see 11.3) give maps from $K^*(M)$ to $\mathbb{Z}$. The computation of these maps are interesting index theorems. A slightly more general index theorem is obtained when computing the map $RK_*(BG)\to\mathbb{Z}$ obtained by composition of $\mu:RK_*(BG)\to K_*(\mathbb{C},C^*(M,F))$ with the K-theory map defined by an element of $KK(C^*(M,F),\mathbb{C})$ constructed above. This index theorem can be used to prove the non vanishing of $\mu(x)$ for certain elements $x\in RK_*(BG)$ hence it is a step towards injectivity of the Baum-Connes map μ . Actually a more general result concerning the injectivity of μ (taking into account the Gel'fand Fuchs cohomology) was obtained by Connes using the cyclic cohomology (cf. [29]).

12. Application to the Novikov conjecture

As a major success of his theory Kasparov proved the Novikov conjecture for a large class of groups. This is a twenty years old conjecture and many mathematicians have contributed to proving it for various classes of groups (cf. [23,47,60,74,76,88]). It turns out that Kasparov's approach covers all these classes (cf.[69,72] see also [64,80]).

Let us recall the Novikov conjecture in a K-homological setting; the equivalence with the (more usual) cohomological setting will be discussed at the end of the section.

The Novikov conjecture deals with classifying spaces of discrete groups. These spaces are only defined up to homotopy equivalence and in general they are not locally compact. They are rather constructed as inductive limits of compact spaces (more precisely as CW complexes). The natural K-groups to be considered for such spaces are the **representable K-homology** groups or **K-homology with compact supports** (cf.[69,99]).

12.1. Definition: *Let* Y *be a Hausdorff topological space. Its K-homology with compact supports* $RK_*(Y)$ *is the inductive limit group* $RK_*(Y)=\varinjlim K_*(X)$ *where* X *runs over all compact subspaces of* Y.

Recall that for a compact space X, $K_*(X)=K^*(C(X))=KK_*(C(X),\mathbb{C})$ is the K-homology group of X (we note K_* for $K_0\oplus K_1$, K^* for $K^0\oplus K^1$ and KK_* for $KK_0\oplus KK_1$).

Let M be an oriented compact manifold. Let D_M denote the signature operator of M. As we discussed in §11, it defines an element $[D_M]\in K_*(M)$.

12.2. Novikov conjecture: *The discrete group* Γ *is said to satisfy the Novikov conjecture if given an oriented compact manifold* M *and a continuous map* $f:M\to B\Gamma$ *from* M *to the classifying space* $B\Gamma$ *of* Γ, *the element* $f_*([D_M])\in RK_*(B\Gamma)$ *only depends up to torsion on the oriented homotopy type of the pair* (M,f).

In other words, if $g:N\to M$ is an orientation preserving homotopy equivalence and $f':N\to B\Gamma$

is homotopic to $f \circ g$, then the images of $f_*([D_M])$ and $f'_*([D_N])$ in $RK_*(B\Gamma)\otimes\mathbb{Q}$ coincide.

Kasparov proved:

12.3. Theorem: *If Γ is either the fundamental group of a complete Riemannian manifold X of non positive sectional curvature or a (closed) discrete subgroup of a connected Lie group G then the Novikov conjecture holds for Γ.*

When X is as above and compact this was proved by Mishchenko (cf.[75]).

Let Γ be a discrete group and let $C^*(\Gamma)$ be its group C^*-algebra (§9). Kasparov constructed a homomorphism $\rho: RK_*(B\Gamma)\to K_*(C^*(\Gamma))$ that we now describe (Mishchenko gave independently essentialy the same construction cf. [69,75]). Let X be a compact space. A homotopy class of a continuous map $f: X\to B\Gamma$ is exactly an isomorphism class of a regular Γ-covering $\tilde{X}$ of X. Let then E_X be the finitely generated projective $C(X)\otimes C^*(\Gamma)$-module which is obtained by completing the space $C_c(\tilde{X})$ (of continuous compactly supported functions on $\tilde{X}$) with respect to the norm $\|\xi\|=\sup\{\|\xi_x\|,\ x\in\tilde{X}\}$ where $\xi_x\in C^*(\Gamma)$ is the element $\xi_x=\sum\xi(xg)u_g$ (the sum is taken over Γ - it is convenient to write the Γ action on $\tilde{X}$ on the right). If $\varphi\in C(X)$ and $h\in C_c(\Gamma)\subset C^*(\Gamma)$, the right action of $\varphi\otimes h$ on E_X is given by:

$\xi(\varphi\otimes h)(x)=\varphi(p(x))\sum_{g\in\Gamma}\xi(xg^{-1})h(g)$ (where $p:\tilde{X}\to X$ is the projection).

The finitely generated projective module E_X defines an element $[E_X]\in K_0(C(X)\otimes C^*(\Gamma))=KK(\mathbb{C},C(X)\otimes C^*(\Gamma))$. Define the homomorphism $\rho_X: K_*(X)\to K_*(C^*(\Gamma))$ by $\rho_X(x)=[E_X]\otimes_{C(X)}x$. Then ρ is defined to be the inductive limit of ρ_X over the compact subspaces of $B\Gamma$ (one checks immediately that if $i:X\to X'$ is the inclusion $i^*(E_{X'})=E_X$ and therefore $\rho_{X'}\circ i_*=\rho_X$).

Kasparov and Mishchenko independently proved:

12.4. Theorem: *(cf: [69]) With the notation of theorem 12.2, the element $\rho(f_*([D_M]))\in K_*(C^*(\Gamma))$ only depends on the homotopy type of the pair (M,f).*

Theorem 12.3 is a consequence of 12.4 and of the following:

12.5. Theorem: *In the situation of theorem 12.3 , the map* $\rho\otimes 1: RK_*(B\Gamma)\otimes\mathbb{Q}\to K_*(C^*(\Gamma))$ *is injective.*

Let me say some words on the proofs of theorems 12.4 and 12.5.

The idea of the proof of 12.4 is that the signature operator D_M defines naturally an element of a surgery group $L_*(C^*(\Gamma))$. The homotopy invariance is proved in this group. But as every positive element in a C*-algebra has a square root, the L-theory and K-theory coincide.

We now come to theorem 12.5. If Γ is the fundamental group of X as in theorem 12.2 , let Y be its universal covering. If Γ is a discrete subgroup of a Lie group G set $Y=G/K$. In this case we will also assume that Γ is torsion free:

If Γ is finitely generated it has a torsion free subgroup Γ' of finite index (cf. [87]) ; then $B\Gamma'$ is a finite cover of $B\Gamma$ hence $RK_*(B\Gamma')\otimes\mathbb{Q}=RK_*(B\Gamma)\otimes\mathbb{Q}$; also the map $K_*(C^*(\Gamma'))\to K_*(C^*(\Gamma))$ is rationaly injective ; therefore theorem 12.5 for Γ' implies theorem 12.5 for Γ. In general Γ is a union of Γ_n with Γ_n finitely generated. Then $RK_*(B\Gamma)=\varinjlim RK_*(B\Gamma_n)$ and $K_*(C^*(\Gamma))=\varinjlim K_*(C^*(\Gamma_n))$. Hence theorem 12.5 for Γ_n implies theorem 12.5 for Γ.

If Γ is a torsion free discrete subgroup of the Lie group G then the action of Γ in $Y=G/K$ is proper(9) and free. In both cases of theorem 12.5 we then have $B\Gamma=Y/\Gamma=X$.

We assume to simplify the notation that Y has even dimension and is endowed with an equivarient $\mathrm{Spin}^{\mathbb{C}}$ structure. In this case we have a Poincaré duality which reads $RK_*(X)=K^*(X)=K_*(C_0(X))$ (cf. 6.9).

Also $C_0(X)$ is Morita equivalent to $C_0(Y)\times\Gamma$. The homomorphism ρ is the composition $RK_*(X)\xrightarrow{\sim}K^*(X)\xrightarrow{\sim}KK_*(\mathbb{C},C_0(Y)\times\Gamma)\to KK_*(\mathbb{C},C^*(\Gamma))=K_*(C^*(\Gamma))$ where the map $KK_*(\mathbb{C},C_0(Y)\times\Gamma)\to KK_*(\mathbb{C},C^*(\Gamma))$ is the Kasparov product by an element

(9) An action of a locally compact group on a locally compact space Y is said to be proper if the map $(g,y)\to(y,gy)$ from $G\times Y$ to $Y\times Y$ is proper i.e. the inverse image of compact sets are compact.

$\alpha' \in KK(C_0(Y)\times\Gamma, C^*(\Gamma))$ which is the image $j_\Gamma(\alpha)$ of the class $\alpha\in KK_\Gamma(C_0(Y),\mathbb{C})$ of the equivarient Dirac operator on Y.

To prove the injectivity of ρ it is enough to construct an element $\beta\in KK_\Gamma(\mathbb{C},C_0(Y))$ such that $\alpha\otimes_{\mathbb{C}}\beta=1_{C_0(Y)}$. In the case $\Gamma\subset G$ this is a consequence of theorem 9.10 ; in the case of non positive sectional curvature it is a consequence of the proof of theorem 9.10 .

To end this section we reformulate the Novikov conjecture:

12.6. Cohomological formulation of the Novikov conjecture

We begin by recalling some facts about the **Chern character.**

Let X be a finite CW complex. The Chern character is a rational isomorphism $ch:K^*(X)\otimes\mathbb{Q}\to H^*(X;\mathbb{Q})$. It is **natural** which means that for every continuous map $f:X\to Y$ the following diagram commutes:

$$\begin{array}{ccc} K^*(Y)\otimes\mathbb{Q} & \to & H^*(Y;\mathbb{Q}) \\ f^*\downarrow & & f^*\downarrow \\ K^*(X)\otimes\mathbb{Q} & \to & H^*(X;\mathbb{Q}) \end{array}$$

Dual to it there is also a Chern character in homology. It is a natural isomorphism $ch:K_*(X)\otimes\mathbb{Q}\to H_*(X;\mathbb{Q})$ given by the equality:

$$\langle ch(x),ch(y)\rangle=\langle x,y\rangle \quad (\in\mathbb{Q}\ ,\ x\in K^*(X)\otimes\mathbb{Q}\ ,\ y\in K_*(X)\otimes\mathbb{Q}\ ).$$

Using naturality of the Chern character, we may pass to inductive limits and still get an isomorphism $ch:RK_*(Y)\otimes\mathbb{Q} \to H_*(Y;\mathbb{Q})$ for general CW complexes.

The cohomological form of the Atiyah-Singer index theorem (cf. [11]) computes for compact manifolds the Chern character in homology: given an elliptic pseudodifferential operator P the Chern character of the K-homology class of P is computed (in terms of the Chern character of the K-theory class of its symbol). For the signature operator on M we get $ch([D_M]) = L(M)\cap[M]$, where $[M]\in H_n(M;\mathbb{Z})$ is the fundamental class of M (n=dimension of M) and $L(M)\in H^*(M;\mathbb{Q})$ is a distinguished class called the **Pontryagin-Hirzebruch**

characteristic class (or L-genus) of M .

The Novikov conjecture then states:

The element $f_*(L(M) \cap [M]) \in H_*(B\Gamma;\mathbb{Q})$ *only depends on the oriented homotopy type of the pair* (M,f) .

Using duality between homology and cohomology it is equivalent to saying:

For any cohomology class $\xi \in H^*(B\Gamma;\mathbb{Q})$ *the number* $\langle \xi, f_*(L(M) \cap [M])\rangle \in \mathbb{Q}$ *only depends on the oriented homotopy type of the pair* (M,f) .

The number $\langle \xi, f_*(L(M) \cap [M])\rangle = \langle f^*(\xi), L(M) \cap [M]\rangle$ is called a **higher signature** of M (recall that $\langle L(M),[M]\rangle$ is the signature of M). The Novikov conjecture says:

All higher signatures are invariant under oriented homotopy equivalence.

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