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# The coarse Baum–Connes conjecture and groupoids

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**Abstract**

To every discrete metric space with bounded geometry $X$ we associate a groupoid $G(X)$ for which the coarse assembly map for $X$ is equivalent to the Baum–Connes assembly map for $G(X)$ with coefficients in the $C^*$-algebra $\ell^\infty(X, \mathscr{K})$. We thus obtain a new proof of the fact that if $X$ admits a uniform embedding into Hilbert space, the coarse assembly map is an isomorphism. If furthermore $X$ is a discrete group $\Gamma$ with a translation-invariant metric, we show, using Higson's descent technique, that $\Gamma$ also satisfies the Novikov conjecture. This removes the finiteness condition in (Yu, Invent. Math. 139 (2000) 201–204).


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## 1. Introduction

The Baum–Connes conjecture deals primarily with discrete groups, but can be generalized to group actions, foliations, and even locally compact, $\sigma$-compact Hausdorff groupoids endowed with a Haar system [18,19]. If $G$ is such a groupoid and $B$ is a $C^*$-algebra acted upon by $G$, there is an assembly map

$$\mu_r : K_*^{\mathrm{top}}(G; B) \to K_*(B \rtimes_r G).$$

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It had been conjectured for a long time that this map is always an isomorphism, but counter-examples to the conjecture with coefficients for discrete groups, and to the conjecture without coefficients for groupoids (and even group actions and foliations) have been constructed by Higson, Lafforgue, Skandalis, Ozawa and Yu. See [2,7] for details about the conjecture and its applications.

Let $Y$ be a metric space. One constructs [11,17,20] the "coarse assembly map"

$$A : KX_*(Y) \to K_*(C^*(Y)).$$

The advantage of working in the coarse category is that it is much more flexible, in that the groups $KX_*(Y)$ and $K_*(C^*(Y))$ do not depend on the local topology of $Y$ but only on its macroscopic structure. In particular, every metric space in this category is isomorphic to a discrete one. It was conjectured that for every metric space with bounded geometry, i.e. such that for every $R > 0$ every ball of radius $R$ has at most $N(R)$ elements for some real number $N(R) > 0$, its coarse assembly map is an isomorphism; however, a counterexample has been recently given by Higson. In this article we associate to every discrete metric space with bounded geometry $X$ a locally compact groupoid $G(X)$. We show that the assembly map for the metric space $X$ identifies with the Baum–Connes assembly map for the groupoid $G(X)$ with coefficients in the $C^*$-algebra $\ell^\infty(X, \mathscr{K})$

$$K_*^{\text{top}}(G(X); \ell^\infty(X; \mathscr{K})) \to C^*(\ell^\infty(X; \mathscr{K}) \rtimes_r G(X)).$$

If $\Gamma$ is a countable group, all its proper left-invariant distances are coarse-equivalent, hence $\Gamma$ is endowed with an intrinsic coarse structure; word metrics on finitely generated groups are examples of such distances. We show

**Proposition 3.4.** *If $\Gamma$ is a discrete group, then $G(|\Gamma)|) = \beta(|\Gamma|) \rtimes \Gamma$.*

($|\Gamma|$ is the coarse space underlying $\Gamma$ and $\beta(|\Gamma|)$ its Stone–Čech compactification.) In particular, we obtain a new proof of Yu's result [21] that the coarse assembly map for $\Gamma$ is the same as the Baum–Connes map for $\Gamma$ with coefficients in $\ell^\infty(\Gamma, \mathscr{K})$

$$K_*^{\text{top}}(\Gamma; \ell^\infty(\Gamma, \mathscr{K})) \to K_*(\ell^\infty(\Gamma, \mathscr{K}) \rtimes_r \Gamma).$$

In the second half of this paper we apply our results to spaces which admit a uniform embedding into Hilbert space. Let $X$ be a metric space. Recall that a map $f$ from $X$ to a separable Hilbert space $H$ is said to be a uniform embedding [5] if there exist two non-decreasing functions $\rho_1$ and $\rho_2$ on $[0, +\infty)$ such that

(1) $\lim_{r \to +\infty} \rho_i(r) = +\infty$;
(2) $\rho_1(d(x, y)) \leqslant \|f(x) - f(y)\| \leqslant \rho_2(d(x, y))$ for all $x, y \in X$.

Yu [22] introduced a property on metric spaces, called property A (see Definition 5.1), which guarantees the existence of such an embedding.

Gromov [5,6] raised the question whether every finitely generated group, as a metric space with a word length metric, admits a uniform embedding into Hilbert space and suggested that this should be used to study the Novikov conjecture; however, he recently discovered that this property does not hold for every countable group. Higson and Kasparov proved the Novikov conjecture for groups which act properly and isometrically on Hilbert space [10]. The class of groups which act properly and isometrically on Hilbert space contains all amenable groups [3] although it does not contain any infinite property T group. In [22], Yu proved:

**Theorem 5.5** (Yu [22]). *Let $X$ be a metric space with bounded geometry that admits a uniform embedding into Hilbert space. Then the coarse assembly map for $X$ is an isomorphism.*

In this paper, we obtain a new proof of Theorem 5.5 by observing that $X$ admits a uniform embedding into Hilbert space if and only if the groupoid $G(X)$ admits a proper action on a continuous field of affine Hilbert spaces (cf. Section 5), and using the fact from [19] that for such a groupoid, the Baum–Connes map with coefficients is an isomorphism.

Using the descent principle [17], Theorem 5.5 implies that for a finitely generated group which admits as a metric space with word metric a uniform embedding into Hilbert space, and whose classifying space is a finite CW-complex, the Novikov conjecture holds [22, Corollary 1.2]. In the case of groups with property A, Higson strengthened this result by removing the finiteness assumption on the classifying spaces [8]. Higson's proof is based on an elegant descent technique [8] and Tu's theorem on Baum–Connes conjecture on amenable groupoids [19]. In this paper, we generalize both Higson's result and Corollary 1.2 of [22] as follows:

**Theorem 6.1.** *Let $\Gamma$ be a countable group with a proper left-invariant metric $d$. If $\Gamma$ admits a uniform embedding into Hilbert space, then the Baum–Connes assembly map with coefficients*

$$\mu_r : K_*^{\mathrm{top}}(\Gamma; A) \to K_*(A \rtimes_r \Gamma)$$

*is injective for any separable $\Gamma$-$C^*$-algebra $A$.*

## 2. Preliminaries

### 2.1. Coarse structures

Let $X$ be a set. If $A \in X \times X$ and $B \in X \times X$, we will use the following notation:

$$A^{-1} = \{(y,x) \mid (x,y) \in A\},$$

$$A \circ B = \{(x,z) \mid \exists y \in X, (x,y) \in A \text{ and } (y,z) \in B\},$$

that is, $A^{-1}$ is the groupoid inverse of the set $A$ and $A \circ B$ is the groupoid product of $A$ and $B$ in the groupoid $X \times X$. Let $r$ and $s$ be the maps from $X \times X$ to $X$ defined by $r(x,y) = x$ and $s(x,y) = y$. Recall [9, Definition 2.1] the

**Definition 2.1.** A coarse structure on $X$ is a collection of subsets of $X \times X$, called *entourages*, that have the following properties:

(a) For any entourages $A$ and $B$, $A^{-1}$, $A \circ B$, and $A \cup B$ are entourages;

(b) Every finite subset of $X \times X$ is an entourage;
(c) Any subset of an entourage is an entourage.
If $\Delta = \{(x,x) \mid x \in X\}$ is an entourage, then the coarse structure is said to be *unital*.

This definition is slightly different from [9, Definition 2.1]: equivalence classes of coarse structures as defined in [9] are coarse structures in our sense.

The intersection of a family of coarse structures is a coarse structure. Every set $\mathscr{C}$ of subsets of $X \times X$ generates a coarse structure.

The coarse structure is said to be *countably generated* if there exists a sequence $(E_n)$ of entourages such that any entourage is contained in a finite union of sets of the form $E_{n_1}^{\pm 1} \circ \cdots \circ E_{n_k}^{\pm 1}$. Equivalently, there exists an increasing sequence $(E_n)$ of entourages such that every entourage is contained in one of the $E_n$. If the coarse structure is unital and countably generated, we may assume that $E_0$ is the diagonal $\Delta$, and define the distance $d(x,y)$ as the infimum of all sums $\sum_{i=0}^{k-1} n_i$ where $x_0,\dots,x_k$ are elements of $X$ such that $x_0 = x$, $x_k = y$, and $(x_i,x_{i+1}) \in E_{n_i}^{\pm 1}$ for all $0 \leqslant i < k$. Conversely, given any distance on $X$, we get a coarse structure on $X$ by saying that a set $E$ is an entourage if and only if $\{d(x,y) \mid (x,y) \in E\}$ is bounded. That coarse structure is countably generated, since every entourage is contained in one of the sets

$$\Delta_R = \{(x,y) \in X \mid d(x,y) \leqslant R\}.$$

Consequently, there is a one-to-one correspondence between unital countably generated coarse structures on $X$, and coarse equivalence classes of metrics on $X$ (we say that two distances $d$ and $d'$ on $X$ are coarse-equivalent if there exist increasing functions $k$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $d \leqslant k(d')$ and $d' \leqslant h(d)$).

Let $X, Y$ be sets and $f : X \to Y$ a map. Let $\mathscr{E}_Y$ be a coarse structure on $Y$. We denote by $f^*\mathscr{E}_Y$ the set $\{E \mid (f \times f)(E) \in \mathscr{E}_Y\}$ (where $f \times f : X \times X \to Y \times Y$). It is a coarse structure on $X$. If $X$ is also endowed with a coarse structure $\mathscr{E}_X$, we say $f$ is *coarse* if $\mathscr{E}_X \subset f^*(\mathscr{E}_Y)$. Two coarse maps $f$ and $g : X \to Y$ are said to be *bornotopic* if for every $E \in \mathscr{E}_X$, $(f \times g)(E) \in \mathscr{E}_Y$. Bornotopy is obviously an equivalence relation.

**Definition 2.2.** Let $X$ and $Y$ be sets with coarse structures $\mathscr{E}_X$ and $\mathscr{E}_Y$. A coarse *correspondence* from $X$ to $Y$ is a coarse structure on $X \amalg Y$ which restricts to $\mathscr{E}_Y$ on $Y$, contains $\mathscr{E}_X$, and is generated by its entourages contained in $Y \times (X \amalg Y)$. A coarse *equivalence* between $X$ and $Y$ is a coarse structure on $X \amalg Y$ which is a coarse correspondence from $X$ to $Y$ and from $Y$ to $X$.

**Proposition 2.3.** *Let $X$ and $Y$ be sets with coarse structures $\mathscr{E}_X$ and $\mathscr{E}_Y$. Let $f : X \to Y$. Let $\mathscr{E}_{YX}$ be the set of subsets $E \subset Y \times X$ such that $(\mathrm{id}_Y \times f)(E) \in \mathscr{E}_Y$. Let $\mathscr{E}(f)$ consist of sets of the form $E_X \cup E_Y \cup E_{XY} \cup E_{YX}$, where $E_X \in f^*(\mathscr{E}_Y)$, $E_Y \in \mathscr{E}_Y$, $E_{XY}^{-1} \cup E_{YX} \in \mathscr{E}_{YX}$. Then $\mathscr{E}(f)$ is a coarse correspondence from $X$ to $Y$ if and only if $f$ is coarse. If $\mathscr{E}_X$ is unital, any coarse correspondence is defined by a coarse map, which is unique up to bornotopy.*

**Proof.** Suppose $\mathscr{E}(f)$ is a coarse correspondence. Since $f^*\mathscr{E}_Y$ must be a coarse structure on $X$ containing $\mathscr{E}_X$, $f$ is necessarily coarse.

Conversely, if $f$ is coarse, let $h: X \amalg Y \to Y$ be the map which restricts $f$ in $X$ and to the identity of $Y$. By definition $\mathscr{E}(f) = h^*\mathscr{E}_Y$. It follows that $\mathscr{E}(f)$ is a coarse structure on $X \amalg Y$. Its restriction to $Y$ is $\mathscr{E}_Y$; if $E \in \mathscr{E}_X$, put $E' = E \cup (E \circ E^{-1})$ and $F = (f \times \mathrm{id}_X)(E')$; then $E' \in \mathscr{E}_X \subset f^*(\mathscr{E}_Y)$, whence $F \in \mathscr{E}_{YX}$; moreover, $E \subset (E')^{-1} \circ E' \subset F^{-1} \circ F$.

Now, assume $\mathscr{E}_X$ is unital, and let $\mathscr{C}$ be a coarse correspondence. There exists $E \in \mathscr{C}$, such that $E \subset X \times Y$ and $\Delta \subset E \circ E^{-1}$, where $\Delta = \{(x,x) \mid x \in X\}$. Then $E$ contains the graph of a function $f$. Now $\mathscr{E}(f)$ is generated by $\mathscr{E}_Y$ and the graph $\mathrm{Gr}(f)$ of $f$, therefore, $\mathscr{C}$ contains $\mathscr{E}(f)$.

Conversely, if $E \in \mathscr{C}$, $E \subset Y \times X$, then $E \subset (E \circ \mathrm{Gr}(f)^{-1}) \circ \mathrm{Gr}(f)$ and since the restriction of $\mathscr{C}$ to $Y$ is $\mathscr{E}_Y$, $E \circ \mathrm{Gr}(f)^{-1} \in \mathscr{E}_Y$, whence $E \in \mathscr{E}(f)$. As $\mathscr{C}$ is generated by its entourages contained in $Y \times (X \amalg Y)$, we conclude that $\mathscr{C} = \mathscr{E}(f)$.

Finally, if $\mathscr{E}(f) = \mathscr{E}(g)$ with $f$ and $g$ coarse maps, then $\mathrm{Gr}(f)^{-1} \circ \mathrm{Gr}(g) \in \mathscr{E}_Y$, so $f$ and $g$ are bornotopic. □

### 2.2. Locally finite coarse structures

A coarse structure $\mathscr{E}$ on a set $X$ is said to be *locally finite* if for every $x \in X$ and any entourage $E$, $E \cap s^{-1}(x)$ is finite.

**Definition 2.4.** For every $E \subset X \times X$, let

$$N(E) = \sup_{x \in X} \max(\#(r^{-1}(x) \cap E), \#(s^{-1}(x) \cap E)).$$

We shall say a the coarse structure is *uniformly locally finite* if $N(E) < \infty$ for every entourage $E$. We say that a coarse structure on a set $Y$ has *bounded geometry* if $Y$ is coarse equivalent to a set $X$ with a uniformly locally finite coarse structure.

**Note 1.** *What we call here uniformly locally finite, is usually also called bounded geometry. As there may be some confusion between bounded geometry in the discrete case and in the general case, we chose to change the denomination.*

The coarse structure of a metric space $(X,d)$ is locally finite if and only if $X$ is discrete and $d$ is proper (i.e. closed balls are compact); it is uniformly locally finite, if and only if $(X,d)$ is uniformly locally finite, in the sense that for every $R > 0$, there exists $N(R) > 0$ such that every set of diameter $\leqslant R$ has at most $N(R)$ elements.

**Example 2.5.** Let $\Gamma$ be a discrete group. The left coarse structure on $\Gamma$ is such that $E \subset \Gamma \times \Gamma$ is an entourage if and only if $\{x^{-1}y \mid (x,y) \in E\}$ is finite. It is obviously uniformly locally finite.

**Remark 2.6.** (a) Let $\mathscr{E}_u(X)$ be the set $\{E \subset X \times X \mid N(E) < \infty\}$. Then $\mathscr{E}_u(X)$ is a uniformly locally finite coarse structure, which contains any uniformly locally finite coarse structure on $X$.

(b) Let $\mathscr{E}$ be a coarse structure on $X$. Then $\mathscr{E} \cap \mathscr{E}_u(X)$ is a uniformly locally finite coarse structure on $X$.

**Lemma 2.7.** *Let $X$ and $Y$ be sets, $E \subset X \times Y$. Denote by $r$ and $s$ the projections from $E$ to $X$ and $Y$, respectively. Assume that there exist $m, n \in \mathbb{N}$ such that $\forall x \in X$, $\#(r^{-1}(x) \cap E) \leqslant m$ and $\forall y \in Y$, $\#(s^{-1}(y) \cap E) \leqslant n$.*

(a) *Put $p = m(n-1)+1$. There exists a partition $(B_i)_{1 \leqslant i \leqslant p}$ of $X$ such that $s$ is injective on $E \cap (B_i \times Y)$ for every $i$.*
(b) *The map $\bar{E} \to \beta X \times \beta Y$ is injective, where $\bar{E}$ denotes the closure of $E$ in $\beta(X \times Y)$.*

**Proof.** (a) Let $(B_i)_{1 \leqslant i \leqslant p}$ be pairwise disjoint subsets of $X$ such that $s_{|E \cap (B_i \times Y)}$ is injective, whose union is maximal with this property. If there exists an element $x$ outside $B_1 \cup \cdots \cup B_p$, then $\forall i$, $s_{|E \cap (B_i \cup \{x\} \times Y)}$ is non-injective. Set $A = \{y \in Y \mid (x, y) \in E\}$; there exist $x_i' \in B_i$, $y_i \in A$ such that $(x_i', y_i) \in E$. But, $\#A \leqslant m$ and, for all $y \in A$, $\#\{i, B_i \times \{y\} \cap E\} \leqslant n-1$, which yields an impossibility. Therefore, $(B_i)_{1 \leqslant i \leqslant p}$ is a partition of $X$ that satisfies the required properties.

(b) Let $(B_i)_{1 \leqslant i \leqslant p}$ be as in (a). The restriction of $s : \bar{E} \to \beta Y$ to each $\overline{E \cap (B_i \times Y)}$ is one to one. Furthermore, these subsets have disjoint images under the map $r : \bar{E} \to \bar{X}$. □

Let $\mathscr{E}$ be a coarse structure on a set $X$. Let $\Gamma_{\mathscr{E}}$ be the set of entourages $A$ such that $r$ and $s$ are injective on $A$.

**Lemma 2.8.** *The coarse structure on $X$ is generated by $\Gamma_{\mathscr{E}}$ if and only if it is uniformly locally finite.*

**Proof.** Suppose that $N(E) = n$. From Lemma 2.7(a), there exists a partition $(B_i)_{1 \leqslant i \leqslant n^2 - n + 1}$ of $X$ such that $s$ is injective on $E \cap (B_i \times X)$ for all $i$. It is clear that each $E \cap B_i$ is the union of $n$ entourages in $\Gamma_{\mathscr{E}}$, so $E$ is the union of at most $n^3$ entourages in $\Gamma_{\mathscr{E}}$. Conversely, if $E$ is the union of $m$ elements of $\Gamma_{\mathscr{E}}$, then $N(E) \leqslant m$. □

### 2.3. The coarse assembly map

We fix a set $X$ endowed with a uniformly locally finite coarse structure.

Let $H = \ell^2(\mathbb{N})$, $\hat{H} = H \oplus H$ with the usual grading, $\mathscr{K} = \mathscr{K}(H)$ and $\hat{\mathscr{K}} = \mathscr{K}(\hat{H})$. If $B$ is a $C^*$-algebra, let $H_B = B \otimes H$ and $\hat{H}_B = B \hat{\otimes} \hat{H}$. For an entourage $E$ of $X$, let us denote by $P_E(X)$ the Rips' complex such that a finite subset $F \subset X$ spans a simplex in $P_E(X)$ if and only if for every $x, y \in F$, $(x, y) \in E$. Let $B$ be a $C^*$-algebra, and

$$KX_*(X; B) = \lim_{E} KK_*(C_0(P_E(X)), B)$$

be the coarse homology group of $X$ with coefficients in $B$ (the limit is taken along the directed set of entourages of $X$). Let $C^*(X; B)$ be the closure of the algebra of operators in $\ell^2(X) \otimes H_B$

which are locally compact and whose support is an entourage. The assembly map

$$KX_*(X;B) \xrightarrow{A} K_*(C^*(X;B)).$$

is defined as follows [11,20]:

**Definition 2.9.** Let $B$ be a $C^*$-algebra, $Y$ a proper metric space and $\mathscr{C}$ a coarse correspondence from $Y$ to $X$. There exists a partition of unity $(\lambda_x)_{x\in X}$ of $Y$ such that $\{(x,y) \in X \times Y \mid y \in \operatorname{Supp}\lambda_x\} \in \mathscr{C}$. For every $u \in KK_*(C_0(Y),B)$, let $(\hat{H}_B, \varphi, F) \in \mathbf{E}(C_0(Y),B)$ whose $KK$-class is $u$. Let $V_0 : C_0(Y) \hookrightarrow \ell^2(X) \otimes C_0(Y)$ be the isometry $f \mapsto (\lambda_x^{1/2} f)_{x\in X}$, $V = V_0 \otimes_\varphi 1$, and $P = VV^*$. Then the assembly map $A : KK_*(C_0(Y),B) \to K_*(C^*(X;B))$ is defined by

$$(\hat{H}_B, \varphi, F) \mapsto (P(\ell^2(X)\hat{\otimes}\hat{H}_B), P(1\hat{\otimes}F)P).$$

Note that the operator $F' = \sum_x \lambda_x^{1/2} F \lambda_x^{1/2}$ has bounded propagation and is a compact perturbation of $F$ in the sense that for every $a \in C_0(Y)$, $\varphi(a)(F - F')$ is compact. Since $P(1\hat{\otimes}F)P = V(V^*(1\hat{\otimes}F)V)V^* = VF'V^*$, the element $(P(\ell^2(X)\hat{\otimes}\hat{H}_B), P(1\hat{\otimes}F)P)$ defines a $K$-theory element of $C^*(X;B)$ as claimed.

In particular, for $Y = P_E(X)$, every $y \in P_E(X)$ can be written as a finite convex combination, $\sum_x \lambda_x(y)x$, and $(\lambda_x)_{x\in X}$ is a partition of unity satisfying the above condition.

There is another interpretation of the assembly map. For every pair of $C^*$-algebras $A$ and $B$ there is a natural transformation

$$\tilde{\sigma}_X : KK(A,B) \to KK(C^*(X;A), C^*(X;B)).$$

Indeed, let $(\hat{H}_B, \varphi, F)$ be a $A$, $B$-Kasparov bimodule. Define $\tilde{F} = 1 \otimes F$ acting on $\ell^2(X) \otimes H\hat{\otimes}\hat{H}_B$. The map $T \mapsto T \otimes_\varphi 1$ from $\mathscr{L}((\ell^2(X) \otimes H) \otimes A)$ to $\mathscr{L}((\ell^2(X) \otimes H) \otimes \hat{H}_B)$ induces $\tilde{\varphi} : C^*(X;A) \to M(C^*(X;B\hat{\otimes}\hat{\mathscr{K}}))$. The bimodule $(C^*(X;B\hat{\otimes}\hat{\mathscr{K}}), \tilde{\varphi}, \tilde{F})$ defines an element of $KK(C^*(X;A), C^*(X;B))$, since $C^*(X;B\hat{\otimes}\hat{\mathscr{K}}) \simeq C^*(X;B \otimes \mathscr{K}\hat{\otimes}\mathscr{L}(\mathbb{C} \oplus \mathbb{C})) \simeq C^*(X;B \otimes \mathscr{K})\hat{\otimes}\mathscr{L}(\mathbb{C} \oplus \mathbb{C}) \simeq C^*(X;B) \otimes \mathscr{L}(\mathbb{C} \oplus \mathbb{C})$ where $\mathbb{C} \oplus \mathbb{C}$ has the usual grading.

Let $Y$ be a locally compact space which is coarse-equivalent to $X$, and let $(\lambda_x)_{x\in X}$ be a partition of unity as above. The projection $V_0V_0^*$ defines a canonical element that we shall denote by $\lambda_{Y,X} \in K_0(C^*(X;C_0(Y)))$. The assembly map is then the composition

$$KK^*(C_0(Y),B) \xrightarrow{\tilde{\sigma}_X} KK^*(C^*(X,C_0(Y)), C^*(X;B)) \xrightarrow{\lambda_{Y,X}\otimes\cdot} K_*(C^*(X;B)).$$

### 2.4. Groupoids

Let $G$ be a locally compact groupoid (for detailed definitions about locally compact groupoids, see [16,14]). We denote by $G^{(0)}$ and $G^{(2)}$ the set of units and the set of composable pairs of $G$, respectively. Let $r$, $s : G \to G^{(0)}$ be the range and the source maps. The groupoid $G$ is said to be

(a) *principal* if $(r,s) : G \to G^{(0)} \times G^{(0)}$ is injective;
(b) *proper* if $(r,s) : G \to G^{(0)} \times G^{(0)}$ is proper;

(c) *étale*, or *r-discrete*, if the range map $r: G \to G^{(0)}$ is a local homeomorphism, i.e. if every $x \in G$ admits an open neighborhood such that $r(U)$ is an open subset of $G^{(0)}$ and $r: U \to r(U)$ is a homeomorphism. In this case, $s$ is also a local homeomorphism, as well as the composition map $G^{(2)} \to G$ and $G^{(0)}$ is an open subset of $G$.

For instance, if $Y$ is a locally compact space, then $G = Y \times Y$ is endowed with the structure of a principal and proper groupoid, with unit space $Y$, range and source maps $r(y,z) = y$ and $s(y,z) = z$, composition $(y,z)(z,w) = (y,w)$ and inverse $(y,z)^{-1} = (z,y)$.

Suppose from now on that $G$ is Hausdorff and has a Haar system $(\lambda^y)_{y \in G^{(0)}}$. A *cutoff function* on $G$ is a continuous function $c: G^{(0)} \to \mathbb{R}_+$ such that for every $y \in G^{(0)}$, $\int_{G^y} c(s(g))\lambda^x(\mathrm{d}g) = 1$, and for every compact $K \subset G^{(0)}$, $\operatorname{supp}(c) \cap s(G^K)$ is compact. Such a function exists if and only if $G$ is proper [18, Propositions 6.10, 6.11].

If $G$ is proper, the quotient $G^{(0)}/G$ of the space $G^{(0)}$ by the equivalence relation

$$x \sim y \quad \Leftrightarrow \quad \exists g \in G, \qquad s(g) = x \quad \text{and} \quad r(g) = y$$

is locally compact and Hausdorff.

An action (on the right) of $G$ on a space $Z$ is given by a map $s_Z: Z \to G^{(0)}$, called the source map, and a continuous map from $Z \times_{G^{(0)}} G = \{(z,g) \mid s_Z(z) = r(g)\}$ to $Z$, denoted by $(z,g) \mapsto zg$, such that

(a) $s_Z(zg) = s(g)$;
(b) $(zg)g' = z(gg')$ whenever $s_Z(z) = r(g)$ and $s(g) = r(g')$;
(c) $zy = y$ if $y = s_Z(z) \in G^{(0)} \subset G$.

The groupoid $Z \rtimes G$ with space of units $Z$ is defined as the subgroupoid of $(Z \times Z) \times G$ consisting of elements $(z, z', \gamma)$ such that $z' = z\gamma$. Equivalently, it is $\{(z,g) \in Z \times G \mid s_Z(z) = r(g)\}$ with source and range maps $s(z,g) = zg$, $r(z,g) = z$, and composition $(z,g)(zg,h) = (z,gh)$. The action of $G$ on $Z$ is free (resp. proper) if and only if the groupoid $Z \rtimes G$ is principal (resp. proper). A space $Z$ endowed with an action of $G$ is called a $G$-space. It is said to be $G$-compact if the action is proper and the quotient $Z/G$ is compact.

One can define the notions of continuous actions of locally compact groupoids on $C^*$-algebras and on Hilbert $C^*$-modules [15,14]. For instance, a $G$-equivariant continuous field of $C^*$-algebras over $G^{(0)}$ is a $G$-algebra.

Suppose now that $G$ is a locally compact, Hausdorff, $\sigma$-compact groupoid with Haar system acting on a $C^*$-algebra $A$. Then one defines the full and the reduced crossed-products of $A$ by $G$, denoted by $A \rtimes G$ and $A \rtimes_r G$, respectively. Let us sketch the definition of these algebras (cf. [16]). Let us denote by $C_c(G; r^*A)$ the space of functions with compact support $g \mapsto \phi(g) \in A_{r(g)}$ which are continuous in a sense defined in [14]. The product and adjoint are defined respectively by

$$\phi * \psi(g) = \int_{h \in G^{r(g)}} \phi(h)\alpha_h(\psi(h^{-1}g))\,\mathrm{d}\lambda^{r(g)}(h),$$

$$\phi^*(g) = \alpha_g(\phi(g^{-1}))^*.$$

One denotes by $L^1(G,r^*A)$ the completion of $C_c(G;r^*A)$ for the norm $||\phi|| = \max(|\phi|_1, |\phi^*|_1)$, where $|\phi|_1 = \sup_{x\in G^{(0)}} \int_{g\in G^x} ||\phi(g)||\, \mathrm{d}\lambda^x(g)$. Then, $A \rtimes G$ is the enveloping $C^*$-algebra of $L^1(G,r^*A)$, and $A \rtimes_r G$ is the closure of $L^1(G,r^*A)$ in $\mathscr{L}(L^2(G;r^*A))$.

### 2.5. The Baum–Connes assembly map for groupoids

Le Gall [15,14] defines, for every pair $(A,B)$ of graded $G$-algebras, a bifunctor $KK_G(A,B)$ that generalizes Kasparov's [13]. If $G$ is $\sigma$-compact, there is a product $KK_G(A,B)\times KK_G(B,D) \to KK_G(A,D)$ that satisfies the same naturality properties as the non-equivariant $KK$-functor. Modeled on Kasparov's descent morphisms there are natural maps

$$j_G : KK_G(A,B) \to KK(A \rtimes G, B \rtimes G),$$

$$j_{G,r} : KK_G(A,B) \to KK(A \rtimes_r G, B \rtimes_r G).$$

Suppose that $G'$ is proper with $G'^{(0)}/G'$ compact, and let $c$ be a cutoff function for $G'$. The function $g \mapsto c(r(g))^{1/2}c(s(g))^{1/2}$ defines a projection in $C^*(G') = C^*_r(G')$ whose homotopy class is independent of the choice of the cutoff function, hence defines a canonical $K$-theory element $\lambda_{G'} \in K_0(C^*(G'))$. If $Z$ is a $G$-compact proper space, and $B$ is a $G$-algebra, the map

$$KK^*_G(C_0(Z),B) \overset{j_{G,r}}{\to} KK^*(C^*(Z \rtimes G), B \rtimes_r G) \overset{\lambda_{Z\rtimes G}\otimes \cdot}{\to} K_*(B \rtimes_r G)$$

induces the Baum–Connes map

$$\mu_r : K^{\mathrm{top}}_*(G;B) = \lim_{\substack{Z\subset \underline{E}G \\ Z\ G\text{-compact}}} KK^*_G(C_0(Z),B) \to K_*(B \rtimes_r G),$$

where $\underline{E}G$ is the classifying space for proper actions of $G$ [2,18]. The so-called Baum–Connes conjecture with coefficients (to which counterexamples have recently been obtained) states that $\mu_r$ is an isomorphism.

### 2.6. Pseudogroups and groupoids

Let $X$ be a Hausdorff topological space. By *partial transformation* on $X$, we mean a homeomorphism $\varphi$: $\mathrm{Dom}\,\varphi \to \mathrm{Im}\,\varphi$ between open subsets of $X$. (More generally, a partial transformation on a $C^*$-algebra $A$ is a $C^*$-isomorphism between ideals of $A$.) If $\varphi$ and $\psi$ are partial transformations, the composition $\varphi\circ\psi$ is the ordinary composition of functions from $\mathrm{Dom}\,\psi \cap \psi^{-1}(\mathrm{Dom}\,\varphi)$ onto $\varphi(\mathrm{Dom}\,\varphi \cap \mathrm{Im}\,\psi)$. The inverse $\varphi^{-1}$ has domain $\mathrm{Im}\,\varphi$ and range $\mathrm{Dom}\,\varphi$. We call *unit elements* partial transformations of the form $\mathrm{Id}_U$, $U$ open.

A *pseudogroup* of partial transformations on $X$ is a set $\mathscr{G}$ of partial transformations on $X$ which is stable by composition and inverse (note that the empty transformation may belong to $\mathscr{G}$). It is *unital* if it contains the identity map $\mathrm{Id} : X \to X$. If $\mathscr{G}$ is a pseudogroup, and $\mathscr{U}$ denotes the set of all domains of elements of $\mathscr{G}$, then for all $U \in \mathscr{U}$, $\mathrm{Id}_U$ necessarily belongs to $\mathscr{G}$. It follows that for all $\varphi \in \mathscr{G}$ and for all $U \in \mathscr{U}$, the restriction of $\varphi$ to $\mathrm{Dom}\,\varphi \cap U$ belongs to $\mathscr{G}$, and that $\mathscr{U}$ is stable by finite intersection.

Let us introduce the following terminology: given a transformation pseudogroup $\mathcal{G}$, we say that $x \in X$ is $\mathcal{G}$-strongly fixed by $\varphi$ if there exists $U \in \mathcal{U}$ such that $\varphi_{|U} = \mathrm{Id}_U$ and $U \ni x$. Let us emphasize that this notion depends not only on $\varphi$, but also on the ambient pseudogroup $\mathcal{G}$. In particular, $\varphi$ may fix a neighborhood of a point which is not $\mathcal{G}$-strongly fixed by $\varphi$.

Given any pseudogroup $\mathcal{G}$, we associate a groupoid $G(\mathcal{G})$ in the following way: as a set, $G(\mathcal{G})$ is the quotient of

$$\{(\varphi,x) \in \mathcal{G} \times X \mid x \in \mathrm{Dom}\,\varphi\}$$

by the equivalence relation: $(\varphi,x) \sim (\psi,x)$ if and only if $\varphi(x)$ is strongly fixed by $\psi \circ \varphi^{-1}$ (that this actually constitutes an equivalence relation is an easy exercise). Using the 'charts' $U_\varphi = \{(\varphi,x) \mid x \in \mathrm{Dom}\,\varphi\}$ defined by the partial transformations, we endow $G(\mathcal{G})$ with a structure of a topological space such that the map $(\varphi,x)$ from $U_\varphi$ to $\mathrm{Dom}\,\varphi$ is a homeomorphism. The source and range maps of the groupoid are $s(\varphi,x) = x$, $r(\varphi,x) = \varphi(x)$. The inverse and the composition are defined by

$$(\varphi,x)^{-1} = (\varphi^{-1}, \varphi(x)),$$

$$(\varphi, \psi(x)) \circ (\psi, x) = (\varphi \circ \psi, x).$$

We note that $(\varphi,x)$ is a unit element of $G(\mathcal{G})$ if and only if $x$ is $\mathcal{G}$-strongly fixed by $\varphi$.

The set of units is $\bigcup_{U \in \mathcal{U}} U$. We now on assume that $X = \bigcup_{U \in \mathcal{U}} U$. Note that $U_\varphi$ is an open subset of $G(\mathcal{G})$ on which $r$ and $s$ are homeomorphisms onto open subsets of $X$, therefore $G(\mathcal{G})$ is étale.

**Example 2.10.** If $\Gamma$ is a subgroup of Homeo(X) endowed with the discrete topology, then the groupoid $G(\Gamma)$ is simply the crossed-product $X \rtimes \Gamma$.

**Lemma 2.11.** *Let $\mathcal{G}$ be a transformation pseudogroup on $X$. Then $G(\mathcal{G})$ is Hausdorff if and only if for every $\varphi \in \mathcal{G}$, the set of $\mathcal{G}$-strongly fixed points by $\varphi$ is closed in* $\mathrm{Dom}\,\varphi$. (*Note that it is open.*)

**Proof.** An étale groupoid $G$ is Hausdorff if and only if $G^{(0)}$ is Hausdorff and closed. Whence, $G(\mathcal{G})$ is Hausdorff if and only if $G(\mathcal{G})^{(0)} \cap U_\varphi$ is closed in $U_\varphi$ for every $\varphi \in \mathcal{G}$. The homeomorphism $U_\varphi \xrightarrow{\simeq} \mathrm{Dom}\,\varphi$ maps $G(\mathcal{G})^{(0)} \cap U_\varphi$ to the set of $\mathcal{G}$-strongly fixed points by $\varphi$. □

Let us note that in the case of a group of transformations $\Gamma \subset$ Homeo(X), since $\mathcal{U} = \{X\}$, an element $\varphi$ has $\mathcal{G}$-strongly fixed points if and only if $\varphi$ is the identity map, hence we find that $X \rtimes \Gamma$ is Hausdorff, as expected.

Finally, we observe that actions of $G(\mathcal{G})$ correspond to actions of $\mathcal{G}$ in the sense we define below:

A left *action* of $\mathcal{G}$ on a space $Z$ is given by a map $p: Z \to X$, and a homomorphism $\alpha$ from $\mathcal{G}$ to a transformation pseudogroup on $Z$ such that for every element $\varphi: U \to V$ of $\mathcal{G}$, we have

$\operatorname{Dom}\alpha_\varphi = p^{-1}(U)$, $\operatorname{Im}\alpha_\varphi = p^{-1}(V)$ and the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\alpha_\varphi"] \arrow[d, "p"'] & p^{-1}(V) \arrow[d, "p"] \\ U \arrow[r, "\varphi"] & V \end{tikzcd}$$

commutes. More generally, a left action of $\mathscr{G}$ on a $C^*$-algebra $A$ is given by a structure $\psi : C(X) \to Z(M(A))$ of $C(X)$-space on $A$, and a homomorphism $\alpha$ from $\mathscr{G}$ to a transformation pseudogroup on $A$ such that for every element $\varphi : U \to V$ of $\mathscr{G}$, we have $\operatorname{Dom}\alpha_\varphi = A_U = C_0(U)A$, $\operatorname{Im}\alpha_\varphi = A_V = C_0(V)A$ and the diagram

$$\begin{tikzcd} M(A_U) \arrow[r, "\alpha_\varphi"] & M(A_V) \\ C_0(U) \arrow[u, "\psi"] & C_0(V) \arrow[l, "\varphi^*"'] \arrow[u, "\psi"'] \end{tikzcd}$$

commutes.

## 3. The groupoid of a set with uniformly locally finite coarse structure

### 3.1. The pseudogroup $\mathscr{G}(X)$

**Definition 3.1.** Let $X$ be a set with a uniformly locally finite coarse structure. Every $A \in \Gamma_{\mathscr{E}}$ defines a partial transformation $\varphi_A$ on the Stone–Čech compactification $\beta X$ of $X$. We denote by $\mathscr{G}(X)$ the pseudogroup $\{\varphi_A \mid A \in \Gamma_{\mathscr{E}}\}$.

It is immediate that if $A \in \Gamma_{\mathscr{E}}$, $\varphi_A$ has clopen, compact domain and range. The pseudogroup $\mathscr{G}(X)$ is countably generated (resp. unital) if and only if the coarse structure is. Therefore, $\mathscr{G}(X)$ is countably generated and unital if and only if the coarse structure on $X$ comes from a (uniformly locally finite) metric.

### 3.2. The groupoid $G(X)$

Let us first introduce some notation. If $A$ is a subset of a set $Y$, we shall denote by $\bar{A}$ the closure of $A$ in the Stone–Čech compactification $\beta Y$ of $Y$.

Let $X$ be a set with a uniformly locally finite coarse structure $\mathscr{E}$. Define

$$G(\mathscr{E}) = \bigcup_{E \text{ entourage}} \bar{E} \subset \beta(X \times X).$$

We will also use the notation $G(X)$ instead of $G(\mathscr{E})$ if there is no ambiguity. Let us note that $G(X)$ is also the spectrum of the abelian $C^*$-subalgebra of $\ell^\infty(X \times X)$ generated by the characteristic functions $\chi_E$ of entourages $E$.

Recall that $X \times X$ is endowed with a structure of groupoid with source and range maps $s(x,y)=y$ and $r(x,y)=x$. The maps $r$ and $s$ extend to maps from $\beta(X\times X)$ to $\beta X$, hence to maps from $G(X)$ to $\beta X$.

**Proposition 3.2.** *Let $X$ be a set with a uniformly locally finite coarse structure. Recall that to $X$ is associated a unital pseudogroup $\mathscr{G}(X)$. There is a homeomorphism $G(\mathscr{G}(X)) \to G(X)$ such that the induced groupoid structure on $G(X)$ has source and range maps $r$ and $s$ defined above, and extends the groupoid structure on $X \times X$. The groupoid $G(X)$ is étale, locally compact, Hausdorff and principal.*

**Proof.** Denote by $G'(X)$ the groupoid $G(\mathscr{G}(X))$. The space $G(X)$ is locally compact Hausdorff because it is open in the compact space $\beta(X\times X)$. The groupoid $G'(X)$ is étale by construction.

Let us prove that $G(X)$ and $G'(X)$ are homeomorphic. Recall the notation: $\Gamma_{\mathscr{E}}$ is the set of entourages on which $r$ and $s$ are injective. Since any entourage is a finite union of elements of $\Gamma_{\mathscr{E}}$ (see Lemma 2.8 by $\mathrm{E}_0$), we have

$$G(X) = \bigcup_{A\in\Gamma_{\mathscr{E}}} \bar{A}.$$

The groupoid $G'(X)$ is a quotient of $\amalg_{A\in\Gamma_{\mathscr{E}}} \bar{A}$. By definition, we have a surjective, continuous, open map $\psi: \amalg_{A\in\Gamma_{\mathscr{E}}} \bar{A} \to G(X)$. We just have to show that $\psi$ induces a bijection from $G'(X)$ to $G(X)$.

By Lemma 2.7(b), the map $(r,s): G(X) \to \beta X \times \beta X$ is one to one; as $(r,s)$ is defined on the groupoid $G'(X)$, the map $\psi$ passes to the quotient.

Let $A$ and $B \in \Gamma_{\mathscr{E}}$. Let $\bar{x} \in \overline{s(A)\cap s(B)}$; if $g=(\varphi_A,\bar{x})$ and $h=(\varphi_B,\bar{x})$ have distinct images in the groupoid $G(X)$, then $g \notin \overline{A\cap B}$. Then $g \in \overline{A\setminus B}$ and $h \in \bar{B}$ are distinct in $G(X)$.

The fact that $G(X)$ is principal follows from Lemma 2.7(b). $\square$

The set of objects $G(X)^{(0)}$ of $G(X)$ is $\bigcup_A \bar{A} \subset \beta(X)$, the union being taken over subsets $A$ of $X$ such that $\Delta_A = \{(x,x) \mid x\in A\} \in \mathscr{E}$. If $\mathscr{E}$ is unital $G(X)^{(0)} = \beta X$.

### 3.3. Expressing $G(X)$ as a crossed-product

In what follows, $X$ is a set with a uniformly locally finite metric. A unital sub-pseudogroup $\mathscr{A}$ of $\mathscr{G}(X)$ will be called *admissible* if it covers $G(X)$, i.e., identifying elements of $\mathscr{G}(X)$ with open subspaces of $G(X)$, we have $\bigcup_{\varphi\in\mathscr{A}} \varphi = G(X)$. Let $\mathscr{A}$ be a countable admissible pseudogroup of $\mathscr{G}(X)$. Its existence follows from the fact that the coarse structure is countably generated. Let $G_{\mathscr{A}}$ be the spectrum of the sub-$C^*$-algebra of $C_0(G(X))$ generated by $\{\chi_A \mid \varphi_A \in \mathscr{A}\}$, and let $X_{\mathscr{A}}$ be the spectrum of the sub-$C^*$-algebra of $C(\beta X) = \ell^\infty(X)$ generated by $\{\chi_{s(A)} \mid \varphi_A \in \mathscr{A}\}$.

**Lemma 3.3.** *Let $X$ be a set with a uniformly locally finite metric. Then*

(a) *There exists a countable admissible sub-pseudogroup of $\mathscr{G}(X)$.*

(b) *If $\mathscr{A}$ is as in* (*a*), *then $G_{\mathscr{A}}$ is a metrizable* (*as a topological space*), *locally compact, Hausdorff, σ-compact groupoid with unit space $X_{\mathscr{A}}$. The groupoid $G_{\mathscr{A}}$ naturally acts on $\beta X = G(X)^{(0)}$ and $G(X) = \beta X \rtimes G_{\mathscr{A}}$.*

**Proof.** (a) Follows from the fact that the coarse structure is countably generated. To prove (b), let $p:\beta X \twoheadrightarrow X_{\mathscr{A}}$ be the map induced by $C(X_{\mathscr{A}}) \hookrightarrow C_b(X)$. For every $A \in \mathscr{A}$, let $\Omega_A = \chi_A C_0(G_{\mathscr{A}})$ and $\Omega'_{s(A)} = \chi_{s(A)} C(X_{\mathscr{A}})$. These sets are clopen in $G_{\mathscr{A}}$ and $X_{\mathscr{A}}$, respectively. It is easy to see that $\Omega_A$ is the spectrum of $\{\chi_B \,|\, B \subset A\}$ and $\Omega'_{s(A)}$ is the spectrum of $\{\chi_{s(B)} \,|\, B \subset A\}$, so the source map $s: G(X) \to G(X)^{(0)}$ passes through the quotient and defines $s: G_{\mathscr{A}} \to X_{\mathscr{A}}$ which is a homeomorphism on each $\Omega_A$, $A \in \mathscr{A}$.

Since $\chi_A\chi_{A'} = \chi_{A\cap A'}$, we have $\Omega_A \cap \Omega_{A'} = \Omega_{A\cap A'}$. One can also show that if $B$, $B' \in \mathscr{A}$ and $B$, $B' \subset \Delta$, then $\Omega'_B \cap \Omega'_{B'} = \Omega'_{B\cap B'}$, so we get the following composition of isomorphisms $\Omega_A \times_{X_{\mathscr{A}}} \Omega_B \overset{s o pr_2}{\to} \Omega'_{s(A\circ B)} \overset{\to}{s^{-1}} \Omega_{A\circ B}$.

The action of $G_{\mathscr{A}}$ on $\beta X$ is defined as follows: let $\bar{x} \in \beta X$, $g \in \Omega_A$ such that $r(g) = p(\bar{x})$. There exists a unique $\tilde{g} \in \bar{A} \subset G(X)$ such that $r(\tilde{g}) = \bar{x}$. We define $\bar{x}g = s(\tilde{g})$.

The crossed-product $\beta X \rtimes G_{\mathscr{A}}$ is the quotient of $(\amalg \beta(r(A)) \times_{X_{\mathscr{A}}} \Omega_A)/\sim$ by the identification of $\beta(r(A)) \times_{X_{\mathscr{A}}} \Omega_A$ and $\beta(r(B)) \times_{X_{\mathscr{A}}} \Omega_B$ on $\beta(r(A\cap B)) \times_{X_{\mathscr{A}\cap\mathscr{B}}} \Omega_{A\cap B}$. Noting that $\beta(r(A)) \times_{X_{\mathscr{A}}} \Omega_A$ is homeomorphic to the closure $\bar{A}$ of $A$ in $\beta(X\times X)$, we conclude $\beta X \rtimes G_{\mathscr{A}} \simeq \bigcup_{A\in\mathscr{A}} \bar{A} \simeq G(X)$ since by assumption $\mathscr{A}$ covers $G(X)$. □

If for example $X$ is the metric space $|\Gamma|$ underlying a countable group $\Gamma$ and $\mathscr{A}$ is the group $\Gamma$ acting on the right on itself, then $X_{\mathscr{A}}$ is a point and $G_{\mathscr{A}} = \Gamma$, so we get

**Proposition 3.4.** *If $\Gamma$ is a countable group endowed with the left coarse structure*, *then $G(|\Gamma|) = (\beta|\Gamma|) \rtimes \Gamma$.*

### 3.4. Coarse correspondences and groupoid homomorphisms

Let $X$ and $Y$ be sets with uniformly locally finite coarse structures $\mathscr{E}_X$ and $\mathscr{E}_Y$ and $\mathscr{C}$ be a uniformly locally finite coarse structure on $X \amalg Y$, which is a coarse correspondence from $X$ to $Y$. We say that $\mathscr{C}$ is a *uniformly locally finite coarse correspondence*. Let $G(X \amalg Y)$ be the groupoid associated with the uniformly locally finite coarse structure $\mathscr{C}$. The restriction of $G(X\amalg Y)$ to the clopen subset $G(Y)^{(0)}$ of $G(X\amalg Y)^{(0)}$ is $G(Y)$. Moreover, if $A \subset X$ is such that $\Delta_A \in \mathscr{C}$, then by restriction we have a coarse correspondence from the restriction of $\mathscr{E}_X$ to $A$ and $\mathscr{E}_Y$. By Proposition 2.3, there is a coarse map $f: A \to Y$ whose graph is in $\mathscr{C}$, hence $\mathrm{Gr}(f)$ is a finite union of elements of $\Gamma_{\mathscr{C}}$ (Lemma 2.8). It follows that the $G(X\amalg Y)$-orbits of elements of $G(Y)^{(0)}$ cover $\bar{A}$, and thus they cover all of $G(X\amalg Y)^{(0)}$. Therefore the groupoids $G(X\amalg Y)$ and $G(Y)$ are Morita equivalent. Now, the inclusion $G(X) \to G(X \amalg Y)$ is a generalized groupoid homomorphism $G(X) \to G(Y)$. In this way we proved:

**Proposition 3.5.** *Let $X$ and $Y$ be sets with uniformly locally finite coarse structures $\mathscr{E}_X$ and $\mathscr{E}_Y$. Any uniformly locally finite coarse correspondence between $\mathscr{C}$ from $X$ to $Y$ defines a* (*generalized*) *groupoid homomorphism* $G(X) \to G(Y)$. *This construction is compatible with the composition of correspondences and groupoid homomorphisms.*

**Corollary 3.6.** *If $X$ and $Y$ are coarse-equivalent sets with uniformly locally finite coarse structures, then the groupoids $G(X)$ and $G(Y)$ are Morita-equivalent.*

## 4. Relation between the coarse assembly map and the Baum–Connes assembly map for groupoids

Since the groupoid $G(X)$ is not metrizable, one may wish to replace it by a metrizable groupoid, thanks to Lemma 3.3. The following lemma shows that this can be done without altering the Baum–Connes assembly map.

**Lemma 4.1.** *Let $G$ be a locally compact, Hausdorff, $\sigma$-compact groupoid with Haar system, $Z$ a locally compact $G$-space with corresponding source map $p: Z \to G^{(0)}$. We suppose that $p$ is proper. Let $G'$ be the groupoid $Z \rtimes G$, and $B$ a $G'$-algebra. Then, we have a commutative diagram, where vertical maps are isomorphisms*

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G';B) & \xrightarrow{\mu_r} & K_*(B \rtimes_r G') \\ \simeq \downarrow & & \downarrow = \\ K_*^{\mathrm{top}}(G;B) & \xrightarrow{\mu_r} & K_*(B \rtimes_r G) \end{array}$$

**Proof.** First, $B \rtimes_r G' = B \rtimes_r G$ is an immediate consequence of the definition of a crossed-product. Next, we have $\underline{E}G' = Z \times_{G^{(0)}} \underline{E}G$, and for every $G$-compact ($G$-invariant) subspace $T$ of $\underline{E}G$, the following diagram commutes:

$$\begin{array}{ccc} KK_{G'}^*(C_0(Z \times_{G^{(0)}} T), B) & \xrightarrow{\simeq} & KK_G^*(C_0(T), B) \\ j_{G',r} \downarrow & & \downarrow j_{G,r} \\ KK_*(C_0(Z \times_{G^{(0)}} T) \rtimes G, B \rtimes_r G) & \xrightarrow{i^*} & KK_*(C_0(T) \rtimes G, B \rtimes_r G) \\ \downarrow \lambda_{(Z \times_{G^{(0)}} T) \rtimes G} \otimes & & \downarrow \lambda_{T \rtimes G} \otimes \\ K_*(B \rtimes_r G) & \xrightarrow{=} & K_*(B \rtimes_r G) \end{array}$$

The proper map $p$ induces a proper map $q: Z \times_{G^{(0)}} T \to T$. We therefore have a $G$-equivariant embedding $C_0(T) \hookrightarrow C_0(Z \times_{G^{(0)}} T)$, whence a $*$-morphism $i: C_0(T) \rtimes G \to C_0(Z \times_{G^{(0)}} T) \rtimes G$.

Let $c \in C_c(T)$ such that $\forall t \in T$, $\int_{g \in G^{s(t)}} c(tg)\, \lambda^{s(t)}(\mathrm{d}g) = 1$. Then $\lambda_{T \rtimes G}$ is the projection of $C_0(T) \rtimes G$ defined by $c(t)^{1/2} c(tg)^{1/2}$, and $\lambda_{(Z \times_{G^{(0)}} T) \rtimes G}$ is the image by $i$ of $\lambda_{T \rtimes G}$ in $C_0(Z \times_{G^{(0)}} T) \rtimes G$. $\square$

**Lemma 4.2.** *Let $G$ be a locally compact, Hausdorff, $\sigma$-compact groupoid with Haar system, $A$ and $B$ two $G$-algebras, $\mathscr{E}_1 \subset \mathscr{E}_2$ two $G$-equivariant $B$-modules, $\varphi: A \to \mathscr{L}(\mathscr{E}_2)$ a* $G$-equivariant

*$*$-homomorphism, $F = F^* \in \mathscr{L}(\mathscr{E}_2)$. We suppose that $\mathscr{E}_1$ is stable by $F$ and by $\varphi(a)$ for all $a \in A$. Assume that $\forall a \in A$, $\varphi(a)(F^2-1) \in \mathscr{K}(\mathscr{E}_1)$, $[\varphi(a),F] \in \mathscr{K}(\mathscr{E}_1)$ and $\forall a' \in r^*A$, $a'(Vs^*FV^* - r^*F) \in r^*\mathscr{K}(\mathscr{E}_1)$ where $V \in \mathscr{L}(s^*\mathscr{E}_2, r^*\mathscr{E}_2)$ is the unitary that defines the action of $G$ on $\mathscr{E}_2$.*

*Then, $[(\mathscr{E}_1,\varphi,F)] = [(\mathscr{E}_2,\varphi,F)]$ in $KK_G(A,B)$.*

**Proof.** Let $\mathscr{E} = \{\xi \in C([0,1],\mathscr{E}_2) \mid \xi(0) \in \mathscr{E}_1\}$. Then $\mathscr{E}$ is stable by $F\otimes 1$ and $\varphi(a)\otimes 1 \in \mathscr{L}(\mathscr{E}_2 \otimes C([0,1]))$, and the restriction $(\mathscr{E},\varphi\otimes 1, F\otimes 1)$ defines a homotopy between $(\mathscr{E}_1,\varphi,F)$ and $(\mathscr{E}_2,\varphi,F)$. $\square$

Let $G$ be a locally compact, Hausdorff, $\sigma$-compact groupoid with Haar system. The completion of $C_c(G)$ endowed with the $C_0(G^{(0)})$-valued scalar product

$$\langle \xi,\eta\rangle(x) = \int_{g\in G^x} \bar{\xi}(x)\eta(x)\,\mathrm{d}\lambda^x(g)$$

is a Hilbert $C_0(G^{(0)})$-module that we shall denote by $L^2(G)$. More generally, for any $C^*$-algebra $B$ with an action $\alpha$ of $G$ and any non-$G$-equivariant $B$-module $\mathscr{E}$, there is a $G$-equivariant $B$-module, denoted by $L^2(G,\mathscr{E})$, obtained by completion of $C_c(G)\odot_{C_0(G^{(0)})}\mathscr{E}$ with the $B$-valued scalar product

$$\langle \xi_1\otimes\eta_1, \xi_2\otimes\eta_2\rangle(x) = \int_{G^x} \overline{\xi_1(g)}\xi_2(g)\alpha_g(\langle\eta_1,\eta_2\rangle_{s(g)})\,\mathrm{d}\lambda^x(g).$$

The action of $B$ is $[(\xi\otimes\eta)b](g) = \xi(g)\eta_{s(g)}\alpha_{g^{-1}}(b_{r(g)})$, and the action of the groupoid is defined by $\alpha_g(\xi\otimes\eta) = \alpha_g(\xi)\otimes\eta$, where $(\alpha_g(\xi))(h) = \xi(g^{-1}h)$. If now $\mathscr{E}$ is endowed with an action $\alpha$ of $G$, then we have an isomorphism $L^2(G,\mathscr{E}) \to L^2(G)\otimes_{C_0(G^{(0)})}\mathscr{E}$ defined by $e \mapsto e'$, $e \in C_c(G,\mathscr{E})$, $e'(g) = \alpha_g(e(g))$.

**Lemma 4.3.** *Let $G$ be a locally compact, Hausdorff groupoid with Haar system, $D$ a $G$-algebra and $J$ a $G$-equivariant ideal of $D$. Assume that $D \to M(J)$ is faithful. Then $D\rtimes_r G$ is faithfully represented in $L^2(G,J)$.*

**Proof.** By definition, $D\rtimes_r G$ is faithfully represented in $\mathscr{E} = L^2(G,D)$. Since the map $T\mapsto T\otimes 1$ from $\mathscr{L}(\mathscr{E})$ to $\mathscr{L}(\mathscr{E}\otimes_D J)$ is isometric, the result follows. $\square$

**Lemma 4.4.** *Let $X$ be a set with a uniformly locally finite coarse structure, and $B$ a $C^*$-algebra. There is a natural isomorphism $C^*(X;B) \simeq \ell^\infty(X, B\otimes\mathscr{K})\rtimes_r G(X)$.*

**Proof.** Let $D = \ell^\infty(X,B\otimes\mathscr{K})$. Then $G(X)$ obviously acts on $D$. Indeed, any $A\in\Gamma_{\mathscr{E}}$ defines a $*$-isomorphism between $D_{s(A)}$ and $D_{r(A)}$.

From Lemma 4.3 with $D = \ell^\infty(X,B\otimes\mathscr{K})$ and $J = C_0(X,B\otimes\mathscr{K})$, $D\rtimes_r G(X)$ is faithfully represented in $L^2(G(X),C_0(X,B\otimes\mathscr{K})) \simeq C_0(X)\otimes\ell^2(X)\otimes B\otimes\mathscr{K}$, where $G(X)$ and $D$ act on the factor $\ell^2(X)\otimes B\otimes\mathscr{K}$. Since the map $T\mapsto 1\otimes T$ from $\mathscr{L}(\ell^2(X)\otimes B\otimes\mathscr{K})$ to $\mathscr{L}(C_0(X)\otimes\ell^2(X)\otimes B\otimes\mathscr{K})$ is isometric, we have $D\rtimes_r G(X)\hookrightarrow \mathscr{L}(\ell^2(X)\otimes B\otimes\mathscr{K})$.

We show that $D\rtimes_r G(X)$ and $C^*(X)$ are equal as subalgebras of $\mathscr{L}(\ell^2(X)\otimes H_B)$. The algebra $D\rtimes_r G(X)$ is generated by elements of the form $d\otimes\chi_A$, where $A\in\Gamma_{\mathscr{E}}$ and $d\in\ell^\infty(X,B\otimes\mathscr{K})$. Put $T_{x,y}=d(x)$ if $(x,y)\in A$, and $T_{x,y}=0$ otherwise. Then, $T=(T_{x,y})_{(x,y)\in X\times X}$ is in $C^*(X;B)$, and is equal to $d\otimes\chi_A$. Conversely, $C^*(X;B)$ is generated by operators $T=(T_{x,y})_{(x,y)\in X\times X}$ with support on some $A\in\Gamma_{\mathscr{E}}$, hence we prove likewise that $C^*(X;B)\subset D\rtimes_r G(X)$. □

**Lemma 4.5.** *Let $A$ and $B$ be $C^*$-algebras and $X$ a set with a uniformly locally finite coarse structure. We have a commutative diagram*

$$\begin{array}{ccc} KK(A,B) & \xrightarrow{\bar\sigma_X} & KK(C^*(X;A),C^*(X;B)) \\ \big\downarrow k & & \big\downarrow \simeq \\ KK_{G(X)}(\ell^\infty(X,A),\ell^\infty(X,B)) & \xrightarrow{J_{G(X)}} & KK(A\rtimes_r G(X),B\rtimes_r G(X)), \end{array}$$

*where $k(\mathscr{E},\varphi,F)=(\ell^\infty(X,\mathscr{E}),\tilde\varphi,\tilde F)$, $\tilde\varphi((a_x)_{x\in X})=(\varphi(a_x))_{x\in X}$, $\tilde F((\xi_x)_{x\in X})=(F(\xi_x))_{x\in X}$.*

**Proof.** Straightforward. □

**Lemma 4.6.** *Let $G$ be a locally compact, Hausdorff, $\sigma$-compact groupoid with Haar system, $Z$ a proper $G$-space, $B$ a $G$-algebra. Then, every element of $KK_G(C_0(Z),B)$ can be represented by a Kasparov bimodule of the form $(L^2(G)\hat\otimes_{C_0(G^{(0)})}\otimes\hat H_B,\varphi,F)$ with $F=F^*$ a $G$-invariant operator.*

**Proof.** Let $u$ be an element of $KK_G(C_0(Z),B)$, represented by a $G$-equivariant $C_0(Z)$, $B$-bimodule $(\mathscr{E},\varphi,F)$. Let $c$ be a cutoff function on $Z$. We can replace $F$ with $(F'+F'^*)/2$, where $F'_x=\int_{G^x}\alpha_g(c)\alpha_g(F_{s(g)})\,\lambda^x(\mathrm{d}g)$, so we can suppose that $F$ is $G$-invariant. The isometry

$$f\mapsto f'(z,g)=f(z)c(zg)^{1/2}$$

embeds $C_0(Z)$ as a direct factor of $C_0(Z)\otimes_{C_0(G^{(0)})}L^2(G)$. By applying $\cdot\otimes_{C_0(Z)}\mathscr{E}$, we see that $\mathscr{E}$ is a direct factor of $L^2(G)\otimes_{C_0(G^{(0)})}\mathscr{E}\simeq L^2(G,\mathscr{E})$. By Kasparov's stabilization Theorem, $\mathscr{E}$ is (non-equivariantly) a direct factor of $\hat H_B$, therefore we can represent $u$ in $L^2(G,\hat H_B)\simeq L^2(G)\hat\otimes_{C_0(G^{(0)})}\hat H_B$. □

In what follows, identifying elements of $P_E(X)\times\{x\}$ with probability measures on $s^{-1}(x)\subset G_{\mathscr{A}}$, $P_E(G_{\mathscr{A}})$ will (abusively) denote the closure of the image of $P_E(X)\times X$ in the set of probability measures on $G_{\mathscr{A}}$, endowed with the weak topology of the dual of $C_c(G_{\mathscr{A}})$.

**Lemma 4.7.** *Let $X$ be a discrete uniformly locally finite metric space, $B$ a $C^*$-algebra, and $d>0$. With the above notations, we have a commutative diagram where vertical maps* *are isomorphisms*

$$\begin{array}{ccc} KK^*_{G_{\mathscr{A}}}(C_0(P_E(G_{\mathscr{A}})),\ell^\infty(X,B)) & \xrightarrow{\mu_r} & K_*(\ell^\infty(X,B)\rtimes_r G_{\mathscr{A}}) \\ \downarrow i^* & & \downarrow = \\ KK_*(C_0(P_E(X)),B) & \xrightarrow{A} & K_*(C^*(X;B)) \end{array}$$

*$i^*$ is defined as follows: let $x$ be an arbitrary point on $X$. Then $i:\{x\}\hookrightarrow G(X)$ is a groupoid homomorphism, and $i^*$ is the natural transformation $KK_{G_{\mathscr{A}}}\to KK$ induced by $i$.*

**Proof.** To prove that $i^*$ is an isomorphism, let us define a map $j$ in the other direction: for every Kasparov bimodule $(\hat{H}_B,\varphi,F)\in \mathbf{E}(C_0(P_E(X)),B)$, let $\mathscr{E}=\ell^\infty(X,B\otimes\hat{\mathscr{K}})$, $((\tilde{\varphi}(a))(\xi))(x)=\varphi(a(x))\xi(x)$, $(\tilde{F}\xi)(x)=F\xi(x)$. It is obvious that $i^*\circ j=\mathrm{Id}$. Conversely, if $(\mathscr{E},\varphi,F)\in \mathbf{E}_{G_{\mathscr{A}}}(C_0(P_E(G_{\mathscr{A}})),\ell^\infty(X,B))$, then we can suppose from Lemma 4.6 that $F$ is $G$-equivariant, and that $\mathscr{E}=\ell^\infty(X,B)\otimes L^2(G_{\mathscr{A}})\hat{\otimes}\hat{H}$. But $\mathscr{E}_1$ is a submodule of $\mathscr{E}'=\ell^\infty(X,B\otimes\ell^2(X)\hat{\otimes}\hat{H})\simeq \ell^\infty(X,B\otimes\hat{\mathscr{K}})$, so we can suppose by Lemma 4.2 that $\mathscr{E}=\ell^\infty(X,B\otimes\hat{\mathscr{K}})$, and then it is obvious that $j\circ i^*=\mathrm{Id}$.

To prove that the diagram is commutative, it suffices to note that, for $A=C_0(P_E(X))\otimes\mathscr{K}$,

$$\begin{array}{ccc} KK_*(A,B) & \xrightarrow{\bar{\sigma}_X} & K_*(C^*(X;A),C^*(X;B)) \\ \downarrow k & & \downarrow \simeq \\ KK^*_{G_{\mathscr{A}}}(\ell^\infty(X,A),\ell^\infty(X,B)) & \xrightarrow{j_G} & KK_*(\ell^\infty(X,A)\rtimes G_{\mathscr{A}},C^*(X;B)) \\ \downarrow & & \downarrow \\ KK^*_{G_{\mathscr{A}}}(C_0(P_E(G_{\mathscr{A}})),\ell^\infty(X,B)) & \xrightarrow{j_G} & KK_*(C_0(P_E(G_{\mathscr{A}}))\rtimes G_{\mathscr{A}},C^*(X;B)) \end{array}$$

commutes (see Lemma 4.5), and that the composition of the two vertical maps on the left is $j$. □

**Proposition 4.8.** *Let $X$ be a uniformly locally finite metric space, $B$ a $C^*$-algebra. We have a commutative diagram, whose vertical maps are isomorphisms*:

$$\begin{array}{ccc} K^{\mathrm{top}}_*(G(X);\ell^\infty(X,B)) & \xrightarrow{\mu_r} & K_*(\ell^\infty(X,B)\rtimes_r G(X)) \\ \downarrow \simeq & & \downarrow = \\ K^{\mathrm{top}}_*(G_{\mathscr{A}};\ell^\infty(X,B)) & \xrightarrow{\mu_r} & K_*(\ell^\infty(X,B)\rtimes_r G_{\mathscr{A}}) \\ \downarrow \simeq & & \downarrow = \\ KK_*(X;B) & \xrightarrow{A} & K_*(C^*(X;B)) \end{array}$$

**Proof.** The assertion for the upper square follows from Lemmas 3.3 and 4.1. For the lower square, it results from Lemma 4.7. □

**Corollary 4.9** (Yu [21]). *Let $\Gamma$ be a discrete countable group, and $|\Gamma|$ its underlying metric space endowed with the coarse structure of Example* 2.5. *The Baum–Connes assembly map for $\Gamma$ with coefficients in $\ell^\infty(\Gamma, \mathscr{K})$ is equivalent to the coarse assembly map for $|\Gamma|$.*

## 5. Property A and uniform embedding

Let $X$ be a uniformly locally finite metric space and let $G(X)$ be the associated groupoid. In this section we relate some properties of $X$ with properties of the groupoid. More precisely, we prove that

1. Property (A) introduced by Yu in [22] is equivalent to the amenability of the groupoid $G(X)$. This generalizes the main result of [12], and actually the proof is almost the same as in [12].
2. The space $X$ admits a uniform embedding into Hilbert space if and only if the groupoid $G(X)$ has a proper affine action on a Hilbert bundle (in the sense of [19]).

### *5.1. Property A*

Let us recall property A of [22] and some definitions related with amenability taken from [12] (cf. also [12]). We need some notation:

- recall that $\Delta_r = \{(x, y) \mid d(x, y) \leqslant r\}$.
- Let $Y$ be a subset of $X \times X \times \mathbb{N}$; for $x$, set $Y^x = \{(z, n) \in X \times \mathbb{N} \mid (x, z, n) \in Y\}$.

**Definition 5.1** (Yu [22]). The space $X$ is said to satisfy property (A) if there exists a sequence of subsets $Y_n \subset X \times X \times \mathbb{N}$ such that

(a) For every $n \in \mathbb{N}$, $\Delta_0 \times \{0\} \subset Y_n$; moreover, for every $x \in X$, the set $Y_n^x$ is finite;
(b) For every $n \in \mathbb{N}$, there exists $r \in \mathbb{R}_+$ such that $Y_n$ is contained in $\Delta_r \times \mathbb{N}$;
(c) The sequence of functions $g_n(x, y) = \#(Y_n^x \cap Y_n^y)/\#(Y_n^x \cup Y_n^y)$ converges to 1 uniformly on every $\Delta_r$.

We next recall the definition of amenability. Note that if $G$ is an étale groupoid, an $r$-system in the sense of [1] is just a function on $G$.

**Definition 5.2** (Anantharaman-Delaroche and Rehault [1]). An étale groupoid $G$ is said to be amenable if given compact subsets $K \subset G$, $C \subset G^{(0)}$ and $\epsilon > 0$, there exists $f \in C_c(G)$ such that

(a) $\forall u \in C$, we have $\sum_{x \in G^u} f(x) = 1$.
(b) $\forall x \in K$, we have $\sum_{y \in G^{s(x)}} |f(xy) - f(y)| \leqslant \varepsilon$.

Of course, if $G^{(0)}$ is compact, one just takes $C = G^{(0)}$. Furthermore, if $G$ is $\sigma$-compact, one finds a sequence $f_n$ of functions such that

(a) $\forall u \in G^{(0)}$ and every $n \in \mathbb{N}$, we have $\sum_{x \in G^u} f_n(x) = 1$.
(b) The sequence $\sum_{y \in G^{s(x)}} |f_n(xy) - f_n(y)|$ converges to 0 uniformly on compact subsets of $G$.

As in [12], denote by $C_u^*(X)$ the completion of the $*$-algebra of operators in $\ell^2(X)$ whose support is an entourage. Generalizing a result of Higson and Roe [12], we get:

**Theorem 5.3.** *Let $X$ be a uniformly locally finite metric space. The following are equivalent*:

(i) *The space $X$ satisfies property $(A)$*;
(ii) *There is a sequence of functions $f_n : X \times X \to [0,1]$ such that*
 (a) *The support of $f_n$ is an entourage*;
 (b) *For all $x \in X$, $\sum_y f_n(x,y) = 1$*;
 (c) *For every $r$, we have $\lim_{n\to\infty} \sup\{\sum_z |f_n(x,z) - f_n(y,z)|;\ (x,y) \in \Delta_r\} = 0$.*
(iii) *The groupoid $G(X)$ is amenable.*
(iv) *There is a countable admissible sub-pseudogroup $\mathscr{A} \subset \mathscr{G}(X)$ such that the associated groupoid $G(\mathscr{A})$ is amenable.*
(vi) *$C_u^*(X)$ is nuclear.*

**Proof.** The proof is almost exactly the same as the one in [12].

We prove (ii) $\Rightarrow$ (i). Assume that $f_n$ has support in some $\Delta_{r_n}$; let $R_n = \sup_{x\in X} \#\{y \mid (x,y) \in \Delta_{r_n}\}$. Put then

$$Y_n = (\Delta_0 \times \{0\}) \cup \{(x,y,k) \in X \times X \times \mathbb{N} \mid k < nR_n f_n(x,y)\}.$$

It is clear that if $(x,y,k) \in Y_n$, then $f_n(x,y) \neq 0$, whence $(x,y) \in \Delta_{r_n}$; moreover, since $f_n \leqslant 1$, if $(y,k) \in Y_n^x$, then $(x,y) \in \Delta_{r_n}$ and $k < nR_n$; whence $Y_n^x$ has at most $nR_n^2$ points.

Note that for every $x,z \in X$, setting $B(x,r_n) = \{z \in X \mid d(x,z) \leqslant r_n\}$, we have

$$\#\{k \in \mathbb{N} \mid (x,z,k) \in Y_n\} \geqslant nR_n f(x,z) - 1,$$

hence

$$\#Y_n^x \geqslant \sum_{z\in B(x,r_n)} (nR_n f(x,z) - 1) = nR_n - \#B(x,r_n) \geqslant (n-1)R_n.$$

Moreover, for every $x,y,z \in X$,

$$\#\{k \in \mathbb{N} \mid (z,k) \in Y_n^x \triangle Y_n^y\} \leqslant 1 + nR_n |f(x,z) - f(y,z)|,$$

whence

$$\begin{aligned}\#(Y_n^x \triangle Y_n^y) &\leqslant \sum_{z\in B(x,r_n)\cup B(y,r_n)} (1 + nR_n|f(x,z) - f(y,z)|)\\ &\leqslant 2R_n + nR_n \sum_{z\in X} |f(x,z) - f(y,z)|.\end{aligned}$$

It follows that condition (c) is satisfied.

- To prove that (ii) $\Leftrightarrow$ (iii), one just needs to notice that, since $X \times X$ is dense in $G(X)$, $C_c(G(X))$ is the set of bounded functions defined on $X \times X$ with support in some $\Delta_r$.
- To prove that (iv) $\Rightarrow$ (iii), just note that, since $G(\mathscr{A})$ is a quotient of $G(X)$, $C_c(G(\mathscr{A})) \subset C_c(G(X))$. Therefore, the sequence of functions $f_n \in C_c(G(\mathscr{A}))$ giving the amenability of the groupoid $G(\mathscr{A})$, also gives the amenability of the groupoid $G(X)$.

- To prove that (i) $\Rightarrow$ (iv), for $n, k \in \mathbb{N}$, $k > 0$, let $Z_n^k$ be the set of $(x, y) \in X \times X$, such that $\#\{j \in \mathbb{N} \mid (x, y, j) \in Y_n\} = k$; there exist a countable set $D$ of partial transformations such that each $Z_n^k$ is a finite disjoint union of elements in $D$. Let $\mathscr{A}$ be the subpseudogroup generated by $D$. The functions on $X \times X$ given by $f_n(x, y) = \#\{j \in \mathbb{N} \mid (x, y, j) \in Y_n\}/\#(Y_n^x)$, are restrictions of elements of $C_c(G(\mathscr{A}))$ which in turn imply amenability of the groupoid $G(\mathscr{A})$.
- (iii) $\Leftrightarrow$ (v) follows from the obvious fact that $C_u^*(X) \simeq C_r^*(G(X))$. $\square$

Of course, one may now express the amenability of $G(\mathscr{A})$ using many equivalent conditions (cf. [1]).

### 5.2. Uniform embedding

Let $G$ be a groupoid. Recall that a negative type function on $G$ is a function $f : G \to \mathbb{R}$ such that

(a) $f_{|G^{(0)}} = 0$.
(b) $\forall x \in G,\ f(x^{-1}) = f(x)$.
(c) Given $x_1, x_2, \ldots, x_n \in G$ all having the same range and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum \lambda_k = 0$, we have $\sum_{j,k} f(x_j^{-1} x_k)\lambda_j \lambda_k \leqslant 0$.

Moreover, if $G$ is a locally compact (Hausdorff) second countable groupoid with compact unit space, the following are equivalent (cf. [19]):

(i) There exists a proper negative type function on $G$.
(ii) There is a continuous field of Hilbert spaces over $G^{(0)}$ with a proper affine action of $G$.

**Theorem 5.4.** *Let $X$ be a uniformly locally finite metric space. The following are equivalent*:

(i) *The space $X$ admits a uniform embedding into a Hilbert space*;
(ii) *There exists a negative type function $f : X \times X \to \mathbb{R}$ such that*
   (a) *$f$ is bounded on every entourage*;
   (b) *For all $r \in \mathbb{R}_+$, $\{(x, y) \in X \times X \mid |f(x, y)| \leqslant r\}$ is an entourage.*
(iii) *There is a continuous proper negative type function $f : G(X) \to \mathbb{R}$.*
(iv) *There is a countable admissible sub-pseudogroup $\mathscr{A} \subset \mathscr{G}(X)$ and a continuous proper negative type function $f : G(\mathscr{A}) \to \mathbb{R}$.*

**Proof.** We prove (i) $\Rightarrow$ (ii): if $g : X \to H$ is a uniform embedding into a Hilbert space, set $f(x, y) = ||g(x) - g(y)||^2$. Conditions (a) and (b) are obvious. Let $x \mapsto \lambda_x$ be a function on $X$ with finite support such that $\sum \lambda_x = 0$. We have

$$\sum_{x,y} \lambda_x \lambda_y ||g(x) - g(y)||^2$$

$$=\sum_{x,y}\lambda_x\lambda_y(||g(x)||^2+||g(y)||^2-2\langle g(x),g(y)\rangle)$$

$$=\left(\sum_x\lambda_x||g(x)||^2\right)\left(\sum_y\lambda_y\right)+\left(\sum_x\lambda_x\right)\left(\sum_y\lambda_y||g(y)||^2\right)$$

$$-2\left\langle\sum_x\lambda_x g(x),\sum_y\lambda_y g(y)\right\rangle$$

$$\leqslant 0.$$

- We prove (ii) $\Rightarrow$ (iii): note that a function $f:X\times X\to\mathbb{R}$ is the restriction of a continuous function $F:G(X)\to\mathbb{R}$ if and only if it is bounded on every entourage; moreover $F$ is proper if and only if $f^{-1}([-r,r])$ is an entourage for every $r\in\mathbb{R}_+$; finally, if $f$ is of negative type and $x_1,\dots,x_n\in G(X)$ have the same range $a\in\beta X$, there exist partial transformations $g_1,\dots,g_n$ of $X$ such that $x_i$ is in the closure of the graph of $g_i$. Up to restricting the $g_i$, we may further assume that they all have the same image. Given $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ with sum 1, and $u$ in the image of the $g_i$, we have $\sum_{j,k}f(g_j^{-1}(u),g_k^{-1}(u))\lambda_j\lambda_k\leqslant 0$. Letting $u$ converge to $a$, we find $\sum_{j,k}F(x_j^{-1}x_k)\lambda_j\lambda_\ell\leqslant 0$.
- To prove (iii) $\Rightarrow$ (iv), let $f:G(X)\to\mathbb{R}$ be a negative type proper function. For every pair $(s,t)$ of rational numbers such that $s\leqslant t$, the entourage $\{(x,y)\in X\times X\mid s\leqslant f(x,y)\leqslant t\}$ is a finite disjoint union of entourages $E_{s,t,i}$ in $\Gamma_{\mathscr{E}}$. Let $\mathscr{A}$ be the pseudogroup generated by the partial transformations $\varphi_{E_{s,t,i}}$. Then $\mathscr{A}$ is a countable admissible sub-pseudogroup of $\mathscr{G}(X)$ and $f$ defines a continuous proper negative type function on $G(\mathscr{A})$.
- To prove (iv) $\Rightarrow$ (i), let $\mathscr{A}\subset\mathscr{G}(X)$ be a countable admissible sub-pseudogroup and let $\alpha$ be a continuous proper isometric affine action of $G(\mathscr{A})$ on a Hilbert bundle $H$ over $G(\mathscr{A})^{(0)}$. Let then $x_0\in X\subset G(\mathscr{A})^{(0)}$ and put $\mathscr{H}=H_{x_0}$. For $x\in X$, we put $g(x)=\alpha(x_0,x)0_x$, where $(x_0,x)\in X\times X\subset G(\mathscr{A})$ and $0_x$ is the $0_x$ element of the Hilbert space $H_x$. For $x,y\in X$, we have $||g(y)-g(x)||=||\alpha(x_0,x)\alpha(x,y)0_y-\alpha(x_0,x)0_x||=||\alpha(x,y)0_y||$ since $\alpha(x_0,x)$ is isometric. The function $\gamma\mapsto||\alpha(\gamma)0_{r(\gamma)}||$ is a continuous proper function on $G(\mathscr{A})$; the entourages are the subsets of $X\times X$ whose closure in $G(\mathscr{A})$ is compact. It follows that $x\mapsto g(x)$ is a uniform embedding of $X$ into $\mathscr{H}$. □

Now, a new proof of Yu's theorem [22] follows easily from Theorem 5.4 and Tu's generalization of Higson and Kasparov's theorem [19,10].

**Theorem 5.5** (Yu [22]). *Let $X$ be a uniformly locally finite metric space that admits a uniform embedding into Hilbert space. Then the coarse assembly map for $X$ is an isomorphism.*

**Proof.** From Theorem 5.4 (i $\Rightarrow$ iii), there is a proper negative type function $f:G(X)\to\mathbb{R}$, hence, from [19], the Baum–Connes assembly map with coefficients for $G(X)$ is an isomorphism. The conclusion follows from Corollary 4.9. □

## 6. The Novikov conjecture for groups which admit a uniform embedding into Hilbert space

The main purpose of this section is to prove

**Theorem 6.1.** *Let $\Gamma$ be a countable group with a proper left-invariant metric. If $\Gamma$ admits a uniform embedding into Hilbert space, then the Baum–Connes assembly map*

$$\mu_r\colon K_*^{\mathrm{top}}(\Gamma; A) \to K_*(A \rtimes_r \Gamma)$$

*is injective for any separable $\Gamma$-$C^*$-algebra $A$.*

### *6.1. Proper affine actions and negative type functions of transformation groupoids, Tu's theorem*

For the convenience of readers who are not familiar with groupoids, we shall briefly discuss the concept of proper affine action and its relation to negative type function for transformation groupoids, even though we have discussed the same material in the general groupoid case.

Let $\Gamma$ be a countable discrete group. Denote by $e$ its identity element. Assume that $\Gamma$ acts on the right on a compact Hausdorff space $X$ by homeomorphisms. Recall that the product and the inverse operations of the transformation groupoid $X \rtimes \Gamma$ is given by: $(x,g)(x',g') = (x, gg')$ for all $(x,g)$ and $(x',g')$ in $X \times \Gamma$ satisfying $x' = xg$, and $(x,g)^{-1} = (xg, g^{-1})$ for all $(x,g) \in X \times \Gamma$.

**Definition 6.2.** Let $H$ be a continuous field of Hilbert spaces over $X$ [4, pp. 210–211]. We say that the transformation groupoid $X \rtimes \Gamma$ acts on $H$ by affine isometries if, for every $(x,g) \in X \times \Gamma$, there is an affine isometry $U_{(x,g)}\colon H_{xg} \to H_x$ such that

(1) $U_{(x,e)}\colon H_x \to H_x$ is the identity map;
(2) $U_{(x,g)} U_{(x',g')} = U_{(x,gg')}$ if $x' = xg$;
(3) for every continuous vector field $h(x)$ in $H$ and every $g \in \Gamma$, $U_{(x,g)}(h(xg))$ is a continuous vector field in $H$.

**Definition 6.3** (Tu [19]). Let $X \rtimes \Gamma$ act on $H$ as in Definition 6.2. The action is said to be proper if for any $R > 0$, the number of elements in $\{g \in \Gamma \mid \exists x \in X \mid U_{(x,g)}(B_{H_{xg}}(R)) \cap B_{H_x}(R) \neq \emptyset\}$ is finite, where $B_{H_x}(R) = \{h \in H_x \mid \|h\| \leqslant R\}$.

Let us also recall [19] that $X \rtimes \Gamma$ admits a proper action on a continuous field of affine Hilbert spaces if and only if it admits a continuous, negative type function in the sense of Definition 6.4 below:

**Definition 6.4.** Let $X \rtimes \Gamma$ be a transformation groupoid. A continuous function $\psi\colon X \times \Gamma \to \mathbb{R}$, is said to be a negative type function if

(1) $\psi(x,e) = 0$ for all $x \in X$;
(2) $\psi(x,g) = \psi(xg, g^{-1})$ for all $(x,g) \in X \times \Gamma$;
(3) $\sum_{i,j=1}^n t_i t_j \psi(xg_i, g_i^{-1} g_j) \leqslant 0$ for all $\{t_i\}_{i=1}^n \subseteq \mathbb{R}$ satisfying $\sum_{i=1}^n t_i = 0$, $g_i \in \Gamma$ and $x \in X$.

Let us also state the following result which is a particular case of Theorem 5.4.

**Proposition 6.5.** *Let $\Gamma$ be a countable group with a proper left-invariant metric $d$. The following are equivalent*:

(i) *there exists a uniform embedding $f:\Gamma \to H$.*
(ii) *there exists a proper negative type function on $\beta\Gamma \rtimes \Gamma$;*
(iii) *there exists a compact, Hausdorff, second countable space $Y$ with an action of $\Gamma$ which admits a proper negative type function on $Y \rtimes \Gamma$.*

**Proof.** Follows from the proof of Theorem 5.4, in which one can suppose that the sub-pseudogroup $\mathscr{A}$ contains the group $\Gamma$. Then, letting $C(Y) = C^*\{\chi_A \mid \mathrm{Id}_A \in \mathscr{A}\}$, one has $G(\mathscr{A}) = Y \rtimes \Gamma$. $\square$

The following is a particular case of a theorem of Tu [19] which generalizes a theorem of Higson and Kasparov [10]:

**Theorem 6.6** (Tu [19]). *Let $X$ be a compact, second countable Hausdorff space. If the transformation groupoid $X \rtimes \Gamma$ acts properly on some continuous field of Hilbert spaces by affine isometries, then the Baum–Connes assembly map for the transformation groupoid*

$$\mu_r : K_*^{\mathrm{top}}(\Gamma; C(X) \otimes A) \to K_*((C(X) \otimes A) \rtimes_r \Gamma)$$

*is an isomorphism for any separable $\Gamma$-$C^*$-algebra $A$.*

Indeed, by Lemma 4.1, the Baum–Connes assembly map for the groupoid $X \rtimes \Gamma$ with coefficients in $C(X) \otimes A$ is the same as the one for the group $\Gamma$ with coefficients in $C(X) \otimes A$.

### 6.2. Proof of Theorem 6.1

Suppose from now on that $\psi$ is a continuous proper negative type function on $Y \rtimes \Gamma$. As in [8], we consider $\mathrm{Prob}(Y)$, the space of all Borel probability measures on $Y$ with the weak$^*$ topology. Denote $\mathrm{Prob}(Y)$ by $X$. Notice that $X$ is a compact, second countable and Hausdorff space. The $\Gamma$ action on $Y$ induces a $\Gamma$ action on $X$. We define $\phi : X \times \Gamma \to \mathbb{R}$ by

$$\phi(m,g) = \int_Y \psi(y,g)\,\mathrm{d}m$$

for all $(m,g) \in X \times \Gamma$.

**Lemma 6.7.** *$\phi$ is a proper negative type function on the transformation groupoid $X \rtimes \Gamma$.*

**Proof.** Condition (1) in Definition 6.4 is clear. Let us verify condition (2). For every $(m,g) \in X \times \Gamma$, we have

$$\begin{aligned}\phi(mg,g^{-1}) &= \int_Y \psi(y,g^{-1})\,\mathrm{d}(mg)\\ &= \int_Y \psi(yg,g^{-1})\,\mathrm{d}m = \int_Y \psi(y,g)\,\mathrm{d}m\\ &= \phi(m,g).\end{aligned}$$

Next, we verify condition (3) in Definition 6.4. If $\{t_i\}_{i=1}^n \subseteq \mathbb{R}$ and $\sum_{i=1}^n t_i = 0$, we have

$$\begin{aligned}\sum_{i,j=1}^n t_i t_j \phi(mg_i, g_i^{-1}g_j) &= \sum_{i,j=1}^n t_i t_j \int_Y \psi(y, g_i^{-1}g_j)\,\mathrm{d}(mg_i)\\ &= \int_Y \left(\sum_{i,j=1}^n t_i t_j \psi(yg_i, g_i^{-1}g_j)\right) \mathrm{d}m \leqslant q0,\end{aligned}$$

where the last inequality follows from the fact that $\psi$ is a negative type function on the transformation groupoid $Y \rtimes \Gamma$. The properness of $\phi$ follows from the definition of $\phi$ and the fact that $f$ is a uniform embedding. $\square$

**Proof of Theorem 6.1.** We consider the following Higson descent diagram (cf. the proof of Theorem 3.2 in [8]):

$$\begin{tikzcd} K_*^{\mathrm{top}}(\Gamma, A) \arrow[r, "\mu_r"] \arrow[d] & K_*(A \rtimes_r \Gamma) \arrow[d] \\ K_*^{\mathrm{top}}(\Gamma, C(X) \otimes A) \arrow[r, "\mu_r"] & K_*((C(X) \otimes A) \rtimes_r \Gamma), \end{tikzcd}$$

where the vertical maps are induced by the inclusion of $\mathbb{C}$ into $C(X)$. By Lemma 6.7, the transformation groupoid $X \rtimes \Gamma$ acts properly on a continuous field of Hilbert spaces by affine isometries. Hence, by Tu's Theorem, the bottom horizontal map is an isomorphism. By [8, Proposition 3.7], the left vertical map is an isomorphism since, for any finite subgroup $H$ of $\Gamma$, $X$ is $H$-equivariantly homotopy equivalent to a point. It follows from the commutativity of the above diagram that the top horizontal map is split injective. $\square$

## Appendix A. The groupoid of the universal coarse structure

Let $X$ be countable discrete set. From Proposition 3.2, there is a groupoid $\Omega_u$ associated with the universal uniformly locally finite coarse structure $\mathscr{E}_u(X) = \{E \subset X \times X \mid N(E) < \infty\}$. Up to isomorphism, $\Omega_u$ does not depend on $X$. It is clear that for every uniformly locally finite coarse structure $\mathscr{E}$ on X, $G(\mathscr{E})$ is a subgroupoid of $\Omega_u$. The objective of this appendix is to prove that the correspondence $\mathscr{E} \to G(\mathscr{E})$ is a bijection between uniformly locally finite coarse structures on $X$ and open subgroupoids of $\Omega_u$ which contain $X \times X$, and to characterize these groupoids among subsets of $\beta(X \times X)$.

For every open set $V \subset \beta Y$, let us introduce the notation

$$\mathscr{C}(V) = \{A \subset Y \mid \bar{A} \subset V\}. \tag{A.1}$$

**Lemma A.1.** *Let $Y$ be a set, and denote by $\mathscr{P}(Y)$ the set of its subsets.*

(a) *Clopen subsets of $\beta Y$ are $\bar{A}$, where $A \subset Y$.*
(b) *For any open set $V \subset \beta Y$ we have $V = \bigcup_{A \in \mathscr{C}(V)} \bar{A}$. In other words, clopen sets constitute a basis for the topology of $\beta Y$.*

(c) *$V \mapsto \mathscr{C}(V)$ is a bijective correspondence between open subsets of $\beta Y$ and sets $\mathscr{C} \subset \mathscr{P}(Y)$ which are stable by taking finite unions and subsets.*

**Proof.** (a) Clopen subsets of $\beta Y$ correspond to projections of $C(\beta Y)=\ell^\infty(Y)$, hence to subsets of $Y$.

(b) Let $y \in V$. There exist $U$ open and $K$ compact such that $y \in U \subset K \subset V$. Since $U \cap Y$ is dense in $U$ and $K$ is closed, one has $y \in U \subset \overline{U \cap Y} \subset K$, hence $y \in \bar{A}$ where $A = U \cap Y \in \mathscr{C}(V)$.

(c) Let $\varphi$ be the map $V \mapsto \mathscr{C}(V)$ and $\psi$ the map $\mathscr{C} \mapsto \bigcup_{A \in \mathscr{C}} \bar{A}$. From (b), $\psi \circ \varphi$ is the identity map. To prove that $\varphi \circ \psi$ is the identity, let $\mathscr{C} \subset \mathscr{P}(Y)$ be stable by finite unions and subsets, and $V = \psi(\mathscr{C})$. Obviously, $V$ is open and $\mathscr{C} \subset \mathscr{C}(V)$. Conversely, if $\bar{A} \subset V$, by compactness it is covered by $\bar{A}_1 \cup \cdots \cup \bar{A}_n$, $A_i \in \mathscr{C}$, so $A \subset A_1 \cup \cdots \cup A_n$. Since $\mathscr{C}$ is stable by finite unions and subsets, $A \in \mathscr{C}$. □

**Proposition A.2.** *The correspondence $\mathscr{E} \to G(\mathscr{E})$ is a bijection between uniformly locally finite coarse structures on $X$ and open subgroupoids of $\Omega_u$ which contain $X \times X$. Moreover, $G(\mathscr{E})$ is $\sigma$-compact if and only if $\mathscr{E}$ is countably generated.*

**Proof.** In view of Lemma A.1(c), we just need to show that, if $G$ is a groupoid, $\mathscr{C}(G)$ is a uniformly locally finite coarse structure. Note that by Lemma A.1(c), we have $\mathscr{C}(\Omega_u) = \mathscr{E}_u$. Therefore, if $E \in \mathscr{C}(G) \subset \mathscr{C}(\Omega_u)$, then $N(E) < +\infty$. Conditions (b) and (c) of Definition 2.1 are obviously satisfied by $\mathscr{C}(G)$. Finally, let $E, F \in \mathscr{C}(G)$. Then $(\bar{E})^{-1}$ and $\bar{E}.\bar{F}$ are compact subsets of $G$ and contain respectively $E^{-1}$ and $E \circ F$. Therefore $E^{-1}, E \circ F \in \mathscr{C}(G)$.

If there exists a sequence $E_n$ of entourages of $X$ such that every entourage is contained in one of the $E_n$, then $G(X) = \bigcup_{n \in \mathbb{N}} \bar{E}_n$ is $\sigma$-compact. Conversely, if $G(X) = \bigcup_{n \in \mathbb{N}} K_n$ with $K_n$ compact, and since $\bar{E}$ is open for every entourage $E$, $K_n \subset \bar{E}_n$ for some entourage $E_n$. It follows that every entourage is contained in one of the $E_n$. □

Recall that $X \times X$ is endowed with a groupoid structure with space of units $X$. If $X$ is infinite, that structure does not extend continuously to a groupoid structure on $\beta(X \times X)$, but the following proposition says that an open subset of $\Omega$ of $\beta(X \times X)$ containing $X \times X$ is a groupoid if and only if it comes from a uniformly locally finite coarse structure on $X$.

**Proposition A.3.** *Let $\Omega$ be an open subset of $\beta(X \times X)$ containing $X \times X$. Denote by $r : \beta(X \times X) \to \beta X$ and $s : \beta(X \times X) \to \beta X$ the continuous maps whose restriction to $X \times X$ is, respectively, $(x, y) \mapsto x$ and $(x, y) \mapsto y$. Then, the following assertions are equivalent.*

(i) *The groupoid structure on $X \times X$ extends continuously to a groupoid structure on $\Omega$.*
(ii) *The maps $p:((x, y), (y, z)) \mapsto (x, z)$ and $\kappa:(x, y) \mapsto (y, x)$ extend to continuous maps $\bar{p}:\{(u, v) \in \Omega \times \Omega \mid s(u) = r(v)\} \to \Omega$ and $\bar{\kappa} : \Omega \to \Omega$.*
(iii) *$\Omega$ is a subgroupoid of $\Omega_u$.*
(iv) *$\mathscr{C}(\Omega)$ is a uniformly locally finite coarse structure.*
(v) *$\mathscr{C}(\Omega)$ is a coarse structure and $\Omega \xrightarrow{(r,s)} \beta X \times \beta X$ is injective.*

**Proof.** The equivalence (iii) $\Leftrightarrow$ (iv) comes from Proposition A.2. As the groupoid associated to a coarse structure is principal (Proposition 3.2), (iv) $\Rightarrow$ (v).

The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are obvious.

In order to prove the implications (v)$\Rightarrow$(ii)$\Rightarrow$(iv), we need a few lemmas.

**Lemma A.4.** *Let $f : X \to Y$ be a map between discrete sets. Then the induced map $f : \beta X \to \beta Y$ is open.*

**Proof.** For every subset $A$ of $X$, as $\bar{A}$ is compact $f(\bar{A}) = \overline{f(A)}$. The conclusion follows from Lemma A.1(b). $\square$

In the following lemma, (a) and (b) are simple exercises and (c) is an immediate consequence of (a) and (b) by taking $U = f^{-1}(V)$.

**Lemma A.5.** *Let $f : T \to T'$ be an open continuous map between topological spaces, $D \subset T$ a dense subset, and $V \subset T'$ open. Then*

(a) *For every open subset $U$ of $T$, one has $\overline{U \cap D} = \bar{U}$.*
(b) $f^{-1}(\bar{V}) = \overline{f^{-1}(V)}$.
(c) $f^{-1}(\bar{V}) = \overline{f_{|D}^{-1}(V)}$.

**Lemma A.6.** *Let $X$ be a set, and $E_1$, $E_2$ be subsets of $X \times X$. Denote by $\bar{E}_i$ the closure of $E_i$ in $\beta(X \times X)$. Then $E_1 \times_X E_2$ is dense in $\bar{E}_1 \times_{\beta X} \bar{E}_2 \subset \beta(X \times X) \times \beta(X \times X)$.*

**Proof.** Let $s(x, y) = y$ and $r(x, y) = x$. From Lemma A.4, $s : \bar{E}_1 \to \beta X$ and $r : \bar{E}_2 \to \beta X$ are open. Therefore, the map $f = (s, r) : \bar{E}_1 \times \bar{E}_2 \to \beta X \times \beta X$ is open. Let $\Delta$ be the diagonal of $X \times X$. Using Lemma A.5, $\overline{E_1 \times_X E_2} = \overline{f_{|E_1 \times E_2}^{-1}(\Delta)} = f^{-1}(\bar{\Delta}) = \bar{E}_1 \times_{\beta X} \bar{E}_2$. $\square$

Let $\Omega$ be an open subset of $\beta(X \times X)$ containing $X \times X$. Lemma A.6 shows that if the groupoid structure of $X \times X$ extends continuously to $\Omega$, this extension is unique.

**Lemma A.7.** *Let $\Omega$ be an open subset of $\beta(X \times X)$ containing $X \times X$ and satisfying condition* (ii) *of Proposition A.*3. *Then $\forall E, F \in \mathscr{C}(\Omega)$, we have $E \circ F \in \mathscr{C}(\Omega)$ and $E^{-1} \in \mathscr{C}(\Omega)$.*

**Proof.** The sets $\bar{p}(\{(u, v) \in \bar{E} \times \bar{F} | s(u) = r(v)\})$ and $\bar{\kappa}(\bar{E})$ are compact subsets of $\Omega$ which, respectively, contain $E \circ F$ and $E^{-1}$. $\square$

**Lemma A.8.** *Let $X$ and $Y$ be sets, and $Z$ a subset of $X \times Y$. Let $\varphi : \beta Z \to \beta X \times \beta Y$ be the natural map. Let $K$ be a clopen subset in $\varphi(\beta Z)$. Then there exists $n \in \mathbb{N}$ and subsets $B_1, \ldots, B_n \subset X$, $C_1, \ldots, C_n \subset Y$ such that $K = \bar{A}$ with $A = \bigcup_{1 \leqslant i \leqslant n} (B_i \times C_i) \cap Z$.*

**Proof.** As $Z=\varphi(Z)$ is dense in $\varphi(\beta Z)$, a clopen subset $K$ of $\varphi(\beta Z)$ is of the form $\bar{A}$ where $A=Z\cap K$ (Lemma A.5(a)).

The sets $\bar{B}\times\bar{C}$, $B\subset X$, $C\subset Y$ form a basis for the topology of $\beta X\times\beta Y$ (Lemma A.1). As $K$ is open in $\varphi(\beta Z)$ there exist a set $I$ and families $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ such that $K=\cup_{i\in I}(\bar{B}_i\times\bar{C}_i)\cap\varphi(\beta Z)$. Since $K$ is compact, one may replace the set $I$ by a finite subset. □

**Lemma A.9.** *Let $E,X$ be sets, and $f:E\to X$ a map. Set $F=E\times_X E=\{(y,z)\in E\times E|f(y)=f(z)\}$. Let $\varphi:\beta F\to\beta E\times\beta E$ be the map extending the inclusion of $F$ into $E\times E$. If there exists a clopen subset $K$ in $\varphi(\beta F)$ such that $F\cap K=\{(y,y)|y\in E\}$, then* $\sup_{x\in X}\#f^{-1}(\{x\})<+\infty$.

**Proof.** Let $A=\{(y,y)\,|\,y\in E\}\subset F$. Applying Lemma A.8, we find $n\in\mathbb{N}$ and subsets $B_1,\dots,B_n,C_1,\dots,C_n\subset E$ such that $A=\bigcup_{1\leqslant i\leqslant n}(B_i\times C_i)\cap F$. Note that $(B_i\times C_i)\cap A=\{(y,y)\,|\,y\in B_i\cap C_i\}$; we therefore have $A=\bigcup_{1\leqslant i\leqslant n}(D_i\times D_i)\cap F$, with $D_i=B_i\cap C_i$. Now, if $y,z\in D_i$ satisfy $f(y)=f(z)$, then $(y,z)\in(D_i\times D_i)\cap F\subset A$, whence $y=z$. It follows that the restriction of $f$ to $D_i$ is injective. As $A=\bigcup_{1\leqslant i\leqslant n}(D_i\times D_i)\cap F$, the $D_i's$ cover $E$, whence $f$ is at most $n$ to 1. □

**Proof.** End of the proof of Proposition A.3 (ii) $\Rightarrow$ (iv): Assume (ii) is satisfied. Let $s:\beta(X\times X)\to\beta X$ be the source map. Let $E\in\mathscr{C}(\Omega)$. Set $F=\{(u,v)\in E\times E\,|\,s(u)=s(v)\}$. By condition (ii), the map $q:(u,v)\mapsto uv^{-1}=p(u,\kappa(v))$ extends continuously to the closed subset $\{(u,v)\in\beta E\times\beta E|s(u)=s(v)\}$. We therefore get a map $\bar{q}:\varphi(\beta F)\to\beta(X\times X)$, where $\varphi:\beta F\to\beta E\times\beta E$ is the continuous map extending the inclusion $F\to E\times E$. Let $\Delta=\{(x,x)|\,x\in X\}$ be the diagonal of $X$; its closure is a clopen subset $\bar{\Delta}\subset\beta(X\times X)$. Put $K=\bar{q}^{-1}(\bar{\Delta})$. It is clopen in $\varphi(\beta F)$. Moreover, $F\cap K=\{(u,v)\in F|uv^{-1}\in\Delta\}=\{(u,u)|u\in E\}$. By Lemma A.9, $\sup_{x\in X}\#(E\cap s^{-1}(\{x\}))<+\infty$. Now it follows from Lemma A.7 that $\mathscr{C}(\Omega)$ is a uniformly locally finite coarse structure.

We show (v) $\Rightarrow$ (ii): Assume (v) is satisfied and let $E,F\in\mathscr{C}(\Omega)$. Since $E^{-1}\in\mathscr{C}(\Omega)$, $\kappa$ extends continuously to $\bar{E}$. We need to prove that the map $p$ extends to a continuous map from $K=\{(u,v)\in\bar{E}\times\bar{F}\,|\,s(u)=r(v)\}$ into $\Omega$. Put $H=E\circ F$. By (v), the map $(r,s):\bar{H}\to\beta X\times\beta X$ is a homeomorphism $\psi$ from $\bar{H}$ onto the closure $\psi(\bar{H})$ of $H$ in $\beta X\times\beta X$. Now, as $\{(u,v)\in E\times F|s(u)=r(v)\}$ is dense in $K$ (Lemma A.6), the set $\{(r(u),s(v))|(u,v)\in K\}$ is contained in $\psi(\bar{H})$. Finally, the map $(u,v)\mapsto\psi^{-1}(r(u),s(v))$ is the desired extension of $p$ from $K$ into $\bar{H}\subset\Omega$. □

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