J. reine angew. Math. **546** (2002), 47—72

Journal für die reine und angewandte Mathematik

Regular representation of groupoid C^*-algebras and applications to inverse semigroups

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Abstract. The analogue of the left regular representation of a locally compact groupoid is constructed in the Hausdorff as well as in the non-Hausdorff case. A necessary and sufficient condition for a locally compact groupoid with a cocycle to be Morita equivalent to a group action is obtained. As an application, the C^*-algebras of a class of inverse semigroups is shown to be Morita-equivalent to crossed products of groups by abelian C^*-algebras.

Introduction

This paper is devoted to some questions concerning groupoid C^*-algebras—in particular in the non-Hausdorff case—and the relations between groupoids and inverse semigroups.

Groupoid C^*-algebras have been studied for years. A systematic development of the fundamentals of the theory of groupoid C^*-algebras was provided by Jean Renault in [17], which is the classical reference for the subject. More or less at the same time, A. Connes ([2]) showed how groupoid C^*-algebras have to be used in the study of geometric objects as natural as foliations. In Connes' work ([4]), various groupoids arise, in order to explain all kinds of geometric phenomena: foliations, Penrose tilings, deformations ... Furthermore, groupoids turn out to take into account, somewhat unexpectedly, various kinds of geometric phenomena (e.g. coarse geometry—cf. [21]).

In this paper, we give a necessary and sufficient condition for a locally compact groupoid with a cocycle to be Morita equivalent to a group action, by showing that if the cocycle is faithful, closed and transverse (see Definition 1.6), then the groupoid is Morita equivalent to a group action.

Actually the only really new difficulty in dealing with groupoids rather than group actions on spaces, is that groupoids need not be Hausdorff. Moreover, non-Hausdorff

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*) Partially supported by an NSERC grant.

groupoids actually occur in many important examples of foliations, such as the Reeb foliation. Alain Connes ([3], [4]) explained which is the right modification to be made in treating the non-Hausdorff groupoids (at least in the foliation case): one just needs to change the definition of the algebra of continuous compactly supported functions $C_c(G)$. However, a few facts, known in the Hausdorff case have to be clarified in the construction of faithful representations and faithful families of representations for the reduced C^*-algebra.

We view the analogue of $L^1(G)$ as a quotient, with a quotient norm and this allows us to clarify its properties and give an alternate way of describing the full C^*-algebra in the non-Hausdorff case (cf. [17], [18]). We also construct 'the regular representation' of a groupoid: in the Hausdorff case, this representation takes place on a natural Hilbert module $L^2(G)$ over $C_0(X)$ where X is the space $G^{(0)}$ of units of G. In the non-Hausdorff case, we construct a natural locally compact space Y, which contains X as a dense subset: Y is the spectrum of a suitable C^*-algebra of bounded Borel functions on X. If G is Hausdorff, X and Y coincide. In general, we construct a Hilbert $C_0(Y)$-module on which the reduced C^*-algebra $C_r^*(G)$ of G acts faithfully.

This representation of $C_r^*(G)$ points to the following subtle point. If D is a dense subset of X, the family of regular representations associated with the elements of D needs not be faithful. For these representations to be faithful one has to assume that D is dense in Y, which is a strictly stronger assumption. This is due to the fact that the inclusion map $X \to Y$ is not continuous and thus a dense subset of X is not always dense in Y (see Corollary 2.11). This rather surprising appearence of a "breaking" of the spectrum turns out to be an important input of the first counterexamples to the Baum-Connes conjecture ([9]). Applying these results to the foliation case, we give a refinement of the main result of [7]: we show that if (X, F) is a minimal foliation such that the set of points with trivial holonomy group is dense *for the topology of* Y, then the foliation C^*-algebra is simple (Remark 2.12).

As an application of our construction we investigate connections with inverse semigroup C^*-algebras. The study of inverse semigroup C^*-algebras was initiated by J. Duncan and A. Paterson ([5]) and has since attracted the attention of a number of authors ([8], [20], [6]). By the very definition of inverse semigroups, their C^*-algebras are closely related to the important class of C^*-algebras generated by isometries and partial isometries such as Toeplitz algebras, Cuntz algebras and Cuntz-Krieger algebras, just to mention a few examples. Here we focus our attention on the intrinsic connections between inverse semigroup C^*-algebras and groupoid C^*-algebras. Given an inverse semigroup S, let X be the spectrum of the commutative C^*-algebra generated by the idempotents of S. Then S can be viewed as a pseudogroup of partial transformations of X. Associated to it is a locally compact groupoid—the groupoid of S. The inverse semigroup S and its groupoid have the same C^*-algebras. Using this fact, we prove that the C^*-algebra of a large class of E-unitary inverse semigroups (the class contains all F-inverse semigroups) are Morita equivalent to the crossed product of an abelian C^*-algebra by a group.

A summary of the paper is as follows.

- In section one, we discuss groupoid C^*-algebras, especially in the non-Hausdorff case. Moreover, we prove a result (Theorem 1.8) on group valued cocycles, which allows us to answer the question of when is a locally compact groupoid with a cocycle Morita equivalent to a group action.

• In section two, natural Hilbert modules associated with groupoids are introduced and the regular representation of a groupoid is constructed. The main results of this section are Theorem 2.3 and Theorem 2.10 showing that the regular representation of G is a faithful representation of $C_r^*(G)$. Moreover, we find a sufficient condition on a subset $D \subset X$, the unit space of G, so that the family of regular representations given by D is faithful on $C_r^*(G)$. We end with the above mentioned application to foliations.

• Section three consists of applications of the previous sections to inverse semigroups. As noted above to an inverse semigroup S a groupoid G_S is associated such that $C^*(G_S) \cong C^*(S)$ and $C_r^*(G_S) \cong C_r^*(S)$. Our construction of G_S follows that of A. Paterson ([16]), but most arguments are simplified and are shorter. In particular, thanks to Lemma 1.4 we give an independent and more conceptual proof of the isomorphism $C^*(G_S) \cong C^*(S)$ than the one given in [16]. As well we fix a small gap in the proof of the fact that $C_r^*(G_S) \cong C_r^*(S)$ in the case G_S is non-Hausdorff (we will elaborate on this in section 2 and 3). We then shift our attention to E-unitary inverse semigroups. For such inverse semigroups, the natural cocycle from G_S onto the maximal group homomorphic image of S is faithful. We find algebraic conditions (similar conditions appear in the algebraic theory of inverse semigroups) on S for the cocycle to be closed and transverse—resulting in Corollary 3.11.

1. Groupoid C^*-algebras

We begin this section by recalling basic definitions, notation and terminology from groupoids and groupoid C^*-algebras. For the details and the proofs of the basic facts we refer to [17], [14] and [1].

Let G be a groupoid, then

– $G^{(0)}$ will denote its space of units;

– $s: G \to G^{(0)}$ and $r: G \to G^{(0)}$ denote respectively the source and range maps;

– $G^{(2)}$ denotes the set $\{(\gamma, \gamma') \in G \times G; s(\gamma) = r(\gamma')\}$ of composable elements;

– given subsets $A, B \subset G^{(0)}$, we set $G_A = \{\gamma \in G; s(\gamma) \in A\}$, $G^B = \{\gamma \in G; r(\gamma) \in B\}$ and $G_A^B = G_A \cap G^B$; for $x \in G^{(0)}$, we write G_x and G^x instead of $G_{\{x\}}$ and $G^{\{x\}}$.

A *locally compact groupoid* is a groupoid endowed with a locally compact topology such that the groupoid operations (composition, inversion, source and range maps) are continuous. Throughout the paper, we will further assume the source map to be open; in this case, the range map is also open.

We will be interested in not necessarily Hausdorff locally compact groupoids. Before giving the precise definitions of these objects, let us briefly review the construction of the full and reduced C^*-algebras of a Hausdorff locally compact groupoid G. Associated with G is a function space to be denoted by $\mathscr{A}$ defined as follows.

A. The Hausdorff case. If G is Hausdorff, $\mathscr{A}$ is $C_c(G)$ the space of continuous complex valued functions with compact support on G. In order to turn $\mathscr{A}$ into an algebra, we need what is called a *Haar system*. A Haar system on G is a collection $\nu = \{\nu_x\}_{x \in G^{(0)}}$ of positive regular Borel measures on G satisfying the following conditions:

a) *Support*: For every $x \in G^{(0)}$, the support of ν_x is contained in G_x.

b) *Invariance*: For all $\gamma_1 \in G$ and $f \in \mathscr{A}$, $\int f(\gamma\gamma_1)\,d\nu_x(\gamma) = \int f(\gamma)\,d\nu_y(\gamma)$, where $x = r(\gamma_1)$ and $y = s(\gamma_1)$.

c) *Continuity*: For each $f \in \mathscr{A}$ the map $x \mapsto \int_{G_x} f(\gamma)\,d\nu_x(\gamma)$ is continuous.

If G is r-discrete, which means that the range and source maps are local homeomorphisms, then a possible choice for ν_x is the counting measure on G_x.

Denote, by (G, ν) a locally compact groupoid together with a fixed Haar system. Then $\mathscr{A}$ is going to be a normed $*$-algebra under the following operations. For $f, g \in \mathscr{A}$, let

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \tag{1}$$

and

$$f \star g(\gamma) = \int_{G_x} f(\gamma\gamma_1^{-1})g(\gamma_1)\,d\nu_x(\gamma_1), \tag{2}$$

where $x = s(\gamma)$. The norm on $\mathscr{A}$ is defined by

$$\|f\|_1 = \sup_{x \in G^{(0)}} \left\{ \max\left(\int_{G_x} |f(\gamma)|\,d\nu_x(\gamma), \int_{G_x} |f(\gamma^{-1})|\,d\nu_x(\gamma) \right) \right\}. \tag{3}$$

The *full groupoid C^*-algebra* $C^*(G, \nu)$ (or $C^*(G)$ when there is no ambiguity on the Haar system) is defined to be the enveloping C^*-algebra of the Banach $*$-algebra obtained by completion of $\mathscr{A}$ with respect to the norm $\| \ \|_1$.

Given $x \in G^{(0)}$, $f \in \mathscr{A}$ and $\xi \in L^2(G_x, \nu_x)$, we set

$$\lambda_x(f)\xi(\gamma) = \int_{G_x} f(\gamma\gamma_1^{-1})\xi(\gamma_1)\,d\nu_x(\gamma_1). \tag{4}$$

One shows that, for every $f \in \mathscr{A}$ and $x \in G^{(0)}$, the operator $\lambda_x(f)$ is bounded and we have $\|\lambda_x(f)\| \leqq \|f\|_1$. We thus get a bounded $*$-representation λ_x of $\mathscr{A}$ on $L^2(G_x, \nu_x)$. The reduced norm on $\mathscr{A}$ is

$$\|f\|_r = \sup_{x \in G^{(0)}} \{\|\lambda_x(f)\|\} \tag{5}$$

which is a C^*-norm.

The *reduced C^*-algebra* $C_r^*(G, \nu)$ (or $C_r^*(G)$ when there is no ambiguity on the Haar system) is defined to be the C^*-algebra obtained by completion of $\mathscr{A}$ with respect to $\| \|_r$. Clearly, $C_r^*(G)$ is a quotient of $C^*(G)$.

The full and reduced C^*-algebras don't depend—up to Morita equivalence on the Haar system ν ([17]).

Remark (cf. [17]). It may be worth noticing that the above 'usual' conventions for groupoid algebras are slightly different from the conventions for groups (in the non unimodular case). The main difference is equation (1) for the adjoint. In fact, if G is a locally compact group, there is a natural $*$-homomorphism from the algebra $L^1(G)$ for G as seen as a groupoid to $L^1(G)$ for G seen as a group given by $f \mapsto \Delta^{1/2} f$. It is then not difficult to see that, although this homomorphism is not onto, those two algebras have the same enveloping C^*-algebra—because they have the same representations: unitary representations of G.

B. The non-Hausdorff case. Many natural examples of groupoids, such as some interesting foliation groupoids, are non-Hausdorff. Let us fix our setting and then explain the modifications that have to be made in order to cover this case.

1.1. Definition. A *locally compact groupoid* is a groupoid G endowed with a topology such that

a) the groupoid operations (composition, inversion, source and range maps) are continuous;

b) the space of units $G^{(0)}$ is Hausdorff;

c) each point of G has a compact (Hausdorff) neighborhood;

d) for each $x \in G^{(0)}$, the space G_x is Hausdorff[1)];

e) the range and source maps are open.

Furthermore all the groupoids here will be assumed to be σ-compact.

These conditions are satisfied by all important examples (such as holonomy groupoids). The following example may seem somewhat unnatural; however it illustrates the kind of singularities which naturally occur in non-Hausdorff foliation groupoids.

1.2. Example. Let Γ be a non trivial discrete group, X a compact space and $x_0 \in X$ a non isolated point. Then, $\Gamma \times X$ is a groupoid with $r(g,x) = s(g,x) = x$ and $(g,x)(h,x) = (gh,x)$. Define an equivalence relation on $\Gamma \times X$ by $(g,x) \sim (h,x)$ if $x \neq x_0$. The quotient space $G = \Gamma \times X/\sim$, endowed with the quotient topology and groupoid structure is a non-Hausdorff locally compact groupoid in the above sense. Indeed, denote by $\overline{(g,x)}$ the class in G of $(g,x) \in \Gamma \times X$. Then, $\overline{(g,x)}$ converges to $\overline{(g,x_0)}$ as $x \to x_0$ for any $g \in \Gamma$.

[1)] Jean-Louis Tu pointed out to us that this is automatic, since the diagonal of G_x is the set $\{(\gamma,\gamma') \in G_x \times G_x; r(\gamma') = r(\gamma), \gamma^{-1}\gamma' = x\}$ and by c) points are closed subsets of G.

As a set, G is the disjoint union of $\Gamma \times \{x_0\}$ and $X \backslash \{x_0\}$.

Let G be a, not necessarily Hausdorff, locally compact groupoid. As pointed out by Alain Connes ([2]), one has to modify the choice of $\mathscr{A}$; indeed $C_c(G)$ has no natural convolution algebra structure and is too small to capture the topological or differential structure of G.

For instance, in the case of Example 1.2, $C_c(G)$ consists of functions that are constant on $\Gamma \times \{x_0\}$ and therefore vanish if Γ is infinite. The space $C_c(G)$ therefore contains very little information on Γ.

Following [2], we define $\mathscr{A}$ to be the space of complex valued functions on G spanned by functions which vanish outside a compact (Hausdorff) subset K of G and are continuous on a neighborhood of K. Since in a non-Hausdorff space a compact set may not be closed, members of $\mathscr{A}$ are not necessarily continuous on G. In the case of Example 1.2, the algebra $\mathscr{A}$ consists of pairs $(\varphi, \psi) \in C_c(\Gamma) \times C(X)$ such that $\psi(x_0) = \sum_{g \in \Gamma} \varphi(g)$.

The following result gives a practical way to describe all the elements of $\mathscr{A}$.

1.3. Lemma. *Let $(U_i)_{i \in I}$ be a covering of G by open Hausdorff subsets. Then $\mathscr{A}$ is the set of finite sums $\sum_i f_i$ where f_i is a continuous compactly supported function on U_i.*

Proof. We just need to show that if U is an open Hausdorff subset of G any function $f \in C_c(U)$ can be decomposed as above. Let K be a compact neighborhood of the support of f. Let $(\varphi_i)_i$ be a finite partition of the identity of K associated with its open covering $U_i \cap \overset{\circ}{K}$. Then $f = \sum_i f_i$ with $f_i = f\varphi_i$, which is a continuous compactly supported function in $\overset{\circ}{K} \cap U_i$. $\square$

In other words, setting $\Omega = \coprod_{i \in I} U_i$, the space $\mathscr{A}$ is a quotient of $C_c(\Omega)$.

We now show how the various objects associated with a Hausdorff groupoid are defined in the non-Hausdorff case thanks to the modified definition of $\mathscr{A}$.

- The Haar system of a non-Hausdorff groupoid is defined as before: the support and invariance conditions are the same as in the Hausdorff case; the continuity condition c) is exactly the same as in the non-Hausdorff case with respect to the modified $\mathscr{A}$. Note that if G is r-discrete (as for instance in Example 1.2), the family v_x of counting measure on G_x satisfies the above conditions.

- One obviously sees that the adjoint operation is well defined on $\mathscr{A}$ by formula (1).

- To show that the convolution is also well defined, take U and V being open Hausdorff subsets of G and let $f \in C_c(U)$ and $g \in C_c(V)$. We want to show that $f \star g$ as defined by formula (2) is still an element of $\mathscr{A}$.

Using compactness of the supports of f and g and partitions of the identity, we may further assume that U and V are small enough so that the open subset

$$U.V = \{\gamma_1\gamma_2; \gamma_1 \in U, \gamma_2 \in V, s(\gamma_1) = r(\gamma_2)\}$$

of G is Hausdorff. Set $B = \{(\gamma_1, \gamma_2) \in U \times V; s(\gamma_1) = r(\gamma_2)\}$; it is a closed subset of $U \times V$. The map $(\gamma_1, \gamma_2) \mapsto (\gamma_1\gamma_2, \gamma_2)$ is a homeomorphism from B into an open subset B' of $\{(\gamma_3, \gamma_4) \in U.V \times V; s(\gamma_3) = s(\gamma_4)\}$. Therefore, there exists $\varphi \in C_c(B')$ such that, for all $(\gamma_1, \gamma_2) \in B$ we have $f(\gamma_1)g(\gamma_2) = \varphi(\gamma_1\gamma_2, \gamma_2)$.

Formula (2) gives

$$(f \star g)(\gamma) = \int \varphi(\gamma, \gamma_2)\, d\nu_{s(\gamma)}(\gamma_2). \tag{6}$$

Note that by the continuity condition on the Haar system, and since the map s is open, formula (6) shows that $f \star g \in C_c(U.V) \subset \mathscr{A}$.

- The norm $\| \;\|_1$ on $\mathscr{A}$ is defined in the same way as in the Hausdorff case by formula (3). The *full groupoid C^*-algebra* $C^*(G)$ is defined in the same way as in the Hausdorff case: it is the enveloping C^*-algebra of the Banach $*$-algebra obtained by completion of $\mathscr{A}$ with respect to the norm $\| \;\|_1$.

- In the same way as in the Hausdorff case (cf. [17]), we see that formula (4) defines a bounded operator $\lambda_x(f)$ on $L^2(G_x)$ for any $f \in \mathscr{A}$ and any $x \in G^{(0)}$ and that $\|\lambda_x(f)\| \leqq \|f\|_1$.

- We finally define $\| \;\|_r$ and $C_r^*(G)$ as in the Hausdorff case.

In the case of Example 1.2, a Haar system consists of: the Dirac measure for $x \neq x_0$ and the Haar measure of Γ at $x = x_0$. The full C^*-algebra of G consists of pairs $(\varphi, \psi) \in C^*(\Gamma) \times C(X)$ such that $\psi(x_0) = \varepsilon(\varphi)$, where ε denotes the trivial representation of Γ. The reduced C^*-algebra of G is the quotient of $C^*(G)$ under the family of representations $\lambda_{x_0}\colon (\varphi, \psi) \mapsto \lambda(\varphi)$ (where λ is the left regular representation of Γ) and, for $x \neq x_0$, $\lambda_x\colon (\varphi, \psi) \mapsto \psi(x)$. It follows that

– if Γ is amenable, $C_r^*(G) = C^*(G)$;

– if Γ is not amenable, $C_r^*(G) = C_r^*(\Gamma) \oplus C(X)$.

We will use the following complement to Lemma 1.3:

For $f \in \mathscr{A}$ and $x \in G^{(0)}$, put $N_f(x) = \int\limits_{G_x} |f(\gamma)|\, d\nu_x(\gamma)$ and $N^f(x) = \int\limits_{G_x} |f(\gamma^{-1})|\, d\nu_x(\gamma)$.

Let $(U_i)_{i \in I}$ be a covering of G by open Hausdorff subsets and set

$$\Omega = \coprod_{i \in I} U_i = \{(\gamma, i) \in G \times I; \gamma \in U_i\}.$$

For $g \in C_c(\Omega)$ we let $\varphi(g) \in \mathscr{A}$ be the function $\gamma \mapsto \sum_i g(\gamma, i)$ (this is a finite sum). Lemma 1.3 states that φ is onto.

For $x \in G^{(0)}$ we put

$$N_g(x) = \sum_i \int\limits_{G_x} |g(\gamma, i)|\, d\nu_x(\gamma) \quad \text{and} \quad N^g(x) = \sum_i \int\limits_{G_x} |g(\gamma^{-1}, i)|\, d\nu_x(\gamma).$$

Finally, we put $\|g\|_{1,\ell} = \sup\{N_g(x); x \in G^{(0)}\}$, $\|g\|_{1,r} = \sup\{N^g(x); x \in G^{(0)}\}$ and

$$\|g\|_1 = \max(\|g\|_{1,\ell}, \|g\|_{1,r}).$$

1.4. Lemma. *Let $(U_i)_{i \in I}$ be a covering of G by open Hausdorff subsets and set $\Omega = \coprod_{i \in I} U_i$. Let $f \in \mathscr{A}$; we have the following equalities*:

a) $N_f(x) = \inf\{N_g(x); g \in C_0(\Omega), \varphi(g) = f\}$, *for all* $x \in G^{(0)}$;

b) $\sup\{N_f(x); x \in G^{(0)}\} = \inf\{\|g\|_{1,\ell}; g \in C_0(\Omega), \varphi(g) = f\}$;

c) $\|f\|_1 = \inf\{\|g\|_1; g \in C_0(\Omega), \varphi(g) = f\}$.

Proof. For every $g \in C_c(\Omega)$ and every $x \in G^{(0)}$ we have

$$N_{\varphi(g)}(x) = \int \left| \sum_i g(\gamma, i) \right| d\nu_x(\gamma) \leqq \int \sum_i |g(\gamma, i)| \, d\nu_x(\gamma) = N_g(x).$$

This establishes the inequalities $\leqq$.

Write f as a finite sum of functions h_j where the h_j are continuous with compact supports in open subsets V_j; put $W = \coprod V_j$ and let $h \in C_c(W)$ be the function $(\gamma, j) \mapsto h_j(\gamma)$. Set $U_i' = W \cap (U_i \times J)$. Given a partition of the identity $\chi = (\chi_i)$ of the support of h adapted with the covering U_i', we define $g_\chi \in C_0(\Omega)$ by setting $g_\chi(\gamma, i) = \sum_j \chi_i(\gamma, j) h_j(\gamma)$ (where $i \in I$ and $\gamma \in U_i$; the sum is taken over all j's such that $\gamma \in V_j$). Obviously $\varphi(g_\chi) = f$.

To prove a) and b), we will show

a′) For every $x \in X$, there exists a partition of the identity χ as above such that $N_f(x) = N_{g_\chi}(x)$.

b′) For every $\varepsilon > 0$, there exists a partition of the identity χ as above such that $\sup\{N_f(x); x \in G^{(0)}\} \geqq \|g_\chi\|_{1,\ell} - \varepsilon$.

a′) Since G_x is Hausdorff, the space $K = \left(\bigcup_j \operatorname{Supp}(h_j) \right) \cap G_x$ is compact and Hausdorff; let (ω_i) be a finite partition of the identity of K adapted to the covering $(U_i \cap K)$. For $(\gamma, j) \in \operatorname{Supp}(h)$ such that $s(\gamma) = x$ and $i \in I$, put $\chi_i(\gamma, j) = \omega_i(\gamma)$. Extend (χ_i) to a partition of the identity of $\operatorname{Supp}(h)$. An elementary calculation shows that $N_f(x) = N_{g_\chi}(x)$.

b′) Let $\varepsilon > 0$. By a′) and compactness of the support of h, there are partitions $\chi_1, \dots, \chi_n$ of the identity such that the open sets $W_k = \left\{ x \in G^{(0)}; N_{g_{\chi_k}}(x) < \varepsilon + \sup_y N_f(y) \right\}$ $(k = 1, \dots, n)$ form an open covering of $G^{(0)}$. Take a partition of the identity $\psi_1, \dots, \psi_n$ of $G^{(0)}$ adapted to the covering W_k. It is now enough to put $\chi(\gamma) = \sum_k \chi_k(\gamma) \psi_k(s(\gamma))$.

c) Let $\varepsilon > 0$. By b), there exists $h \in C_0(\Omega)$ satisfying $\varphi(h) = f$ and

$$\sup\{N_f(x); x \in G^{(0)}\} \geqq \|h\|_{1,\ell} - \varepsilon.$$

Apply then b′) replacing f, the U_i's and h by $\tilde{f}: \gamma \mapsto f(\gamma^{-1})$ and $\tilde{U}_i = \{\gamma^{-1}; \gamma \in U_i\}$ and $\tilde{h}: (\gamma, i) \mapsto h(\gamma^{-1}, i)$; we find a partition of the identity χ such that

$$\sup\{N^f(x); x \in G^{(0)}\} \geqq \|g_\chi\|_{1,r} - \varepsilon.$$

But by construction of g_χ it follows that $N_{g_\chi} \leqq N_h$. Therefore

$$\sup\{N_f(x); x \in G^{(0)}\} \geqq \|g_\chi\|_{1,\ell} - \varepsilon. \quad \square$$

From a), it follows that, for every $f \in \mathscr{A}$, the function N_f is lower semi-continuous.

C. Group valued cocycles. We end this section by examining in which cases a locally compact groupoid with a cocycle is Morita equivalent to a group action.

We first consider a general situation where a Morita equivalence occurs. Recall that the locally compact groupoids G and H with spaces of units X and Y are said to be *Morita equivalent* if there exists a locally compact groupoid $\mathscr{G}$ with space of units $X \coprod Y$ such that both X and Y meet all $\mathscr{G}$-orbits and the restrictions of $\mathscr{G}$ to X and Y are respectively G and H.

A groupoid homomorphism $h: G \to H$ is a *Morita equivalence* if it is invertible in the category of generalized homomorphisms in the sense of [14]; this means that there exists a $\mathscr{G}$ as above endowed with a continuous section ξ of $s: \mathscr{G}_X^Y \to X$ such that, for all $\gamma \in G$, we have $h(\gamma)\xi(s(\gamma)) = \xi(r(\gamma))\gamma$.

1.5. Lemma. *Let H be a locally compact groupoid with space of units Y. Let X be a locally compact space and $f: X \to Y$ a continuous map. Set*

$$Z = \{(x, g) \in X \times H; f(x) = r(g)\}.$$

Assume that the map $(x, g) \mapsto s(g)$ from $Z \to Y$ is open and surjective. Then the set $G = \{(x, g, x') \in X \times H \times X; r(g) = f(x), s(g) = f(x')\}$ is a locally compact groupoid with space of units X, source and range maps $(x, g, x') \mapsto x'$ and $(x, g, x') \mapsto x$, composition $(x, g, x')(x', h, x'') = (x, gh, x'')$; in particular the source map $G \to X$ is open. Moreover, the groupoid homomorphism $(x, g, x') \mapsto g$ is a Morita equivalence from G to H.

Proof. The fact that the source map of G is open is due to the following observation: if $g: Z \to Y$ is open and $f: X \to Y$ is continuous, the map $(z, x) \mapsto x$ from $\{(z, x); g(z) = f(x)\}$ to X is open.

Let us prove the last statement on Morita equivalence. Obviously, if $X = Y$ and f is the identity, then $H = G$. In the general case, let $\mathscr{X}$ be the disjoint union of X and Y and $f': \mathscr{X} \to Y$ be f on X and the identity on Y. The associated groupoid $\mathscr{G}$ with space of objects $\mathscr{X}$ realizes the desired Morita equivalence, since both X and Y are open subsets of $\mathscr{X}$ which meet all the $\mathscr{G}$-orbits. One sees that $\mathscr{G}_X^Y = \{(g, x) \in H \times X; f(x) = r(g)\}$; the desired section $X \to \mathscr{G}_X^Y$ is $x \mapsto (f(x), x)$, where $f(x) \in Y = H^{(0)} \subset Y$. $\square$

1.6. Definition. Let G be a locally compact groupoid, Γ a locally compact group and $\rho: G \to \Gamma$ a cocycle (a continuous groupoid homomorphism). Set $X = G^{(0)}$. We will say that the cocycle ρ is

- *faithful* if the map $\gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$ is injective from G into $X \times \Gamma \times X$;
- *closed* if the map $\gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$ is closed;
- *transverse* if the map $(g, \gamma) \mapsto (g\rho(\gamma), s(\gamma))$ from $\Gamma \times G$ to $\Gamma \times X$ is open.

The cocycle ρ is faithful and closed, if and only if the map $\gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$ is a homeomorphism from G into a closed subset of $X \times \Gamma \times X$.

Note that if the group Γ is discrete, the cocycle ρ is transverse if and only if the set $\{(\rho(\gamma), s(\gamma)); \gamma \in G\}$ is an open subset of $\Gamma \times X$.

Let $\rho: G \to \Gamma$ be a faithful, closed and transverse cocycle. On the space $\Gamma \times X$ define an equivalence relation by $(g, x) \sim (h, y)$ if there exists $\gamma \in G$ such that $r(\gamma) = x$, $s(\gamma) = y$ and $\rho(\gamma) = g^{-1}h$. Let $Y = \Gamma \times X/\sim$ be the quotient space.

1.7. Lemma. *Let G be a locally compact groupoid, Γ a locally compact group and $\rho: G \to \Gamma$ be a faithful, closed and transverse cocycle.*

a) *The quotient map $p: \Gamma \times X \to Y$ is open.*

b) *The graph $\{(\xi, \eta); \xi \sim \eta\}$ of $\sim$ is closed.*

c) *The quotient space Y is a locally compact Hausdorff space.*

d) *The formula $g\overline{(h, x)} = \overline{(gh, x)}$ defines a continuous action of Γ on Y.*

Proof. Denote by $\varphi: \Gamma \times G \to \Gamma \times X$ the map $(g, \gamma) \mapsto (g\rho(\gamma), s(\gamma))$ and by $\psi: G \to X \times \Gamma \times X$ the map $\gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$.

a) Let O be an open subset of $\Gamma \times X$. Set $\Omega = \{(g, \gamma) \in \Gamma \times G; (g, r(\gamma)) \in O\}$. It is an open subset of $\Gamma \times G$. The saturation of O for the relation $\sim$ is the set $p^{-1}(p(O)) = \varphi(\Omega)$. Since ρ is transverse, it is an open subset of $\Gamma \times X$. By definition of the quotient topology, the map p is open.

b) By definition of $\sim$, a pair $((g, x), (h, y))$ is in the graph of $\sim$ if and only if $(x, g^{-1}h, y)$ is in the image of ψ which is closed.

c) Follows from a) and b).

d) The continuous action of Γ on $\Gamma \times X$ by left translation on Γ obviously permutes the equivalence classes of $\sim$. Assertion d) follows. □

1.8. Theorem. *Let G be a locally compact groupoid, Γ a locally compact group and $\rho: G \to \Gamma$ be a faithful, closed and transverse cocycle. There exists a locally compact space Y endowed with a continuous action of Γ and a homomorphism from G to the groupoid $Y \rtimes \Gamma$ which is a Morita equivalence.*

Proof. Let Y be as above. Recall that the groupoid $Y \rtimes \Gamma$ is the set

$$\{(y, g, y') \in Y \times \Gamma \times Y; g(y') = y\}.$$

For $x \in X$, let $f(x)$ denote the class of $(1, x)$ in Y. The map $h: \gamma \mapsto \big(f\big(r(\gamma)\big), \rho(x), f\big(s(\gamma)\big)\big)$ is the desired homomorphism. □

1.9. Corollary. *Let G, Γ, and ρ be as in Theorem* 1.7. *Then*:

a) *The groupoid G is Hausdorff.*

b) *The C^*-algebras $C^*(G)$ and $C_r^*(G)$ are respectively Morita-equivalent to $C_0(Y) \rtimes \Gamma$ and $C_0(Y) \rtimes_r \Gamma$.*

Proof. Clear from Theorem 1.8. □

Remarks. a) The converse of Theorem 1.8 is also true: if $h: G \to Y \rtimes \Gamma$ is a homomorphism and a Morita equivalence, one checks easily that its composition with the homomorphism $Y \rtimes \Gamma \to \Gamma$ is faithful, closed and transverse.

b) The closedness condition is not automatic: For instance, let X be a compact space, $x_0 \in X$ a non isolated point and Γ a discrete group. Set $H = \Gamma \times X - \{x_0\} \cup \{1\} \times X$. It is an open subgroupoid of the groupoid $\Gamma \times X$ (see Example 1.2). The associated quotient space Y is not Hausdorff in this case: it is actually equal (as a topological space) to the non-Hausdorff groupoid G described in Example 1.2.

2. Groupoids and Hilbert modules

The main purpose of this section is to define the groupoid analogue of the regular representation. This is done by associating with a groupoid G a Hilbert module on which $C_r^*(G)$ acts faithfully. Thus giving a concrete picture of $C_r^*(G)$ on a natural space. More precisely, we construct in this section for a groupoid what may be regarded as the Hilbert space $L^2(G)$ (when G is a group) and its left regular representation. In the case of a groupoid, the space $L^2(G)$ is no longer a Hilbert space, but rather a Hilbert module over an abelain C^*-algebra B. If G is Hausdorff, the algebra B is just the algebra $C_0(G^{(0)})$ of continuous complex valued functions vanishing at infinity on the base space of G. In the non-Hausdorff case, this algebra has to be replaced by a bigger algebra which contains some Borel non continuous functions on $G^{(0)}$.

We first need a general lemma.

Let A be a C^*-algebra, $\mathscr{E}$ a Hilbert A-module and $\pi: A \to \mathscr{L}(H)$ a representation of A. We denote by $I_\pi: \mathscr{L}(\mathscr{E}) \to \mathscr{L}(\mathscr{E} \otimes_\pi H)$ the representation $T \mapsto T \otimes 1$.

2.1. Lemma. *Let $\{\rho_i\}$ be a family of representations of A.*

a) *If the representation σ of A is weakly contained in the family $\{\rho_i\}$, then I_σ is weakly contained in the family $\{I_{\rho_i}\}$.*

b) *If $\{\rho_i\}$ is a faithful family, then $\{I_{\rho_i}\}$ is also a faithful family.*

Proof. Let ρ be a representation of A and $T \in \mathscr{L}(\mathscr{E})$. For every $\xi \in \mathscr{E}$, $\eta \in H_\rho$, we have $\langle T(\xi \otimes \eta), T(\xi \otimes \eta)\rangle = \big\langle \rho\big(\langle T(\xi), T(\xi)\rangle^{1/2}\big)(\eta), \rho\big(\langle T(\xi), T(\xi)\rangle^{1/2}\big)(\eta)\big\rangle$. Therefore,

$$\begin{aligned} T \in \ker I_\rho &\Leftrightarrow \forall \xi \in \mathscr{E}, \quad \forall \eta \in H_\rho, \quad T(\xi \otimes \eta) = 0 \\ &\Leftrightarrow \forall \xi \in \mathscr{E}, \quad \forall \eta \in H_\rho, \quad \langle T(\xi \otimes \eta), T(\xi \otimes \eta)\rangle = 0 \\ &\Leftrightarrow \forall \xi \in \mathscr{E}, \quad \rho\big(\langle T(\xi), T(\xi)\rangle\big) = 0. \end{aligned}$$

The lemma follows immediately. □

The Hausdorff case. Define a $C_0(X)$-valued scalar product on $\mathscr{A} = C_c(G)$ by

$$\langle \xi, \eta\rangle(x) = \int_{G_x} \overline{\xi(\gamma)}\eta(\gamma)\, d\nu_x(\gamma). \tag{7}$$

In other words, $\langle \xi, \eta\rangle \in C_0(X)$ is the restriction to $X \subset G$ of $\xi^* \star \eta \in \mathscr{A} = C_c(G)$. Given $f \in C_0(X)$ and $\xi \in \mathscr{A}$ define the right action by $\xi f(\gamma) = \xi(\gamma) f\big(s(\gamma)\big)$. With these operations $\mathscr{A}$ is a pre-Hilbert $C_0(X)$-module. Let $L^2(G, \nu)$ be its Hilbert-module completion. We will show that $C_r^*(G)$ acts faithfully on $L^2(G, \nu)$. With this notation we have:

2.2. Lemma. *If $f, \xi \in C_c(G)$, then $\|f \star \xi\|_{L^2(G,\nu)} \leqq \|f\|_r \|\xi\|_{L^2(G,\nu)}$.*

Proof. First note that for $\xi \in C_c(G)$, we have

$$\|\xi\|^2_{L^2(G,\nu)} = \sup_x \langle \xi, \xi\rangle(x) = \sup_x \int_{G_x} |\xi(\gamma)|^2\, d\nu_x(\gamma) = \sup_x \|\xi|_{G_x}\|_2^2.$$

Then, using the above, we have

$$\|f \star \xi\|_{L^2(G,\nu)} = \sup_x \|f \star \xi|_{G_x}\|_2 = \sup_x \|\lambda_x(f)(\xi)\|_2 \leqq \|f\|_r \|\xi\|_{L^2(G,\nu)}. \quad \square$$

Recall that a Hilbert $C(X)$-module $\mathscr{E}$ is the space of continuous sections of a continuous field $(\mathscr{E}_x)_{x \in X}$ of Hilbert spaces. An element T of $\mathscr{L}(\mathscr{E})$ is a ($*$-strongly continuous) field $(T_x)_{x \in X}$ of operators on the field $(\mathscr{E}_x)_{x \in X}$, and $\|T\| = \sup_x \|T_x\|$.

2.3. Theorem. *There exists a $*$-representation λ: $C^*(G) \to \mathscr{L}\big(L^2(G, \nu)\big)$ such that, for $f, \xi \in \mathscr{A}$ we have $\lambda(f)(\xi) = f \star \xi$. For all $f \in C^*(G)$ and for all $x \in X$, we have $\lambda(f)_x = \lambda_x(f) \in \mathscr{L}\big(L^2(G, \nu)_x\big) \cong \mathscr{L}\big(L^2(G_x, \nu_x)\big)$. For all $f \in C^*(G)$, we have $\|\lambda(f)\| = \|f\|_r$. In this way, we may identify $C_r^*(G)$ with $\lambda\big(C^*(G)\big)$.*

Proof. For $f, \xi \in \mathscr{A}$, let $\lambda(f)\xi = f \star \xi$. Then, by Lemma 2.2

$$\|\lambda(f)\xi\|_{L^2(G,\nu)} \leqq \|f\|_r \|\xi\|_{L^2(G,\nu)}.$$

It follows:

1. For all f, $\lambda(f)$ extends to a bounded operator on $L^2(G,\nu)$.

2. λ extends to a $*$-representation of $C_r^*(G)$.

Now, for $f \in \mathscr{A}$, the formula $\big(\lambda(f)\big)_x = \lambda_x(f)$ (using the canonical identification of $L^2(G,\nu)_x$ with $L^2(G_x,\nu_x)$) is obvious since these operators are defined on $\mathscr{A}$ with the same convolution formula. Moreover $\|f\|_r = \sup\|\lambda(f)_x\| = \|\lambda(f)\|$. □

If μ is a measure on X, let $\mathrm{Ind}_\mu(f) = \int_X^{\oplus} \lambda_x(f)\, d\mu(x)$.

2.4. Corollary. a) *For a dense subset $D \subset X$ we have* $\|f\|_r = \sup_{x\in D} \|\lambda_x(f)\|$.

b) *Let μ be a positive measure on X. If $x \in \mathrm{Supp}\,\mu$, then λ_x is weakly contained in* Ind_μ. *In particular, if the support of μ is X, then* Ind_μ *is a faithful representation of $C_r^*(G)$.*

Proof. a) Since D is dense in X, the maps $f \mapsto f(x)$, from $C_0(X)$ onto $\mathbb{C}$ form a faithful family $(\varepsilon_x)_{x\in D}$ of representations of $C_0(X)$. Now, the corollary follows from Lemma 2.1.

b) Let π_μ be the representation of $C(X)$ by multiplication in $L^2(X,\mu)$. If $x \in \mathrm{Supp}\,\mu$, then by Lemma 2.1, $I_{\varepsilon_x} = \lambda_x$ is weekly contained in $I_{\pi_\mu} = \mathrm{Ind}_\mu$. □

The non-Hausdorff case. The following example shows that in the non-Hausdorff case Corollary 2.4 is not correct as stated above. The example also shows that in the statement of [16], Proposition 3.1.2, Hausdorffness of the groupoid is needed.

2.5. Example. Let G be the groupoid of Example 1.2. Then, $D = X\backslash\{x_0\}$ is dense in X. But, it is clear from the example, that for $T \in \mathscr{A}$, $\|T\|_r \neq \sup_{x\in D} \|\lambda_x(T)\|$. Moreover, if μ is a measure on X with dense support, the representation Ind_μ is faithful if and only if the point $\{x_0\}$ has nonzero mass.

We now examine which modifications should be done.

Let G be a non-Hausdorff groupoid and let $\mathscr{A}$ be as in section 1. Set $D = \{f|_X; f \in \mathscr{A}\}$. Since we are in the non-Hausdorff case, elements of $\mathscr{A}$ are *not continuous*, therefore D does not consist of continuous functions. However, $D \subset B(X)$ where $B(X)$ is the C^*-algebra of bounded Borel functions on X. Let B be the C^*-subalgebra of $B(X)$ generated by D. By Gelfand-Naimark theorem, $B \cong C_0(Y)$ for some locally compact Hausdorff space Y. Since, X is an open and Hausdorff subset of G, $C_c(X) \subset D \subset B$. We summarize the important properties of the space Y in the following proposition:

2.6. Proposition. *Let X and Y be as above. Then:*

a) *The inclusion $C_0(X) \subset C_0(Y)$ yields a continuous map $j\colon Y \to X$ which is proper and onto.*

b) *The inclusion* $C_0(Y) \subset B(X)$ *yields a Borel map* $i: X \to Y$ *such that* $i(X)$ *is dense in* Y *and* $i \circ j$ *is the identity map on* X.

Proof. From $C_0(X) \subset C_0(Y)$ we obtain a continuous surjection $j: Y^+ \to X^+$ satisfying the conditions of a), where X^+ and Y^+ denote the one point compactification of X and Y. Moreover, for every $f \in \mathscr{A}$, there exists $g \in C_0(X)$, such that $(g \circ r)f = f$; we have $\big((g \circ r)f\big)|_X = f|_X g$. Therefore, $C_0(X)C_0(Y) = C_0(Y)$. It follows that j maps the point at infinity of Y^+ to that of X^+ and Y to X, and hence it maps Y onto X.

b) is clear. □

Let G be the groupoid of Example 1.2 with $|\Gamma| \neq 1$. Then $Y = X \times \{0\} \cup \{(x_0, 1)\}$; in this case $i(x) = (x, 0)$ if $x \neq x_0$ and $i(x_0) = (x_0, 1)$. Note the fact that i is not continuous at x_0.

2.7. A construction of a Hilbert module. Before proceeding we need a Hilbert module construction based on Kasparov's generalized Stinespring theorem ([11]).

Let A be a C^*-algebra and let $\mathscr{E}$ be a vector space with a completely positive A-valued scalar product, i.e. a sesquilinear map $\mathscr{E}^2 \ni (\xi, \eta) \mapsto \langle \xi, \eta \rangle \in A$ such that for every n-tuple $(\xi_1, \dots, \xi_n) \in \mathscr{E}^n$ the $n \times n$ matrix $(\langle \xi_i, \xi_j \rangle)_{i,j}$ is an element of $M_n(A)_+$. Form the algebraic tensor product $\mathscr{E} \odot A$ of linear spaces and define an A-valued scalar product on the elementary tensors by

$$\langle \xi \otimes a, \eta \otimes b \rangle = a^* \langle \xi, \eta \rangle b \quad (\xi, \eta \in \mathscr{E}, a, b \in A).$$

This product is obviously A-sesquilinear with respect to the right action of A on $\mathscr{E} \odot A$ given by $(\xi \otimes a)b = \xi \otimes ab$. This product is also positive. Indeed, let $\xi_1, \xi_2, \dots, \xi_n \in \mathscr{E}$ and $a_1, a_2, \dots, a_n \in A$; then

$$\left\langle \sum_i \xi_i \otimes a_i, \sum_i \xi_i \otimes a_i \right\rangle = \sum_{i,j} a_i^* (\langle \xi_i, \xi_j \rangle) a_j = \langle \eta, C\eta \rangle \in A_+,$$

where, $\eta \in A^n$ is the column matrix given by $a_1, a_2, \dots, a_n$ and $C \in M_n(A)_+$ is the matrix $(\langle \xi_i, \xi_j \rangle)_{i,j}$. Hence, $\mathscr{E} \odot A$ with the right action $(\xi \otimes a)b = \xi \otimes ab$ is a pre-Hilbert A-module. The Hausdorff completion, denoted by $\mathscr{E} \otimes A$, is a Hilbert A-module.

We apply the above construction to the case of a non-Hausdorff groupoid G. Given $\xi, \eta \in \mathscr{A}$, let $\langle \xi, \eta \rangle = (\xi^* \star \eta)|_X \in D \subset C_0(Y)$.

2.8. Lemma. *The inner product* $\langle \xi, \eta \rangle = (\xi^* \star \eta)|_X$ *is completely positive.*

Proof. Given $\xi_1, \xi_2, \dots, \xi_n \in \mathscr{A}$, we must show that for each $y \in Y$, the scalar matrix $\big(\langle \xi_i, \xi_j \rangle (y)\big)$ is positive. It is therefore enough to see that, for all $z_1, \dots, z_n \in \mathbb{C}^n$ and every $y \in Y$ we have $\big\langle z, \big(\langle \xi_i(y), \xi_j(y) \rangle\big) z \big\rangle \in \mathbb{R}_+$ which means that $\langle \xi, \xi \rangle \in C_0(Y)_+$ where $\xi = \sum_i z_i \xi_i \in \mathscr{A}$.

But for all $x \in X$, we have $\langle \xi, \xi \rangle \big(i(x)\big) = \int_{G_x} |\xi(\gamma)|^2 \, d\nu_x(\gamma) \geqq 0$; the lemma follows by the density of $i(X)$ in Y. □

Using Lemma 2.8, we form the Hilbert $C_0(Y)$-module, $\mathscr{A} \otimes C_0(Y)$, as described in 2.7. This module will be denoted by $L^2(G,\nu)$. Of course, if G is Hausdorff, $Y = X$ and the definitions of $L^2(G,\nu)$ coincide.

2.9. Lemma. *For each $x \in X$, let $\hat{x}: C_0(Y) \to \mathbb{C}$ be the character defined by $\hat{x}(f) = f\big(i(x)\big)$. Then there is an isomorphism $U_x: L^2(G,\nu) \otimes_{\hat{x}} \mathbb{C} \to L^2(G_x, \nu_x)$ such that, for all $\xi \in \mathscr{A}$, $f \in C_0(Y)$ and $z \in \mathbb{C}$ we have $U_x\big((\xi \otimes f) \otimes z\big) = \big(zf\big(i(x)\big)\big)\xi|_{G_x}$.*

Proof. We first check that the inner product is preserved. Let $\xi, \eta \in \mathscr{A}$, $f, g \in C_0(Y)$ and $z, w \in \mathbb{C}$. We have

$$\begin{aligned}
\langle \xi \otimes f \otimes_{\hat{x}} z, \eta \otimes g \otimes_{\hat{x}} w \rangle &= \bar{z}\hat{x}(\langle \xi \otimes f, \eta \otimes g \rangle)w \\
&= \bar{z}\hat{x}\big(\bar{f}(\xi^* \star \eta)|_X g\big)w \\
&= \bar{z}\overline{f\big(i(x)\big)}(\xi^* \star \eta)(x) g\big(i(x)\big)w \\
&= \int_{G_x} \overline{zf\big(i(x)\big)\xi(\gamma)} w g\big(i(x)\big)\eta(\gamma)\, d\nu_x(\gamma) \\
&= \langle zf\xi, wg\eta \rangle.
\end{aligned}$$

This shows that U_x defines an isometry from $L^2(G,\nu) \otimes_{\hat{x}} \mathbb{C}$ to $L^2(G_x, \nu_x)$. Moreover, since G_x is a closed Hausdorff subset of G, the restriction to G_x defines a surjection from $\mathscr{A}$ to $C_c(G_x)$; it follows that U_x is onto. $\square$

2.10. Theorem. *There exists a $*$-representation $\lambda: C^*(G) \to \mathscr{L}\big(L^2(G,\nu)\big)$ such that, for $f, \xi \in \mathscr{A}$ and $g \in C_0(Y)$ we have $\lambda(f)(\xi \otimes g) = (f \star \xi) \otimes g$. For all $f \in C^*(G)$ and for all $x \in X$, we have $\big(\lambda(f)\big)_{i(x)} = \lambda_x(f) \in \mathscr{L}\big(L^2(G,\nu)_x\big) = \mathscr{L}\big(L^2(G_x,\nu_x)\big)$. For all $f \in C^*(G)$, we have $\|\lambda(f)\| = \|f\|_r$. In this way, we may identify $C_r^*(G)$ with $\lambda\big(C^*(G)\big)$.*

Proof. Given $f \in \mathscr{A}$ define $T_f: \mathscr{A} \odot C_0(Y) \to \mathscr{A} \odot C_0(Y)$ setting

$$T_f(\xi \otimes g) = (f \star \xi) \otimes g.$$

For every $x \in X$, we have

$$\begin{aligned}
U_x\big(T_f(\xi \otimes g) \otimes_{\hat{x}} 1\big) &= U_x(f \star \xi \otimes g \otimes_{\hat{x}} 1) \\
&= (f \star \xi)|_{G_x} g\big(i(x)\big) \\
&= \lambda_x(f) U_x(\xi \otimes g \otimes_{\hat{x}} 1).
\end{aligned}$$

Let $\zeta \in \mathscr{A} \odot C_0(Y)$; the norm of $T_f(\zeta)$ in $L^2(G,\nu)$ is the supremum over $x \in X$ of its image in $L^2(G,\nu)_x \cong L^2(G_x,\nu_x)$. This image is $U_x\big(T_f(\zeta) \otimes 1\big) = \lambda_x(f)\big(U_x(\zeta \otimes 1)\big)$, whose norm is less than $\|f\|_r\|\zeta\|$.

In particular, T_f extends to a continuous linear map $\lambda(f): L^2(G,\nu) \to L^2(G,\nu)$. Furthermore, one sees easily that $\lambda(f)$ is adjointable with adjoint $\lambda(f^*)$.

The above computations show that, for all $x \in X$, we have $\big(\lambda(f)\big)_{i(x)} = \lambda_x(f)$ in $\mathscr{L}\big(L^2(G,\nu)_x\big)$ identified with $\mathscr{L}\big(L^2(G_x,\nu_x)\big)$ thanks to Lemma 2.9.

Finally,

$$\begin{aligned}\|\lambda(f)\| &= \sup_{x\in X}\big\|\big(\lambda(f)\big)_{i(x)}\big\| \quad \text{(by density of } i(X) \text{ and Lemma 2.1)}\\ &= \sup_{x\in X}\|\lambda_x(f)\|\\ &= \|f\|_r. \quad \square\end{aligned}$$

2.11. Corollary. a) *Let $D \subset X$ be such that $i(D)$ is dense in $i(X)$, then for every $T \in C^*(G)$ we have $\|T\|_r = \sup_{x\in D}\{\|\lambda_x(T)\|\}$.*

b) *Let μ be a probability measure on X, and let $i(x) \in \operatorname{Supp} i(\mu)$. Then, λ_x is weakly contained in $\operatorname{Ind}_{i(\mu)}$.*

Proof. Clear from Lemma 2.1, and Theorem 2.10. □

2.12. Remark. Let (V,F) be a minimal foliation. In [7], it is shown that, if the holonomy groupoid is Hausdorff, then the C^*-algebra of (V,F) is simple. In the non-Hausdorff case, this is no longer true (cf. [19]). However, from the proof of [7] and Corollary 2.11 we get:

Let $Z \subset V$ be the set of points of V with trivial holonomy group. If $i(Z)$ is dense in $i(V)$, then the foliation C^*-algebra is simple.

Indeed, let π be a nonzero representation of $C^*(V,F)$ and $x \in Z$. It follows from [7] that π weakly contains λ_x. By Corollary 2.11, the family $(\lambda_x)_{x\in L}$ is faithful, whence π is faithful.

3. Applications to inverse semigroups

The relations between inverse semigroups and groupoids are well known. A. Paterson ([16]) gives an extensive account of the connections between the two theories through their operator algebras.

In particular, with an inverse semigroup S is naturally associated a groupoid G_S, such that $C^*(S) \cong C^*(G_S)$ and $C^*_r(S) \cong C^*_r(G_S)$. It turns out that, in the non-Hausdorff case there is a small difficulty in the latter isomorphism that had been overseen. Equipped with the result of section 2, we give a new proof of the isomorphism $C^*(S) \cong C^*(G_S)$ and show that, even in the non-Hausdorff case, this isomorphism passes to the reduced C^*-algebras.

Moreover, using Theorem 1.8 and Corollary 1.9, we prove that the full and the reduced C^*-algebras of an E-unitary inverse semigroup satisfying an extra assumption (see Prop. 3.9) is Morita-equivalent to the crossed product of its maximal group homomorphic image with an abelian C^*-algebra.

Inverse semigroups. We begin by recalling some basic facts about inverse semigroups.

A semigroup S is called an *inverse semigroup* if for each $u \in S$, there exists a unique element, denoted by u^* in S such that $uu^*u = u$, and $u^*uu^* = u^*$.

The *set of idempotents* $E_S = \{u^*u \in S; u \in S\}$ of S plays a crucial role in the study of inverse semigroups. It is a well known fact that E_S is a commutative sub-semigroup of S. Moreover, E_S is a semi-lattice under the partial ordering $e \leqq f$ if $ef = e$.

For $u, v \in S$, set $u \sim v$ if there exists $e \in E_S$ such that $eu = ev$. This is an equivalence relation on S. The quotient $S/\sim$ is a group denoted by Γ_S called the *maximal group homomorphic image* of S, in that any group homomorphic image of S is a quotient of Γ_S. Let $\sigma\colon S \to \Gamma_S$ denote the quotient map. Note that the set E_S maps to the identity of Γ_S. For these facts and more on the algebraic theory of inverse semigroups we refer to [9].

3.1. Lemma. *For $u, v \in S$ the following conditions are equivalent.*

(i) $u^*v = u^*u$; (i)′ $v^*u = u^*u$; (ii) $vu^* = uu^*$; (ii)′ $uv^* = uu^*$;

(iii) $u = uu^*v$; (iii)′ $\exists p \in E, u = pv$; (iv) $u = vu^*u$; (iv)′ $\exists q \in E, u = vq$.

Proof. Note that (i) $\Leftrightarrow$ (i)′ and (ii) $\Leftrightarrow$ (ii)′ are proved by passing to adjoints; (iii) $\Rightarrow$ (iii)′ and (iv) $\Rightarrow$ (iv)′ are obvious; (i) $\Rightarrow$ (iii) (resp. (ii) $\Rightarrow$ (iv)) is obtained by left (resp. right) multiplication by u; finally, if (iv)′ (resp. (iii)′) holds, we find

$$u^*u = qv^*vq = v^*vq^2 = v^*u$$

(resp. $uu^* = pvv^*p = vv^*p^2 = vu^*$). $\square$

For $u, v \in S$, write $u \prec v$ if one (and hence all) of the equivalent conditions of Lemma 3.1 holds. This is a partial ordering on S.

The C^*-algebras of an inverse semigroup. An inverse semigroup S can be represented as a semigroup of partial isometries on a Hilbert space H: a *representation* of an inverse semigroup S on a Hilbert space H is a map $\pi\colon S \to \mathscr{L}(H)$ such that $\pi(uv) = \pi(u)\pi(v)$, and $\pi(u^*) = \pi(u)^*$.

For $f, g \in \ell^1(S)$, we set

$$(8) \qquad f \star g(w) = \sum_{uv=w} f(u)g(v), \quad f^*(w) = \overline{f(w^*)}, \quad \|f\| = \sum_{u \in S} |f(u)|.$$

Endowed with these operations, $\ell^1(S)$ is a Banach-$*$-algebra. Each representation of S extends linearly to a $*$-representation of $\ell^1(S)$. The enveloping C^*-algebra of $\ell^1(S)$ is called the *C^*-algebra of S* and is denoted by $C^*(S)$. In particular, there exists a one to one correspondence between the representations of S and those of $C^*(S)$.

A very important representation of S, the left regular representation, is

$$\Lambda\colon S \to B(\ell^2(S))$$

defined by

$$\Lambda(u)(\delta_v) = \begin{cases} \delta_{uv} & \text{if } u^*uv = v, \\ 0 & \text{otherwise,} \end{cases}$$

where $(\delta_u)_{u\in S}$ denotes the canonical orthonormal basis of $\ell^2(S)$. The extension to $\ell^1(S)$ of Λ is known to be faithful (cf. [22]), but Λ is not in general faithful on $C^*(S)$. The *reduced C^*-algebra of S* is by definition the image of $C^*(S)$ under Λ and will be denoted by $C_r^*(S)$.

The groupoid of an inverse semigroup. We recall a construction which associates with each inverse semigroup S, in a natural and explicit way, a groupoid G. For more on this construction, the basic properties of the groupoid, and examples see [16].

Let S be an inverse semigroup and $E = E_S$ the set of idempotents of S. Let X be the space of multiplicative linear functionals on $\ell^1(E)$ with the relative weak*-topology. In other words, X is the spectrum of the abelian C^*-algebra $C^*(E)$. We consider the elements of E as continuous functions on X in the obvious way. In fact, each $p \in E$ corresponds to the characteristic function of the set

$$F_p = \{x \in X; p(x) = 1\},$$

a clopen subset of X. The space X is totally disconnected, locally compact, and Hausdorff with the sets F_p forming a sub-basis consisting of clopen sets for the topology of X.

Moreover, the semigroup S acts on the space X by local homeomorphisms (i.e. the pair (X, S) is a localization in the sense of [12], [16]) in the following manner. If $x \in F_{u^*u}$, let $u.x$ be the character $e \mapsto x(u^*eu)$ for $e \in E$. Then, $u.x \in F_{uu^*} \subset X$ and this defines a homeomorphism from F_{u^*u} onto F_{uu^*}.

The groupoid of S is the groupoid associated to this pseudogroup of partial homeomorphisms. Namely, let $\hat{S} = \{(u,x) \in S \times X; (u^*u)(x) = 1\}$ with the relative product topology. Define an equivalence relation on $\hat{S}$ by $(u,x) \sim (v,x)$ if there exists $e \in E$ such that $e(x) = 1$ and $ue = ve$.

Let G_S be the quotient space of $(\hat{S}, \sim)$ equipped with the quotient topology and $(u,x) \mapsto \widetilde{(u,x)}$ the quotient map. Then G_S is a groupoid with $G_S^{(0)} = X$ under the following operations:

$$(9) \quad s\widetilde{(u,x)} = x, \quad r\widetilde{(u,x)} = u.x, \quad \widetilde{(u,x)}^{-1} = \widetilde{(u^*, u.x)}, \quad \text{and} \quad \widetilde{(u,x)}\widetilde{(v,y)} = \widetilde{(uv,y)},$$

for $v.y = x$. The groupoid G_S will be called *the groupoid of S*. It is a locally compact r-discrete groupoid which is not in general Hausdorff. The sets $O_u = \{\widetilde{(u,x)}; x \in F_{u^*u}\}$ form an open covering of G_S; the source and range maps restrict to homeomorphisms $s\colon O_u \to O_{u^*u}$ and $r\colon F_u \to F_{uu^*}$.

Embedding S in G_S. Each $p \in E$ defines an element ε_p of X by $q(\varepsilon_p) = 1$ if $p \leqq q$ (i.e. $pq = p$) and zero otherwise. Furthermore, each $u \in S$ defines an element $\varepsilon_u = \widetilde{(u, \varepsilon_{u^*u})} \in G_S$.

Since $C_0(X) = C^*(E)$ is generated by E, an element $x \in X$ is determined by the set $\mathscr{F}_x = \{p \in E; p(x) = 1\}$. Conversely, the characteristic function of a nonempty subset $F \subset E$ such that

$$(e \in F \text{ and } f \in F) \Leftrightarrow ef \in F \quad \text{for every } e, f \in E,$$

is a nonzero character χ_F of $C^*(E)$, and therefore determines an element of X. Moreover, $\mathscr{F}_x$ is a directed ordered set by $p \ll q$ if $q \leqq p$.

We will use the following result—which is a slightly stronger reformulation of Proposition 4.3.1 in [16].

3.2. Proposition. a) *For $x \in X$ the net $\{\varepsilon_p\}_{p \in \mathscr{F}_x}$ converges to x. In particular, $\{\varepsilon_p\}_{p \in E}$ is dense in X.*

b) *For $e \in E$, we have $\{\gamma \in G_S; s(\gamma) = \varepsilon_e\} = \{\varepsilon_u; u \in S, u^*u = e\}$.*

c) *The map $u \mapsto \varepsilon_u$ is an injection with dense range from S into G_S.*

Proof. a) By definition, $C^*(E)$ is spanned by the functions $p \in E$. Therefore, there exists $p_0 \in E$ such that $p_0(x) \neq 0$.

Let V be an open neighborhood of x in X. There exist $p_1, \ldots, p_n \in E$ and $q_1, \ldots, q_m \in E$ such that $x \in \left(\bigcap_i F_{p_i}\right) \Big\backslash \left(\bigcup_i F_{q_i}\right) \subset V$. Let p be the product of the p_i's from 0 to n.

As $x \in F_{p_i}$, we find $p(x) = 1$, whence $p \in \mathscr{F}_x$.

Let $e \in \mathscr{F}_x$ such that $p \ll e$;

- as $p_i \ll p \ll e$, we have $p_i(\varepsilon_e) = 1$;
- as $x \in F_e \backslash F_{q_j}$, we find $F_e \not\subset F_{q_j}$, whence $e \not\leqq q_j$; we draw $q_j(\varepsilon_e) = 0$.

It follows that $\varepsilon_e \in V$. Assertion a) follows.

b) Let $e \in E$. An element $\gamma \in G_S$ with source ε_e is the class of an element (u, ε_e) with $e \leqq u^*u$, and therefore $\gamma = \varepsilon_{ue}$.

c) If $u, v \in S$ are such that $u^*u = v^*v = e$ and $\widetilde{(u, \varepsilon_e)} = \widetilde{(v, \varepsilon_e)}$, then there exists $f \in E$ with $uf = vf$ and $f(\varepsilon_e) = 1$; the last condition yields $e \leqq f$, therefore $u = ue = uf = vf = v$. Injectivity of $u \mapsto \varepsilon_u$ follows.

It follows easily from a) and b) that $\varepsilon(S)$ is dense in G_S. □

Full C^*-algebras. We now show that the full C^*-algebra of an inverse semigroup is equal to the full C^*-algebra of the associated groupoid. More precisely, we have:

3.3. Theorem. *Let S be an inverse semigroup and G_S the associated groupoid. Let $\mathscr{A}$ be the function algebra associated with the groupoid G_S (cf. Section 1). For $u \in S$, denote by $f_u \in \mathscr{A}$ the characteristic function of the compact open set O_u. Then there is an isomorphism $C^*(S) \cong C^*(G_S)$ mapping $\delta_u \in \ell^1(S) \subset C^*(S)$ to $f_u \in \mathscr{A} \subset C^*(G_S)$.*

Proof. One checks immediately the equalities $f_u f_v = f_{uv}$ and $f_u^* = f_{u^*}$. By the universal property of $C^*(S)$, there exists a $*$-representation π: $C^*(S) \to C^*(G_S)$ such that $\pi(\delta_u) = f_u$.

Let $f = \sum_{u \in S} g_u$ be an element of $\mathscr{A}$, where g_u is a continuous function with compact support on O_u (Lemma 1.3). Now one may write $g_u = f_u h_u$ where h_u is a continuous function on F_{u^*u} such that $g_u = h_u \circ s$.

It follows from the definition of X, that $C_0(X)$ may be viewed as the subalgebra $C^*(E)$ of $C^*(S)$. Moreover, for $f \in C_c(X)$, $\pi(f)$ is the element $f \in C_c(X) \subset \mathscr{A} \subset C^*(G_S)$: this can be checked for $f = p \in E$. Since the image of π contains $C_c(X)$ and the f_u, it contains $\mathscr{A}$. Therefore π is onto.

More precisely, the element $\sum_{u \in S} f_u h_u \in \mathscr{A}$ is the image of the element $\sum_{u \in S} \delta_u h_u \in C^*(S)$.

To show that π is injective, we may use [17], [18]: each representation σ of $C^*(S)$ is a representation of S by partial isometries. The latter, restricts to a representation of E, and therefore to a representation of $C_c(X)$, which defines a measure class μ and a measurable field $(H_x)_{x \in X}$ of Hilbert spaces over X. Each $u \in U$, defines a partial isometry, and looking at its compatibility with the action of $C(X)$, it follows that μ is invariant under the partial transformation of X associated with u; moreover, we may disintegrate u to a measurable family $V_{u,x}$: $H_x \to H_{u(x)}$ over $\{x \in X; (u^*u)(x) = 1\}$. We see immediately that (H, V) is a representation of G_S in the sense of [17] (cf. [18] for the non-Hausdorff case). It follows now straightforwardly from [17], [18] that σ is a representation of $C^*(G_S)$, i.e. factors through π. □

Remark. We may also show in a more direct way the injectivity of π, by using Lemma 1.4. We just need to show that the norm $\left\| \sum_{u \in S} \delta_u h_u \right\|$ of $\sum_{u \in S} \delta_u h_u$ in $C^*(S)$ is less than or equal to the norm $\left\| \sum_{u \in S} \delta_u h_u \right\|_1$ (where the norm $\| \ \|_1$ on $\mathscr{A}$ is given by formula (3)): this will show that we have a morphism σ: $\mathscr{A} \to C^*(S)$ continuous with respect to the norm $\| \ \|_1$; as $\pi \circ \sigma$ is the inclusion of $\mathscr{A} \subset C^*(G_S)$, this σ will be a $*$-homomorphism of $\mathscr{A}$ and will therefore extend to a $*$-homomorphism $\tilde{\sigma}$: $C^*(G_S) \to C^*(S)$. Looking at generators of $C^*(S)$, we find that $\tilde{\sigma}$ is the inverse of π.

Now, by Lemma 1.4, the L^1-norm of $R = \sum_{u \in S} \delta_u h_u$ is an infimum over all writings R as a sum $\sum_{u \in S} \delta_u h_u$. It is therefore enough to show that the norm in $C^*(S)$ of $\sum_{u \in S} \delta_u h_u$ is $\leqq \max\left(\sup_{x \in X} \sum_u |h_u(x)|, \sup_{x \in X} \sum_u |h'_u(x)| \right)$ where $h'_u = \delta_u h_u \delta_{u^*}$. To that end, write $\sum_{u \in S} \delta_u h_u = \sum_{u \in S} k_u \delta_u \ell_u$, with $k_u, \ell_u \in C_0(X)$ given by $k_u = \delta_u \sqrt{|h_u|} \delta_u^* = \sqrt{h'_u}$ and $\ell_u = \varphi \circ h_u$ where φ: $\mathbb{C} \to \mathbb{C}$ is the function defined by $\varphi(0) = 0$ and $\varphi(z) = z|z|^{-1/2}$ for $z \neq 0$. The fact that $\delta_u h \delta_u^* \in C_0(X)$ for all $f \in C_0(X)$ is checked on generators δ_e of $C_0(X)$. Now use the well known elementary matrix calculation: $\sum_{u \in S} k_u \delta_u \ell_u = L\Delta C$, where L is the line matrix with coefficients k_u, Δ is the diagonal matrix with coefficients δ_u and C is the column matrix with coefficients ℓ_u; the δ_u's are nonzero partial isometries whence $\|\delta_u\| = 1$, so that $\|\Delta\| = 1$, moreover $\|L\|^2 = \|LL^*\| = \sup_{x \in X} \sum_u |h'_u(x)|$ and $\|C\|^2 = \|C^*C\| = \sup_{x \in X} \sum_u |h_u(x)|$. We get

$$\left\| \sum_{u \in S} \delta_u h_u \right\| \leqq \|L\| \, \|\Delta\| \, \|C\| = \left(\sup_{x \in X} \sum_u |h'_u(x)| \right)^{1/2} \left(\sup_{x \in X} \sum_u |h_u(x)| \right)^{1/2}$$
$$\leqq \max\left(\sup_{x \in X} \sum_u |h_u(x)|, \sup_{x \in X} \sum_u |h'_u(x)| \right).$$

Reduced C^*-algebras. To establish the isomorphism of the reduced C^*-algebras, $C_r^*(G_S) \cong C_r^*(S)$, we will use an equality

$$\|T\|_r = \sup_p \{ \|\lambda_p(T)\|_r ; p \in E \}.$$

A priori, as seen by Example 2.5, this does not follow from the density of E in $X = G_S^{(0)}$; instead, by Corollary 2.11, we must show that $i(E)$ is dense in the space Y of section 2. The proof of $C_r^*(G_S) \cong C_r^*(S)$ given in [16] uses [16], Proposition 3.1.2, which as alluded to require Hausdorffness (see Example 2.5).

3.4. Lemma. *The set $\{i(\varepsilon_p); p \in E\}$ is dense in Y.*

Proof. Since $i(X)$ is dense in Y, it is enough to show that the closure of $\{i(\varepsilon_p); p \in E\}$ contains $i(X)$.

Let $x \in X$; we will show with the terminology of Proposition 3.2, that the net $\big(i(\varepsilon_p)\big)_{p \in \mathscr{F}_x}$ converges to $i(x)$. One just needs to show that, for all $f \in C_0(Y)$, the net $\big(f\big(i(\varepsilon_p)\big)\big)_{p \in \mathscr{F}_x}$ converges to $f\big(i(x)\big)$. As $C_0(Y)$ is generated by the set D of restrictions to X of elements of $\mathscr{A}$, it suffices to show that for each $g \in \mathscr{A}$ the net $\big(g(\varepsilon_p)\big)_{p \in \mathscr{F}_x}$ converges to $g(x)$.

Moreover, we only need to prove this for elements of the form $f|_X$ where f is a continuous function in a compact open set $O_u = \{\widetilde{(u, y)}; \hat{y}(u^*u) = 1\}$ for some $u \in S$ (Lemma 1.3). We may write $f = f_u g$, where g is a continuous function on F_{u^*u}. Then, for $z \in X$, we have

$$f|_X(z) = \begin{cases} 0 & \text{if } z \notin F_{u^*u}, \\ 0 & \text{if } z \in F_{u^*u},\ \widetilde{(u, z)} \neq z, \\ g(z) & \text{if } z \in F_{u^*u},\ \widetilde{(u, z)} = z. \end{cases}$$

If $x \notin F_{u^*u}$, then as F_{u^*u} is closed in X, the function $f|_X$ is continuous at x; therefore $\big(f(\varepsilon_p)\big)_{p \in \mathscr{F}_x}$ converges to $f(x)$ by Proposition 3.2.

Second, if $x \in F_{u^*u}$, but $\widetilde{(u,x)} \neq x$. Then, for all $p \in \mathscr{F}_x$, we have $\varepsilon_p \notin F_{u^*u}$, for otherwise we get that $\widetilde{(u,x)} = \widetilde{(p,x)}$ which is in contradiction with $\widetilde{(u,x)} \neq x$; therefore $\big(f(\varepsilon_p)\big)_{p \in \mathscr{F}_x}$ is the 0 family, and converges to $f(x)$.

Finally, assume $\widetilde{(u,x)} = x$. As the set $\{z \in F_{u^*u}; \widetilde{(u,z)} = z\}$ is open, $f|_X$ is continuous at x; therefore $\big(f|_X(\varepsilon_p)\big)_{p \in \mathscr{F}_x}$ converges to $f(x)$ by Proposition 3.2. □

3.5. Theorem. *Let S be an inverse semigroup and G_S its associated groupoid. Then $C_r^*(S) \cong C_r^*(G_S)$.*

Proof. The left regular representation of S decomposes into a direct sum $\Lambda = \bigoplus_{e \in E} \Lambda_e$ under the decomposition $\ell^2(S) = \bigoplus_{e \in E} \ell^2(S_e)$, where for $e \in E$, $S_e = \{u \in S; u^*u = e\}$.

From Corollary 2.11, and Lemma 3.4, the family $\{\lambda_{\varepsilon_e}; e \in E\}$ is a faithful family of representations for $C_r^*(G)$, where λ_{ε_e}: $C_r^*(G) \to B\big(\ell^2(G_{\varepsilon_e})\big)$.

By Proposition 3.2 b), one has $G_{\varepsilon_e} = \{\varepsilon_u; u \in S_e\}$. Hence, there is an isomorphism V_e: $\ell^2(S_e) \to \ell^2(G_{\varepsilon_e})$ such that, for $u \in S_e$, we have $V_e(\delta_u) = \delta_{\widetilde{(u,\varepsilon_e)}}$.

Let π: $C^*(G) \to C^*(S)$ be the isomorphism given by Theorem 3.3. Then, for each $T \in C^*(S)$, one gets $\Lambda_e(T) = U_e^* \lambda_{\varepsilon_e}\big(\pi(T)\big) U_e$, which is easily verified by checking on the generators of $C^*(S)$. This completes the proof. □

The *E*-unitary case. Let S be an inverse semigroup S. Denote by σ: $S \to \Gamma_S$ the homomorphism of S onto its maximal group homomorphic image Γ_S.

Recall that S is said to be *E-unitary* if every element $u \in S$ satisfying $\sigma(u) = 1$ is an idempotent of S. This is a very important class of inverse semigroups whose algebraic theory is well developed.

We will show that if S is E-unitary and satisfies certain algebraic condition, then its C^*-algebras are Morita equivalent to crossed products of its maximal group homomorphic image by a commutative C^*-algebra.

3.6. Proposition. *Let S be an inverse semigroup, Γ its maximal group homomorphic image and G_S its groupoid.*

a) *There is a continuous, transverse cocycle ρ: $G_S \to \Gamma$ that maps the class of (u,x) in G_S to $\sigma(u)$.*

b) *The map (ρ, s): $\gamma \mapsto \big(\rho(\gamma), s(\gamma)\big)$ from G_S to $\Gamma_S \times X$ is injective if and only if S is E-unitary.*

Proof. a) If $\widetilde{(u,x)} = \widetilde{(v,x)}$, then for some idempotent $e \in E$, we have $ue = ve$. Whence, $\sigma(u) = \sigma(v)$ which shows that ρ is well defined. It is clear that ρ is a continuous cocycle. To show that ρ is transverse, since Γ_S is discrete, we only need to prove that $\big\{\big(\rho(\gamma), s(\gamma)\big); \gamma \in G_S\big\}$ is an open subset of $\Gamma_S \times X$. In other words, it is enough to show that for any $g \in \Gamma_S$, the set $\{s(\gamma); \gamma \in G_S, \rho(\gamma) = g\}$ is open in X. But $\{s(\gamma); \gamma \in G_S, \rho(\gamma) = g\} = \bigcup_{u \in S, \sigma(u) = g} F_{u^*u}$: it is an open subset of X.

b) If (ρ, s) is injective, then the map $(\rho, s) \circ \varepsilon \colon S \to X \times \Gamma$ is injective; it follows immediately that S is E-unitary.

Assume S is E-unitary. Let $u, v \in S$, $x, y \in X$ such that $x \in F_{u^*u}$ and $y \in F_{v^*v}$. Suppose that $(\rho, s)(\widetilde{u, x}) = (\rho, s)(\widetilde{v, y})$. Then $x = s(\widetilde{u, x}) = s(\widetilde{v, y}) = y$ and $\sigma(u) = \sigma(v)$. Since S is E-unitary, uv^* is an idempotent of S. Then, $(\widetilde{u, x})(\widetilde{v, x})^{-1} = (\widetilde{uv^*, v.x})$ is a unit of G_S since uv^* is an idempotent of S. It follows that $(\widetilde{v, x}) = (\widetilde{u, x})$. □

3.7. Corollary. *The groupoid of an E-unitary inverse semigroup is Hausdorff.*

Proof. By Proposition 3.6, the map $(\rho, s) \colon \gamma \mapsto (\rho(\gamma), s(\gamma))$ is a continuous injective map from G_S into the Hausdorff space $\Gamma_S \times X$. □

Note that every element u of S defines a compact open subset O_u (its graph) in G_S, i.e. the set of the elements $(\widetilde{u, x}) \in G_S$. We have the following lemma:

3.8. Lemma. *Let $U \subset S$ and $u \in S$. If $O_u \subset \bigcup_{v \in U} O_v$, then there exists $v \in U$ such that $u \prec v$.*

Proof. Indeed, the element ε_u has to belong to an O_v which implies $u \prec v$. □

Our aim is to give a condition on S which is equivalent to the fact that the image of G_S by the map $\psi \colon \gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$ has closed range. For $g \in \Gamma_S$, set $S_g = \sigma^{-1}(\{g\})$.

3.9. Proposition. *Let S be an E-unitary semigroup. The map $\psi \colon G \to X \times \Gamma_S \times X$ defined by $\psi(\gamma) = (r(\gamma), \rho(\gamma), s(\gamma))$ has closed range if and only if, for every $p, q \in E$ and every $g \in \Gamma_S$, there exists a finite subset $U \subset pS_gq$ such that*

$$pS_gq = \{u \in S; \exists v \in U, u \prec v\}.$$

Proof. Since Γ_S is discrete, ψ has closed range if and only if, for all $g \in \Gamma_S$, the set $\mathrm{Gr}(g) = \bigcup_{u \in S_g} O_u$ is closed in $X \times X$.

Since the F_p $(p \in E)$ form a compact open cover of X, the set $\mathrm{Gr}(g)$ is closed in $X \times X$ if and only if, for every $p, q \in E$, the set $\mathrm{Gr}(g) \cap (F_p \times F_q)$ is compact. Now the O_u, $u \in pS_gq$ form a compact open cover of $\mathrm{Gr}(g)$; whence $\mathrm{Gr}(g) \cap (F_p \times F_q)$ is compact if and only if there exists a finite subset $U \subset S_g$ such that $\mathrm{Gr}(g) \cap (F_p \times F_q) = \bigcup_{u \in U} O_u$, which by Lemma 3.8 is equivalent to $pS_gq = \{u \in S; \exists v \in U, u \prec v\}$. □

3.10. Theorem. *If S is an E-unitary inverse semigroup which satisfies the condition of Proposition* 3.9, *then $\rho \colon G_S \to \Gamma_S$ is a faithful, closed and transverse cocycle.*

Proof. By Proposition 3.6, ρ is a faithful, transverse, continuous cocycle and by Proposition 3.9, it is closed. □

3.11. Corollary. *If S is as in Theorem* 3.10, *then there exists a locally compact space Y endowed with an action of Γ_S such that $C^*(S)$ and $C^*_r(S)$ are respectively $*$-isomorphic to $C_0(Y) \rtimes \Gamma_S$ and $C_0(Y) \rtimes_r \Gamma_S$.*

Proof. Apply Corollary 1.7, Theorems 3.2, 3.5 and Theorem 3.10. □

We now give two examples of inverse semigroups. The first is E-unitary but the cocycle ρ is not closed; the second, is not E unitary, but the cocycle $\rho: G_S \to \Gamma_S$ is faithful, closed and transverse.

3.12. Examples. a) Let $E = \{p_n; n \in \mathbb{N} \cup \{\infty\}\}$ with $p_n p_m = p_m$ if $n \geqq m$. Let Γ be a locally compact group and define S by

$$S = \Gamma \times \{p_n; n \in \mathbb{N}\} \cup \{(1, \infty)\}.$$

It is a sub-semigroup of $\Gamma \times E$ and therefore it is E-unitary. But the range of the map ψ is not closed. To see this one notes that $X = \mathbb{N} \cup \{\infty\}$ with its natural compact topology (since p_∞ is the identity element of E, X is compact). The range of the map ψ is $\{(m, g, m); m \in \mathbb{N}\} \cup \{(\infty, 1, \infty)\}$, but the elements (m, g, m) converge to the point (∞, g, ∞) which is not in the range if $g \neq 1$. Notice that, $\Gamma_S \cong \Gamma$ and for each $g \in \Gamma$ with $g \neq 1$,

$$S_g = \{(g, p_m); m \in \mathbb{N}\}$$

which has no upper bound for the relation $\prec$. It follows that S does not satisfy the condition of Proposition 3.9.

b) Let A be a set with at least two elements and let S be the union of $A \times A$ and an extra element o. Define the semigroup operations in S by setting $(a, b)(b, c) = (a, c)$ and all other products equal to o. One sees immediately that S is an inverse semigroup, $(a, b)^* = (b, a)$ and $o^* = o$. Its set of idempotents is $E = \{(a, a); a \in A\} \cup \{o\}$. A subset F of E satisfying $ef \in F \Leftrightarrow e, f \in F$ either contains o and is therefore equal to E, or it doesn't and therefore can contain only one (a, a). It follows that $\varepsilon: E \to X$ is onto, whence $\varepsilon: S \to G_S$ is a bijection. It follows that G_S identifies with the subgroupoid $A \times A \cup \{(o, o)\}$ of $(A \cup \{o\}) \times (A \cup \{o\})$. It is obvious, that Γ_S is the trivial group, whence S is not E-unitary. Since X and Γ are discrete, the cocycle $\rho: G_S \to \Gamma_S$ is closed and transverse; since G_S is a subgroupoid of $X \times X$, the map $(r, s): G_S \to X \times X$ is one to one, whence ρ is faithful.

It follows from this second example that we may construct a Morita equivalence $G_S \to Y \rtimes \Gamma_S$ in a somewhat more general situation than that of Corollary 3.11. Note that in the case of Corollary 3.11, the Morita equivalence $G_S \to Y \rtimes \Gamma_S$ is moreover injective, i.e. X is an open subset of Y.

We end this section with a discussion on a well known class of inverse semigroups for which the transverse cocycle ρ is faithful and closed and by an algebraic condition equivalent to that of Proposition 3.9. We thus get classes of inverse semigroups whose C^*-algebras are Morita-equivalent to the crossed product of an abelian C^*-algebra by their maximal group homomorphic image.

***F*-inverse semigroups.** Let S be an inverse semigroup. With the above notation, S is said to be *F-inverse*, if for every $g \in \Gamma_S$, the set S_g has a maximal element for the partial ordering $\prec$. It is easily seen that an F-inverse semigroup is E-unitary and unital. In the non unital case, Mark Lawson ([13]) gave a more general definition of F-inverse semigroups. Keeping our notation, S is said to be *F-inverse* in the sense of [13], if it is E-unitary and for every $g \in \Gamma_S$ and every $e, f \in E$, the set $eS_g f$ has a maximal element for the partial ordering $\prec$.

The condition of Proposition 3.9 is obviously satisfied for F-inverse semigroups even in the weaker sense.

It is in fact quite easy to express the condition of Proposition 3.9 in the terms of [13]. To a E-unitary semigroup S is naturally associated a partially ordered set Z: Z is the quotient of $\Gamma_S \times E_S$ by the equivalence relation $\sim$ defined by $(g, e) \simeq (h, f)$ if there exists $u \in S$ such that $\sigma(u) = g^{-1}h$, $uu^* = e$ and $u^*u = f$; denote by $g.e \in Z$ the class of $(g, e) \in \Gamma_S \times E_S$. The partial order of Z is given by: $g.e \prec h.f$ if there exists $u \in S$ such that $\sigma(u) = g^{-1}h$, $uu^* = e$ and $u^*u \leqq f$.

Recall that an order ideal of Z is a set I such that, if $x \in Z$ and $y \in I$ are such that $x \prec y$ then $x \in I$. Any subset A of Z generates an ideal of Z: the smallest order ideal containing A.

Recall (cf. [13]) that S is F-inverse (in the weaker sense—[13]) if and only if Z is a semilattice, i.e. if the intersection of two singly generated order ideals is singly generated. It is easy to see that S satisfies the condition of Proposition 3.9 if and only if the intersection of two finitely generated order ideals of Z is finitely generated.

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Eingegangen 29. Januar 2001