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Some Remarks on Kasparov Theory

GEORGES SKANDALIS

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K7L 3N6, Canada

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INTRODUCTION

This work consists of the simplification of two central points of Kasparov's paper [4]: the homotopy invariance of the "Ext" bifunctor [4, Sect. 6, Theorem 1] and the associativity of the Kasparov (intersection) product [4, Sect. 4, Theorem 4].

The proof of homotopy invariance given here, is based upon the following remark: the Kasparov product may be defined for the groups where only "operatorial homotopy" is allowed (Theorem 12). When looking at it carefully, one sees that this proof is very similar to the one given by Kasparov in [4], but, I believe that it may seem more conceptual.

The associativity of the Kasparov product is seen through the notion of connexion introduced in [2, Appendix A]. Both of these proofs use in a crucial manner the technical part of Kasparov's work [4, Sect. 3], especially Theorem 4 therein.

NOTATION

All gradings are $Z/2$ gradings. All tensor products are graded and minimal (spacial) tensor products. All commutators are graded commutators ($[a, b] = ab - (-1)^{\partial a \partial b} ba$, where ∂a is the degree of a). The reference for graded Hilbert C^*-modules, endomorphisms, compact endomorphisms of Hilbert C^*-modules, graded tensor products, etc, is [4, Sects. 1, 2] (see also [3]). As in [4] if $\mathscr{E}$ is a graded Hilbert C^*-module $\mathscr{L}(\mathscr{E})$ (resp. $\mathscr{K}(\mathscr{E})$) denotes the graded C^*-algebra of endomorphisms (resp. compact endomorphisms) of $\mathscr{E}$.

If x is an element of a graded C^*-algebra, or a graded Hilbert C^*-module, $x = x^{(0)} + x^{(1)}$ represents its decomposition in the even and odd parts. The grading operator (resp. the grading automorphism) of a graded C^*-module (resp. C^*-algebra) is the operator $x^{(0)} + x^{(1)} \to x^{(0)} - x^{(1)}$. If A is a graded C^*-algebra, $\tilde{A}$ denotes the graded C^*-algebra with an added unit of degree 0.

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Let us begin by recalling Kasparov's Theorem 4 of [4, Sect. 3], in the form it will be used several times here. Let B be a graded C^*-algebra and $\mathscr{E}$ a *countably generated* graded Hilbert C^*-module over B.

THEOREM 1 [4, Sect. 3, Theorem 4]. *Let $E_1, E_2 \subset \mathscr{L}(\mathscr{E})$ be graded subalgebras $\mathscr{F} \subset \mathscr{L}(\mathscr{E})$ a graded vector subspace. Assume that*:

(i) *E_1 has a countable approximate unit and $\mathscr{K}(\mathscr{E}) \subset E_1$,*

(ii) *$\mathscr{F}, E_2$ are separable,*

(iii) *$E_1 \cdot E_2 \subset \mathscr{K}(\mathscr{E})$, $[\mathscr{F}, E_1] \subset E_1$.*

Then there exist M, N of degree 0 (for the grading) such that

$$M + N = 1, \qquad M \geqslant 0, \qquad N \geqslant 0,$$

$$M \cdot E_1 \subset \mathscr{K}(\mathscr{E}), \qquad N \cdot E_2 \subset \mathscr{K}(\mathscr{E}), \qquad [\mathscr{F}, M] \subset \mathscr{K}(\mathscr{E}).$$

(*Notice that $\mathscr{E}$ being countably generated, $\mathscr{K}(\mathscr{E})$ has a countable approximate unit.*)

Let us also recall the basic definitions and notations of the KK groups [4, Sect. 4, Definitions 1–3].

DEFINITION 2. (1) Let A, B be graded C^*-algebras. An A, B bimodule is a *countably generated graded* Hilbert C^*-module $\mathscr{E}$ over B acted upon by A through a grading preserving * homomorphism $A \to \mathscr{L}(\mathscr{E})$.

(2) A Kasparov A, B bimodule is a pair $(\mathscr{E}, F)$, where $\mathscr{E}$ is an A, B bimodule, $F \in \mathscr{L}(\mathscr{E})$ is of degree 1 and satisfies

$$\forall a \in A, \quad [a, F] \in \mathscr{K}(\mathscr{E}), \qquad a(F^2 - 1) \in \mathscr{K}(\mathscr{E}), \qquad a(F - F^*) \in \mathscr{K}(\mathscr{E}).$$

$\mathbb{E}(A, B)$ denotes the set of all Kasparov A, B bimodules.

(3) A Kasparov A, B bimodule $(\mathscr{E}, F)$ will be said to be degenerate iff $[a, F] = a(F^2 - 1) = a(F - F^*) = 0$, $\forall a \in A$. $\mathscr{D}(A, B)$ denotes the set of degenerate Kasparov A, B bimodules.

(4) An operatorial homotopy through Kasparov bimodules, is a homotopy $(\mathscr{E}, F_t)$, where $t \to F_t$ is norm continuous.

(5) An element of $\mathbb{E}(A, B \otimes C([0, 1]))$ is given by a family $(\mathscr{E}_t, F_t) \in \mathbb{E}(A, B)$ which will be called a homotopy between $(\mathscr{E}_0, F_0)$ and $(\mathscr{E}_1, F_1)$.

(6) The addition of two Kasparov bimodules $(\mathscr{E}_1, F_1)$ $(\mathscr{E}_2, F_2)$ is defined by $(\mathscr{E}_1, F_1) \oplus (\mathscr{E}_2, F_2) = (\mathscr{E}_1 \oplus \mathscr{E}_2, F_1 \oplus F_2)$.

(7) The set $KK(A, B)$ is defined as the quotient of $\mathbb{E}(A, B)$ by the equivalence relation given by homotopy.

(8) The set $\widetilde{KK}(A, B)$ is defined as the quotient of $\mathbb{E}(A, B)$ by the equivalence relation generated by addition of elements of $\mathscr{D}(A, B)$ and operatorial homotopy.

Remark 3. (a) If $(\mathscr{E}, F) \in \mathscr{D}(A, B)$ then $(\mathscr{E}, F)$ is homotopic to the Kasparov bimodule (0, 0), because $(\mathscr{E} \otimes C_0([0, 1)), F \otimes 1) \in \mathbb{E}(A, B \otimes C([0, 1]))$.

(b) The definition given here of $KK(A, B)$ is different from the one given in [4, Sect. 4 Definition 3]. However, using the stabilization theorem [3, Theorem 2] these two definitions coincide when B has a countable approximate unit.

One has

PROPOSITION 4 [4, Sect. 4, Theorem 1]. *$KK(A, B)$ and $\widetilde{KK}(A, B)$ are abelian groups when equipped with addition as in Definition* 2(6). *$KK(A, B)$ is a quotient of $\widetilde{KK}(A, B)$.*

Proof. The second assertion follows from Remark 3(a). For the first one, we just recall that $-(\mathscr{E}, F) = (-\mathscr{E}, -UFU^{-1})$, where $-\mathscr{E}$ is the same Hilbert B module with opposite grading, $U \in \mathscr{L}_B(\mathscr{E}, -\mathscr{E})$ is the identity, and the action of A is given by $aU\xi = U(\alpha(a)\xi)$, where α is the grading automorphism of A). Then

$$\begin{bmatrix} \cos\theta \cdot F & \sin\theta U^{-1} \\ \sin\theta \cdot U & -\cos\theta UFU^{-1} \end{bmatrix}, \qquad \theta \in [0, \Pi/2]$$

defines an operatorial homotopy joining $(\mathscr{E}, F) \oplus (-(\mathscr{E}, F))$ to a degenerate element. ∎

We may notice that if $(\mathscr{E}, F)$ and $(\mathscr{E}', F')$ are unitarily equivalent [4, Sect. 4, Definition 2], then $(\mathscr{E}, F) \oplus (-(\mathscr{E}', F'))$ is operatorially homotopic to a degenerate element. Hence the classes of $(\mathscr{E}, F)$ and $(\mathscr{E}', F')$ in $\widetilde{KK}$ and KK coincide.

FUNCTORIAL PROPERTY 5. (1) Let $f: A_2 \to A_1$ be a homomorphism of graded C^*-algebras. Let $(\mathscr{E}, F)$ be a Kasparov A_1, B bimodule. Then $\mathscr{E}$ may be looked at as an A_2, B bimodule $f^*\mathscr{E}$ through the A_2 action $A_2 \to^f A_1 \to \mathscr{L}(\mathscr{E})$. This defines a map $f^*: \mathbb{E}(A_1, B) \to \mathbb{E}(A_2, B)$, $f^*(\mathscr{E}, F) = (f^*\mathscr{E}, F)$.

(2) Let $g: B_1 \to B_2$ be a homomorphism of graded C^*-algebras, and $(\mathscr{E}, F) \in \mathbb{E}(A, B_1)$. Then put $g_*(\mathscr{E}, F) = (\mathscr{E} \hat{\otimes}_{B_1} B_2, F \hat{\otimes} 1)$ [4, Sects. 2, 8]. This defines a map $g_*: \mathbb{E}(A, B_1) \to \mathbb{E}(A, B_2)$. (Note that as $\mathscr{E}$ is countably generated, the same holds for $\mathscr{E} \hat{\otimes}_{B_1} B_2$).

(3) Both of these maps pass to the quotients KK and $\widetilde{KK}$. We keep the notations f^*, g_* for these quotient homomorphisms.

(4) For a graded C^*-algebra D with a *countable approximate unit* one defines the map $\tau_D : \mathbb{E}(A, B) \to \mathbb{E}(A \hat{\otimes} D,\ B \hat{\otimes} D)$ by putting $\tau_D(\mathscr{E}, F) = (\mathscr{E} \hat{\otimes} D,\ F \hat{\otimes} 1)$.

Again this map passes to the quotients KK and $\widetilde{KK}$, and gives homomorphisms still called τ_D. One has obviously [4, Sect. 4, Theorem 3].

PROPOSITION 6. *The bifunctor $KK(A, B)$ is homotopy invariant in both entries.*

The homotopy invariance theorem given here is in fact the equality $KK = \widetilde{KK}$.

LEMMA 7. *Let $(\mathscr{E}, F)$ be a Kasparov A, B bimodule. Let $f : D_1 \to D_2$ be a homomorphism of graded C^*-algebras.*

(a) *If D_1 and D_2 are unital, and $f(1) = 1$, then*

$$f^*(\tau_{D_2}(\mathscr{E}, F)) = f_*(\tau_{D_1}(\mathscr{E}, F))$$

(b) *In general, this equality holds in $KK(A \hat{\otimes} D_1, B \hat{\otimes} D_2)$.*

Proof. (a) is obvious.

(b) Put $J = D_1 \hat{\otimes}_{D_1} D_2$; it is the right ideal in D_2 generated by $f(D_1)$. Then $f_*(\tau_{D_1}(\mathscr{E}, F)) = (\mathscr{E} \hat{\otimes}_C J, F \hat{\otimes} 1)$ and $f^*(\tau_{D_2}(\mathscr{E}, F)) = (\mathscr{E} \hat{\otimes}_C D_2, F \hat{\otimes} 1)$. Let $\mathscr{E}'$ be the $D_1, D_2 \otimes C([0, 1])$ bimodule $\mathscr{E}' \subseteq D_2 \otimes C([0, 1])$, $\mathscr{E}' = \{f : [0, 1] \to D_2 \mid f(1) \in J\}$. Then $(\mathscr{E} \hat{\otimes}_C \mathscr{E}',\ F \hat{\otimes} 1) \in (A \hat{\otimes} D_1,\ B \hat{\otimes} D_2 \otimes C([0, 1]))$ realizes a homotopy between $f^*(\tau_{D_2}(\mathscr{E}, F))$ and $f_*(\tau_{D_1}(\mathscr{E}, F))$. ∎

DEFINITION 8 [2. Appendix, Definition A.1]. Let $\mathscr{E}_2$ be a D, B bimodule. Let $\mathscr{E}_1$ be a Hilbert D module. Put $\mathscr{E} = \mathscr{E}_1 \hat{\otimes}_D \mathscr{E}_2$. Let $F_2 \in \mathscr{L}(\mathscr{E}_2)$. An element $F \in \mathscr{L}(\mathscr{E})$ is said to be *an F_2 connexion for* $\mathscr{E}_1$ iff $\forall \xi \in \mathscr{E}_1$,

$$[\tilde{T}_\xi, F_2 \oplus F] \in \mathscr{L}(\mathscr{E}_2 \oplus \mathscr{E}), \qquad \text{where} \quad \tilde{T}_\xi = \begin{bmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{bmatrix} \in \mathscr{L}(\mathscr{E}_2 \oplus \mathscr{E}),$$

$T_\xi \in \mathscr{L}(\mathscr{E}_2, \mathscr{E})$ being defined by $T_\xi(\eta) = \xi \hat{\otimes} \eta \in \mathscr{E}$.

Let us gather some easy results about connexions in

PROPOSITION 9 [2, Appendix, Proposition A.2]. (a) *If F_2 satisfies*

$[F_2, d] \in \mathscr{L}(\mathscr{E})$, $\forall d \in D$, *then there exists an* F_2 *connexion for any countably generated* $\mathscr{E}_1$.

(b) *If* F *is an* F_2 *connexion, then* F^* *(resp.* $F^{(0)}, F^{(1)}$*) is an* F_2^* *(resp.* $F_2^{(0)}, F_2^{(1)}$*) connexion.*

(c) *If* F *is an* F_2 *connexion and* F' *is an* F_2' *connexion, then* $F + F'$ *(resp.* FF'*) is an* $F_2 + F_2'$ *(resp.* $F_2 \cdot F_2'$*) connexion.*

(d) *The space of* 0 *connexions for* $\mathscr{E}_1$ *is* $\Omega = \{T \in \mathscr{L}(\mathscr{E}) \mid \forall x \in \mathscr{K}(\mathscr{E}_1) \hat{\otimes} 1, \ Tx \in \mathscr{K}(\mathscr{E}), \ xT \in \mathscr{K}(\mathscr{E})\}$.

(e) *If* F *is an* F_2 *connexion for some* F_2, *then* $[F, x] \in \mathscr{K}(\mathscr{E})$, $\forall x \in \mathscr{K}(\mathscr{E}_1) \hat{\otimes} 1$.

(f) *If* $[F_2, d] = 0$, $\forall d \in D$, $1 \hat{\otimes} F_2$ *makes sense in* $\mathscr{L}(\mathscr{E})$, *and* $1 \hat{\otimes} F_2$ *is an* F_2 *connexion. Moreover for any* $T \in \mathscr{L}(\mathscr{E}_1)$, $[T \hat{\otimes} 1, 1 \hat{\otimes} F_2] = 0$.

(g) *If* $F_2 \in \mathscr{L}(\mathscr{E}_2)$ *and* $F \in \mathscr{L}(\mathscr{E})$ *are normal and* F *is an* F_2 *connexion,* $f(F)$ *is an* $f(F_2)$ *connexion* ($f\colon \mathbb{C} \to \mathbb{C}$ *continuous).*

(h) *If* $(\mathscr{E}_2, F_2)$ *is a Kasparov* D, B *bimodule and* $\mathscr{E}_1$ *is countably generated, and if* F *is an* F_2 *connexion of degree* 1, *then* $(\mathscr{E}, F)$ *is a Kasparov* $\mathscr{K}(\mathscr{E}_1) \hat{\otimes} 1$, B *bimodule.*

(i) *Assume* $\mathscr{E}_2$ *is a* D, E *bimodule and* $\mathscr{E}_3$ *is an* E, B *bimodule. Let* $F \in \mathscr{L}(\mathscr{E}_1 \hat{\otimes}_D \mathscr{E}_2 \hat{\otimes}_E \mathscr{E}_3)$, $F_2 \in \mathscr{L}(\mathscr{E}_2 \hat{\otimes}_E \mathscr{E}_3)$, $F_3 \in \mathscr{L}(\mathscr{E}_3)$.

If F_2 *is an* F_3 *connexion for* $\mathscr{E}_2$ *and* F *is an* F_2 *connexion for* $\mathscr{E}_1$, *then* F *is an* F_3 *connexion for* $\mathscr{E}_1 \hat{\otimes}_D \mathscr{E}_2$.

Recall that the proof of (a) is an easy consequence of the stabilization theorem [2, Theorem 2]: If $\mathscr{E}_1 = P\mathscr{H}_{\tilde{D}}$, where $P \in \mathscr{L}(\mathscr{H}_{\tilde{D}})$ is a degree 0 projection, then $\mathscr{E}_1 \hat{\otimes} \mathscr{E}_2 = (P \hat{\otimes} 1) \cdot (\mathscr{H}_{\tilde{D}} \hat{\otimes}_{\tilde{D}} \mathscr{E}_2) = (P \hat{\otimes}_D 1)(\mathscr{H} \hat{\otimes}_{\mathbb{C}} \mathscr{E}_2)$. Then $(P \hat{\otimes}_D 1)(1 \hat{\otimes}_{\mathbb{C}} F_2)(P \hat{\otimes}_D 1)$ is an F_2 connexion (the grassmann connexion). The other statements are obvious.

Definition 10 [2, Appendix, Theorem A.3]. Let A, B, D be graded C^*-algebras $(\mathscr{E}_1, F_1) \in \mathbb{E}(A, D)$, $(\mathscr{E}_2, F_2) \in \mathbb{E}(D, B)$. Call $\mathscr{E}$ the A, B bimodule $\mathscr{E}_1 \hat{\otimes}_D \mathscr{E}_2$. The pair $(\mathscr{E}, F)$, $F \in \mathscr{L}(\mathscr{E})$ is called a Kasparov product of $(\mathscr{E}, F_1)$ by $(\mathscr{E}_2, F_2)$ (one writes $F \in F_1 \#_D F_2$) if and only if

(a) $(\mathscr{E}, F)$ is a Kasparov A, B bimodule (Definition 2(2)),

(b) F is an F_2 connexion,

(c) $\forall a \in A$, $a[F_1 \hat{\otimes} 1, F]\, a^* \geqslant 0$ modulo $\mathscr{K}(\mathscr{E})$.

Note that $\mathscr{E}_1, \mathscr{E}_2$ being countably generated, $\mathscr{E}$ is countably generated.

We will need

Lemma 11. *Let* $\mathscr{E}$ *be an* A, B *bimodule. Let* $F, F' \in \mathscr{L}(\mathscr{E})$ *be such that*

$(\mathscr{E}, F) \in \mathbb{E}(A, B)$, $(\mathscr{E}, F') \in \mathbb{E}(A, B)$, and $\forall a \in A$, $a[F, F']a^* \geqslant 0$ modulo $\mathscr{K}(\mathscr{E})$. Then $(\mathscr{E}, F)$ and $(\mathscr{E}, F')$ are operatorially homotopic.

Proof. Let $\mathcal{O}$ be the subalgebra of $\mathscr{L}(\mathscr{E})$

$$\mathcal{O} = \{T \in \mathscr{L}(\mathscr{E}) / [T, a] \in \mathscr{K}(\mathscr{E}), \forall a \in A\}$$

and J be the ideal of $\mathcal{O}$

$$J = \{T \in \mathcal{O} / Ta \in \mathscr{K}(\mathscr{E}), \forall a \in A\}.$$

Then $[F, F'] \in \mathcal{O}$ and is positive modulo J. Write $[F, F'] = P + K$, where $P \in \mathcal{O}$, $P \geqslant 0$, and $K \in J$; P and K being of degree 0. Note that as $F^2 - 1$ and $F'^2 - 1 \in J$, $[F, P]$ and $[F', P] \in J$. Put $F_t = (1 + \cos t \cdot \sin tP)^{-1/2}$ $(\cos tF + \sin tF')$ $(t \in [0, \Pi/2])$. Then $F_t \in \mathcal{O}$, $F_t - F_t^* \in J$, and $F_t^2 - 1 \in J$. Hence $(\mathscr{E}, F_t)$ realizes the desired operatorial homotopy. ∎

Note that in the above proof it is enough to assume that $[F, F'] \geqslant \lambda$ modulo J, where $\lambda \in \mathbb{R}$, $\lambda > -2$.

THEOREM 12 [4, Sect. 4, Theorem 4; 2, Theorem A.3]. *Assume A is separable, $(\mathscr{E}_1, F_1)$ is a Kasparov A, D bimodule, $(\mathscr{E}_2, F_2)$ is a Kasparov D, B bimodule.*

(a) *There exists a Kasparov product $(\mathscr{E}, F)$ of $(\mathscr{E}_1, F_1)$ by $(\mathscr{E}_2, F_2)$ unique up to operatorial homotopy.*

(b) *The map $(\mathscr{E}_1, F_1)$, $(\mathscr{E}_2, F_2) \to (\mathscr{E}, F)$ passes to the quotients, and defines maps $KK(A, D) \otimes KK(D, B) \to KK(A, B)$ and $\widetilde{KK}(A, D) \otimes \widetilde{KK}(D, B) \to \widetilde{KK}(A, B)$. The quotient maps are noted $\otimes_D$, $(x, y) \to x \otimes_D y$.*

Proof. (a) *Existence.* Let $G \in \mathscr{L}(\mathscr{E})$, of degree 1, be an F_2 connexion (Proposition 9(a)). Put $E_1 = \mathscr{K}(\mathscr{E}_1) \hat{\otimes} 1 + \mathscr{K}(\mathscr{E})$ and let E_2 be the subalgebra of $\mathscr{L}(\mathscr{E})$ generated by $(G^2 - 1)$, $[G, a]$ $(a \in A)$, $G - G^*$, $[G, F_1 \hat{\otimes} 1]$. Let $\mathscr{F}$ be the vector space spaned by $F \hat{\otimes} 1$, G, A. One checks that Theorem 1 applies to give M and N such that,

$$M + N = 1, \qquad ME_1 \subset \mathscr{K}(\mathscr{E}), \qquad NE_2 \subset \mathscr{K}(\mathscr{E}), \qquad [M, \mathscr{F}] \subset \mathscr{K}(\mathscr{E}).$$

Then put $F = M^{1/2}(F_1 \hat{\otimes} 1) + N^{1/2}G$. Then $(\mathscr{E}, F)$ is obviously seen to be a Kasparov A, B bimodule. Note that M is a zero-connexion (Proposition 9(d)). As $[F_1 \hat{\otimes} 1, M] \in \mathscr{K}(\mathscr{E})$, $M^{1/2}(F_1 \hat{\otimes} 1)$ is also a zero-connexion. And, hence (Proposition 9(c) and (g)) F is an F_2 connexion. Finally $[F_1 \hat{\otimes} 1, F] = M^{1/2}[F_1 \hat{\otimes} 1, \ F_1 \hat{\otimes} 1] \mod \mathscr{K}(\mathscr{E})$ and hence $a[F_1 \hat{\otimes} 1, F]a^* = 2aM^{1/2}(F_1^2 \hat{\otimes} 1)a^* \mod \mathscr{K}(\mathscr{E}) = 2aM^{1/2}a^* \mod \mathscr{K}(\mathscr{E})$. Thus $F \in F_1 \#_D F_2$.

Uniqueness. Let $F, F' \in F_1 \#_D F_2$. Put $E_1 = \mathscr{K}(\mathscr{E}_1) \tilde{\otimes} 1 + \mathscr{K}(\mathscr{E})$. $E_2 =$ subalgebra generated by $[F_1 \tilde{\otimes} 1, F]$, $[F_1 \tilde{\otimes} 1, F']$, $F - F'$. $\mathscr{F}$ = vector space generated by A, $F_1 \tilde{\otimes} 1$, F, F'. Then take M, N satisfying the conclusion of Theorem 1 and put $F'' = M^{1/2}(F_1 \tilde{\otimes} 1) + N^{1/2}F$. One has $(\mathscr{E}, F'') \in \mathbb{E}(A, B)$ and $\forall a \in A$, $a[F, F'']a^* \geqslant 0 \bmod \mathscr{K}(\mathscr{E})$ and $a[F', F'']a^* \geqslant 0 \bmod \mathscr{K}(\mathscr{E})$. The conclusion follows by Lemma 11.

(b) Let $(\mathscr{E}_1, F_1) \in \mathbb{E}(A, D \otimes C([0, 1]))$ be a homotopy. Let $(\mathscr{E}, F)$ be a Kasparov product of $(\mathscr{E}_1, F_1)$ by $\tau_{C([0,1])}(\mathscr{E}_2, F_2)$. Then, $(\mathscr{E}, F)$ realizes a homotopy between a Kasparov product of $(\mathscr{E}_1^0, F_1^0)$ by $(\mathscr{E}_2, F_2)$ and a Kasparov product of $(\mathscr{E}_1^1, F_1^1)$ by $(\mathscr{E}_2, F_2)$. In the same way, if $(\mathscr{E}_2, F_2) \in \mathbb{E}(D, B \otimes C([0, 1]))$ is a homotopy, a Kasparov product of $(\mathscr{E}_1, F_1)$ by $(\mathscr{E}_2, F_2)$ realizes a homotopy between a Kasparov product of $(\mathscr{E}_1, F_1)$ by $(\mathscr{E}_0^0, F_2^0)$ and a Kasparov product of $(\mathscr{E}_1, F_1)$ by $(\mathscr{E}_2^1, F_2^1)$. This carries over the KK case.

If $(\mathscr{E}_1, F_1)$ is degenerate, then $(\mathscr{E}, F_1 \tilde{\otimes} 1) \in \mathscr{L}(A, B)$ and is operatorially homotopic to any Kasparov product of $(\mathscr{E}_1, F_1)$ by $(\mathscr{E}_2, F_2)$ by Lemma 11. If $(\mathscr{E}_2, F_2)$ is degenerate, then $1 \tilde{\otimes} F_2$ has an obvious meaning in $\mathscr{L}(\mathscr{E}_1 \tilde{\otimes}_D \mathscr{E}_2)$, and satisfies the conditions to be a (degenerate) Kasparov product of $(\mathscr{E}_1, F_1)$ by $(\mathscr{E}_2, F_2)$ (Proposition 9(f)).

Let $(\mathscr{E}_i, F_i^t)$ $(t \in [0, 1],\ i = 1, 2)$ be operatorial homotopies. Let G^t, $t \in [0, 1]$, be a norm continuous family such that each G^t is an F_2^t connexion for $\mathscr{E}_1$. (Such a family exists; take, e.g., the "grassmann connexions" $G^t = (P \tilde{\otimes}_D 1)(1 \tilde{\otimes}_C F_2^t)(P \tilde{\otimes}_D 1)$ for some trivialization $\mathscr{E}_1 = P\mathscr{H}_{\tilde{D}}$.) Put $E_1 = \mathscr{K}(\mathscr{E}_1) \tilde{\otimes} 1 + \mathscr{K}(\mathscr{E})$ $(\subseteq \mathscr{L}(\mathscr{E}))$. E_2 = the C^*-algebra generated by $[G^t, A]$, $(G_t^2 - 1)$, $G_t - G_t^*$, $[F^t \tilde{\otimes} 1, G^t]$, $t \in [0, 1]$. $\mathscr{F}$ = the subspace generated by A, $F_1^t \tilde{\otimes} 1$, G^t, $t \in [0, 1]$. One checks that Theorem 1 applies.

Let M, N satisfy the conclusions of this theorem. Then $(\mathscr{E}, M^{1/2}(F_1^t \tilde{\otimes} 1) + N^{1/2}G^t)$ is the desired operatorial homotopy. ∎

PROPOSITION 13 (Functoriality of the Kasparov product). (a) *If A_1 and A_2 are separable, $f: A_2 \to A_1$ is a homomorphism $x_1 \in KK(A_1, D)$, $x_2 \in KK(D, B)$ (or $\widetilde{KK}$), then $f^*(x_1) \otimes_D x_2 = f^*(x_1 \otimes_D x_2)$.*

(b) *If $h: D_1 \to D_2$ is a homomorphsim $x_1 \in KK(A, D_1)$, $x_2 \in KK(D_2, B)$ (or $\widetilde{KK}$), then $h_*(x_1) \otimes_{D_2} x_2 = x_1 \otimes_{D_1} h^*(x_2)$.*

(c) *If $g: B_1 \to B_2$ is a homomorphism $x_1 \in KK(A, D)$, $x_2 \in KK(D, B_1)$ (or $\widetilde{KK}$), then $g_*(x_1 \otimes_D x_2) = x_1 \otimes_D g_*(x_2)$.*

Proof. (a) If $(\mathscr{E}_1, F_1) \in \mathbb{E}(A_1, D)$ and $(\mathscr{E}_2, F_2) \in \mathbb{E}(D, B)$, then one has $f^*(F_1 \#_D F_2) \subset f^*(F_1) \#_D F_2$.

(b) If $(\mathscr{E}_1, F_1) \in \mathbb{E}(A, D_1)$, $(\mathscr{E}_2, F_2) \in \mathbb{E}(D_2, B)$, then one has $h_*(F_1) \#_{D_2} F_2 = F_1 \#_{D_1} h^*(F_2)$.

(c) If $(\mathscr{E}_1, F_1) \in \mathbb{E}(A, D)$, $(\mathscr{E}_2, F_2) \in \mathbb{E}(D, B_1)$, then one has $g_*(F_1 \#_D F_2) \subset F_1 \#_D g_*(F_2)$. ∎

Let A_1, A_2 be separable. Let $x_1 \in KK(A_1, B_1 \hat{\otimes} D)$ (resp. $\widetilde{KK}(A_1, B_1 \hat{\otimes} D)$) and $x_2 \in KK(D \hat{\otimes} A_2, B_2)$ (resp. $\widetilde{KK}(D \hat{\otimes} A_2, B_2)$). Consider the product $\tau_{A_2}(x_1) \otimes_{B_1 \hat{\otimes} D \hat{\otimes} A_2} i^*(\tau_{\tilde{B}_1}(x_2))$ ($i: B_1 \to \tilde{B}_1$ is inclusion). One sees that it gives in fact an element of $KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$ (resp. $\widetilde{KK}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$). This, because, if $(\mathscr{E}, F)$ is a Kasparov bimodule which defines this product, one has

$$\forall \xi, \eta \in \mathscr{E}, \qquad \langle \xi, \eta \rangle \in B_1 \hat{\otimes} B_2 \subset \tilde{B}_1 \hat{\otimes} B_2.$$

Definition 15. The Kasparov product $x_1 \otimes_D x_2$ is defined as the element of $KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$ (resp. $\widetilde{KK}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$) corresponding to the Kasparov product $\tau_{A_1}(x_1) \otimes_{B_1 \hat{\otimes} D \hat{\otimes} A_2} i^*(\tau_{\tilde{B}_1}(x_1))$.

Remark 16. (a) If B_1 has a countable approximate unit, $\tau_{B_1}(x_2)$ makes sense and one has $x_1 \otimes_D x_2 = \tau_{A_2}(x_1) \otimes_{B_1 \hat{\otimes} D \hat{\otimes} A_2} \tau_{B_1}(x_1)$.

(b) More generally, one may take, instead of $\tilde{B}_1$ any algebra with countable approximate unit in which B_1 is an ideal. The result will not change.

Proposition 16 (Part of [4, Theorem 4.4]). *The product $x_1 \otimes_D x_2$ is bilinear, contravariantly functorial in A_1, and covariantly functorial in B_1 and B_2. Moreover, it is contravariantly functorial in A_2* in *KK and with respect to* unital maps *in $\widetilde{KK}$. If $h: D_1 \to D_2$, $x_1 \in KK(A_1, B_1 \hat{\otimes} D_1)$, $x_2 \in KK(D_2 \hat{\otimes} A_2, B_2)$ (or in the corresponding $\widetilde{KK}$) $h_*(x_1) \otimes_{D_2} x_2 = x_2 \otimes_{D_1} h^*(x_2)$.*

Proof. Follows obviously from Proposition 13. The A_2 functoriality uses Lemma 7. ∎

Let $1 \in \widetilde{KK}(\mathbb{C}, \mathbb{C}) (= KK(\mathbb{C}, \mathbb{C}))$ be given by the $\mathbb{C}, \mathbb{C}$ bimodule $\mathbb{C}$, trivially (zero) graded and the zero operator.

Proposition 17 [4, Sect. 4, Theorem 5]. *Let A be separable, $x \in KK(A, B)$ or $\widetilde{KK}(A, B)$. Then $x \otimes_{\mathbb{C}} 1 = x$. If A is unital, then $1 \otimes_{\mathbb{C}} x = x$. In general, this equality holds in $KK(A, B)$.*

Proof. That $x \otimes_{\mathbb{C}} 1 = x$ is obvious. Assume A is unital. Let $(\mathscr{E}, F)$ be a Kasparov A, B bimodule, and let $P \in \mathscr{L}(\mathscr{E})$ be the image of $1 \in A$. The product $1 \otimes_{\mathbb{C}} (\mathscr{E}, F)$ is given by $(P\mathscr{E}, PFP)$. But $(\mathscr{E}, F)$ is operatorially homotopic to $(\mathscr{E}, PFP) = (P\mathscr{E}, PFP) + ((1-P)\mathscr{E}, 0)$ (this last term being degenerate). Let $(\mathscr{E}, F) \in \mathbb{E}(A, B)$ (A non unital). One may extend the action on $\mathscr{E}$ to $\tilde{A}$ sending $1 \in \tilde{A}$ to $1 \in \mathscr{L}(\mathscr{E})$. (This does not in general give a Kasparov $\tilde{A}, B$ bimodule!)

Let $\mathscr{E}'$ be the $A, \tilde{A} \otimes C([0,1])$ bimodule, $\mathscr{E}' \subseteq \tilde{A} \otimes C([0,1])$, $\mathscr{E}' = \{f: [0,1] \to \tilde{A},\ f(1) \in A\}$. Put $\tilde{\mathscr{E}} = \mathscr{E}' \otimes_{\tilde{A} \otimes C([0,1])} (\mathscr{E} \otimes C([0,1]))$. Let $\tilde{F} \in \mathscr{L}(\tilde{\mathscr{E}})$ be an $F \otimes 1$ connexion for $\mathscr{E}'$. Then $(\tilde{\mathscr{E}}, \tilde{F}) \in \mathbb{E}(A, B \otimes C([0,1]))$ and $(\mathscr{E}^1, F^1)$ is a Kaparov product $1 \otimes_{\mathbb{C}} (\mathscr{E}, F)$ and $(\mathscr{E}^0, F^0)$ is operatorially homotopic to $(\mathscr{E}, F)$. ∎

Now let $f_t: C([0,1]) \to \mathbb{C}$ be evaluation at time t.

LEMMA 18. *One has* $f_0^*(1) = f_1^*(1)$ *in* $\widetilde{KK}(C([0,1]), \mathbb{C})$.

We do not want to get into the proof. It is a consequence of "homotopy invariance" in the abelian case, for the Ext functor [1, Theorem 2.14] see, also, [4, Sect. 6, Theorem 1, beginning of proof].

THEOREM 19 (Homotopy invariance [4, Sect. 6, Theorem 1]. *Assume A is separable. Then, the map $\widetilde{KK}(A, B) \to KK(A, B)$ is an isomorphism.*

Proof. Let $(\mathscr{E}^t, F^t) \in (A, B \otimes C([0,1]))$ be a homotopy. Let x be its class in $\widetilde{KK}(A, B \otimes C([0,1]))$. Let x_t be the class of $(\mathscr{E}^t, F^t)$ in $\widetilde{KK}(A, B)$. One has

$$\begin{aligned} x_0 &= f_{0*}(x) = f_{0*}(x) \otimes_{\mathbb{C}} 1 && \text{(Proposition 17)} \\ &= x \otimes_{C([0,1])} f_0^*(1) && \text{(Proposition 16)} \\ &= x \otimes_{C([0,1])} f_1^*(1) && \text{(Lemma 18)} \\ &= x_1. \end{aligned}$$

Let us recall

PROPOSITION 20 [4, Sect. 4, Theorem 4]. *The Kasparov product satisfies the following commutation relations with the functor τ_D:*

(a) $\tau_{D_2}(x_1) \otimes_{D_1 \hat{\otimes} D \hat{\otimes} D_2} \tau_{D_1}(x_2) = x_1 \otimes_D x_2$ $(x \in KK(A_1, B \hat{\otimes} D_1 \hat{\otimes} D)$, $y \in KK(D \hat{\otimes} D_2 \hat{\otimes} A_2, B_2)$; *$A_1, A_2, D_2$ separable, D_1 with countable approximate unit*).

(b) $\tau_{D_1}(x_1 \otimes_D x_2) = \tau_{D_1}(x_1) \otimes_{D \hat{\otimes} D_1} \tau_{D_1}(x_2)$ $(x_1 \in KK(A_1, B_1 \hat{\otimes} D)$, $x_2 \in KK(D \hat{\otimes} A_2, B_2)$; *$A_1, A_2, D_1$ separable*).

Proof. It follows from Definition 15 and Remark 16. ∎

Let us now pass to associativity:

THEOREM 21 (Part of [4, Sect. 4, Theorem 4]. *Let A_1, A_2, A_3, D_1 be separable and let $x_1 \in KK(A_1, B_1 \hat{\otimes} D_1)$, $x_2 \in KK(D_1 \hat{\otimes} A_2, B_2 \hat{\otimes} D_2)$, $x_3 \in KK(D_2 \hat{\otimes} A_3, B_3)$. Then $(x_1 \otimes_{D_1} x_2) \otimes_{D_2} x_3 = x_1 \otimes_{D_1} (x_2 \otimes_{D_2} x_3)$.*

Proof. Put $A = A_1 \hat{\otimes} A_2 \hat{\otimes} A_3$, $D = B_1 \hat{\otimes} D_1 \hat{\otimes} A_2 \hat{\otimes} A_3$, $E = B_1 \hat{\otimes} B_2 \hat{\otimes} D_2 \hat{\otimes} A_3$, $B = B_1 \hat{\otimes} B_2 \hat{\otimes} B_3$. Replace the x_i's by Kasparov bimodules representing them. Let $i_1 : B_1 \to \tilde{B}_1$, $i_2 : B_2 \to \tilde{B}_2$ be the inclusions and put

$$(\mathscr{E}_1, F_1) = \tau_{A_2 \hat{\otimes} A_3}(x_1) \in \mathbb{E}(A, D),$$
$$(\mathscr{E}_2, F_2) = i_1^* \tau_{\tilde{B}_1 \hat{\otimes} A_3}(x_2) \in \mathbb{E}(D, E),$$
$$(\mathscr{E}_3, F_3) = i_2^* \tau_{\tilde{B}_1 \hat{\otimes} \tilde{B}_2}(x_3) \in \mathbb{E}(E, B).$$

Due to Proposition 20, Theorem 21 is a consequence of

LEMMA 22. *Let A be separable $(\mathscr{E}_1, F_1) \in \mathbb{E}(A, D)$, $(\mathscr{E}_2, F_2) \in \mathbb{E}(D, E)$, $(\mathscr{E}_3, F_3) \in \mathbb{E}(E, B)$. Take $G_1 \in F_1 \#_D F_2$ and $H \in G_1 \#_E F_3$. Assume that $F_2 \#_E F_3$ is nonempty and $G_2 \in F_2 \#_E F_3$. Take then $F \in F_1 \#_E G_2$. Then $(\mathscr{E}_1 \hat{\otimes}_D \mathscr{E}_2 \hat{\otimes}_E \mathscr{E}_3, F)$ and $(\mathscr{E}_1 \hat{\otimes}_D \mathscr{E}_2 \hat{\otimes}_E \mathscr{E}_3, H)$ are operatorially homotopic.*

Notice that G_1 and H are unique up to norm homotopy by Theorem 12, but G_2 and hence F are not apriori unique. Notice also that for the proof of Theorem 21, we know that $F_2 \#_E F_3 \neq \varnothing$ because $(\mathscr{E}_2, F_2)$ and $(\mathscr{E}_3, F_3)$ are of the form τ_{B_1} of something, for which Theorem 12 applies.

Proof of Lemma. Put $\mathscr{E}'_1 = \mathscr{E}_1 \hat{\otimes}_D \mathscr{E}_2$, $\mathscr{E}'_2 = \mathscr{E}_2 \hat{\otimes}_E \mathscr{E}_3$, $\mathscr{E} = \mathscr{E}_1 \hat{\otimes}_D \mathscr{E}'_2 = \mathscr{E}'_1 \hat{\otimes}_E \mathscr{E}_3$. As G_1 is an F_2 connexion for $\mathscr{E}_1$ and F is a G_2 connexion, $[G_1 \hat{\otimes} 1, F]$ is an $[F_2 \hat{\otimes} 1, G_2]$ connexion for $\mathscr{E}_1$. To see this, write

$$\tilde{F} = F \oplus G_2 \in \mathscr{L}(\mathscr{E} \oplus \mathscr{E}'_2), \qquad \tilde{G} = G_1 \oplus F_2 \in \mathscr{L}(\mathscr{E}'_1 \oplus \mathscr{E}_2).$$

Then $\forall \xi \in \mathscr{E}_1$, $[[\tilde{F}, \tilde{G} \hat{\otimes} 1], \tilde{T}_\xi] = [\tilde{G} \hat{\otimes} 1,\ [\tilde{F}, \tilde{T}_\xi]] - [[\tilde{T}_\xi, \tilde{G} \hat{\otimes} 1], \tilde{F}]$. The first term belongs to $\mathscr{K}(\mathscr{E} \oplus \mathscr{E}'_2)$ because F is a G_2 connexion. Also $[\tilde{T}_\xi, \tilde{G} \hat{\otimes} 1] \in \mathscr{K}(\mathscr{E}'_1 \oplus \mathscr{E}_2) \hat{\otimes} 1$, and hence $[[\tilde{T}_\xi, \tilde{G} \hat{\otimes} 1], \tilde{F}] \in \mathscr{K}(\mathscr{E} \oplus \mathscr{E}'_2)$ because $\tilde{F}$ is an F_3 connexion for $(\mathscr{E}'_1 \oplus \mathscr{E}_2)$ (Proposition 9(i) and (e)). Hence $[G_1 \hat{\otimes} 1, F]$ is a 0 connexion for $\mathscr{E}'_1$ (Proposition 9(i)) and $[G_1 \hat{\otimes} 1, F] - \mathrm{Re}[G_1 \hat{\otimes} 1, F]^+$ is a 0 connexion for $\mathscr{E}_1$ (Proposition 9(g)).

Put $E_1 = \mathscr{K}(\mathscr{E}) + \mathscr{K}(\mathscr{E}_1) \hat{\otimes}_D 1 + \mathscr{K}(\mathscr{E}'_1) \hat{\otimes}_E 1 \subset \mathscr{L}(\mathscr{E})$. E_2 = subalgebra of $\mathscr{L}(\mathscr{E})$ generated by $([G_1 \hat{\otimes} 1, F] - \mathrm{Re}[G_1 \hat{\otimes} 1, F]^+)$, $[F_1 \hat{\otimes}_D 1, F]$. $\mathscr{F}$ = subspace of $\mathscr{L}(\mathscr{E})$ generated by $F, F_1 \hat{\otimes} 1, G \hat{\otimes} 1, A$.

We may apply Theorem 1. Let M, N satisfy the conclusions of Theorem 1. Put $F' = M^{1/2}(F_1 \hat{\otimes} 1) + N^{1/2}F$. Then $(\mathscr{E}, F') \in \mathbb{E}(A, B)$ and $\forall a \in A$, $a[F, F']a^* \geqslant 0 \mod \mathscr{K}(\mathscr{E})$. Hence $(\mathscr{E}, F)$ and $(\mathscr{E}, F')$ are operatorially homotopic (Lemma 11). On the other hand, $\forall a \in A$, $a[F', G_1 \hat{\otimes} 1]a^* \geqslant 0 \mod \mathscr{K}(\mathscr{E})$ and F' is an F_3 connexion for $\mathscr{E}'_1$. Hence $F' \in G_1 \#_D F_3$. ∎

Remark 23. Let $(\mathscr{E}_1, F_1) \in \mathbb{E}(A_1, B_1)$, $(\mathscr{E}_2, F_2) \in \mathbb{E}(A_2, B_2)$, where A_1, A_2 are separable. Let $\mathscr{E}$ be the $A_1 \hat{\otimes} A_2$, $B_1 \hat{\otimes} B_2$ bimodule $\mathscr{E}_1 \hat{\otimes}_C \mathscr{E}_2$. Using [4, Sect. 3, Theorem 3], one finds $M, N \in \mathscr{L}(\mathscr{E})$ of degree 0, $M \geqslant 0$, $N \geqslant 0$,

$M+N=1$, such that $M(\mathscr{K}(\mathscr{E}_1)\hat{\otimes} 1)\subseteq \mathscr{K}(\mathscr{E})$, $N(1\hat{\otimes}\mathscr{K}(\mathscr{E}_2))\subseteq \mathscr{K}(\mathscr{E})$, and $[M,A]\subseteq \mathscr{K}(\mathscr{E})$, $[M,F_1\hat{\otimes} 1]\in \mathscr{K}(\mathscr{E})$, $[M,1\hat{\otimes} F_2]\in \mathscr{K}(\mathscr{E})$. Then $M^{1/2}(F_1\hat{\otimes} 1)+N^{1/2}(1\hat{\otimes} F_2)$ is a Kasparov product of $(\mathscr{E}_1,F_1)$ by $(\mathscr{E}_2,F_2)$ and also a Kasparov product of $(\mathscr{E}_2,F_2)$ by $(\mathscr{E}_1,F_1)$.

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