On the Strong Ext Bifunctor

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This text is an (almost) exact copy of a preprint written during the Academic year 1983-84 while invited at the Mathematics department of Queen's University, Kingston, Ontario. It was part of my ''Thèse d'État''. I don't really know why, I did not publish it at that time, although I liked the results of this paper, gave a few talks on it and circulated it quite widely.

Introduction

In Brown Douglas and Fillmore theory [1] two groups were defined: the weak and the strong Ext groups. In the subsequent developments and in particular in Kasparov's bivariant theory [7], the weak Ext played a far more important role than the strong Ext.

We present here a strong Ext bifunctor $\mathrm{Ext}_s(A, B)$ which reduces to $\mathrm{Ext}_s(A)$ when $B = \mathbb{C}$. We compute it in terms of the Kasparov groups (*i.e.* the weak Ext bifunctor). Our main result is a six term exact sequence:

$$\begin{array}{ccccc} K_0(B) & \longrightarrow & \mathrm{Ext}_s(A,B) & \longrightarrow & \mathrm{Ext}(A,B) \\ \uparrow & & & & \downarrow \\ \mathrm{Ext}(A, B\otimes C_0(\mathbb{R})) & \longleftarrow & \mathrm{Ext}_s(A, B\otimes C_0(\mathbb{R})) & \longleftarrow & K_1(B) \end{array}$$

We next remark that the group $\mathrm{Ext}_s(A, B)$ is nothing else than a relative group $KK^1(i, B)$ where $i : \mathbb{C} \to A$ is the unital inclusion.

The presentation of this paper is the following:

In the first section we give the definitions relative to the strong Ext bifunctor.

In the second one we prove the above mentioned six term exact sequence and draw some consequences.

In the third one we give a definition of the relative KK-theory and identify the strong Ext as a relative group.

Our results are presented in the complex case. They remain however true in the real and the equivariant cases.

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1 The strong-Ext bifunctor

Let us first recall some definitions relative to Kasparov's bifunctor $\mathrm{Ext}(A, B)$. Let A, B be C^*-algebras.

1.1 Definition. a) An (A, B) extension is a $*$-homomorphism $\varphi : A \to \mathcal{L}(E)/\mathcal{K}(E)$ where E is a countably generated Hilbert B-module (*cf.* [6, Def. 1-5]).

b) The extensions φ and φ' are said to be unitarily equivalent (write $\varphi \sim_u \varphi'$ if there exists a unitary $U \in \mathcal{L}(E, E')$ with $\varphi'(a) = U\varphi(a)U^*$ (with obvious meaning) for all a in A.

c) The extension φ is said to be trivial if it admits a lifting which is a morphism $\pi : A \to \mathcal{L}(E)$.

d) The sum of extension φ and φ' is defined by the map $\varphi \oplus \varphi' : A \to \mathcal{L}(E \oplus E')/\mathcal{K}(E \oplus E')$.

e) $\mathrm{Ext}(A, B)$ denotes the semi-group of extensions divided by the equivalence relation: $\varphi \sim \varphi'$ if there exist trivial extensions σ and σ' with $\varphi \oplus \sigma \sim_u \varphi' \oplus \sigma'$.

Assume now that A has a unit. Let $\varphi : A \to \mathcal{L}(E)/\mathcal{K}(E)$ be a unital extension ($\varphi(1) = 1$). It may happen that φ admits a lifting $\pi : A \to \mathcal{L}(E)$ but $\pi(1) \neq 1$.

This motivates the following:

1.2 Definition. a) The unital extension φ is said to be unitarily trivial if it admits a unital lifting $\pi : A \to \mathcal{L}(E)$.

b) $\mathrm{Ext}_s(A, B)$ denotes the semi-group of unital extensions divided by the equivalence relation: $\varphi \sim_s \varphi'$ if there exist unitarily trivial extensions σ and σ' with $\varphi \oplus \sigma \sim_u \varphi' \oplus \sigma'$.

1.3 Remarks. a) One may look at extensions as short exact sequences $0 \to \mathcal{K}(E) \longrightarrow D \xrightarrow{p} A \to 0$. In this presentation, the unital extension are those for which the algebra D is unital; the trivial ones are those for which p admits a cross-section s which is a $*$-homomorphism; the unitarily trivial ones are those for which the above s can be chosen unital.

b) D. Voiculescu's theorem ([12]) asserts that if $B = \mathbb{C}$ and A is separable, all unitarily trivial faithful([1]) extensions σ are unitarily trivial and satisfy $\varphi \oplus \sigma \sim_u \varphi$ for any faithful extension φ.

[1] *i.e.* the map $\varphi : A \to \mathcal{L}(E)/\mathcal{K}(E)$ is one to one. In the presentation with a short exact sequence this means that the ideal $\mathcal{K}(E)$ is essential in D.

Hence one has $\mathrm{Ext}_s(A, \mathbb{C}) = \mathrm{Ext}_s(A)$, where $\mathrm{Ext}_s(A)$ is the Brown-Douglas-Fillmore semi-group (*i.e.* the semi-group of faithful unital extensions divided by unitary equivalence [1]).

c) G.G. Kasparov's generalization of Voiculescu's theorem ([6, theorem 6]) asserts that, if A is separable B has a countable approximate unit and either A or B is nuclear, each unital extension is strongly equivalent to a "unitarily absorbing" extension. An extension is unitarily absorbing iff $\varphi \oplus \sigma \sim_u \varphi$ for any unitarily trivial extension σ (in particular the Hilbert B-module E has to be $\mathcal{H}_B$). In that case, $\mathrm{Ext}_s(A, B)$ is the semi-group of unitarily absorbing extensions divided by unitary equivalence.

d) Obviously, $\mathrm{Ext}_s(\widetilde{A}, B) = \mathrm{Ext}(A, B)$ where $\widetilde{A}$ is obtained from A by adjoining a unit.

e) The definition that we use for $\mathrm{Ext}(A, B)$ is slightly different from the one given by Kasparov in [7, §7]. However, when A is separable and B has a countable approximate unit, using Kasparov's stabilization theorem ([6, theorem 2]) one sees that these definitions coincide.

We will be mainly interested in the groups $\mathrm{Ext}(A, B)^{-1}$ and $\mathrm{Ext}_s(A, B)^{-1}$ of invertible extensions.

Let us end this section by recalling some facts that we use in section 2:

1.4 Recalls. a) ([7, §7]) When A is separable, the invertible extensions are those which admit completely positive liftings $s : A \to \mathcal{L}(E)$ (by Kasparov's generalization of Stinespring's theorem ([6, theorem 3]).

Thus, if A is nuclear and separable, using the Choi-Effros lifting theorem [2], we get that $\mathrm{Ext}(A, B)$ and $\mathrm{Ext}_s(A, B)$ are groups (*i.e.* $\mathrm{Ext}(A, B)^{-1} = \mathrm{Ext}(A, B)$ and $\mathrm{Ext}_s(A, B)^{-1} = \mathrm{Ext}_s(A, B)$).

b) ([7, §7]) If A is separable, $\mathrm{Ext}(A, B)^{-1} = KK^1(A, B)$ ([7, §5 def. 1]). In particular $\mathrm{Ext}(A, B)^{-1}$ is a homotopy invariant functor in both variables.

c) ([7, §6 theorem 3] or [3, Lemma 3.2]). We have $\mathrm{Ext}(\mathbb{C}, B) = K^1(B)$.

2 The six term exact sequence

2.1 Construction of the map $\tau : K_0(B) \to \mathrm{Ext}_s(A, B)$. Let A be unital and separable.

An element of $K_0(B)$ is given by a triple (E_0, E_1, U) where E_0, E_1 are countably generated Hilbert B-modules and $U \in \mathcal{L}(E_0, E_1)$ satisfies $U^*U - 1 \in \mathcal{K}(E_0)$ and $UU^* - 1 \in \mathcal{K}(E_1)$ (thanks to the equality $KK(\mathbb{C}, B) = K_0(B)$ - [7, §7, theorem 3]).

Replacing (E_0, E_1, U) by $(E_0 \oplus E', E_1 \oplus E', U \oplus 1_{E'})$, we may moreover assume that there exists a unital $*$-homomorphism $\pi : A \to \mathcal{L}(E_1)$ (for example, if $\pi_0 : A \to \mathcal{L}(H)$ is a $*$-representation where H is a separable Hilbert space, put $E_1 \oplus E' = E_1 \otimes H$ and $\pi :$

$A \to \mathcal{L}(E_1 \otimes H)$ given by $\pi(a) = 1_{E_1} \otimes \pi_0(a)$). Let then $\pi_U : A \to \mathcal{L}(E_0)$ be given by $\pi^U(a) = q(U^*\pi(a)U)$ ($q : \mathcal{L}(E_0) \to \mathcal{L}(E_0)/\mathcal{K}(E_0)$ is the quotient map).

Obviously, π^U is a unital extension.

2.2 Lemma. *The class of π^U in $\mathrm{Ext}_s(A, B)$ only depends upon the class of (E_0, E_1, U) in $K_0(B)$ (in particular, it does not depend on π). Moreover the map $\tau : K_0(B) \to \mathrm{Ext}_s(A, B)$ given by $\tau([E_0, E_1, U]) = [\pi^U]$ is a homomorphism.*

Proof. Notice first that if U is unitary, π^U is unitarily trivial. Moreover, π^U doesn't change if we perturb U by an element of $\mathcal{K}(E_0, E_1)$.

Take now a triple (E_0, E_1, U) and let $\pi_1, \pi_1' : A \to \mathcal{L}(E_1)$ be two unital $*$-homomorphisms. We may assume moreover that there exists a unital morphism $\pi_0 : A \to \mathcal{L}(E_0)$.

Then $(\pi_1 \oplus \pi_0)^{U \oplus U^*}$ and $(\pi_1' \oplus \pi_0)^{U \oplus U^*}$ are unitarily trivial (since $U \oplus U^*$ is equal modulo $\mathcal{K}(E_0 \oplus E_1)$ to a unitary), and as $(\pi_1')^U \oplus (\pi_1 \oplus \pi_0)^{U \oplus U^*}$ and $\pi_1^U \oplus (\pi_1' \oplus \pi_0)^{U \oplus U^*}$ are unitarily equivalent, we get $\pi_1^U \sim_s (\pi_1')^U$.

All the rest of the proof is routine checks. □

Let us now state our main result:

2.3 Theorem. *Assume that A is separable. Then the sequence*

$$\begin{array}{ccccc}
K_0(B) & \xrightarrow{\ \tau\ } & \mathrm{Ext}_s(A, B)^{-1} & \xrightarrow{\ I\ } & \mathrm{Ext}(A, B)^{-1} \\
{\scriptstyle i^*}\uparrow & & & & \downarrow{\scriptstyle i^*} \\
\mathrm{Ext}(A, B(\mathbb{R}))^{-1} & \xleftarrow{\ I\ } & \mathrm{Ext}_s(A, B(\mathbb{R}))^{-1} & \xleftarrow{\ \tau\ } & K_1(B)
\end{array}$$

is exact, where τ is defined above, I is the obvious forgetful map and i^* is induced by the unital $*$-homomorphism $i : \mathbb{C} \to A$.

Proof. **Exactness at $\mathrm{Ext}_s(A, B)^{-1}$.** The following computation makes it clear that $I \circ \tau = 0$: $\pi^U \oplus 0^{U^*} = (\pi \oplus 0)^{U \oplus U^*}$ is trivial, where $0 : A \to \mathcal{L}(E_0)$ is the 0 map.

Let φ be a unital extension whose class in $\mathrm{Ext}(A, B)$ is zero. This means that there exists a $*$-homomorphism $\pi : A \to \mathcal{L}(E')$ such that $\varphi \oplus q \circ \pi$ admits a lifting $\pi' : A \to \mathcal{L}(E \oplus E')$. Let $\pi_0 : A \to \mathcal{L}(\pi(1)E)$ be the restriction of π. Put $E_0 = E \oplus \pi(1)E'$ and $E_1 = \pi'(1)(E \oplus E')$. Let $U \in \mathcal{L}(E_0, E_1)$ be the map $\xi \mapsto \pi'(1)\xi$. As $q \circ \pi' = \varphi \oplus q \circ \pi$, the triple (E_0, E_1, U) defines an element of $K_0(B)$ and $(\pi')^U = \varphi \oplus \sigma$ where $\sigma = q \circ \pi_0$ is unitarily trivial, whence $[\varphi] = \tau([E_0, E_1, U])$.

Exactness at $\mathrm{Ext}(A, B)^{-1}$. Let φ be a unital extension. Then $\varphi(1)$ lifts to the projection 1. It follows that the extension $\varphi \circ i$ is trivial, *i.e.* the zero element of $\mathrm{Ext}(\mathbb{C}, B) = K_1(B)$. This shows that $i^* \circ I = 0$.

Conversely, if the image of $\varphi(1)$ in $K_1(B)$ is zero, there exists a countably generated Hilbert B-module E' and projections $P \in \mathcal{L}(E')$ and $Q \in \mathcal{L}(E \oplus E')$ such that $q(Q) = \varphi(1) \oplus q(P)$.

We may further assume that there exists a morphism $\pi : A \to \mathcal{L}(E')$ with $\pi(1) = P$ (replacing if necessary E' by $E' \oplus (PE' \otimes H)$ and P by $P \oplus 1$).

Let then $\varphi_0 : A \to \mathcal{L}(Q(E \oplus E'))/\mathcal{K}(Q(E \oplus E'))$ be the unital extension given by

$$\varphi_0(a) = q(Q)(\varphi(a) \oplus q \circ \pi(a))q(Q).$$

As $\varphi_0 \oplus 0_{(1-Q)(E\oplus E')} \sim_u \varphi \oplus q \circ \pi$, we find $[\varphi] = [I(\varphi_0)]$. Moreover, if φ is invertible, it admits a completely positive lifting. Hence so does φ_0.

Exactness at $K_0(B)$. We write $\mathrm{Ext}(A, B)^{-1} = KK(A, B)$ ([6, §7]). We consider the elements of $KK(A, B)$ as triples (π_0, π_1, U), where $\pi_i : A \to \mathcal{L}(E_i)$ are *unital* *-homomorphisms (the E_i are countably generated Hilbert B-modules, $i = 0, 1$) and $U \in \mathcal{L}(E_0, E_1)$ satisfies $U\pi_0(a)U^* - \pi_1(a) \in \mathcal{K}(E_1)$ for all $a \in A$ and $U^*U - 1 \in \mathcal{K}(E_0)$. The image by i^* of (π_0, π_1, U) in $K_0(B)$ is the class of the triple (E_0, E_1, U).

If (π_0, π_1, U) is in $KK(A, B)$, then $\pi_1^U = q \circ \pi_0$. We thus obtain $\tau \circ i^* = 0$.

Conversely, let (E_0, E_1, U) be in $K_0(B)$ and let $\pi_1 : A \to \mathcal{L}(E_1)$ be a unital $*$-homomorphism. If π_1^U defines the zero element of $\mathrm{Ext}_s(A, B)$, there exists a countably generated Hilbert B-module E' and unital $*$-homomorphisms $\pi' : A \to \mathcal{L}(E')$ and $\pi_0 : A \to \mathcal{L}(E_0 \oplus E')$ such that $\pi_1^U \oplus (q \circ \pi') = q \circ \pi_0$. Then $(\pi_0, \pi_1 \oplus \pi', U \oplus 1)$ defines an element of $KK(A, B)$ whose image by i^* is the class of (E_0, E_1, U).

Exactness at all other points is obtained using these ones together with Bott periodicity of the functor $B \to \mathrm{Ext}(A, B)^{-1} = KK^1(A, B)$. □

We now use this theorem together with the Puppe exact sequence ([4]) in order to compute the bifunctor Ext_s.

Let again $i : \mathbb{C} \to A$ denote the unital inclusion and let $C_i = \{f \in A[0, 1;\ f(0) \in i(\mathbb{C})\}$ denote the mapping cone of i.

Let $0 \to \mathcal{K}(E) \to D \to A \to 0$ be a unital extension and let $i' : \mathbb{C} \to D$ be the unital inclusion. We get the exact sequence $0 \to \mathcal{K}(E)(0, 1) \to C_{i'} \to C_i \to 0$. This construction yields a homomorphism

$$\mathrm{Ext}_s(A, B) \xrightarrow{\Phi} \mathrm{Ext}(C_i, B(0, 1)).$$

2.4 Corollary. *If A is separable, the map Φ induces an isomorphism from $\mathrm{Ext}_s(A, B)^{-1}$ to $\mathrm{Ext}(C_i, B(0, 1))^{-1} = KK(C_i, B)$.*

Proof. It is a consequence of [7, §7], theorem 2.3, the Puppe exact sequence [4] and the five lemma □

This corollary shows that the bifunctor $\mathrm{Ext}_s(A, B)^{-1}$ is homotopy invariant, that the functor $B \to \mathrm{Ext}_s(A, B)^{-1}$ is stable, periodic, half exact *etc.*

2.5 Remarks. a) We could use the dual Puppe sequence ([4]) to get an isomorphism $\mathrm{Ext}_s(A, B)^{-1} \simeq \mathrm{Ext}(\hat{C}_i, B)^{-1}$. The map from $\mathrm{Ext}_s(A, B)$ to $\mathrm{Ext}(\hat{C}_i, B)$ is described as follows:
Let $0 \to \mathcal{K}(E) \to D \to A \to 0$ be a unital extension and let $i' : \mathbb{C} \to D$ be the unital inclusion. We get the exact sequence $0 \to \mathcal{K} \otimes \mathcal{K}(E) \to \hat{C}_{i'} \to \hat{C}_i \to 0$.

b) Let N be a countably decomposable II_∞ factor. One defines analogously to $\mathrm{Ext}_s(A,B)$ the functor $\mathrm{Ext}_s^N(A)$ (*cf.* [10, §1]). One then proves either as in [10] or by constructing a six term exact sequence analogous to theorem 2.3 that $\mathrm{Ext}_s^N(A)^{-1}$ is isomorphic to $\mathrm{Ext}_s(A,M)^{-1}$ where M is a II_1 factor with $N = M\overline{\otimes}\mathcal{L}(H)$ (*cf.* [10, theorem 2.2]).

c) We may also derive from theorem 2.3 a "universal coefficient formula" (*cf.* [8]) as follows.

An element of $\mathrm{Ext}_s(A,B)$ gives a short exact sequence $0 \to J \xrightarrow{j} D \xrightarrow{p} A \to 0$. It therefore gives an exact sequence

$$\begin{array}{ccc} K_*(B) & \xrightarrow{\quad j_* \quad} & K_*(D) \\ & {}_{\delta}\nwarrow \quad \swarrow_{p_*} & \\ & K_*(A) & \end{array} \qquad \text{with} \quad p_*[1] = [1].$$

We get a homomorphism $\mathrm{Ext}_s(A,B) \xrightarrow{\gamma_s} \mathrm{Hom}_{[1]}(K_*(A), K_*(B))$ which to a unital extension of A by B associates the connecting map $\delta : K_*(A) \to K_*(B)$. Here, $\mathrm{Hom}_{[1]}(K_*(A), K_*(B)) = \mathrm{Hom}(K_*(A)/\mathbb{Z}[1], K_*(B))$ is the group of homomorphisms from $K_*(A)$ to $K_*(B)$ vanishing on the element $[1] \in K_0(A)$.

Each element of the kernel of γ_s yields a pointed exact sequence

$$0 \to K_*(B) \to (K_*(D), [1]) \to (K_*(A), [1]) \to 0.$$

Thus we get a morphism $\kappa : \ker \gamma_s \to \mathrm{Ext}_{[1]}(K_*(A), K_*(B))$ where $\mathrm{Ext}_{[1]}(K_*(A), K_*(B))$ is the group of pointed extensions $0 \to K_*(B) \to (G,x) \to (K_*(A),[1]) \to 0$.

Applying then theorem 2.3 together with the Rosenberg and Schochet Universal coefficient formula ([8, theorem 4.2]) and exactness of the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{[1]}(K_*(A), K_*(B)) & \longrightarrow & \mathrm{Hom}(K_*(A), K_*(B)) & & \\ & & & & & \searrow & \\ & & & & & & K_*(B) \\ & & & & & \swarrow & \\ 0 & \longleftarrow & \mathrm{Ext}(K_*(A), K_*(B)) & \longleftarrow & \mathrm{Ext}_{[1]}(K_*(A), K_*(B)) & & \end{array}$$

we get an exact sequence

$$0 \to \mathrm{Ext}_{[1]}(K_*(A), K_*(B)) \to \mathrm{Ext}_s^*(A,B) \to \mathrm{Hom}_{[1]}(K_*(A), K_*(B)) \to 0$$

(when the algebra A is in some distinguished class of algebras).

Note that this Universal Coefficient Formula coincides with the one we would obtain by applying Corollary 2.4 and [8, theorem 4.2] only if the map $i_* : \mathbb{Z} = K_0(\mathbb{C}) \to K_0(A)$ is injective.

3 The relative KK-theory

The strong ext bifunctor is a particular case of the following relative KK-theory, or KK-theory for maps.

In this section, all algebras are $\mathbb{Z}/2\mathbb{Z}$ graded. We will use freely the notations of [9] for the Kasparov groups.

Let $\varphi : A \to B$ be a grading preserving $*$-homomorphism and let D be a graded C^*-algebra.

3.1 Definition. a) Let $\mathbb{E}(\varphi, B)$ denote the set of pairs $((E, F), (\widetilde{E}, \widetilde{F}))$ where $(E, F) \in \mathbb{E}(B, D)$ and $(\widetilde{E}, \widetilde{F}) \in \mathbb{E}(A, D[0, 1))$ together with an identification (unitary equivalence) of $\varphi^*(E, F)$ with $(\widetilde{E}_0, \widetilde{F}_0)$ (they are both in $\mathbb{E}(A, D)$).

b) A homotopy is an element of $\mathbb{E}(\varphi, D[0, 1])$.

c) $KK(\varphi, D)$ denotes the group of homotopy classes of elements in $\mathbb{E}(\varphi, D)$.

Consider the maps
$I : KK(\varphi, D) \to KK(B, D)$ induced by the map $((E, F), (\widetilde{E}, \widetilde{F})) \mapsto (E, F)$
$I : KK(A, D(0, 1)) \to KK(\varphi, D)$ induced by the map $(\widetilde{E}, \widetilde{F}) \mapsto ((0, 0), (\widetilde{E}, \widetilde{F}))$.

One easily gets:

3.2 Theorem. *The sequence*

$$KK(B, D(0, 1)) \xrightarrow{\varphi^*} KK(A, D(0, 1)) \xrightarrow{J} KK(\varphi, D) \xrightarrow{I} KK(B, D) \xrightarrow{\varphi^*} KK(A, D)$$

is exact. □

3.3 Remarks. a) In [5, definitions 5 and 6 of §1], Kasparov defines also a relative K-group $K_{p,q}(\omega)$ which corresponds to $KK(\omega, \mathcal{C}_{p-1,q})$ ($\omega : B_2 \to B_1$ an epimorphism). By [5, §6, theorem 2] and theorem 3.2 above, we find that the two definitions coincide.

b) If φ is an isomorphism $KK(\varphi, D) = 0$. If $\varphi_0 : A \to 0$ and $\varphi_1 : 0 \to B$ are the zero maps, we have $KK(\varphi_0, D) = KK(A(0, 1), D)$ and $KK(\varphi_1, D) = KK(B, D)$ (by definition).

Let $\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[\varphi']{} & B' \end{array}$ be a commutative square. One may naturally construct a group homomorphism $f^* : KK(\varphi', D) \to KK(\varphi, D)$.

The maps I and J of theorem 3.2 are the f^* corresponding to the commutative squares

$$\begin{array}{ccc} 0 & \xrightarrow{\varphi_1} & B \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Id}_B} \\ A & \xrightarrow[\varphi]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \mathrm{Id}_A}\downarrow & & \downarrow \\ A & \xrightarrow[\varphi_0]{} & 0 \end{array}$$

c) When constructing the group $KK(D,\varphi)$ one naturally finds $\mathbb{E}(D,\varphi) = \mathbb{E}(D, C_\varphi)$ and $KK(D,\varphi) = KK(D, C_\varphi)$ where C_φ is the mapping cone of φ.

Let now A, B be trivially graded and assume that A is unital and separable and let $i : \mathbb{C} \to A$ be the unital inclusion. Let $\rho : A \to \mathcal{L}(E)/\mathcal{K}(E)$ be a unital invertible extension. Let then ρ' be such that $\rho + \rho'$ is unitarily trivial, say $\rho \oplus \rho' = q \circ \pi$ where $\pi : A \to \mathcal{L}(E \oplus E')$ is a unital $*$-homomorphism. Put $E'' = E \oplus E'$ and let $P \in \mathcal{L}(E'')$ be the projection to the summand A.

Let $\mathcal{C}_1$ be the first Clifford algebra and let $e \in \mathcal{C}_1$ be such that $e^2 = 1$, $e = e^*$, $\partial e = 1$. Let A act on the Hilbert $B \otimes \mathcal{C}_1$ module $E'' \otimes \mathcal{C}_1$ by $\pi \otimes 1 : A \to \mathcal{L}(E'' \otimes \mathcal{C}_1) = \mathcal{L}(E'') \otimes \mathcal{C}_1$ and let $F = (2P-1) \otimes e$. The pair $(E'' \otimes \mathcal{C}_1, F)$ is a Kasparov $(A, B \otimes \mathcal{C}_1)$ module. Let $(\widetilde{E}, \widetilde{F})$ be the Kasparov $(\mathbb{C}, (B \otimes \mathcal{C}_1)[0,1))$ module $\widetilde{E} = (E'' \otimes \mathcal{C}_1)[0,1)$, $\widetilde{F} = F \otimes 1$, where $\mathbb{C}$ acts unitarily on $\widetilde{E}$. Then $((E'' \otimes \mathcal{C}_1, F), (\widetilde{E}, \widetilde{F}))$ is in $\mathbb{E}(i, B \otimes \mathcal{C}_1)$. In this way we construct a morphism $\lambda : \mathrm{Ext}_s(A,B)^{-1} \to KK(i, B \otimes \mathcal{C}_1)$.

3.4 Corollary. *The map λ is an isomorphism from $\mathrm{Ext}_s(A,B)^{-1}$ to $KK(i, B \otimes \mathcal{C}_1)$.*

Proof. Follows immediately from theorem 2.3, theorem 3.2 and the five lemma. □

One may also prove this result by using Corollary 2.4 and:

3.5 Corollary. *Let A and B be separable graded C^*-algebras and le $\varphi : A \to B$ be a grading preserving $*$-homomorphism. Then, for every C^*-algebra D, $KK(\varphi, D)$ is isomorphic to $KK(C_\varphi, D(0,1))$.*

Proof. Using theorem 3.2, the Puppe exact sequence ([4]) and the five lemma, it is enough to construct a natural map $KK(\varphi, D) \to KK(C_\varphi, D(0,1))$. It is in fact convenient to construct this map from $KK(\varphi, D)$ to $KK(C_\varphi, D(-1,1))$.

Let $((E,F), (\widetilde{E}, \widetilde{F})) \in \mathbb{E}(\varphi, B)$. Put then $E' = \{(\xi, \eta) \in E(-1,0] \times E';\ \xi(0) = \eta(0)\}$. It is naturally a countably generated Hilbert $D(-1,1)$-module. Put $F'(\xi,\eta) = ((F \otimes 1)\xi, \widetilde{F}\eta)$, and let C_φ act on E' by $(a,f)(\xi,\eta) = (\pi(f)\xi, a\eta)$, where $a \in A$ and $f \in B[0,1)$ satisfy $f(0) = \varphi(a)$ and $(\pi(f)\xi)(t) = f(-t)\xi(t)$ (for $t \in (-1,0]$). Then $(E', F') \in \mathbb{E}(C_\varphi, D(-1,1))$. One immediately checks that this defines a morphism $KK(\varphi, D)$ to $KK(C_\varphi, D(-1,1))$ and that that it relates the exact sequence of theorem 3.2 with the Puppe exact sequence. □

Using theorem 3.2 and [7, §, Theorem 2] (*cf.* also [11, Th. 1.1] for the graded case), we get:

3.6 Corollary. *Let $0 \to J \xrightarrow{j} A \xrightarrow{p} A/J \to 0$ be a short exact sequence of separable (graded) C^*-algebras. Assume that p admits a completely positive cross-section. Then $KK(j, D) \simeq KK(A/J, D)$ and $KK(p, D) \simeq KK(J, D(0,1))$.* □

The maps $KK(A/J, D) \to KK(j, D)$ and $KK(p, D) \to KK(J, D(0,1))$ are very easily described:

- Let $(E,F) \in \mathbb{E}(A/J,D)$. Let $(\widetilde{E},\widetilde{F}) \in \mathbb{E}(J,D[0,1))$ be given by $\widetilde{E} = E[0,1)$, $\widetilde{F} = F \otimes 1$ and J acts by the zero action. Then $((E,F),(\widetilde{E},\widetilde{F})) \in \mathbb{E}(j,D)$. This defines the first map.

- Let $((E,F),(\widetilde{E},\widetilde{F})) \in \mathbb{E}(p,D)$. Let E' be the Hilbert $D(0,1)$ module $E' = \{\xi \in \widetilde{E};\ \xi(0) = 0\}$ and let F' be the restriction of $\widetilde{F}$ to E'. As the action of J on $\widetilde{E}_0$ is the zero action we get that (E',F') is a Kasparov $(J,D(0,1))$ bimodule. This defines the second map.

3.7 Remarks. a) One may also use the results of [4] to get an isomorphism $KK(\widehat{C}_\varphi,D) \simeq KK(\varphi,D)$.

b) Let ω be a commutative square

$$\omega : \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ A' & \xrightarrow[\varphi']{} & B' \end{array} .$$

One may associate to ω and a C^*-algebra D a group $KK(\omega,D)$. We then would obtain an exact sequence:

$$KK(\varphi',D(0,1)) \xrightarrow{f^*} KK(\varphi,D(0,1)) \longrightarrow KK(\omega,D) \longrightarrow KK(\varphi',D) \xrightarrow{f^*} KK(\varphi,D).$$

In fact, we have a map $\overline{f} : C_\varphi \to C_{\varphi'}$ and, when algebras A, A', B, B' are separable, we get $KK(\omega,D) \simeq KK(C_{\overline{f}}, D((0,1)\times(0,1))) \simeq KK(C_{\overline{f}},D)$.

c) Let (A,B,D) be C^*-algebras. Let $x = (E,F) \in \mathbb{E}(A,B)$.

Define $\mathbb{E}(x,D)$ as the set of pairs $((E',F'),(\widetilde{E},\widetilde{F}))$ where $(E',F') \in \mathbb{E}(B,D)$ and $(\widetilde{E},\widetilde{F}) \in \mathbb{E}(A,D[0,1))$ such that $(\widetilde{E}_0,\widetilde{F}_0)$ is a Kasparov product of (E,F) by (E',F').

Define also $\mathbb{E}(D,x)$ as the set of pairs $((E',F'),(\widetilde{E},\widetilde{F}))$ where $(E',F') \in \mathbb{E}(D,A)$ and $(\widetilde{E},\widetilde{F}) \in \mathbb{E}(D,B[0,1))$ such that $(\widetilde{E}_0,\widetilde{F}_0)$ is a Kasparov product of (E',F') by (E,F).

Define $KK(x,D)$ as the group of homotopy classes of elements in $\mathbb{E}(x,D)$ and $KK(D,x)$ as the group of homotopy classes of elements in $\mathbb{E}(D,x)$.

One then proves easily:

- If A is separable we have the exact sequence:

$$KK(B,D(0,1)) \xrightarrow{x\otimes\cdot} KK(A,D(0,1)) \to KK(x,D) \to KK(B,D) \xrightarrow{x\otimes\cdot} KK(A,D).$$

- If D is separable we have the exact sequence:

$$KK(D,A(0,1)) \xrightarrow{\cdot\otimes x} KK(D,B(0,1)) \to KK(D,x) \to KK(D,A) \xrightarrow{\cdot\otimes x} KK(D,B).$$

- The groups $KK(x,D)$ and $KK(D,x)$ only depend on the class of (E,F) in $KK(A,B)$ (with separability assumptions). If this class is given by an exact sequence (admitting a completely positive cross section): $0 \to B((0,1) \otimes \mathcal{K} \to E \to A \to 0$, we find $KK(D,x) = KK(D,E)$ and $KK(x,D) = KK(E,D(0,1))$.

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