Journée annuelle
, 2003, p. 1–

QUANTUM REPRESENTATIONS OF MAPPING CLASS GROUPS

by

Gregor Masbaum

Résumé. — Le « mapping class group » (ou groupe de difféotopie, ou groupe modulaire) d'une surface de genre g est le groupe de ses auto-difféomorphismes directs, considérés à isotopie près. Ce groupe de présentation finie agit naturellement sur des objets mathématiques attachés à la surface : espace de Teichmüller, espaces de connexions plates, modules skein... Ces actions permettent de construire des représentations de dimension finie de ce groupe, appelées quantiques à cause de leur liens avec la théorie quantique des champs et les invariants quantiques des nœuds. L'exposé se propose de discuter quelques résultats récents concernant ces représentations, notamment leur fidélité asymptotique (J.E.Andersen, Freedman-Walker-Wang).

Contents

Introduction

This talk will be about quantum representations of mapping class groups. We will describe the geometric approach through Hitchin's connection and the combinatorial-topological approach through the skein theory of the Kauffman bracket *à la* [**BHMV**]. The reader should beware that both on the geometric side and on the combinatorial-topological side there are other approaches, which will be mentioned only briefly.

Thus, this text is not meant to be a complete survey of the subject, and I apologize in advance to those whose work should be mentioned but isn't [(1)].

1. Basic notions and notation

Let $\Sigma = \Sigma_g$ be a closed oriented surface of genus g. The *mapping class group* of Σ is the group of isotopy classes of orientation preserving diffeomorphisms of Σ. We will denote it by $\Gamma = \Gamma_g$. The name comes from the German *Abbildungsklassen* for (isotopy) classes of mappings. Other names for this group include *Teichmüller modular group* and *homeotopy (*or *diffeotopy) group*.

For example, in genus one, the mapping class group is simply the group $SL(2,\mathbb{Z})$ of two-by-two integer matrices with determinant one. Indeed, a genus one surface is homeomorphic to a torus $S^1 \times S^1$, which we can also think of as the quotient of $\mathbb{R}^2$ by the lattice $\mathbb{Z}^2$. Any matrix in $SL(2,\mathbb{Z})$ defines an orientation-preserving linear automorphisms of $\mathbb{R}^2$ which preserves the lattice $\mathbb{Z}^2$. This gives a map from $SL(2,\mathbb{Z})$ to the mapping class group Γ_1, which is an isomorphism.

Diffeomorphisms of surfaces may be induced by automorphisms of compact Riemann surfaces (or smooth complex-algebraic curves). However, a given complex structure on Σ will have only finitely many automorphisms, and one of the *rôles* of the mapping class group (which is an infinite group) in geometry is to relate the different complex structures. The picture (which we will need later) is the following. Let $\mathcal{T} = \mathcal{T}_g$ be Teichmüller space, which we think of as the space of complex structures on Σ_g. Here, two complex structures σ and σ' are identified in $\mathcal{T}_g$ if $\sigma' = f^*(\sigma)$ for some diffeomorphism f isotopic to the identity. Thus, the mapping class group acts on $\mathcal{T}_g$, and the stabilizer of a point $\sigma \in \mathcal{T}_g$ is precisely the automorphism group of the complex curve (Σ, σ). Moreover, by Nielsen's theorem, any $f \in \Gamma$ of finite order has a fixed point in Teichmüller space (in fact, f can be represented by an actual diffeomorphism of finite order, and this diffeomorphism fixes an actual complex structure, *i.e.* not just up to isotopy).

The easiest examples of diffeomorphisms of infinite order are *Dehn twists* about simple closed curves on the surface. Here, a *simple closed curve* γ on Σ is an (unparametrized) embedded circle. By definition, the Dehn twist t_γ about γ is the identity on Σ except in an annulus neighborhood of γ, where it can be described as follows. Choose polar coordinates (r,θ) in this neighborhood so that γ corresponds the locus where $r = 1$, and the angle θ varies between 0 and 2π. Then the map sends (r,θ) to $(r, \theta + \varphi(r))$ where φ is a smooth increasing (*i.e.*, $\varphi' \geq 0$) function such that $\varphi(r) = 0$ for $r \leq 1 - \varepsilon$, and $\varphi(r) = 2\pi$ for $r \geq 1 + \varepsilon$, for some small $\varepsilon > 0$. Note that this definition does not require an orientation of the curve γ, but does require an orientation of the surface (the identification of the neighborhood of γ with the standard annulus with coordinates (r,θ) should preserve the orientation). Note also that t_γ only depends on the isotopy class of γ.

[(1)] In order to keep the list of references within reasonable length, we will also not give explicit bibliographic references for results about the mapping class group which are in some sense "classical"; for those the reader is referred to existing surveys such as for example Ivanov's paper [**Iv**].

Dehn twists are important because, by the theorem of Dehn and Lickorish, the mapping class group is generated by Dehn twists, in fact, by a finite number of such. Moreover, Γ has a finite presentation due to Wajnryb based on previous work of Hatcher and Thurston.

A good way to study a mapping class $f \in \Gamma$ is to look at its action on simple closed curves. In fact, mapping classes on closed surfaces are determined by their action on isotopy classes of unoriented simple closed curves, except in genus one and two. Since we will need this fact later, we shall briefly sketch the argument. For every $f \in \Gamma$ and every simple closed curve γ, one has the fundamental relation

$$f t_\gamma f^{-1} = t_{\gamma'} \tag{1}$$

where $\gamma' = f(\gamma)$. Thus, if f acts trivially on the set of (isotopy classes of) unoriented simple closed curves, then f commutes with every Dehn twist, and must therefore be central in the mapping class group. But the center of Γ_g is trivial, except in genus one and two, where the center is $\mathbb{Z}/2$, generated by the hyperelliptic involution. (This is an involution with $2g+2$ fixed points on Σ_g such that the quotient is a sphere S^2. In genus one and two, it is unique up to isotopy.) This shows that mapping classes on closed surfaces are indeed determined by their action on simple closed curves if the genus is at least three, and determined up to multiplication by the hyperelliptic involution if the genus is one or two.

One may also look at how a mapping class f acts on the fundamental group $\pi_1 = \pi_1(\Sigma_g)$. Recall that this group is generated by $2g$ generators $\alpha_1, \beta_1, \alpha_2, \ldots, \beta_g$ with one relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1} = 1.$$

Elements of $\pi_1(\Sigma_g)$ are homotopy classes of loops on Σ, where all loops have to start and end at a given base point. Since a diffeomorphism f need not respect this base point, one gets in this way only an outer automorphism of π_1, that is, an element of $Out(\pi_1) = Aut(\pi_1)/Inn(\pi_1)$, where $Inn(\pi_1)$ stands for the inner automorphisms (*i.e.* conjugations). The Baer-Nielsen theorem states that this gives an isomorphism

$$\Gamma_g \xrightarrow{\approx} Out^+(\pi_1(\Sigma_g))$$

(where the superscript $+$ indicates automorphisms which preserve orientation in an appropriate sense). This result was used by Grossman [**Gr**] to show that the mapping class group is residually finite.

Let us now look at linear representations of the mapping class group. The first example is, of course, the action on the homology of the surface. Recall that $H_1(\Sigma_g;\mathbb{Z}) \simeq \mathbb{Z}^{2g}$ comes equipped with a skew-symmetric non-degenerate bilinear form, the *intersection form.* In terms of generators a_i, b_j corresponding to the generators α_i, β_j of the fundamental group, the intersection form is given by

$$a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij},$$

(all other intersections are zero). This form is preserved by the mapping class group, and so one gets a homomorphism

$$\Gamma_g \;\to\; \mathrm{Aut}(H_1(\Sigma_g;\mathbb{Z}), \cdot) = Sp(2g, \mathbb{Z}). \tag{2}$$

This map is moreover surjective because the symplectic group $Sp(2g,\mathbb{Z})$ is generated by transvections which can be realized by Dehn twists.

In genus one, this is just the isomorphism $\Gamma_1 \xrightarrow{\approx} SL(2,\mathbb{Z})$. But in higher genus, the map (2) is not injective. The kernel is called the *Torelli group* T_g. For example, a Dehn twist about a separating (hence null-homologous) curve acts trivially in homology and therefore lies in the Torelli group. Much of what we know about this group is due to Johnson. For example, he defined the so-called Johnson homomorphism from T_g to $\Lambda^3 H_1(\Sigma_g;\mathbb{Z})$. He also defined higher homomorphisms, each one being defined on the kernel of the preceding one. For more information on the Torelli group, see his survey paper [**J**].

2. Geometric quantization

There are various approaches to quantum representations of mapping class groups. In some sense, they are all quantizations of the natural action of $\Gamma = \Gamma_g$ on the space of representations of the fundamental group $\pi_1(\Sigma)$ into a Lie group G. First, let us describe the geometric approach, which fits into the general framework of *geometric quantization.* For simplicity, assume $G = SU(2)$.

The space

$$M' = \mathrm{Hom}(\pi_1(\Sigma), G)^{irr}/G$$

of irreducible representations up to conjugation is in a natural way a smooth symplectic manifold of dimension $6g-6$. The symplectic form can be described as follows. A point in M' may be represented by an irreducible flat connection A on the trivial $G = SU(2)$-bundle $P = \Sigma \times G$. Then the tangent space to M' at this point is

$$T_{[A]}M' = H^1(\Sigma_g; d_A),$$

where d_A is the induced covariant derivative in the associated adjoint bundle $\mathrm{ad}\,P$. The symplectic form ω is given by

$$\omega(\alpha_1,\alpha_2) = -\int_\Sigma \mathrm{Tr}(\alpha_1 \wedge \alpha_2),$$

where α_1, α_2 are d_A-closed 1-forms with values in $\mathrm{ad}\,P$. If the trace is appropriately normalized, then the symplectic form ω is the curvature form of a connection on a complex line bundle $\mathcal{L}$ over M', whose first Chern class is

$$c_1(\mathcal{L}) = \frac{[\omega]}{2\pi}.$$

Now pick a complex structure σ on Σ. This induces on M' the structure of a complex manifold which we will denote by M'_σ. In fact, by the theorem of Narasimhan and Seshadri, M'_σ is the moduli space of stable holomorphic rank two bundles with trivial determinant over the Riemann surface (or algebraic curve) (Σ,σ). By the theorem of Drézet and Narasimhan, the Picard group of M'_σ is isomorphic to $\mathbb{Z}$, generated by a (holomorphic) line bundle $\mathcal{L}_\sigma$, whose underlying C^∞ bundle is just the line bundle $\mathcal{L}$. Note that multiplying the symplectic form by a positive integer k called the *level*, just amounts to replacing $\mathcal{L}_\sigma$ by its k-th tensor power $\mathcal{L}_\sigma^{\otimes k}$.

We define $Z_k(\Sigma,\sigma)$ to be the space of (holomorphic) sections of $\mathcal{L}_\sigma^{\otimes k}$:

$$Z_k(\Sigma,\sigma)=H^0(M'_\sigma,\mathcal{L}_\sigma^{\otimes k}).$$

These sections are sometimes referred to as *non-abelian theta functions*. They form a finite-dimensional vector space whose dimension is given by the famous Verlinde formula

$$d_g(k)=\left(\frac{k+2}{2}\right)^{g-1}\sum_{j=1}^{k+1}\left(\sin\frac{\pi j}{k+2}\right)^{2-2g}.$$

This is a non-trivial result and the proof of this formula (and of similar formulas for other Lie groups) has involved many famous people; see Sorger's Bourbaki talk [**S**] for more about this.

Note that the dimension is independent of the complex structure σ which we needed to choose in order to define $Z_k(\Sigma,\sigma)$. In fact, as σ varies in Teichmüller space $\mathcal{T}_g$, the vector spaces $Z_k(\Sigma,\sigma)$ fit together to form a vector bundle $\mathcal{Z}_k(\Sigma)$ which will be called the *Verlinde bundle* over Teichmüller space.

If f is a diffeomorphism of Σ fixing the complex structure σ, one may study its induced action on $Z_k(\Sigma,\sigma)$ purely with algebro-geometric means (see [**A1**]). But such diffeomorphisms are necessarily of finite order. To get a representation of the full mapping class group, we need a way to compare $Z_k(\Sigma,\sigma)$ and $Z_k(\Sigma,f^*(\sigma))$. This is achieved by a projectively flat connection on the Verlinde bundle $\mathcal{Z}_k$. Its existence was proved by Axelrod, della Pietra, and Witten [**ADW**] and Hitchin [**H**]. Integrating the connection along a path in Teichmüller space between two points σ_1 and σ_2 gives a linear isomorphism called *parallel transport*

$$P_{\sigma_1,\sigma_2}\ :\ Z_k(\Sigma,\sigma_1)\xrightarrow{\approx}Z_k(\Sigma,\sigma_2).$$

Projective flatness of the connection means that up to multiplication by a non-zero scalar factor, the parallel transport map P_{σ_1,σ_2} is independent of the choice of the path (we are using here that Teichmüller space is contractible). A more invariant way of saying this is that the fibers of the projectified bundle $\mathbb{P}\mathcal{Z}_k(\Sigma)$ are canonically identified; in fact, each fiber is identified with the space of covariant constant sections of $\mathbb{P}\mathcal{Z}_k(\Sigma)$. Let us denote this space by $\mathbb{P}Z_k(\Sigma)$.

Thus we get a projective representation

$$\rho_k\ :\ \Gamma\ \to\mathrm{Aut}(\mathbb{P}Z_k(\Sigma))\ ;$$

the latter group is, of course, isomorphic to the projective-linear group $PGL(d_g(k),\mathbb{C})$. Concretely, if we identify $\mathbb{P}Z_k(\Sigma)$ with $\mathbb{P}Z_k(\Sigma,\sigma)$ for some $\sigma\in\mathcal{T}$, then a mapping class f is represented by the natural map f^* followed by parallel transport back to the fiber over σ:

$$\rho_k(f)=P_{f^*(\sigma),\sigma}\circ f^*.$$

Thus, the "ideology" of geometric quantization has succeeded: The symplectic manifold M' with its natural action of the mapping class group has been "quantized" to a series of vector spaces $Z_k(\Sigma,\sigma)$ which depend on additional structure (the choice of the complex structure σ); however, the associated projective spaces are independent of the additional structure, and carry a (projective) action of the mapping class group. Of course, the key point in this construction is the projectively flat connection on the

Verlinde bundle $\mathcal{Z}_k$. Another approach to this is the construction of Tsuchiya, Ueno, and Yamada [**TUY**] of a connection on the bundle of conformal blocks in the WZW (Wess-Zumino-Witten) model (we again refer to [**S**] for more about this). Laszlo [**La**] has shown that this WZW connection does indeed coincide with Hitchin's connection.

3. The skein-theoretical approach

Recall that a knot (resp. a link) in a 3-manifold is an embedded circle (resp. a collection of disjointly embedded circles). Beginning with the Jones polynomial of links in the 3-sphere, it has proved useful to consider *skein modules*, *i.e.* modules generated by linear combinations of (isotopy classes of) links in the given manifold, modulo so-called *skein relations*, of which there are various types. We will use the Kauffman bracket relations (see Figure 1). They are a convenient variant of the skein relations for the Jones polynomial; however, one must actually consider *banded* (or *ribbon*) links, *i.e.* collections of disjointly embedded annuli $S^1 \times [0,1]$. We denote the Kauffman bracket skein module of a 3-manifold N by $\mathcal{K}(N)$. It is a module over the ring of Laurent polynomials $\mathbb{Z}[A, A^{-1}]$. For a complex number $\xi \in \mathbb{C}^\times$, we denote by $\mathcal{K}_\xi(N)$ the vector space obtained from $\mathcal{K}(N)$ by extending coefficients *via* the ring homomorphism $\mathbb{Z}[A, A^{-1}] \to \mathbb{C}$, $A \mapsto \xi$.

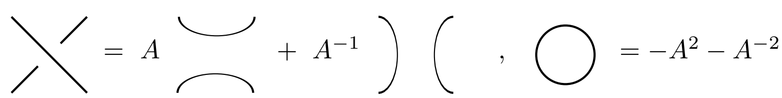

FIGURE 1. The Kauffman bracket relations.

Let us now look at the case where the 3-manifold is the product of a surface Σ by an interval $[0,1]$. The skein module $K_\xi(\Sigma \times [0,1])$ is naturally an algebra, the product being given by stacking one link on top of the other. If $\xi = \pm 1$, this product is commutative, but in general it is non-commutative (if the genus is at least one). Thus we may think of $K_\xi(\Sigma \times [0,1])$ as a deformation of the commutative algebra $K_{-1}(\Sigma \times [0,1])$.

Every diffeomorphism of Σ extends canonically to a diffeomorphism of $\Sigma \times [0,1]$. Thus the mapping class group Γ acts on links in $\Sigma \times [0,1]$. Skein relations are "local", so this induces an action of Γ on the skein algebras $\mathcal{K}_\xi(\Sigma \times [0,1])$. These are infinite-dimensional as vector spaces, and hence not yet the quantum representations we seek. But they can be used to construct those finite-dimensional representations, as we will see.

Before that, let us make a brief digression to remark that the above is closely related to "deformation quantization" of the action of Γ on the $SU(2)$-representation variety M' considered in the previous section, except that one should replace the latter by the $SL(2,\mathbb{C})$-character variety, $M_{\mathbb{C}}$. Note that every simple closed curve γ on Σ (or, more generally, a knot in $\Sigma \times [0,1]$), gives rise to a *holonomy function*

$$h_\gamma \ : M_{\mathbb{C}} \ \to \mathbb{C},$$

which associates to a representation $\rho : \pi_1(\Sigma) \to SL(2,\mathbb{C})$ the trace of the holonomy of ρ along γ. (A matrix in $SL(2,\mathbb{C})$ and its inverse matrix have the same trace; therefore the trace of the holonomy does not depend on an orientation of γ.) The point is that the *coordinate ring* of the $SL(2,\mathbb{C})$-character variety $M_{\mathbb{C}}$ is naturally isomorphic to the skein algebra $\mathcal{K}_{-1}(\Sigma \times [0,1])$ (see [**PS**] and references therein). Under this isomorphism, the holonomy function h_γ (which is an element of the coordinate ring of $M_{\mathbb{C}}$) corresponds to the curve $\gamma \times \frac{1}{2}$ viewed as a knot in $\Sigma \times [0,1]$ and hence as an element of the skein module. This is based on the fact (first observed by Bullock [**Bu**]) that the Kauffman bracket skein relation for $A = -1$ is very similar to the well-known equality

$$\operatorname{Tr}(A)\operatorname{Tr}(B) = \operatorname{Tr}(AB) + \operatorname{Tr}(AB^{-1}),$$

for any two matrices $A, B \in SL(2,\mathbb{C})$. Thus, roughly speaking, the skein algebra $\mathcal{K}_\xi(\Sigma \times [0,1])$ may be viewed as a non-commutative deformation of the representation variety, and, moreover, in a Γ-equivariant way. (There is a lot more one should say about this topic, but we will end our digression here.)

Skein modules were used in [**BHMV**] to construct finite-dimensional representations $V_k(\Sigma)$ of the mapping class group which should correspond, in some sense, to the representations $Z_k(\Sigma)$ discussed in the preceding section. This construction is equivalent to (but logically independent from) Reshetikhin and Turaev's construction using representations of the quantum group $U_q sl(2)$ at roots of unity [**RT**]. As before, only the projective space $\mathbb{P}V_k(\Sigma)$ is canonically associated to the surface. The minimal amount of extra structure needed to get a well-defined vector space (and a well-defined representation of a certain central extension, $\widetilde{\Gamma}$, of the mapping class group Γ on it), is a so-called p_1-structure on the surface (see [**BHMV**]). Another approach put forward by Walker and Turaev (see [**T**]) is to fix a lagrangian subspace in the first homology of the surface. These things are necessary to construct properly a *Topological Quantum Field Theory* (TQFT).

The connection with TQFT's is, of course, an important aspect of quantum representations of mapping class groups. Indeed, historically they were initiated by Witten's paper [**W**], where he "explained" the Jones polynomial of knots from the point of view of Quantum Field Theory. For the connection with Conformal Field Theory, see also Kohno [**Ko**]. But we will not go into this here. Let us just mention that a TQFT contains topological invariants of 3-manifolds called *quantum invariants*. In particular, it associates a vector in $V_k(\Sigma)$ to every "extended" 3-manifold N with boundary $\partial N = \Sigma$. (Here, "extended" means that N must also be equipped with an appropriate extra structure.)

Let us now describe the projective representation of the mapping class group. We follow Roberts [**Ro**], who has found a simple way to construct it from the natural action of Γ on the skein algebra $\mathcal{K}_\xi(\Sigma \times [0,1])$. Here, and from now on, ξ is a primitive root of unity of order $4k+8$. Embed the surface Σ into the 3-sphere so that its complement consists of two handlebodies H and H':

$$H \cup_\Sigma H' = S^3.$$

The skein module $\mathcal{K}_\xi(S^3)$ is naturally isomorphic to $\mathbb{C}$, so this gives a symmetric bilinear form

$$\langle\ ,\ \rangle\ :\ \mathcal{K}_\xi(H)\times\mathcal{K}_\xi(H')\ \to\ \mathcal{K}_\xi(S^3)=\mathbb{C} \tag{3}$$

defined as follows: If $x\in\mathcal{K}_\xi(H)$ is represented by a link L in H, and $x'\in\mathcal{K}_\xi(H')$ by a link L' in H', then $\langle x,x'\rangle$ is simply given by the union $L\cup L'$ viewed as a link in S^3. Let V_k (resp. V_k') be the quotient of $\mathcal{K}_\xi(H)$ (resp. $\mathcal{K}_\xi(H')$) by the left (resp. right) kernel of this form, so that we have a *non-degenerate* pairing

$$V_k\times V_k'\ \to\ \mathbb{C}. \tag{4}$$

which we denote again by $\langle\ ,\ \rangle$. It turns out [**BHMV, Li**] that V_k (and hence also V_k') is a finite-dimensional vector space, of dimension given by the Verlinde formula $d_g(k)$. This V_k will be our definition of the vector space $V_k(\Sigma)$.

Let K be the set of Dehn twists in Γ about curves which bound a disk in the handlebody H. Such Dehn twists extend naturally (and uniquely, up to isotopy) to diffeomorphisms of H, and so act on the skein module $\mathcal{K}_\xi(H)$. It turns out that they preserve the left kernel of the form (3). Thus, we get a linear action of the group $\langle K\rangle$ (the subgroup of Γ generated by Dehn twists in the set K) on the quotient vector space V_k. Let us call this action ρ_k. We claim that it extends to a projective action of the whole mapping class group Γ.

To see this, consider the set K' of Dehn twists which extend to the other handlebody, H'. The set $K\cup K'$ generates Γ (for example, it contains the generators in Wajnryb's presentation of Γ). *A priori,* the group $\langle K'\rangle$ acts not on V_k but on V_k'. But since the form in (4) is non-degenerate, $\langle K'\rangle$ also acts linearly on V_k, *via* the adjoint: For $f'\in K'$, we define $\rho_k(f')$ by the formula

$$\langle\rho_k(f')(x),y\rangle=\langle x,(f')^{-1}(y)\rangle,$$

for all $x\in V_k, y\in V_k'$.

Thus, for every $\varphi\in\Gamma$ we have a candidate for $\rho_k(\varphi)$: just write φ as a product of Dehn twists from K or K' or their inverses, and apply ρ_k to each factor. The claim is that $\rho_k(\varphi)$ is well defined up to a scalar factor, and that this defines a projective representation of Γ. To prove this, we must show the following: If

$$w=f_1f_2f_3\dots f_n$$

is a *relator*, *i.e.* a non-trivial word in Dehn twists from $K\cup K'$ or their inverses which represents the identity element of Γ, then $\rho_k(w)$ (computed as the product of matrices $\rho_k(f_1)\cdots\rho_k(f_n)$) is a multiple of the identity matrix.

Roberts' argument to prove this goes as follows. If $s\in\mathcal{K}_\xi(\Sigma\times[0,1])$ and $x\in\mathcal{K}_\xi(H)$, we define

$$Add_s(x)=x\cup s,$$

where $x\in\mathcal{K}_\xi(H)$, and $H\cup_\Sigma(\Sigma\times[0,1])$ is identified with H in the obvious way. This defines an action of the skein algebra $\mathcal{K}_\xi(\Sigma\times[0,1])$ on the skein module $\mathcal{K}_\xi(H)$ and also on the quotient vector space V_k. Let us denote this action on V_k by

$$s\mapsto Add_s^{(k)}.$$

The key observation is that for $f \in K$, one has

$$\rho_k(f) \circ Add_s^{(k)} \circ \rho_k(f)^{-1} = Add_{f(s)}^{(k)} \tag{5}$$

(note the analogy with Eq. (1)). One can check that this also holds if $f \in K'$. So we can apply this equation to our relator $w = f_1 f_2 f_3 \dots f_n$. We find

$$\rho_k(w) \circ Add_s^{(k)} \circ \rho_k(w)^{-1} = Add_{f_1 f_2 f_3 \dots f_n(s)}^{(k)} = Add_s^{(k)},$$

since the product $f_1 f_2 f_3 \dots f_n$, viewed as a mapping class, is the identity. Thus $\rho_k(w)$ *commutes* with every endomorphism of the form $Add_s^{(k)}$. Roberts now shows that the endomorphism ring $\mathrm{End}(V_k)$ is generated by such endomorphisms. Thus $\rho_k(w)$ is central in $\mathrm{End}(V_k)$, and hence is a scalar multiple of the identity matrix. Thus ρ_k is indeed a projective representation of the mapping class group.

Note that, as for the geometric construction in the preceding section, we needed to make an additional choice to define the vector space $V_k = V_k(\Sigma)$. While it was the choice of complex structure before, it is now the choice of the handlebody H. The fact that the projective representation is independent of this choice follows from the construction of the full TQFT in [**BHMV**].

We remark in passing that it is possible to compute the scalar factors $\rho_k(w)$ for the relators w in a presentation of the mapping class group. In this way, one gets a presentation for the extended mapping class group acting linearly on $V_k(\Sigma)$ [**MR**].

4. Some properties of quantum representations

From now on, we will use the notation ρ_k^{geom} for the representation defined using Hitchin's connection and write ρ_k^{skein} for the one defined through using skein theory. It is expected that the two are the same, but it is not clear to me whether this has been proved. There certainly is no direct proof, but it might be that it can be deduced by putting together various results in the literature: By Laszlo's result [**La**] ρ_k^{geom} is equivalent to the representation coming from conformal field theory, and it is also known that ρ_k^{skein} is equivalent to the representation constructed by Reshetikhin and Turaev [**RT**] using the quantum group $U_q sl_2$. It remains to establish the connection between conformal field theory and Turaev's modular categories constructed from quantum groups; this is discussed in the book [**BK**].

Unitarity. The vector space $V_k(\Sigma)$ carries a hermitian form which is preserved by the representation ρ_k^{skein}. The form is positive definite (and so the representation is unitary) if $\xi = \exp(2\pi i/(4k+8))$ (but not for an arbitrary choice of the root of unity ξ) [**BHMV**, Th. 4.11][(2)]. In fact, the vector space $V_k(\Sigma)$ and the hermitian form on it may be defined over a cyclotomic field, and the signature of that form depends on the embedding of the field into $\mathbb{C}$. For ρ_k^{geom}, unitarity does not seem to be known.

Dehn twists. It is very easy to see that ρ_k^{skein} represents Dehn twists by matrices of finite order (with roots of unity as their eigenvalues). The same is true for ρ_k^{geom},

(2) Our V_k is denoted V_p in [**BHMV**], where $p = 2k + 4$.

using the equivalence with the conformal field theory approach. In genus two, a direct proof using Hitchin's connection was given in [**BJ**].

(In)finiteness. Although the mapping class group Γ is generated by Dehn twists, the preceding statement does *not* imply that its image $\rho_k(\Gamma)$ under the representation is a finite group.(3) Nevertheless, in genus one, Gilmer [**G1**] showed that the image $\rho_k^{skein}(\Gamma_1)$ is finite. But in genus two and higher, it is easy to write down a mapping class f (a product of two Dehn twists) such that $\rho_k^{skein}(f)$ has infinite order except for a few low values of k [**M**].

Integral structure. As already mentioned, the representation ρ_k^{skein} can be defined over a cyclotomic field containing the root of unity ξ. The hermitian form preserved by it is also defined over that field. For the $SO(3)$-variant of the theory, and at roots of unity of prime order, it is known that the mapping class group representation preserves a full lattice defined over the ring of integers in the cyclotomic field [**G2**]. Explicit bases for the lattice are known so far in genus one and two only [**GMW**]. I don't know how to define a similar integral structure from the geometric approach. There is an analogy here with the representation of the mapping class group on the homology of the surface, where the image is the group of automorphisms of the *integral* homology which preserve the intersection form.

Heisenberg group. Consider ρ_k^{skein} as a linear representation of the extended mapping class group $\widetilde{\Gamma}$ as in [**BHMV**]. For every Dehn twist t_γ, one has that

$$\tau_\gamma = \rho_k^{skein}(t_\gamma)^{2k+4}$$

is an involution on $V_k(\Sigma)$, *i.e.* one has $\tau_\gamma^2 = 1$. The group $\mathcal{E}_k(\Sigma)$ generated by these involutions depends only on the value of k mod 4. If $k \equiv 0$ mod 4, the involution τ_γ depends only on the mod 2 homology class of γ, and $\mathcal{E}_k(\Sigma)$ is isomorphic to $H_1(\Sigma;\mathbb{Z}/2)$. Otherwise, depending on k mod 4, the group $\mathcal{E}_k(\Sigma)$ is a central extension of $H_1(\Sigma;\mathbb{Z}/2)$ by $\mu_2 = \{\pm 1\}$ or by μ_4 (the group of fourth roots of unity) which may in both cases be described as a finite Heisenberg group associated to the intersection form on the first homology. (See Section 7 of [**BHMV**] for details.) These involutions play a *rôle* in further developments such as Spin refined TQFT's [**BM**] *etc.* Similar involutions were defined in [**AM**] for ρ_k^{geom}, as follows. Assume Σ is equipped with a complex structure σ. The mod 2 homology of Σ is isomorphic to the group $J^{(2)}$ of 2-torsion points on the Jacobian of Σ. This group acts on the moduli space M'_σ of rank two bundles with trivial determinant by tensoring. One can show that this lifts to an action of the group $\mathcal{E}_k(\Sigma)$ on the line bundle $\mathcal{L}_\sigma^{\otimes k}$ and hence on its space of sections $Z_k(\Sigma,\sigma)$. Here, one uses 4-torsion points to specify elements of the extension $\mathcal{E}_k(\Sigma)$, and this group is now presented in terms of the Weil pairing (which is the algebro-geometric analog of the intersection form). The result is that $V_k(\Sigma)$ and $Z_k(\Sigma,\sigma)$ are canonically isomorphic as representations of the finite Heisenberg group $\mathcal{E}_k(\Sigma)$. I believe this confirms that there should be a natural isomorphism between ρ_k^{geom} and ρ_k^{skein}.

(3)Not even in genus one: for example, the quotient of $PSL(2,\mathbb{Z})$ by the Nth power of a Dehn twist is a triangle group $\Gamma(2,3,N) = \langle a,b | a^2 = b^3 = (ab)^N = 1\rangle$ which is infinite for $N \geq 7$.

5. Asymptotic Faithfulness

This is for me the most striking result about quantum representations of mapping class groups. The statement is as follows:

(AF) *Let $f \in \Gamma_g$ be in the kernel of all the representations ρ_k,* i.e. *such that $\rho_k(f) \in \mathbb{C}\,\mathrm{Id}$ for all k. Then f is the identity mapping class, except in genus one or two where f may also be the hyperelliptic involution.*

This result was discovered by Andersen, who proved it for ρ_k^{geom} [**A2**] using Toeplitz operators. (His argument works in the higher rank case as well, and then the hyperelliptic involution is also detected by the representations.) The proof seems to be rather difficult, and I should say that I am unable to check it completely. The main strategy is, however, rather simple to describe, as we will see below. Prompted by Andersen's announcement, Freedman, Walker, and Wang [**FWW**] then proved asymptotic faithfulness for ρ_k^{skein}. They also point out that this gives another proof that mapping class groups are residually finite.

The proofs in [**A2**] and [**FWW**] are logically independent, of course, since we do not know that the two representations are equivalent. However, the main steps of both proofs are parallel, as we will see.

Andersen's idea was to consider endomorphisms of $Z_k(\Sigma, \sigma) = H^0(M'_\sigma, \mathcal{L}_\sigma^{\otimes k})$ given by Toeplitz operators. They are defined as follows. Fix a complex structure σ on Σ. A vector ψ in $Z_k(\Sigma, \sigma)$ is a (holomorphic) section of the line bundle $\mathcal{L}_\sigma^{\otimes k}$. If h is a C^∞ function on M' with compact support, the product $h\psi$ is only a C^∞ section. But there is a natural orthogonal projection π_σ from $L^2(M'_\sigma, \mathcal{L}_\sigma^{\otimes k})$ back to $H^0(M'_\sigma, \mathcal{L}_\sigma^{\otimes k})$, and the Toeplitz operator $T_{h,\sigma}^{(k)}$ is simply the composite map. In other words, one has

$$T_{h,\sigma}^{(k)}(\psi) = \pi_\sigma(h\psi).$$

The function h might be a holonomy function h_γ along some simple closed curve γ on the surface. As explained above, the skein algebra $\mathcal{K}_\xi(\Sigma \times [0,1])$ is in some sense a deformation of the commutative algebra of (holonomy) functions on the moduli space M'. Thus, it should not come as a surprise that the skein-theoretical analog of the Toeplitz operators $T_{h,\sigma}^{(k)}$ are the endomorphisms $Add_s^{(k)}$ of $V_k(\Sigma)$, where $s \in \mathcal{K}_\xi(\Sigma \times [0,1])$.

Let us first describe the skein-theoretical proof of (AF). It is in three steps. The first step is very easy to formulate in terms of the construction of ρ_k^{skein} we have given above. Observe that

$$\rho_k^{skein}(f) \circ Add_s^{(k)} \circ \rho_k^{skein}(f)^{-1} = Add_{f(s)}^{(k)} \tag{6}$$

for every mapping class f. Indeed, this holds for Dehn twists in the generating set $K \cup K'$ (see Eq. (5)), hence for every $f \in \Gamma$. Thus, if f satisfies the hypothesis of (AF), *i.e.* if $\rho_k^{skein}(f) \in \mathbb{C}\,\mathrm{Id}$ for all k, then

$$Add_s^{(k)} = Add_{f(s)}^{(k)} \tag{7}$$

for every skein element s in $\Sigma \times [0,1]$. (By a skein element in a 3-manifold, one means an element of the skein module of that manifold.)

The main work is done in the second step. Recall that the vector space $V_k(\Sigma)$ was constructed as a quotient of the skein module of the 3-dimensional handlebody H. Thus it contains a vector $v_\varnothing^{(k)} \in V_k(\Sigma)$ (the "ground state") represented by the empty link in the handlebody H. If the skein element s is given by a simple closed curve γ on the surface, we denote the vector $Add_s^{(k)}(v_\varnothing^{(k)})$ by $v_\gamma^{(k)}$. It is represented by the curve γ pushed slightly into the interior of H. Thus, Eq. (7) implies that

$$v_\gamma^{(k)} = v_{f(\gamma)}^{(k)}$$

for every simple closed curve γ. The claim now is that for k big enough, this is only possible if the (unoriented) curves γ and $\gamma' = f(\gamma)$ are isotopic. This follows from the standard description of a basis of the vector space $V_k(\Sigma)$ in terms of decompositions of the surface into pairs of pants, together with a topological lemma(4) asserting the existence of a particular pants decomposition having special properties with respect to a given pair of non-isotopic curves γ and γ'. See [**FWW**] for the details of this.

Finally, the third step is to recall (see Section 1) that if a mapping class f leaves invariant every isotopy class of unoriented simple close curves, then f must be the identity, except for the hyperelliptic involution in genus one and two. This completes the (sketch of) the proof of the asymptotic faithfulness of ρ_k^{skein}.

In Andersen's original proof, Step 1 is more difficult to establish (which is why we have preferred to give the skein-theoretical approach first). Recall that we consider the endomorphism of $Z_k(\Sigma, \sigma) = H^0(M'_\sigma, \mathcal{L}_\sigma^{\otimes k})$ given by the Toeplitz operator $T_h^{(k)}$ associated to the function h on M'. We may consider $T_h^{(k)}$ as a section of the endomorphism bundle $\operatorname{End}(\mathcal{Z}_k)$ over Teichmüller space. Andersen's key theorem is that this section is asymptotically flat with respect to the flat connection induced from the flat connection in the projective bundle $\mathbb{P}(\mathcal{Z}_k)$. More precisely, if we let $P_{\sigma_0,\sigma_1}^{\mathrm{End}}$ denote parallel transport in the endomorphism bundle, then

$$\|P_{\sigma_0,\sigma_1}^{\mathrm{End}} \circ T_{h,\sigma_0}^{(k)} - T_{h,\sigma_1}^{(k)}\| = O(k^{-1}), \tag{8}$$

where $\|\cdot\|$ is the operator norm on $H^0(M'_\sigma, \mathcal{L}_\sigma^{\otimes k})$.

The representation ρ_k^{geom} was defined by $\rho_k^{geom}(f) = P_{f^*(\sigma),\sigma} \circ f^*$ where f^* is the natural map from the fiber of $\mathcal{Z}_k$ over σ to the fiber over $f^*(\sigma)$, and $P_{f^*(\sigma),\sigma}$ is parallel transport back to the fiber over σ. We have

$$\rho_k^{geom}(f) \circ T_{h,\sigma}^{(k)} \circ \rho_k^{geom}(f)^{-1} = P_{f^*(\sigma),\sigma}^{\mathrm{End}} \circ T_{h\circ f, f^*(\sigma)}^{(k)}. \tag{9}$$

(This corresponds to Eq. (6) in the skein-theoretical approach.)

Thus, if f satisfies the hypothesis of (AF), *i.e.* if $\rho_k^{geom}(f) \in \mathbb{C}\,\mathrm{Id}$ for all k, then $T_{h,\sigma}^{(k)}$ is equal to $P_{f^*(\sigma),\sigma}^{\mathrm{End}} \circ T_{h\circ f, f^*(\sigma)}^{(k)}$. By the asymptotic flatness in Eq. (8), we deduce that

$$\|T_{h-h\circ f,\sigma}^{(k)}\| = \|T_{h,\sigma}^{(k)} - T_{h\circ f,\sigma}^{(k)}\| = O(k^{-1}).$$

(This corresponds to Eq. (7) in the skein-theoretical approach, but notice that now the conclusion is weaker, as we only have an asymptotic statement here.)

(4)The proof of this lemma [**FWW**, Lemma 4.1] as given has a gap but the lemma is true and can be proved by considering an additional case.

Andersen's second step is to deduce from this that

$$h - h \circ f = 0.$$

This follows from a theorem of Bordemann, Meinrenken and Schlichenmaier [**BMS**] which says that the operator norm of Toeplitz operators associated to a C^∞ function converges for $k \to \infty$ to the sup norm of that function. (Actually, the theorem of [**BMS**] is for compact Kaehler manifolds, so one needs to adapt it to the current setting. The moduli space M'_σ is non-compact but has a natural (but singular) compactification M_σ of M'_σ given by semi-stable bundles, and Andersen originally used a desingularization of this compactification. Recently, he and Christ have announced a generalization of [**BMS**] which allows one to argue directly on the singular variety M_σ itself.) Thus, f acts by the identity on $C^\infty_c(M')$, and therefore also on the moduli space M' itself.

The third step is to show that if f acts trivially on the space M' of irreducible $SU(2)$-representations, then f must be the identity or the hyperelliptic involution in genus one or two. This seems to be well-known; a proof is also given in [**A2**].

References

[A1] J. E. Andersen. The Witten invariant of finite order mapping tori I. *J. Reine Angew. Math.* (to appear)

[A2] J. E. Andersen. Asymptotic Faithfulness of the quantum $SU(n)$ representations of the mapping class groups. arXiv:math.QA/0204084

[AM] J. E. Andersen, G. Masbaum. Involutions on moduli spaces and refinements of the Verlinde formula. *Math. Ann.* **314** (1999) 2, 291-326.

[ADW] S. Axelrod, S. Della Pietra, E. Witten. Geometric quantization of Chern Simons gauge theory. *J. Diff. Geom.* **33** (1991) 787–902.

[BK] B. Bakalov, A. Kirillov. Lectures on Tensor Categories and Modular Functors. AMS University Lecture Series, 2001.

[BM] C. Blanchet, G. Masbaum. Topological quantum field theories for surfaces with spin structure. *Duke Math. J.* **82** (1996) 229-267.

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology* **34** (1995), 883-927

[BMS] M. Bordeman, E. Meinrenken, M. Schlichenmaier. Toeplitz quantization of Kähler manifolds and $gl(N), N \to \infty$ limit. *Comm. Math. Phys.* **165** (1994), 281–296.

[Bu] D. Bullock. Rings of $\mathrm{SL}_2(C)$-characters and the Kauffman bracket skein module. *Comment. Math. Helv.* 72 (1997), no. 4, 521–542.

[DN] J.-M. Drézet, M.S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. math.* **97** (1989) 53–94.

[BJ] B. van Geemen, A. de Jong. On Hitchin's connection. *J. Amer. Math. Soc.* 11 (1998), no. 1, 189–228.

[FWW] M. Freedman, K. Walker, Z. Wang. Quantum SU(2) faithfully detects mapping class groups modulo center. *Geom. Topol.* 6 (2002) 523-539.

[G1] P. M. Gilmer. On the Witten-Reshetikhin-Turaev representations of mapping class groups. *Proc. AMS* **127** (1999) 2483-2488.

[G2] P. M. Gilmer. Integrality for TQFTs, `arXiv:math.QA/0105059`

[GMW] P. M. Gilmer, G. Masbaum, P. van Wamelen. Integral bases for TQFT modules and unimodular representations of mapping class groups. `arXiv:math.QA/0207093`

[Gr] E. Grossman. On the residually finiteness of certain mapping class groups. *J. London Math. Soc.* V. 9, No. 1 (1974) 160-164.

[H] N. Hitchin. Flat connections and geometric quantization. *Comm. Math. Phys.* **131** (1990) 347–380.

[Iv] N. Ivanov. Mapping class groups. In: Handbook of Geometric Topology. Ed. by R. Daverman and R. Sher. Elsevier 2001, 523-633.

[J] D. Johnson. A survey of the Torelli group. *Contemp. Math.* **20** (1983) 165-179.

[Ko] T. Kohno. Topological invariants for 3-manifolds using representations of mapping class groups I. *Topology* 31 (1992), no. 2, 203–230.

[La] Y. Laszlo. Hitchin's and WZW connections are the same. *J. Diff. Geom.* **49** (1998), no. 3, 547–576.

[Li] W. B. R. Lickorish. Skeins and handlebodies. *Pacific J. Math.* **159** (1993) 337-350.

[M] G. Masbaum. An element of infinite order in TQFT-representations of mapping class groups. *Contemp. Math.* **233** (1999) 137–139.

[MR] G. Masbaum, J. D. Roberts. On central extensions of mapping class groups. *Math. Ann.* **302**, 131-150 (1995)

[PS] J. H. Przytycki, A. S. Sikora. On skein algebras and $\mathrm{Sl}_2(C)$-character varieties. *Topology* 39 (2000), no. 1, 115–148.

[Ro] J. Roberts. Skeins and mapping class groups. *Math. Proc. Camb. Phil. Soc.* 115 (1994) 53-77.

[RT] N. Reshetikhin, V. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* **103** (1991), 547-597.

[TUY] A. Tsuchiya, K. Ueno, Y. Yamada. Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries. *Advanced Studies in Pure Mathematics*, **19** (1989), 459–566.

[S] C. Sorger. La formule de Verlinde. Sém. Bourbaki n^0 794. *Astérisque* **237** (1996), 87–114.

[T] V. G. Turaev. Quantum invariants of knots and 3-manifolds. de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.

[W] E. Witten. Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **121** (1989) 351–98.

Gregor Masbaum, Institut de Mathématiques de Jussieu (UMR 7586 du CNRS), Université Paris 7, Case 7012, 75251 Paris Cedex 05, FRANCE • *E-mail :* `masbaum@math.jussieu.fr`