Hilbert-Samuel formula and equidistribution theorem over adelic curves

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CHAPTER 1

Introduction

In algebraic geometry, Hilbert function measures the growth of graded linear series of a line bundle on a projective variety. Let k be a field, X be an integral projective scheme of dimension $d \in \mathbb{N}$ $(= \mathbb{Z}_{\geqslant 0})$ over $\operatorname{Spec} k$, and L be an invertible O_X-module. The Hilbert function of L is defined as

$$H_L : \mathbb{N} \longrightarrow \mathbb{N}, \quad H_L(n) := \dim_k(H^0(X, L^{\otimes n})).$$

If L is ample, then the following asymptotic estimate holds:

$$H_L(n) = \frac{(L^d)}{d!} n^d + o(n^d). \tag{1.1}$$

This formula, which relates the asymptotic behaviour of the Hilbert function and the auto-intersection number of L, is for example a consequence of Hirzebruch-Riemann-Roch theorem and Serre's vanishing theorem. It turns out that the construction and the asymptotic estimate of Hilbert function have analogue in various contexts, such as graded algebra, local multiplicity, relative volume of two metrics, etc.

In Arakelov geometry, an arithmetic analogue of Hilbert function has been introduced by Gillet and Soulé [**48**] and an analogue of the asymptotic formula (1.1) has been deduced from their arithmetic Riemann-Roch theorem. This result is called an arithmetic Hilbert-Samuel theorem. Let $\mathscr{X}$ be a regular integral projective scheme of dimension $d+1$ over $\operatorname{Spec} \mathbb{Z}$, and $\overline{\mathscr{L}} = (\mathscr{L}, \varphi)$ be a Hermitian line bundle on $\mathscr{X}$, namely an invertible $O_{\mathscr{X}}$-module $\mathscr{L}$ equipped with a smooth metric φ on $\mathscr{L}(\mathbb{C})$. For any integer $n \in \mathbb{N}$, we let $\|\cdot\|_{n\varphi}$ be the norm on the real vector space $H^0(\mathscr{X}, \mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R}$ defined as follows

$$\forall\, s \in H^0(\mathscr{X}, \mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R} \subseteq H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}}^{\otimes n}), \quad \|s\|_{n\varphi} = \sup_{x \in \mathscr{X}(\mathbb{C})} |s|_{n\varphi}(x).$$

Then the couple $(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})$ forms a lattice in a normed vector space. Recall that its arithmetic Euler-Poincaré characteristic is

$$\chi(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi}) = \ln \frac{\operatorname{vol}(\{s \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n}) \otimes_{\mathbb{Z}} \mathbb{R} \,:\, \|s\|_{n\varphi} \leqslant 1\})}{\operatorname{covol}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})}$$

where $\operatorname{vol}(\cdot)$ denotes a Haar measure on the real vector space

$$H^0(\mathscr{X}, \mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R},$$

and $\operatorname{covol}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})$ is the covolume of the lattice $H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$ with respect to the Haar measure $\operatorname{vol}(\cdot)$, namely the volume of any fundamental domain of this lattice. In this setting the arithmetic Hilbert-Samuel theorem shows that, if $\mathscr{L}$ is relatively ample and the metric φ is positive, then the sequence

$$\frac{\chi(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}), \|\cdot\|_{n\varphi})}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1$$

converges to the arithmetic intersection number $(\overline{\mathscr{L}}^{d+1})$. In the case where $\overline{\mathscr{L}}$ is ample, the arithmetic Hilbert-Samuel theorem also permits to relate the asymptotic behaviour (when $n \rightarrow +\infty$) of

$$\operatorname{card}(\{s \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n}) \;:\; \|s\|_{n\varphi} \leqslant 1\})$$

to the arithmetic intersection number of $\overline{\mathscr{L}}$. These results have various applications in arithmetic geometry, such as Vojta's proof of Mordell conjecture, equidistribution problem and Bogomolov conjecture, etc. The arithmetic Hilbert-Samuel theorem has then been reproved in various settings and also been generalized in works such as [**1, 39, 63**].

Recently, a new framework of Arakelov geometry has been proposed in [**36**], which allows to consider arithmetic geometry over any countable field. Let K be a field. A structure of proper adelic curve with underlying field K is given by a family of absolute values $(|\cdot|_\omega)_{\omega\in\Omega}$ of K parametrized by a measure space $(\Omega, \mathcal{A}, \nu)$, which satisfies a product formula of the form

$$\forall\, a \in K^{\times}, \quad \int_\Omega \ln|a|_\omega\, \nu(\mathrm{d}\omega) = 0.$$

We assume that, either K is countable, or the σ-algebra $\mathcal{A}$ is discrete. This notion is a very natural generalization to any countable field of Weil's adelic approche of number theory. The fundament of height theory and Arakelov geometry for projective varieties over an adelic curve have been established in the works of Gubler [**49**] (in a slightly different setting of M-fields) and Chen-Moriwaki [**36**], respectively, see also the model theoretical approach of Ben Yaacov and Hrushovski [**52**]. More recently, the arithmetic intersection theory in the setting of adelic curves have been developed in [**38**]. Note that in general it is not possible to consider global integral models of an adelic curve. Several classic notions and constructions, such as integral lattice and its covolume, do not have adequate analogue over adelic curves. It turns out that a modified and generalized form of normed lattice — adelic vector bundle — has a natural avatar in the setting of adelic curves. An adelic vector bundle consists of a finite-dimensional vector space V over K equipped with a family of norms $(\|\cdot\|_\omega)_{\omega\in\Omega}$ on vector spaces $V_\omega = V \otimes_K K_\omega$ (where K_ω denotes the completion of K with respect to the absolute value $|\cdot|_\omega$), which satisfy dominancy and measurability conditions. The Arakelov degree of the adelic vector bundle

$$\overline{V} = (V, (\|\cdot\|_\omega)_{\omega\in\Omega})$$

is then defined as

$$\widehat{\deg}(\overline{V}) := -\int_\Omega \ln\|s_1 \wedge \cdots \wedge s_r\|_{\omega,\det}\, \nu(\mathrm{d}\omega),$$

where $(s_i)_{i=1}^r$ is an arbitrary basis of E over K. This notion is a good candidate to replace the Euler-Poincaré characteristic.

Let $\pi : X \rightarrow \operatorname{Spec} K$ be a projective scheme over $\operatorname{Spec} K$. For any $\omega \in \Omega$, let $X_\omega = X \times_{\operatorname{Spec} K} \operatorname{Spec} K_\omega$ and let X_ω^{an} be the analytic variety associated with X_ω (in the sense of Berkovich [**9**] if $|\cdot|_\omega$ is non-Archimedean). If E is a vector bundle on X, namely a locally free $\mathcal{O}_X$-module of finite rank, we denote by E_ω the pull-back of E on X_ω. As *adelic vector bundle* on X, we refer to the data $\overline{E} = (E, (\psi_\omega)_{\omega\in\Omega})$ consisting of a vector bundle E on X and a family $(\psi_\omega)_{\omega\in\Omega}$ of continuous metrics on E_ω with $\omega \in \Omega$, which satisfy dominancy and measurability conditions. It turns out that, if X is geometrically reduced, then the vector space of global sections $H^0(X, E)$ equipped with supremum norms $(\|\cdot\|_{\psi_\omega})_{\omega\in\Omega}$ forms an adelic vector bundle $\pi_*(\overline{E})$ on the base adelic curve.

Let $\pi : X \rightarrow \operatorname{Spec} K$ be an integral projective scheme of dimension d over $\operatorname{Spec} K$ and $\overline{L} = (L, \varphi)$ be an adelic line bundle on X, that is, an adelic vector bundle of rank 1 on X.

Assume that the line bundle L is ample. We introduce the notion of χ-volume as

$$\widehat{\mathrm{vol}}_\chi(\overline{L}) = \limsup_{n\to+\infty} \frac{\widehat{\deg}(\pi_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}.$$

In view of the similarity between Arakelov degree and Euler-Poincaré characteristic of Euclidean lattices, the notion of χ-volume is analogous to that of sectional capacity introduced in [**64**], or to that of volume in [**74**]. Moreover, similarly to the number field case, we show in Theorem-Definition 4.2.1 that the above superior limit defining the χ-volume is actually a limit. However, from the methodological view, we do not follow the classic approaches, which are difficultly implantable in the adelic curve setting. Our strategy consists in casting the Arakelov geometry over an adelic curve to that in the particular case where the adelic curve contains a single copy of the trivial absolute value on K, that is, the absolute value $|\cdot|_0$ such that $|a|_0 = 1$ for any $a \in K \setminus \{0\}$. More precisely, to each adelic vector bundle $\overline{V} = (V, (\|\cdot\|_\omega)_{\omega\in\Omega})$, we associate an ultrametric norm $\|\cdot\|_0$ on V (where we consider the trivial absolute value $|\cdot|_0$) via Harder-Narasimhan theory in the form of $\mathbb{R}$-filtrations, such that

$$\Big|\widehat{\deg}(V, (\|\cdot\|_\omega)_{\omega\in\Omega}) - \widehat{\deg}(V, \|\cdot\|_0)\Big| \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(V)\ln(\dim_K(V)),$$

where Ω_∞ denotes the set of $\omega \in \Omega$ such that $|\cdot|_\omega$ is Archimedean. Then the convergence of the sequence defining $\widehat{\mathrm{vol}}_\chi(\overline{L})$ follows from a limit theorem of normed graded linear series as follows (see Theorem 3.4.3 and Corollary 3.4.4 for this result in a more general form and for more details):

Theorem A. *Assume that the graded K-algebra $\bigoplus_{n\in\mathbb{N}} H^0(X, L^{\otimes n})$ is of finite type. For any integer $n \geqslant 1$, let $\|\cdot\|_n$ be a norm on $H^0(X, L^{\otimes n})$ (where we consider the trivial absolute value on K). Assume that*

(a) $\displaystyle\inf_{s\in V_n\setminus\{0\}} \ln\|s\|_n = O(n)$ *when* $n \to +\infty$,

(b) *for any* $(n,m) \in \mathbb{N}^2_{\geqslant 1}$ *and any* $(s_n, s_m) \in V_n \times V_m$, *one has*

$$\|s_n \cdot s_m\|_{n+m} \leqslant \|s_n\|_n \cdot \|s_m\|_m.$$

Then the sequence

$$\frac{\widehat{\deg}(V_n, \|\cdot\|_n)}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N}_{\geqslant 1}$$

converges in $\mathbb{R}$.

In view of the classic Hilbert-Samuel theorems in algebraic geometry and in Arakelov geometry, it is natural to compare the χ-volume to the arithmetic intersection number of adelic line bundles that we have introduced in [**38**] (see also the work [**49**] on heights of varieties over M-fields under the assumption of integrability of local heights). Let $\pi : X \to \operatorname{Spec} K$ be a projective scheme of dimension $d \geqslant 0$ over K and $\overline{L} = (L, \varphi)$ be an adelic line bundle on X such that L is ample and the metrics in the family φ are semi-positive. Then the arithmetic self-intersection number $(\overline{L}^{d+1})$ of $\overline{L}$ is written in a recursive way as

$$\frac{1}{N}\bigg[(\overline{L}|^d_{\mathrm{div}(s)})_S - \int_\Omega \int_{X^{\mathrm{an}}_\omega} \ln|s|_{\varphi_\omega}(x)\, c_1(L_\omega, \varphi_\omega)^d(\mathrm{d}x)\,\nu(\mathrm{d}\omega)\bigg], \tag{1.2}$$

where N is a positive integer, and s is a global section of $L^{\otimes N}$ which intersects properly with all irreducible components of the projective scheme X. One of the main results of the article is then the following theorem (see Theorem 5.5.1).

THEOREM B. *Assume that, either X is geometrically integral, or the field K is perfect. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X such that L is ample and that all metrics in the family φ are semi-positive, then the following equality holds:*

$$\widehat{\mathrm{vol}}_\chi(\overline{L}) = (\overline{L}^{d+1}).$$

Note that, in the literature there exists a local version of the Hilbert-Samuel theorem which establishes an equality between the relative volume of two metrics and the relative Monge-Ampère energy between them. We refer the readers to [**10**] for the Archimedean case and to [**20, 17**] for the non-Archimedean case (see also [**18**]). These results show that, for a fixed ample line bundle L on X, the difference between $\widehat{\mathrm{vol}}_\chi(\overline{L})$ and $(\overline{L}^{d+1})$ does not depend on the choice of the metric family on L (see Proposition 5.1.4 and Remark 5.1.6). Moreover, by an argument of projection to a projective space (on which the arithmetic Hilbert-Samuel theorem can be proved by explicit computation, see Proposition 5.2.5), one can show that the inequality $\widehat{\mathrm{vol}}_\chi(\overline{L}) \geqslant (\overline{L}^{d+1})$ holds (see Step 2 of the proof of Theorem 5.5.1).

In view of the recursive formula (1.2) defining the self-intersection number, a natural idea to prove the above theorem could be an argument of induction, following the approach of [**1**] by using an adaptation to non-Archimedean setting of some technics of complex analytic geometry developed in [**17, 44**]. However, it seems that a refinement in the form of an asymptotic development of the function defining the local relative volume is needed to realize this strategy. Unfortunately such refinement is not yet available. Our approach consists in casting the arithmetic data of $\overline{L}$ to a series of metrics over a trivially valued field. This could be considered as a higher-dimensional generalization of the approach of Harder-Narasimhan $\mathbb{R}$-filtration mentioned above. What is particular in the trivial valuation case is that the local geometry becomes automatically global, thanks to the trivial "product formula". In this case, the arithmetic Hilbert-Samuel theorem follows from the equality between the relative volume and the relative Monge-Ampère energy with respect to the trivial metric (see Theorem 5.3.2). Note that this result also shows that, in the case of a projective curve over a trivially valued field, the arithmetic intersection number defined in [**38**] coincides with that constructed in a combinatoric way in [**37**] (see Remark 5.3.3). The comparison of diverse invariants of $\overline{L}$ with respect to those of its casting to the trivial valuation case provides the opposite inequality $\widehat{\mathrm{vol}}_\chi(\overline{L}) \leqslant (\overline{L}^{d+1})$. As a sequel to the above arguments in terms of trivially valued fields, our way towards the arithmetic Hilbert-Samuel theorem over an adelic curve gives a new approach even for the classical case.

As an application, we prove the following higher dimensional generalization of Hodge index theorem (see Corollaries 6.5.1 and 6.5.2).

THEOREM C. *Assume that, either X is geometrically integral, or the field K is perfect. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. Assume that L is nef and all metrics in the family φ are semi-positive, then the inequality $\widehat{\mathrm{vol}}(\overline{L}) \geqslant (\overline{L}^{d+1})$ holds. In particular, if $(\overline{L}^{d+1}) > 0$, then the line bundle L is big.*

Theorem B naturally leads to the following refinement of the arithmetic Hilbert-Samuel theorem, in introducing a tensor product by an adelic vector bundle on X (see Corollary 5.5.2). As in Theorem B, we assume that, either X is geometrically integral, or the field K is perfect.

THEOREM D. *Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X and $\overline{E} = (E, \psi)$ be an adelic vector bundle on X. Assume that L is ample and the metrics in φ are semi-positive.*

Moreover we suppose that either $\mathrm{rk}(E) = 1$ *or* X *is normal. Then one has*

$$\lim_{n\to+\infty} \frac{\widehat{\deg}\left(H^0(X, L^{\otimes n}\otimes E), (\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega}\right)}{n^{d+1}/(d+1)!} = \mathrm{rk}(E)(\overline{L}^{d+1}).$$

The second part of the article is devoted to the study of positivity conditions of adelic line bundles. Positivity of line bundles is one of the most fundamental and important notions in algebraic geometry. In Arakelov geometry, the analogue of ampleness and Nakai-Moishezon criterion have been studied by Zhang [**80, 81**]. The arithmetic bigness has been introduced in the works [**60, 74, 61**] of Moriwaki and Yuan. These positivity conditions and their properties have various applications in Diophantine geometry.

We assume that the underlying field K of the adelic curve S is perfect. Let X be a projective scheme over $\operatorname{Spec} K$. Given an adelic line bundle $\overline{L}$ on X, we are interested in various positivity conditions of the adelic line bundle $\overline{L}$. We say that the adelic line bundle $\overline{L}$ is *relatively ample* if the invertible $\mathcal{O}_X$-module L is ample and if the metrics of $\overline{L}$ are all semi-positive. The relative nefness can then been defined in a limit form of relative ampleness, similarly to the classic case in algebraic geometry. Recall that the global intersection number of relatively ample adelic line bundles (or more generally, integrable adelic line bundles) can be defined as the integral of local heights along the measure space in the adelic structure (cf. [**49, 38**]). This construction is fundamental in the Arakelov height theory of projective varieties.

We first introduce a numerical invariant — asymptotic minimal slope — to describe the global positivity of an adelic line bundle $\overline{L}$ such that L is ample. This invariant, which is denoted by $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$, describes the asymptotic behaviour (when $n \to +\infty$) of the minimal slopes of the sectional spaces $H^0(X, L^{\otimes n})$ equipped with sup norms (which are adelic vector bundles on S). It turns out that this invariant is super-additive with respect to $\overline{L}$. This convexity property allows to extend the construction of the asymptotic minimal slope to the cone of adelic line bundles with nef underlying invertible $\mathcal{O}_X$-module (see §6.2 for the construction of the asymptotic minimal slope and its properties). The importance of this invariant can be shown by the following height estimate (see Theorem 6.3.2 for the proof and Proposition 6.4.8 for its generalization to the relatively nef case).

THEOREM E. *Assume that the field* K *is perfect. Let* X *be a reduced projective scheme of dimension* $d \geqslant 0$ *over* $\operatorname{Spec} K$*, and* $\overline{L}_0, \ldots, \overline{L}_d$ *be a family of relatively ample adelic line bundles on* X*. For any* $i \in \{0, \ldots, d\}$*, let* δ_i *be the geometric intersection number*

$$(L_0 \cdots L_{i-1} L_{i+1} \cdots L_d).$$

Then the following inequality holds:

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \sum_{i=1}^{d} \delta_i\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i),$$

where $(\overline{L}_0 \cdots \overline{L}_d)_S$ *denotes the arithmetic intersection number of* $\overline{L}_0, \ldots, \overline{L}_d$*.*

The asymptotic minimal slope always increases if one replaces the adelic line bundle by its pullback by a projective morphism (see Theorem 6.6.6): if $g : X \to P$ is a projective morphism of reduced K-schemes of dimension $\geqslant 0$, then for any adelic line bundle $\overline{M}$ on P such that M is nef, one has $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{M})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M})$. Typical situations include a closed embedding of X into a projective space, or a finite covering over a projective space, which allow to obtain lower bounds of $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i)$ in the applications of the above theorem. Note that the particular case where $\overline{L}_0, \ldots, \overline{L}_d$ are all equal to the same adelic line bundle $\overline{L}$ gives the following inequality

$$\frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)} \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}), \tag{1.3}$$

which relates the normalized height of X with respect to $\overline{L}$ and the asymptotic minimal slope of the latter. This inequality is similar to the first part of [**81**, Theorem 5.2]. However, the imitation of the devissage argument using the intersection of hypersurfaces defined by small sections would not work in the setting of adelic curves. This is mainly due to the fact that the analogue of Minkowski's first theorem fails for adelic vector bundles on a general adelic curve. Although (in the case where X is an integral scheme) the inequality (1.3) could be obtained in an alternative way by using the arithmetic Hilbert-Samuel formula of $\overline{L}$ together with the fact that the minimal slope of an adelic vector bundle on S is always bounded from the above by its slope (see Proposition 6.7.1), the proof of Theorem E needs a new idea. Our approach consists in combining an analogue of the slope theory of Bost [**14, 15**] with the height of multi-resultant.

The relative positivity and the Hilbert-Samuel formula have natural applications in equidistribution. In Arakelov geometry, equidistribution of small algebraic points in an arithmetic projective variety has firstly been studied in the work [**66**] of Szipro, Ullmo and Zhang (see also the Bourbaki seminar review [**2**] of Abbes), which has a fundamental importance in the resolution of Bogomolov's conjecture [**67, 79**] by Arakelov geometry method (see [**40**] for another approach to the conjecture using Diophantine geometry). Let us remind the statement of the arithmetic equidistribution theorem in its classic form. Let A be an abelian variety over a number field, $\overline{L}$ be a symmetric ample line bundle L equipped with a positive adelic metric φ such that the Arakelov height function with respect to $\overline{L}$ coincides with the Néron-Tate height. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of algebraic points of A such that the Néron-Tate height of x_n converges to 0 (we say that such a sequence is *small*). Then the Zariski closure X of $(x_n)_{n\in\mathbb{N}}$ is the translation of an abelian subvariety of A by a torsion point. Moreover, if in addition any subsequence of $(x_n)_{n\in\mathbb{N}}$ is Zariski closed in X, then, for any Archimedean place σ of the number field, the Borel measure $\delta_{x_n,\sigma}$ on $X_\sigma(\mathbb{C})$ of taking the average on the Galois orbit of x_n converges weakly to the Monge-Ampère measure $c_1(L_\sigma, \varphi_\sigma)^{\dim(X)}$ on $X_\sigma(\mathbb{C})$. This equidistribution theorem has then been generalized in various contexts. We refer the readers to [**60**] for the case where the base field is a finitely generated extension of $\mathbb{Q}$, to [**25, 55**] for the case of a semi-abelian variety, to [**5, 4**] for equidistribution of a small sequence of sub-varieties, to [**6, 45, 7**] for the case of a dynamical system on a projective line, and to [**26**] for an equidistribution theorem of a small sequence of algebraic points in the analytic variety over a non-Archimedean place. We also refer to [**51, 43**] for similar results over function fields. In [**74**], an arithmetic analogue of Siu's inequality has been proved, which leads to an equidistribution theorem with a weaker condition on the metrics of the adelic line bundle.

We revisit the equidistribution of a small sequence of subvarieties in the setting of Arakelov geometry over an adelic curve. Assume that the underlying field is countable and perfect. Let X be an integral projective scheme over $\operatorname{Spec} K$ and d be the dimension of X. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X, namely an invertible $\mathcal{O}_X$-module L together with a family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ of metrics on L_ω satisfying dominancy and measurability conditions. We assume in addition that L is semi-ample (namely a tensor power of L is generated by global sections), $\deg_L(X) = (L^d) > 0$ and φ is semi-positive. The data $\overline{L}$ permit to construct an arithmetic intersection number $\big(\overline{L}|_Y^{\dim(Y)+1}\big)_S$ for any integral closed subscheme Y of X, which can be written as an integral over Ω of local intersection numbers. In the case where $\deg_L(Y) = \big(L|_Y^{\dim(Y)}\big) > 0$, the *normalized height* of Y with respect to $\overline{L}$ is defined as

$$h_{\overline{L}}(Y) = \frac{\big(\overline{L}|_Y^{\dim(Y)+1}\big)_S}{(\dim(Y)+1)\deg_L(Y)}.$$

Let Y be an integral closed subscheme of X such that $\deg_L(Y) > 0$. For any $\omega \in \Omega$, we denote by $\delta_{\overline{L},Y,\omega}$ the Radon measure on X such that, for any continuous function f on the analytic space X_ω^{an},

$$\int_{X_\omega^{\mathrm{an}}} f(x)\,\delta_{\overline{L},Y,\omega}(\mathrm{d}x) = \frac{1}{\deg_L(Y)}\int_{Y_\omega^{\mathrm{an}}} f(y)\,c_1(L_\omega|_{Y_\omega},\varphi_\omega|_{Y_\omega})^{\dim(Y)}(\mathrm{d}y).$$

In the case where $|\cdot|_\omega$ is non-Archimedean, the Monge-Ampère measure

$$c_1(L_\omega|_{Y_\omega},\varphi_\omega|_{Y_\omega})^{\dim(Y)}(\mathrm{d}y)$$

has been constructed in [**26**, Definition 2.4].

Note that, if one modifies the metrics φ_ω for ω belonging to a set of measure 0, the height of subvarieties of X does not change. However the local Monge-Ampère measure can be modified by this procedure. Hence it is not adequate to consider a local equidistribution problem with respect to a single place ω unless the set $\{\omega\}$ belongs to $\mathcal{A}$ and has a positive measure with respect to ν. We therefore introduce the following global version of Monge-Ampère measure. Let Ω' be an element of $\mathcal{A}$ such that $\nu(\Omega') > 0$. We denote by $X_{\Omega'}^{\mathrm{an}}$ the disjoint union $\coprod_{\omega\in\Omega'} X_\omega^{\mathrm{an}}$ of local analytifications indexed by Ω'. We equipped this set with a suitable σ-algebra $\mathcal{B}_{X,\Omega'}$ so that the canonical projection map $X_{\Omega'}^{\mathrm{an}} \to \Omega'$ sending the elements of X_ω^{an} to ω gives a fibration of measurable spaces. It turns out that local Monge-Ampère measures mentioned above form a disintegration of a measure on $(X_{\Omega'}^{\mathrm{an}},\mathcal{B}_{X,\Omega'})$ over $\nu|_{\Omega'}$: for any integral closed subscheme Y of X such that $\deg_L(Y) > 0$, we denote by $\delta_{\overline{L},Y,\Omega'}$ the measure on $(X_{\Omega'}^{\mathrm{an}},\mathcal{B}_{X,\Omega'})$ which is defined as

$$\int_{X_{\Omega'}^{\mathrm{an}}} f(x)\,\delta_{\overline{L},Y,\Omega'}(\mathrm{d}x) := \int_{\Omega'}\left(\int_{X_\omega^{\mathrm{an}}} f(x)\,\delta_{\overline{L},Y,\omega}(\mathrm{d}x)\right)\nu(\mathrm{d}\omega).$$

It is worth while to say that the global adelic measure determines the local measures almost everywhere, that is, if the global measure $\delta_{\overline{L},Y,\Omega'}$ coincides with another global measure $\delta_{\overline{L}',Y,\Omega'}$, then $\delta_{\overline{L},Y,\omega} = \delta_{\overline{L}',Y,\omega}$ almost everywhere on Ω' (cf. Proposition 7.7.1). From a functional point of view, one can consider $\delta_{\overline{L},Y,\Omega'}$ as a linear form on the vector space of adelic families of continuous functions on X. Denote by $\mathscr{C}_{\mathrm{a}}^0(X)$ the set of families $f = (f_\omega)_{\omega\in\Omega}$ of continuous functions on X such that $(\mathcal{O}_X,(\mathrm{e}^{-f_\omega}|\cdot|_\omega)_{\omega\in\Omega})$ forms an adelic line bundle on X. Note that f yields a measurable function f_Ω on X_Ω^{an} given by $f_\Omega(x) = f_\omega(x)$ for $x \in X_\omega^{\mathrm{an}}$. We denote by $\mathscr{C}_{\mathrm{a}}^0(X;\Omega')$ the vector subspace of $\mathscr{C}_{\mathrm{a}}^0(X)$ consisting of $f \in \mathscr{C}_{\mathrm{a}}^0(X)$ such that $f_\omega = 0$ for any $\omega \in \Omega\setminus\Omega'$. Then

$$(f \in \mathscr{C}_{\mathrm{a}}^0(X;\Omega')) \longmapsto \int_{X_{\Omega'}^{\mathrm{an}}} f(x)\,\delta_{\overline{L},Y,\Omega'}(\mathrm{d}x)$$

defines a linear functional on $\mathscr{C}_{\mathrm{a}}^0(X;\Omega')$. One of the main results of the article is the following (see Theorem 8.11.2).

Theorem F. *Let X be an integral projective scheme of dimension d over* Spec K *and $\overline{L} = (L,\varphi)$ be an adelic line bundle on X such that L is semi-ample, $(L^d) > 0$ and φ is semi-positive. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of integral closed subschemes of X, such that each of its subsequences is Zariski dense in X, and that $h_{\overline{L}}(Y_n)$ is well-defined* *and converges to* $h_{\overline{L}}(X)$ *when* $n \to +\infty$. *Then, for any* $\Omega' \in \mathcal{A}$ *such that* $\nu(\Omega') > 0$, *the sequence of measures* $(\delta_{\overline{L},Y_n,\Omega'})_{n\in\mathbb{N}}$, *viewed as a sequence of linear functionals on* $\mathscr{C}^0_{\mathrm{a}}(X;\Omega')$, *converges pointwisely to* $\delta_{\overline{L},X,\Omega'}$.

The proof of the theorem is inspired by the original work of Szipro, Ullmo and Zhang, the subvariety version of Autissier, together with the differentiability interpretation introduced in [**30**]. The idea relies on the following simple observation. Let V be a real vector space, x_0 be an element of V, and f and g be two real-valued functions on U such that $f(x) \geqslant g(x)$ for any $x \in V$. Assume f is concave on V, g is Gâteaux differentiable at x_0, and $f(x_0) = g(x_0)$. Then the function f is also Gâteaux differentiable at x_0 and its differential identifies with that of g. Concretely in the case of the equidistribution problem, we consider, for any integral closed subscheme Y of X such that $\deg_L(Y) > 0$, the linear functional $\Phi_Y : \mathscr{C}^0_{\mathrm{a}}(X;\Omega') \to \mathbb{R}$ which sends $f \in \mathscr{C}^0_{\mathrm{a}}(X;\Omega')$ to

$$\frac{\widehat{\mathrm{vol}}_\chi((L,\varphi+f)|_Y)}{(\dim(Y)+1)\deg_L(Y)},$$

where $\widehat{\mathrm{vol}}_\chi((L,\varphi+f)|_Y)$ denotes the χ-volume of $(L,\varphi+f)|_Y$, which is defined as

$$\lim_{n\to+\infty}\frac{\widehat{\deg}\Big(H^0(Y,L|_Y^{\otimes n}),\big(\|\cdot\|_{(n\varphi_\omega+nf_\omega)|_{Y_\omega}}\big)_{\omega\in\Omega}\Big)}{n^{\dim(Y)+1}/(\dim(Y)+1)!}.$$

By the arithmetic Hilbert-Samuel formula, the value of Φ_Y at 0 identifies with $h_{\overline{L}}(Y)$. Moreover, this functional is concave. Consider now a generic sequence $(Y_n)_{n\in\mathbb{N}}$ of integral closed subschemes of X as in Theorem F. For any $f \in \mathscr{C}^0_{\mathrm{a}}(X;\Omega')$, let

$$\Phi_{Y_\bullet}(f) := \liminf_{n\to+\infty}\Phi_{Y_n}(f).$$

Since the functionals Φ_{Y_n} are concave, so is $\Phi_{Y_\bullet}$. The sequence $(Y_n)_{n\in\mathbb{N}}$ being generic, the functional Φ is bounded from below by Φ_X. Moreover, the hypothesis that $(h_{\overline{L}}(Y_n))_n$ converges to $h_{\overline{L}}(X)$ shows that $\Phi_{Y_\bullet}(0) = \Phi_X(0)$. Therefore, we deduce from the differentiability of Φ_X the equidistribution result. Note that the equality $\Phi_{Y_\bullet}(0) = \Phi_X(0)$ is not always satisfied. In general, for any generic sequence $(Y_n)_{n\in\mathbb{N}}$, the limit inferior of $\Phi_{Y_n}(f)$ when $n \to +\infty$ is always bounded from below by the asymptotic maximal slope of $(L,\varphi+f)$, which is defined as

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(L,\varphi+f) = \lim_{n\to+\infty}\frac{\widehat{\mu}_{\max}\Big(H^0(Y,L|_Y^{\otimes n}),\big(\|\cdot\|_{(n\varphi_\omega+nf_\omega)|_{Y_\omega}}\big)_{\omega\in\Omega}\Big)}{n}.$$

Moreover, the lower bound $\widehat{\mu}^{\mathrm{asy}}_{\max}(L,\varphi)$ of $\Phi_{Y_\bullet}(0)$ is attained by a certain generic sequence $(Y_n)_{n\in\mathbb{N}}$ (see § 8.10). In particular, if the function

$$(f \in \mathscr{C}^0_{\mathrm{a}}(X;\Omega')) \longmapsto \widehat{\mu}^{\mathrm{asy}}_{\max}(L,\varphi+f)$$

is Gâteaux differentiable at 0, then the following relation holds

$$\lim_{n\to+\infty}\delta_{\overline{L},Y_n,\Omega'}(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\widehat{\mu}^{\mathrm{asy}}_{\max}(L,\varphi+tf).$$

Note that Theorem F gives a partial answer of [**77**, Conjecture 5.4.1] by Yuan-Zhang.

The global adelic space that we use to study the equidistribution problem permits to extend the construction of arithmetic intersection product in allowing one of the adelic line bundle to be possibly not integrable. This construction has applications in the study of weak relative positivity conditions. Bigness is another type of positivity condition which describes the growth of the total graded linear series of a line bundle. In Arakelov geometry of number fields, the arithmetic bigness describes the asymptotic behaviour of the number of small sections in the graded sectional algebra of adelic vector bundles. This notion can be generalized to the setting of Arakelov geometry over adelic curves in replacing the logarithm of the number of small sections by the positive degree of an adelic vector bundle (namely the supremum of the Arakelov degrees of adelic vector subbundles). In [**36**, Proposition 6.4.18], the arithmetic bigness has been related to an arithmetic sectional invariant — asymptotic maximal slope, which is quite similar to asymptotic minimal slope: for any integral projective K-scheme and any adelic line bundle $\overline{L}$ on X such that L is big, we introduce a numerical invariant $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})$ which describes the asymptotic behaviour (when $n \to +\infty$) of the maximal slopes of $H^0(X, L^{\otimes n})$ equipped with sup norms (see §8.3 for its construction and properties). It turns out that this invariant is also super-additive with respect to $\overline{L}$, which allows to extend the function $\widehat{\mu}_{\max}^{\mathrm{asy}}(\cdot)$ to the cone of adelic line bundles $\overline{L}$ such that L is pseudo-effective. Moreover, in the case where L is nef, the inequality $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \leqslant \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})$ holds.

Recall that Fujita's approximation theorem asserts that a big line bundle can be decomposed on a birational modification into the tensor product of two $\mathbb{Q}$-line bundles which are respectively ample and effective, with a good approximation of the volume function. In this article, we establish the following relative version of Fujita's approximation theorem for the asymptotic maximal slope (see Theorem 8.5.6 and Remark 8.5.7).

THEOREM G. *Assume that the field K is perfect and the scheme X is integral. Let $\overline{L}$ be an adelic line bundle on X such that L is big. For any real number $t < \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})$, there exist a positive integer p, a birational projective K-morphism $g : X' \to X$, a relatively ample adelic line bundle $\overline{A}$ and an effective adelic line bundle $\overline{M}$ on X' such that $g^*(\overline{L}^{\otimes p})$ is isomorphic to $\overline{A} \otimes \overline{M}$ and $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) \geqslant pt$.*

As an application, in the case where X is an integral scheme, we can improve the height inequality in Theorem E in relaxing the positivity condition of one of the adelic line bundles and in replacing the asymptotic minimal slope of this adelic line bundle by the asymptotic maximal slope (see Theorem 8.6.1).

THEOREM H. *Assume that the field K is perfect. Let X be an integral projective scheme of dimension d over* Spec K, *and $\overline{L}_0, \ldots, \overline{L}_d$ be adelic line bundles on X such that $\overline{L}_1, \ldots, \overline{L}_d$ are relatively ample and L_0 is big. For any $i \in \{0, \ldots, d\}$, let $\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d)$. Then the following inequality holds:*

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_0) + \sum_{i=1}^{d} \delta_i\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i).$$

In the case where $\overline{L}_0, \ldots, \overline{L}_d$ are all equal to the same adelic line bundle $\overline{L}$, the above inequality leads to

$$\frac{(\overline{L}^{d+1})_S}{(L^d)} \geqslant \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) + d\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}).$$

In the case where the adelic curve S comes from the canonical adelic structure of a number field, if $\overline{L}$ is a relatively ample adelic line bundle, then $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L})$ is equal to the absolue minimum of the Arakelov (absolute) height function $h_{\overline{L}}$ on the set of closed points of X. This is essentially a consequence of [**81**, Corollary 5.7]. Similarly, the asymptotic maximal slope $\widehat{\mu}_{\max}^{\mathrm{sym}}(\overline{L})$ is equal to the essential minimum of the height function $h_{\overline{L}}$. This is a result of Ballaÿ [**8**, Theorem 1.1]. In this article, we show that these results can be extended to the case of general adelic curves if we consider the heights of all integral closed subschemes of X. More precisely, we obtain the following result (see Theorem 8.8.3 and Proposition 8.10.1).

THEOREM I. *Assume that the field K is pefect. Let X be a non-empty reduced projective scheme over* Spec K *and Θ_X be the set of integral closed subschemes of X. For any relatively ample adelic line bundle $\overline{L}$ on X, the following equalities hold:*

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) = \inf_{Y\in\Theta_X} \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)(L|_Y^{\dim(Y)})} = \inf_{Y\in\Theta_X} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y).$$

Moreover, if X is an integral scheme, the following equality holds:

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) = \sup_{\substack{Y\in\Theta_X\\ Y\neq X}} \inf_{\substack{Z\in\Theta_X\\ Z\not\subseteq Y}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Z).$$

We also show that a property similar to Minkowski's first theorem permits to recover the link between the asymptotic maximal and minimal slopes, and the Arakelov height of closed points in the number field case. More precisely, we say that a relatively ample adelic line bundle $\overline{L}$ is *strongly Minkowskian* if for any $Y \in \Theta_X$ one has

$$\lim_{n\to+\infty} \frac{1}{n} \sup_{\substack{s\in H^0(Y,L|_Y^{\otimes n})\\ s\neq 0}} \widehat{\deg}(s) \geqslant \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)(L|_Y^{\dim(Y)})}.$$

This condition is always satisfied notably when the adelic curve S comes from a number field (consequence of Minkowski's first theorem) or the function field of a projective curve (consequence of Riemann-Roch theorem). We then establish the following result (see Corollary 8.9.2).

THEOREM J. *Assume that the field K is pefect. Let X be an integral projective scheme over* Spec K *and $\overline{L}$ be a relatively ample adelic line bundle on X which is strongly Minkowskian. Denote by $X^{(0)}$ the set of closed points of X. Then the equality $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) = \inf_{x\in X^{(0)}} h_{\overline{L}}(x)$ holds.*

Motivated by Theorem I, we propose the following analogue of successive minima for relatively ample adelic line bundles. Let $f : X \to \operatorname{Spec} K$ be an integral projective scheme of dimension d over Spec K and $\overline{L}$ be a relatively ample adelic line bundle on X. For $i \in \{1, \ldots, d+1\}$, let

$$e_i(\overline{L}) = \sup_{\substack{Y\subseteq X \text{ closed}\\ \operatorname{codim}(Y)\geqslant i}} \inf_{\substack{Z\in\Theta_X\\ Z\not\subseteq Y}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Z).$$

With this notation, one can rewrite the assertion of Theorem I as

$$e_1(\overline{L}) = \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}), \quad e_{d+1}(\overline{L}) = \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}).$$

We show in Remark 8.10.2 that, in the number field case, one has

$$\forall\, i\in\{1,\ldots,d+1\}, \quad e_i(\overline{L}) = \sup_{\substack{Y\subseteq X \text{ closed}\\ \operatorname{codim}(Y)\geqslant i}} \inf_{x\in(X\setminus Y)^{(0)}} h_{\overline{L}}(x). \tag{1.4}$$

Thus we recover the definition of successive minima in the sense of [**80**, §5]. We propose several fundamental questions about these invariants:

(1) Do the equalities (1.4) hold in the case of a general adelic curve, under the assumption that $\overline{L}$ is strongly Minkowskian?

(2) What is the relation between the invariants $e_2(\overline{L}),\ldots,e_d(\overline{L})$ and the sectional algebra $\bigoplus_{n\in\mathbb{N}} f_*(\overline{L}^{\otimes n})$?
(3) Does the analogue of some classic results in Diophantine geometry concerning the successive minima, such as the inequality

$$\frac{(\overline{L}^{d+1})_S}{(L^d)} \geqslant \sum_{i=1}^{d+1} e_i(\overline{L}),$$

still holds for general adelic curve?
(4) In the case where (X,L) is a polarized toric variety and the metrics in φ are toric metrics, is it possible to describe in a combinatoric way the positivity conditions of $\overline{L}$, and express the the invariants $e_i(\overline{L})$ in terms of the combinatoric data of $(X,\overline{L})$, generalizing some results of [**21, 22**] for example?

The last chapter of the article is devoted to the study of global positivity of adelic line bundles. Motivated by Nakai-Moishezon criterion of ampleness, we say that an adelic line bundle $\overline{L}$ on X is *ample* if it is relatively ample and if the normalized height with respect to $\overline{L}$ of integral closed subschemes of X has a positive lower bound. We show that this condition is equivalent to the relative ampleness together with the positivity of the invariant $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$. Therefore, we deduced from Theorem E that, if $\overline{L}_0,\ldots,\overline{L}_d$ are ample adelic line bundles on X, where d is the dimension of X, then one has (see Proposition 9.1.3)

$$(\overline{L}_0\cdots\overline{L}_d)_S > 0.$$

In the case where $\overline{L}$ is strongly Minkowskian, $\overline{L}$ is ample if and only if it is relatively ample and the height function $h_{\overline{L}}$ on the set of closed points of X has a positive lower bound (see Proposition 9.1.4). Once the ample cone is specified, one can naturally define the nef cone as its closure. It turns out that the nefness can also be described in a numerical way: an adelic line bundle $\overline{L}$ is nef if and only if it is relatively nef and $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \geqslant 0$ (see Proposition 9.1.6).

Bigness and pseudo-effectivity are also described in a numerical way by the invariant $\widehat{\mu}^{\mathrm{asy}}_{\max}(\cdot)$: an adelic line bundle $\overline{L}$ is big if and only if L is big and $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) > 0$ (which coincides with the bigness in [**36**]); it is pseudo-effective if and only if L is pseudo-effective and $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \geqslant 0$ (see [**36**, Proposition 6.4.18] and Proposition 9.2.6). We deduce from Theorem H that, if $\overline{L}_0,\ldots,\overline{L}_d$ are adelic line bundles on X such that $\overline{L}_0$ is pseudo-effective and that $\overline{L}_1,\ldots,\overline{L}_d$ are nef, then the inequality $(\overline{L}_0,\ldots,\overline{L}_d)_S \geqslant 0$ holds (see Proposition 9.2.7).

As an application of the equidistribution theorem together with the global positivity properties of adelic line bundles, we consider Bogomolov's conjecture over a countable field of characteristic zero (see Theorem 9.4.1). We assume that K is algebraically closed field of characteristic zero, $\nu(\Omega_\infty) > 0$, and $\nu(\mathcal{A}) \not\subseteq \{0,+\infty\}$. The following theorem is a generalization of [**60**, Theorem 8.1].

Theorem K. *Let A be an abelian variety over K, L be an ample and symmetric line bundle on A, and φ be a family of semipositive metrics of A such that (A,φ) is nef and φ_ω is the canonical metric of L_ω for each $\omega\in\Omega$. If the essential minimum of $(L,\varphi)|_X$ is zero, then X is a translation of an abelian subvariety of A by a closed point of Néron-Tate height* 0*, which is a torsion point provided that any finitely generated subfield of K has Northcott's property (cf.* [**38**, Theorem 2.7.18]*).*

We also discuss arithmetic dynamical systems in the adelic curve setting. We assume that K is algebraically closed. Let X be a projective integral scheme over $\operatorname{Spec} K$ and L

be an ample line bundle on X. We denote by $\mathrm{End}(X;L)$ the set of all endomorphisms $f : X \to X$ such that $f^*(L)$ is isomorphic to a tensor power of L with exponent > 1. For any $f \in \mathrm{End}(X;L)$ with $f^*(L) \cong L^{\otimes d}$ for some $d > 1$, there exists a unique metric family φ_f such that (L, φ_f) forms an adelic line bundle and $f^*(\overline{L})$ is isometric to $\overline{L}^{\otimes d}$. We call it the *global canonical compactification* of L. It is easy to see that any f-preperiodic rational point of X is of height 0. The converge is also true if the adelic curve S has Northcott property. We establish the following result (see Theorem 9.5.1).

Theorem L. *Let L be an ample line bundle on X and f and g be two elements of* $\mathrm{End}(X;L)$. *Then the following statement are equivalent:*

1. *The adelic line bundle (L, φ_f) and (L, φ_g) define the same height function on the set of rational points of X.*
2. $\{x \in X(K) \mid h_{(L,\varphi_f)}(x) = 0\} = \{x \in X(K) \mid h_{(L,\varphi_g)}(x) = 0\}$.
3. $\{x \in X(K) \mid h_{(L,\varphi_f)}(x) = h_{(L,\varphi_g)}(x) = 0\}$ *is Zariski dense in $X(K)$.*

Moreover, when these conditions are satisfied, there exist an integrable function ℓ on Ω and $\Omega' \in \mathcal{A}$ such that $\nu(\Omega \setminus \Omega') = 0$ and that

$$\forall\, \omega \in \Omega', \quad \varphi_{g,\omega} = \mathrm{e}^{\ell(\omega)} \varphi_{f,\omega}.$$

The rest of the article is organized as follows. In the remaining of Introduction, we remind the the notation that we use all through the article.

In the second chapter, we consider metric families on vector bundles and discuss their dominancy and measurability.

In the third chapter, we study normed graded linear series over a trivially valued field and prove the limit theorem of their volumes. Then in the fourth chapter we deduce the limit theorem for graded algebra of adelic vector bundles over a general adelic curve, which proves in particular that the sequence defining the arithmetic volume function actually converges. We also show that the arithmetic Hilbert-Samuel theorem in the original form implies the generalized form with tensor product by an adelic vector bundle.

In the fifth chapter, we prove the arithmetic Hilbert-Samuel theorem. We first prove that the difference of the χ-volume and the arithmetic intersection product does not depend on the choice of the metric family. Then we prove the arithmetic Hilbert-Samuel theorem in the particular case where the adelic curve contains a single copy of the trivial absolute value, and we use the method of casting to the trivial valuation case to prove the arithmetic Hilbert-Samuel theorem in general.

The sixth chapter is devoted to the study of relative ampleness and nefness of adelic line bundles. We begin with a discussion on these positivity conditions and its relation with sectional arithmetic invariants. We also deduce the generalized Hodge index theorem from the arithmetic Hilbert-Samuel theorem.

In the seventh chapter, we establish the Gâteaux differentiability with respect to modification of metrics of the χ-volume function at any adelic line bundle $\overline{L} = (L, \varphi)$ such that L is semi-ample and big and φ is semi-positive. We then deduce the measurability of certain fiber integrals. This measurability result is important in the construction of global adelic space.

In the eighth chapter, we study asymptotic maximal slope and its relation with positivity of adelic line bundles. We also prove an equidistribution theorem for a generic sequence of subvarieties in the setting of adelic curves.

In the ninth and last chapter, we discuss global positivity conditions, and deduce Bogomolov's conjecture over a countable field of characteristic zero.

In order to obtain the main results of the article in positive characteristic case, we need to generalize some results of [**37**, Chapter 5] to any characteristic, which we resume in the appendices.

Notation and preliminaries

1.1. Throughout the article, we fix a proper adelic curve

$$S = (K, (\Omega, \mathcal{A}, \nu), \phi),$$

where K is a commutative field, $(\Omega, \mathcal{A}, \nu)$ is a measure space and $\phi = (|\cdot|_\omega)_{\omega\in\Omega}$ is a family of absolute values on K parametrized by Ω, such that, for any $a \in K^\times$, $(\omega \in \Omega) \mapsto \ln|a|_\omega$ is integrable on $(\Omega, \mathcal{A}, \nu)$, and the following "product formula" holds:

$$\forall\, a \in K, \quad \int_\Omega \ln|a|_\omega\, \nu(\mathrm{d}\omega) = 0.$$

For any $\omega \in \Omega$, we denote by K_ω the completion of K with respect to the absolute value $|\cdot|_\omega$. Let Ω_∞ be the set of $\omega \in \Omega$ such that $|\cdot|_\omega$ is Archimedean. Note that $\nu(\Omega_\infty) < +\infty$ (see [**36**, Proposition 3.1.2]). For $\omega \in \Omega_\infty$, we always assume that $|a|_\omega = a$ for any $a \in \mathbb{Q}_{\geqslant 0}$. Denote by Ω_{fin} the set $\Omega \setminus \Omega_\infty$. We assume that, either the σ-algebra $\mathcal{A}$ is discrete, or the field K is countable.

Let K' be an algebraic extension of K. In [**36**, §§3.3-3.4], it has been constructed an adelic curve

$$S \otimes_K K' = (K', (\Omega_{K'}, \mathcal{A}_{K'}, \nu_{K'}), \phi_{K'})$$

together with a measurable fibration $\pi : \Omega_{K'} \to \Omega$ and a family of disintegration probability measures $(\nu_{K',\omega})_{\omega\in\Omega}$ on fibers $\pi^{-1}(\{\omega\})$ for $\nu_{K'}$ over $(\Omega, \mathcal{A}, \nu)$, which is characterized by the following properties:

(1) for any $\omega \in \Omega$, $\pi^{-1}(\{\omega\})$ is the set of absolute values on K' which extend he absolute value $|\cdot|_\omega$ on K, which is equipped with the projective limit of discrete σ-algebra, where we identify $\pi^{-1}(\{\omega\})$ with the projective limit of the sets of extensions of $|\cdot|_\omega$ to finite field extensions of K contained in K',
(2) for any integrable or non-negative $\mathcal{A}'$-measurable function $f : \Omega' \to \mathbb{R}$, one has

$$\int_{\Omega'} f(x)\, \nu_{K'}(\mathrm{d}x) = \int_\Omega \nu(\mathrm{d}\omega) \int_{\pi^{-1}(\{\omega\})} f(x)\, \nu_{K',\omega}(\mathrm{d}x)$$

The adelic curve $S \otimes_K K'$ is called the *algebraic extension* of S by K'. Note that this adelic curve is proper if S is proper.

1.2. Let V be a finite-dimensional vector space over K. As *norm family* on V, we refer to a family $(\|\cdot\|_\omega)_{\omega\in\Omega}$, where $\|\cdot\|_\omega$ is a norm on $V_\omega := V \otimes_K K_\omega$.

Let $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ and $\xi' = (\|\cdot\|'_\omega)_{\omega\in\Omega}$ be norm families on V. For any $\omega \in \Omega$, we denote by $d_\omega(\xi, \xi')$ the following number

$$\sup_{s\in V\setminus\{0\}} \Big| \ln\|s\|_\omega - \ln\|s\|'_\omega \Big|.$$

In the case where $V = \mathbf{0}$, by convention $d_\omega(\xi, \xi') = 0$.

1.3. As *adelic vector bundle* on S, we refer to the data $\overline{V} = (V, \xi)$ which consists of a finite-dimensional vector space V over K and a family of norms $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ on $V_\omega := V \otimes_K K_\omega$, satisfying the following conditions:

(1) the norm family ξ is *strongly dominated*, that is, there exist an integrable function $C : \Omega \to \mathbb{R}_{\geqslant 0}$ and a basis $(e_i)_{i=1}^r$ of V over K, such that, for any $\omega \in \Omega$ and any $(\lambda_1, \ldots, \lambda_r) \in K_\omega^r \setminus \{(0, \ldots, 0)\}$,

$$\Big|\ln \|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\omega - \ln \max_{i\in\{1,\ldots,r\}} |\lambda_i|_\omega\Big| \leqslant C(\omega).$$

(2) the norm family ξ is *measurable*, that is, for any $s \in V$, the function $(\omega \in \Omega) \mapsto \|s\|_\omega$ is $\mathcal{A}$-measurable.

In the article, we only consider adelic vector bundles which are ultrametric over non-Archimedean places, namely we assume that the norm $\|\cdot\|_\omega$ is ultrametric once the absolute value $|\cdot|_\omega$ is non-Archimedean. If in addition the norm $\|\cdot\|_\omega$ is induced by an inner product whenever $|\cdot|_\omega$ is Archimedean, we say that $\overline{V}$ is *Hermitian*. If $\dim_K(V) = 1$, we say that $\overline{V}$ is an *adelic line bundle* (note that an adelic line bundle is necessarily Hermitian).

If $\overline{V}$ is an adelic vector bundle on S, any vector subspace (resp. quotient vector space) of V together with the family of restricted norms (resp. quotient norms) forms also an adelic vector bundle on S, which is called an *adelic vector subbundle* (resp. *quotient adelic vector bundle*) of $\overline{V}$. Note that if $\overline{V}$ is Hermitian, then all its adelic vector subbundles and quotient adelic vector bundles are Hermitian.

1.4. Let $\overline{V} = (V, (\|\cdot\|_\omega)_{\omega\in\Omega})$ be an adelic vector bundle on S, we define the *Arakelov degree* of $\overline{V}$ as

$$\widehat{\deg}(\overline{V}) := -\int_\Omega \ln \|e_1 \wedge \cdots \wedge e_r\|_{\omega,\det}\, \nu(\mathrm{d}\omega),$$

where $(e_i)_{i=1}^r$ is a basis of V over K, and $\|\cdot\|_{\omega,\det}$ denotes the determinant norm of $\|\cdot\|_\omega$, which is defined as (where $r = \dim_K(V)$)

$$\forall\, \eta \in \det(V) = \Lambda^r(V), \quad \|\eta\|_{\omega,\det} = \inf_{\eta=s_1\wedge\cdots\wedge s_r} \|s_1\| \cdots \|s_r\|.$$

Let $\widehat{\deg}_+(\overline{V})$ be the *positive degree* of $\overline{V}$, which is defined as

$$\widehat{\deg}_+(\overline{V}) = \sup_{W\subseteq V} \widehat{\deg}(\overline{W}),$$

where W runs over the set of vector subspaces of V, and in the adelic vector bundle structure of $\overline{W}$ we consider the restricted norms. In the case where V is non-zero, we denote by $\widehat{\mu}(\overline{V})$ the quotient $\widehat{\deg}(\overline{V})/\dim_K(V)$, called the *slope* of V. We define the *minimal slope* of $\overline{V}$ as

$$\widehat{\mu}_{\min}(\overline{V}) := \inf_{V\twoheadrightarrow W\neq\{0\}} \widehat{\mu}(\overline{W}),$$

where $\overline{W}$ runs over the set of all non-zero quotient adelic vector bundles of $\overline{V}$. Similarly, we define the *maximal slope* of $\overline{V}$ as

$$\widehat{\mu}_{\max}(\overline{V}) := \sup_{\{0\}\neq W\hookrightarrow V} \widehat{\mu}(\overline{W}),$$

where $\overline{W}$ runs over the set of all non-zero adelic vector subbundles of $\overline{V}$.

1.5. Let $\overline{E}$ and $\overline{F}$ be two adelic vector bundles on S and $\varphi : E \to F$ be a K-linear map. We define the *height* of φ as

$$h(\varphi) := \int_{\Omega} \ln \|\varphi\|_{\omega}\, \nu(\mathrm{d}\omega),$$

where $\|\varphi\|_{\omega}$ denotes the operator norm of the K_{ω}-linear map $E_{\omega} \to F_{\omega}$ induced by φ. Moreover, if E is non-zero and if φ is injective, then the following slope inequality holds (see [**36**, Proposition 4.3.31]):

$$\widehat{\mu}_{\max}(\overline{E}) \leqslant \widehat{\mu}_{\max}(\overline{F}) + h(\varphi).$$

1.6. Let $\overline{V}$ be a non-zero adelic vector bundle on S. For any $t \in \mathbb{R}$, we let

$$\mathcal{F}^t(\overline{V}) = \sum_{\substack{\{0\}\neq W \subseteq V \\ \widehat{\mu}_{\min}(\overline{W}) \geqslant t}} W,$$

where $\overline{W}$ runs over the set of all non-zero adelic vector subbundles of $\overline{V}$ such that $\widehat{\mu}_{\min}(\overline{W}) \geqslant t$. We call $(\mathcal{F}^t(\overline{V}))_{t\in\mathbb{R}}$ the *Harder-Narasimhan* $\mathbb{R}$*-filtration* of $\overline{V}$. In the case where $\overline{V}$ is Hermitian, the following equality holds (see [**36**, Theorem 4.3.44]):

$$\begin{cases} \widehat{\deg}(\overline{V}) = -\displaystyle\int_{\mathbb{R}} t\, \mathrm{d}(\dim_K(\mathcal{F}^t(\overline{V}))), \\ \widehat{\deg}_+(\overline{V}) = -\displaystyle\int_0^{+\infty} t\, \mathrm{d}(\dim_K(\mathcal{F}^t(\overline{V}))) = \int_0^{+\infty} \dim_K(\mathcal{F}^t(\overline{V}))\, \mathrm{d}t. \end{cases}$$

In general one has the following inequalities (see [**36**, Propositions 4.3.50 and 4.3.51, and Corollary 4.3.52]):

$$\begin{cases} 0 \leqslant \widehat{\deg}(\overline{V}) + \displaystyle\int_{\mathbb{R}} t\, \mathrm{d}(\dim_K(\mathcal{F}^t(\overline{V}))) \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(V)\ln(\dim_K(V)), \\ 0 \leqslant \widehat{\deg}_+(\overline{V}) - \displaystyle\int_0^{+\infty} \dim_K(\mathcal{F}^t(\overline{V}))\, \mathrm{d}t \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(V)\ln(\dim_K(V)). \end{cases}$$

1.7. Let $\overline{V} = (V, (\|\cdot\|_{V,\omega})_{\omega\in\Omega})$ and $\overline{W} = (W, (\|\cdot\|_{W,\omega})_{\omega\in\Omega})$ be adelic vector bundles on S. For any $\omega \in \Omega$ such that $|\cdot|_{\omega}$ is non-Archimedean, let $\|\cdot\|_{\omega}$ be the ε-tensor product on $V_{\omega} \otimes_{K_{\omega}} W_{\omega}$, of the norms $\|\cdot\|_{V,\omega}$ and $\|\cdot\|_{W,\omega}$. Note that, for any $T \in V_{\omega} \otimes_{K_{\omega}} W_{\omega}$, the value of $\|T\|_{\omega}$ is equal to

$$\min\left\{ \max_{i\in\{1,\ldots,n\}} \|e_i\|_{V,\omega}\|f_i\|_{W,\omega} \;:\; \begin{array}{l} n\in\mathbb{N},\ (e_i)_{i=1}^n \in V_{\omega}^n,\ (f_i)_{i=1}^n \in W_{\omega}^n \\ T = e_1\otimes f_1 + \cdots + e_n \otimes f_n \end{array} \right\}.$$

In the case where $|\cdot|_{\omega}$ is Archimedean, let $\|\cdot\|_{\omega}$ be π-tensor product of $\|\cdot\|_{V,\omega}$ of $\|\cdot\|_{W,\omega}$. Recall that for any $T \in V_{\omega} \otimes_{K_{\omega}} W_{\omega}$, the value of $\|T\|_{\omega}$ is equal to

$$\min\left\{ \sum_{i=1}^{n} \|e_i\|_{V,\omega}\|f_i\|_{W,\omega} \;:\; \begin{array}{l} n\in\mathbb{N},\ (e_i)_{i=1}^n \in V_{\omega}^n,\ (f_i)_{i=1}^n \in W_{\omega}^n \\ T = e_1\otimes f_1 + \cdots + e_n \otimes f_n \end{array} \right\}.$$

The pair

$$\overline{V} \otimes_{\varepsilon,\pi} \overline{W} = (V \otimes_K W, (\|\cdot\|_{\omega})_{\omega\in\Omega})$$

is called the ε,π*-tensor product* of $\overline{V}$ and $\overline{W}$.

Assume that $\overline{V}$ and $\overline{W}$ are Hermitian. If $|\cdot|_\omega$ is non-Archimedean, let $\|\cdot\|_\omega^H$ be the ε-tensor product of $\|\cdot\|_{V,\omega}$ and $\|\cdot\|_{W,\omega}$; otherwise let $\|\cdot\|_\omega^H$ be the orthogonal tensor product of the Euclidean or Hermitian norms $\|\cdot\|_{V,\omega}$ and $\|\cdot\|_{W,\omega}$. Then the pair

$$\overline{V} \otimes \overline{W} = (V \otimes_K W, (\|\cdot\|_\omega^H)_{\omega\in\Omega})$$

is called the *Hermitian tensor product* of $\overline{V}$ and $\overline{W}$.

1.8. Let $(k, |\cdot|)$ be a field equipped with a complete absolute value, X be a projective scheme over $\operatorname{Spec} k$. We denote by X^{an} the analytic space associated with X (in the sense of Berkovich if $|\cdot|$ is non-Archimedean). Recall that a point x of X^{an} is of the form $(j(x), |\cdot|_x)$, where $j(x)$ is a scheme point of X, $|\cdot|_x$ is an absolute value on the residue field of $j(x)$, which extends the absolute value $|\cdot|$ on the base field k. We denote by $\widehat{\kappa}(x)$ the completion of the residue field of $j(x)$ with respect to the absolute value $|\cdot|_x$, on which $|\cdot|_x$ extends by continuity. The set X^{an} is equipped with the most coarse topology which makes continuous the map $j : X^{\mathrm{an}} \to X$ and all functions of the form

$$|s| : U^{\mathrm{an}} \longrightarrow \mathbb{R}_{\geqslant 0}, \quad x \longmapsto |s(x)|_x,$$

where U is a non-empty Zariski open subset of X and $s \in \mathcal{O}_X(U)$ is a regular function on U. In particular, if U is a Zariski open subset of X, then U^{an} is an open subset of X^{an}. We call such open subsets of X^{an} *Zariski open subsets*.

1.9. Let $\pi : X \to \operatorname{Spec} K$ be a projective scheme over $\operatorname{Spec} K$. For any $\omega \in \Omega$, let X_ω be $X \times_{\operatorname{Spec} K} \operatorname{Spec} K_\omega$ and let X_ω^{an} be the analytic space associated with X_ω. If L is an invertible $\mathcal{O}_X$-module, we call *metric family* on L any family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$, where φ_ω is a continuous metric on $L_\omega = L|_{X_\omega}$. In the particular case where X is the spectrum of a finite extension K' of K, the invertible $\mathcal{O}_X$-module L is just a one-dimension vector space over K' and a metric family of L could be viewed as a norm family if we consider the adelic curve $S \otimes_K K'$ obtained by algebraic extension of scalars (see [**36**, §3.4]).

If $\overline{E} = (E, (\|\cdot\|_\omega)_{\omega\in\Omega})$ is a finite-dimensional K-vector space E equipped with a norm family, $g : X \to \mathbb{P}(E)$ is a projective K-morphism and $L = g^*(\mathcal{O}_E(1))$, then, for each $\omega \in \Omega$, the norm $\|\cdot\|_\omega$ induces by passing to quotient by the universal surjective homomorphism

$$(g_\omega \circ \pi_\omega)^*(E_\omega) \longrightarrow g_\omega^*(\mathcal{O}_{E_\omega}(1)) = L_\omega$$

a continuous metric φ_ω on L_ω. The metric family $(\varphi_\omega)_{\omega\in\Omega}$ is called a *quotient metric family* induced by $(\|\cdot\|_\omega)_{\omega\in\Omega}$ (and by g).

Let L be an invertible $\mathcal{O}_X$-module and $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be a metric family of L. For any $\omega \in \Omega$, the metric φ_ω induces by passing to dual a metric on L_ω, which we denote by $-\varphi_\omega$. The metric family $(-\varphi_\omega)_{\omega\in\Omega}$ on $L^\vee$ is denoted by $-\varphi$.

Let $L_1, \ldots, L_n$ be invertible $\mathcal{O}_X$-modules. For any $i \in \{1, \ldots, n\}$, let φ_i be a metric family on L_i. Then the metric families $\varphi_1, \ldots, \varphi_n$ induce by tensor product a metric family on $L_1 \otimes \cdots \otimes L_n$, which we denote by $\varphi_1 + \cdots + \varphi_n$ in the additive form. In particular, if all (L_i, φ_i) are equal to the same (L, φ), the metric family $\varphi + \cdots + \varphi$ is denoted by $n\varphi$.

Let Y be a projective K-scheme and $f : Y \to X$ be a K-morphism. If L is an invertible $\mathcal{O}_X$-module and φ is a metric family of L, then φ induces by pullback a metric family $f^*(\varphi)$ on $f^*(L)$: for any $\omega \in \Omega$ and any $y \in Y_\omega^{\mathrm{an}}$, the norm $|\cdot|_{f^*(\varphi)_\omega}(y)$ is induced by $|\cdot|_{\varphi_\omega}(f_\omega^{\mathrm{an}}(y))$ by extension of scalars.

1.10. Let X be a projective K-scheme, L be an invertible $\mathcal{O}_X$-module and φ be a metric family of L. Assume that there exist invertible $\mathcal{O}_X$-modules L_1 and L_2, together with quotient metric families φ_1 and φ_2 on L_1 and L_2 respectively, which are induced by strongly dominated norm families (see §1.3), such that $L \cong L_1 \otimes L_2^\vee$ and that $\varphi = \varphi_1 - \varphi_2$, we say that the metric family φ is *dominated*. We refer to [**36**, §6.1.2] for more details.

1.11. Let Ω_0 be the set of $\omega \in \Omega$ such that the absolute value $|\cdot|_\omega$ is trivial. Let X be a projective scheme over $\operatorname{Spec} K$. For any triplet $x = (K_x, |\cdot|_x, P_x)$, where $(K_x, |\cdot|_x)$ is a valued extension of the trivially valued field $(K, |\cdot|_0)$ and $P_x : \operatorname{Spec} K_x \to X$ is a K-morphism, we denote by S_x the adelic curve

$$(K_x, (\Omega_0, \mathcal{A}_0, \nu_0), (|\cdot|_x)_{\omega \in \Omega_0}),$$

where $\mathcal{A}_0 = \mathcal{A}|_{\Omega_0}$ and ν_0 is the restriction of ν to $(\Omega_0, \mathcal{A}_0)$. If L is an invertible $\mathcal{O}_X$-module and if φ is a metric family of L, we denote by L_x the pullback $P_x^*(L)$ and by $x^*(\varphi)$ the norm family $(|\cdot|_{\varphi_\omega}(P_x^\omega))_{\omega \in \Omega_0}$ on L_x, where P_x^ω denotes the point of X_ω^{an} determined by $(P_x, |\cdot|_x)$.

Assume that the transcendence degree of K_x/K is $= 1$. Then $|\cdot|_x$ is a discrete absolute value on K_x. Let $\operatorname{ord}_x(\cdot) : K_x \to \mathbb{Z} \cup \{+\infty\}$ be the corresponding discrete valuation, which is defined as

$$\operatorname{ord}_x(a) = \sup\{n \in \mathbb{Z} \,:\, a \in \mathfrak{m}_x^n\},$$

where $\mathfrak{m}_x = \{b \in K_x \,:\, |b|_x < 1\}$. Then there is a non-negative real number q such that $|\cdot|_x = \exp(-q \operatorname{ord}_x(\cdot))$. We call it the *exponent* of x.

1.12. Let X be a projective K-scheme, L be an invertible $\mathcal{O}_X$-module, and φ be a metric family of L. We say that the metric family φ is *measurable* if the following conditions are satisfied (see [**36**, §6.1.4] for more details):

(1) for any finite extension K'/K and any K-morphism $P : \operatorname{Spec} K' \to X$, the norm family $P^*(\varphi)$ is measurable,
(2) for any triplet $x = (K_x, |\cdot|_x, P_x)$, where $(K_x, |\cdot|_x)$ is a valued extension of transcendence degree $= 1$ and of rational exponent of the trivially valued field $(K, |\cdot|_0)$, and $P_x : \operatorname{Spec} K_x \to X$ is a K-morphism, the norm family $x^*(\varphi)$ is measurable.

1.13. Let X be a projective scheme over $\operatorname{Spec} K$, L be an invertible $\mathcal{O}_X$-module and φ be a metric family of L. We say that $\overline{L} = (L, \varphi)$ is an *adelic line bundle* on X if the metric family φ is dominated and measurable (see §1.10 and §1.12).

Suppose that X is geometrically reduced. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. We denote by $f_*(\overline{L})$ the couple $(H^0(X, L), (\|\cdot\|_{\varphi_\omega})_{\omega \in \Omega})$, where for $s \in H^0(X_\omega, L_\omega)$,

$$\|s\|_{\varphi_\omega} = \sup_{x \in X_\omega^{\mathrm{an}}} |s|_{\varphi_\omega}(x).$$

It turns out that $f_*(\overline{L})$ is an adelic vector bundle on S (see [**36**, Theorems 6.1.13 and 6.1.32]).

1.14. Let X be a projective scheme over $\operatorname{Spec} K$. Let L be an invertible $\mathcal{O}_X$-module, $\varphi = (\varphi_\omega)_{\omega \in \Omega}$ and $\psi = (\psi_\omega)_{\omega \in \Omega}$ be metric families on L such that (L, φ) and (L, ψ) are both adelic line bundles. Then we define the *distance* between φ and ψ as

$$d(\varphi, \psi) := \int_\Omega \sup_{x \in X_\omega^{\mathrm{an}}} \left| \ln \frac{|\cdot|_{\varphi_\omega}(x)}{|\cdot|_{\psi_\omega}(x)} \right| \nu(\mathrm{d}\omega).$$

If L is semiample and if there exist a positive integer m and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of quotient metric families (where φ_n is a metric family of $L^{\otimes nm}$), such that

$$\lim_{n\to+\infty}\frac{1}{mn}d(nm\varphi,\varphi_n)=0,$$

we say that the metric family φ is *semi-positive*.

1.15. Let X be a projective scheme over $\operatorname{Spec} K$ and d be the dimension of X. An adelic line bundles $\overline{L}$ on X is said to be *integrable* if it can be written in the form $\overline{A}_1\otimes\overline{A}_2^\vee$, where each $\overline{A}_i$ is an ample invertible $\mathcal{O}_X$-module equipped with a semi-positive metric family. We denote by $\widehat{\mathrm{Int}}(X)$ the set of all integrable adelic line bundles on X. In [**38**], we have constructed an *arithmetic intersection product*

$$\big((\overline{L}_0,\ldots,\overline{L}_d)\in\widehat{\mathrm{Int}}(X)^{d+1}\big)\longmapsto(\overline{L}_0\cdots\overline{L}_d)_S\in\mathbb{R},$$

which is multi-linear with respect to tensor product. We have also related the arithmetic intersection number $(\overline{L}_0\cdots\overline{L}_d)_S$ to the height of the multi-resultant of $L_0,\ldots,L_d$.

1.16. Let K^{ac} be an algebraic extension of K. For any $(a_0:\ldots:a_n)\in\mathbb{P}^n(K^{\mathrm{ac}})$, we denote by $h_S(a_0:\ldots:a_n)$ the following real number

$$\int_{\Omega_{K^{\mathrm{ac}}}}\ln\max\{|a_0|_x,\ldots,|a_n|_x\}\,\nu_{K^{\mathrm{ac}}}(\mathrm{d}x).$$

By the product formula, its value does not depend on the choice of the projective coordinate $(a_0,\ldots,a_n)$, we call it the *height* of $(a_0:\ldots:a_n)$.

We say that S has *Northcott property* if, for any $C\geqslant 0$, the set

$$\{a\in K\,:\,h_S(1:a)\leqslant C\}$$

is finite. Note that this condition is satisfied notably by number fields, and more generally by a finitely generated extension of K equipped with the canonical adelic structure.

If S has Northcott property, then a Northcott theorem type result holds for the adelic curve, namely, for any positive constants C and δ, the set

$$\{x\in\mathbb{P}^n(K^{\mathrm{ac}})\,:\,h_S(x)\leqslant C,\ [K(x):K]\leqslant\delta\}$$

is finite (see [**36**, Theorem 3.5.3]). More generally, if X is a projective K-scheme and $\overline{L}=(L,\varphi)$ is an adelic line bundle on X such that L is ample, then, for all positive real numbers C and δ, the set

$$\{P\in X(K^{\mathrm{ac}})\,:\,h_{\overline{L}}(P)\leqslant C,\ [K(P):K]\leqslant\delta\}$$

is finite, where the height $h_{\overline{L}}(P)$ is defined as $\widehat{\deg}(P^*(L),P^*(\varphi))$. We refer to [**36**, Proposition 6.2.3] for more details.

1.17. • Let $(x_\alpha)_{\alpha\in A}$ be a family of indeterminates over $\mathbb{Q}$ and $K=\mathbb{Q}((x_\alpha)_{\alpha\in A})$. We assume that $\#(A)\leqslant\aleph_0$. Let $\mathcal{A}_{[0,1]^A}$ be the product σ-algebra (namely the smallest σ-algebra making measurable the projection maps to the coordinates) and $\nu_{[0,1]^A}$ be the product of the uniform probability measure on $[0,1]$.

• We define Ω_∞ to be

$$\Omega_\infty:=\Big\{(t_\alpha)_{\alpha\in A}\in[0,1]^A\,\Big|\,(\exp(2\pi i t_\alpha))_{\alpha\in A}\text{ is algebraically independent over }\mathbb{Q}\Big\}.$$

Note that $\Omega_\infty \in \mathcal{A}_{[0,1]^A}$ and $\nu_{[0,1]^A}([0,1]^A \setminus \Omega_\infty) = 0$. Let $\mathcal{A}_{\Omega_\infty}$ and ν_{Ω_∞} be the restrictions of $\mathcal{A}_{[0,1]^A}$ and $\nu_{[0,1]^A}$ to Ω_∞, respectively. For $t = (t_\alpha)_{\alpha\in A} \in \Omega_\infty$ and $f \in K$, $|f|_t$ is defined by

$$|f|_t := |f\big((\exp(2\pi i t_\alpha))_{\alpha\in A}\big)|_{\mathbb{C}},$$

where $|\cdot|_{\mathbb{C}}$ is the usual absolute value of $\mathbb{C}$. Note that $|\cdot|_t$ yields an archimedean absolute value of K.

• For $f \in K^\times$, one introduce $\mathbb{\mu}(f)$ as follows:

$$\mathbb{\mu}(f) := \int_{[0,1]^A} \log\left|f\big((\exp(2\pi i t_\alpha))_{\alpha\in A}\big)\right| d\nu_{[0,1]^A}.$$

• Let $\{T_0\} \cup \{T_\alpha\}_{\alpha\in A}$ be a set of indeterminates over $\mathbb{Q}$ such that $x_\alpha = T_\alpha/T_0$. Let Ω_h be the set of all homogeneous, principal, prime and non-zero ideals in $\mathbb{Z}[T_0, (T_\alpha)_{\alpha\in A}]$. For each $\omega \in \Omega_h$, let P_ω be a defining homogeneous polynomial of ω. Note that P_ω is uniquely determined up to ± 1. We fix $\lambda \in \mathbb{R}_{\geqslant 0}$. For $\omega \in \Omega_h$, a nonarchimedean absolute value $|\cdot|_\omega$ on K is defined to be

$$|f|_\omega := C_\omega^{-\operatorname{ord}_\omega(f)} \quad (\forall f \in K),$$

where

$$C_\omega = \exp\left(\lambda \deg(P_\omega) + \mathbb{\mu}(P_\omega(1, \{x_\alpha\}_{\alpha\in A}))\right).$$

• Let Ω_v be the set of all prime number of $\mathbb{Z}$. For $p \in \Omega_v$, let $|\cdot|_p$ be the p-adic absolute value of $\mathbb{Z}$ with $|p|_p = 1/p$. For $f \in \mathbb{Q}[\{x_\alpha\}_{\alpha\in A}]$, let C_f be the set of all coefficients of f. If we set

$$|f|_p = \max_{a\in C_f}\{|a|_p\},$$

then, by Gauss' lemma, one can see that $|fg|_p = |f|_p|g|_p$, so that $|\cdot|_p$ extends to an absolute value of $\mathbb{Q}((x_\alpha)_{\alpha\in A})$.

• We set $\Omega_{\text{fin}} := \Omega_h \coprod \Omega_v$. A measure space $(\Omega_{\text{fin}}, \mathcal{A}_{\text{fin}}, \nu_{\text{fin}})$ is the discrete measure space on Ω_{fin} such that $\nu_{\text{fin}}(\{\omega\}) = 1$ for all $\omega \in \Omega_{\text{fin}}$.

• An adelic structure of $\mathbb{Q}((x_\alpha)_{\alpha\in A})$ given by

$$(\Omega_{\text{fin}}, \mathcal{A}_{\text{fin}}, \nu_{\text{fin}}) \coprod (\Omega_\infty, \mathcal{A}_\infty, \nu_\infty)$$

is denoted by $S_{A,\lambda}$.

• It is known that the following facts hold (cf. [**38**, Proposition 2.7.10 and Proposition 2.7.14]):

(1) $S_{A,\lambda}$ is proper.
(2) If $\lambda > 0$ and A is finite, then $S_{A,\lambda}$ has Northcott property.

1.18. Let K be a countable field. Let $(x_\alpha)_{\alpha\in A}$ be a transcendental basis of K over $\mathbb{Q}$. As described in 1.17, $\mathbb{Q}((x_\alpha)_{\alpha\in A})$ has an adelic structure $S_{A,\lambda}$, which extends to K because K is algebraic over $\mathbb{Q}((x_\alpha)_{\alpha\in A})$. Note that if $\lambda > 0$, then Northcott property holds for any finitely generated subfield of K (cf. [**38**, Theorem 2.7.18]).

CHAPTER 2

Metric families on vector bundles

The purpose of this chapter is to generalize dominancy and measurability conditions in [**36**, Chapter 6] to metrized locally free modules of finite rank, and to develop related topics. These constructions will be useful further in the extension of the arithmetic Hilbert-Samuel formula to the case with a tensor product by a metrized torsion-free sheaf. Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve as introduced in §1.1. In the first section, we introduce the notion of metric family for vector bundles on a quasi-projective scheme over $\operatorname{Spec} K$. In the second section, we discuss the conditions of measurability and dominancy of metric families in making links to the tautological invertible sheaf of the projective bundle. In the third section we prove the dominancy and measurability of the dual metric, which allows to discuss the dual adelic vector bundle and also adelic vector subbundle. In the fourth section, we extend the notion of metric families to the setting of torsion-free sheaves which are locally free on a Zariski dense open subset and discuss the norm family structure on the sectional space.

2.1. Metric family

Let $p : X \to \operatorname{Spec} K$ be a quasi-projective scheme over $\operatorname{Spec} K$. Let E be a vector bundle on X, that is, a locally free $\mathcal{O}_X$-module of finite rank. For any $\omega \in \Omega$, let E_ω be the restriction of E to $X_\omega = X \times_{\operatorname{Spec} K} \operatorname{Spec} K_\omega$ and ψ_ω be a metric on E_ω. By definition ψ_ω is a family $(|\cdot|_{\psi_\omega}(x))_{x \in X_\omega^{\mathrm{an}}}$ parametrized by X_ω^{an}, where each $|\cdot|_{\psi_\omega}(x)$ is a norm on $E_\omega(x) := E_\omega \otimes_{\mathcal{O}_{X_\omega}} \widehat{\kappa}(x)$. We assume that the norm $|\cdot|_{\psi_\omega}(x)$ is ultrametric if the absolute value $|\cdot|_\omega$ is non-Archimedean. Moreover, we assume that the metric ψ_ω is continuous, namely, for any section s of E over a Zariski open subset U of X_ω, the function

$$(x \in U^{\mathrm{an}}) \longmapsto |s|_{\psi_\omega}(x)$$

is continuous. The data $\psi = (\psi_\omega)_{\omega \in \Omega}$ is called a *metric family* on the vector bundle E.

Assume that X is projective and geometrically reduced. For any $\omega \in \Omega$, we denote by $\|\cdot\|_{\psi_\omega}$ the supremum norm on $H^0(X_\omega, E_\omega)$, which is defined as

$$\forall\, s \in H^0(X_\omega, E_\omega), \quad \|s\|_{\psi_\omega} = \sup_{x \in X_\omega^{\mathrm{an}}} |s(x)|_{\psi_\omega}(x).$$

We denote by $p_*(E, \psi)$ the couple $(H^0(X, E), (\|\cdot\|_{\psi_\omega})_{\omega \in \Omega})$.

If φ and ψ are two metric families of E. For any $\omega \in \Omega$ we denote by $d_\omega(\varphi, \psi)$ the element

$$\sup_{x \in X_\omega^{\mathrm{an}}} \sup_{s \in E_\omega(x) \setminus \{0\}} \Big| \ln |s|_{\varphi_\omega}(x) - \ln |s|_{\psi_\omega}(x) \Big| \in [0, +\infty],$$

which is called the *local distance* at ω between φ and ψ.

We denote by $\mathcal{O}_E(1)$ the universal invertible sheaf on the projective bundle $\pi : \mathbb{P}(E) \to \operatorname{Spec} K$. For any $\omega \in \Omega$, the metric ψ_ω induces by passing to quotient a continuous metric on $\mathcal{O}_E(1)_\omega \cong \mathcal{O}_{E_\omega}(1)$, which we denote by $\psi_\omega^{\mathrm{FS}}$. Recall that, if y is an element of $\mathbb{P}(E_\omega)^{\mathrm{an}}$

and $x = \pi_\omega^{\mathrm{an}}(y)$, then the norm $|\cdot|_{\psi_\omega^{\mathrm{FS}}}$ on $\mathcal{O}_E(1)_y$ is the quotient metric induced by the universal surjective homomorphism

$$E_\omega(x) \otimes_{\widehat{\kappa}(x)} \widehat{\kappa}(y) \longrightarrow \mathcal{O}_E(1)_y,$$

where we consider the ε-extension of $|\cdot|_{\psi_\omega}(x)$ to $E_\omega(x) \otimes_{\widehat{\kappa}(x)} \widehat{\kappa}(y)$ if $|\cdot|_\omega$ is non-Archimedean, and π-extension of $|\cdot|_{\psi_\omega}(x)$ if $|\cdot|_\omega$ is Archimedean (see [**36**, §1.3 and §2.2.3]). Note that, if φ and ψ are two metric families of E, then one has (see [**36**, Proposition 2.2.20])

$$\forall\, \omega \in \Omega, \quad d_\omega(\varphi^{\mathrm{FS}}, \psi^{\mathrm{FS}}) \leqslant d_\omega(\varphi, \psi). \tag{2.1}$$

2.2. Dominancy and measurability

Throughout this section, we fix a projective scheme X over $\operatorname{Spec} K$.

Definition 2.2.1. Let E be a vector bundle on X.

(1) We say the metric family $\psi = (\psi_\omega)_{\omega \in \Omega}$ on the locally free $\mathcal{O}_X$-module E is *dominated* (resp. *measurable*) if the metric family $\psi^{\mathrm{FS}} = (\psi_\omega^{\mathrm{FS}})_{\omega \in \Omega}$ on $\mathcal{O}_E(1)$ is dominated (resp. measurable). We refer the readers to [**36**, Definitions 6.1.9 and 6.1.27] for the dominancy and measurability conditions of metrized line bundles.

(2) We say (E, ψ) is an *adelic locally free $\mathcal{O}_X$-module* or an *adelic vector bundle* if ψ is dominated and measurable, or equivalently, $(\mathcal{O}_E(1), \psi^{\mathrm{FS}})$ is an adelic line bundle on $\mathbb{P}(E)$.

Proposition 2.2.2. (1) *If ψ is dominated, then the norm family*

$$\xi_\psi = (\|\cdot\|_{\psi_\omega})_{\omega \in \Omega}$$

on $H^0(X, E)$ is strongly dominated.

(2) *If the metric family ψ on E is measurable, then the norm family ξ_ψ on $H^0(X, E)$ is measurable.*

Proof. If we identify $H^0(X, E)$ with $H^0(\mathbb{P}(E), \mathcal{O}_E(1))$, then for any $\omega \in \Omega$ one has $\|\cdot\|_{\psi_\omega} = \|\cdot\|_{\psi_\omega^{\mathrm{FS}}}$ by [**36**, Remark 2.2.14]. Therefore the assertions follow from [**36**, Theorems 6.1.13 and 6.1.32]. □

Proposition 2.2.3. *Let E be a vector bundle on X, and φ and ψ be two metric families of E. Suppose that φ is dominated and that the local distance function*

$$(\omega \in \Omega) \longmapsto d_\omega(\varphi, \psi)$$

is bounded from above by an integrable function. Then the metric family ψ is also dominated.

Proof. This is a consequence of [**36**, Proposition 6.1.12] and (2.1). □

Definition 2.2.4. Let $f : Y \to X$ be a projective K-morphism from a geometrically reduced projective K-scheme Y to X. Let E be a vector bundle on X and $\psi = (\psi_\omega)_{\omega \in \Omega}$ be a metric family on E. We denote by $f^*(\psi)$ the metric family on $f^*(E)$ such that, for any $y \in Y_\omega^{\mathrm{an}}$, the norm $|\cdot|_{f^*(\psi)_\omega}(y)$ on

$$f^*(E)_\omega(y) = E_\omega(x) \otimes_{\widehat{\kappa}(x)} \widehat{\kappa}(y)$$

is induced by $|\cdot|_{\psi_\omega}(f^{\mathrm{an}}(y))$ by ε-extension of scalars in the case where $|\cdot|_\omega$ is non-Archimedean, and by π-extension of scalars if $|\cdot|_\omega$ is Archimedean.

Proposition 2.2.5. *We keep the notation and the assumptions of Definition 2.2.4. Suppose that the metric family ψ is dominated* (*resp. measurable*)*, then its pull-back $f^*(\psi)$ is also dominated* (*resp. measurable*)*.*

Proof. The universal property of projective bundle induces a projective morphism $F : \mathbb{P}(f^*(E)) \to \mathbb{P}(E)$ such that the following diagramme is cartesian.

$$\begin{array}{ccc} \mathbb{P}(f^*(E)) & \xrightarrow{F} & \mathbb{P}(E) \\ {\scriptstyle \pi_{f^*(E)}}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ Y & \xrightarrow[f]{} & X \end{array}$$

Moreover, one has $\mathcal{O}_{f^*(E)}(1) \cong F^*(\mathcal{O}_E(1))$ and $F^*(\psi^{\mathrm{FS}}) = f^*(\psi)^{\mathrm{FS}}$. Hence the assertion follows from [**36**, Propositions 6.1.12 and 6.1.28]. □

Definition 2.2.6. Let E be a vector bundle on X and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family of E. If F is a vector subbundle of E, for any $\omega \in \Omega$ and any $x \in X_\omega^{\mathrm{an}}$, we denote by $|\cdot|_{\psi_{F,\omega}}(x)$ the restriction of $|\cdot|_{\psi_\omega}(x)$ to $F_\omega(x)$. Note that $\psi_F = (\psi_{F,\omega})_{\omega\in\Omega}$ forms a metric family of F, called the *restriction of* ψ *to* F. Similarly, if G is a quotient vector bundle of E, we denote by $|\cdot|_{\psi_{G,\omega}}(x)$ the quotient norm of $|\cdot|_{\psi_\omega}(x)$ on $G_\omega(x)$. Then $\psi_G = (\psi_{G,\omega})_{\omega\in\Omega}$ is a metric family of G, called the *quotient metric family* of ψ on G.

Proposition 2.2.7. *Let E be a vector bundle on X and G be a quotient vector bundle of E. Let ψ be a metric family on E. If ψ is dominated* (*resp. measurable*)*, then the quotient metric family ψ_G is also dominated* (*resp. measurable*)*.*

Proof. Let $i : \mathbb{P}(G) \to \mathbb{P}(E)$ be the closed embedding induced by the quotient homomorphism $E \to G$. Then one has $i^*(\psi^{\mathrm{FS}}) = \psi_G^{\mathrm{FS}}$. Hence the assertion of the proposition follows from [**36**, Propositions 6.1.12 and 6.1.28]. □

Definition 2.2.8. Let E and F be vector bundles on X, equipped with metric families ψ_E and ψ_F, respectively. For any $\omega \in \Omega$ and any $x \in X_\omega^{\mathrm{an}}$, if $|\cdot|_\omega$ is non-Archimedean, we denote by $|\cdot|_{(\psi_E\otimes\psi_F)_\omega}(x)$ the ε-tensor product of the norms $|\cdot|_{\psi_{E,\omega}}(x)$ and $|\cdot|_{\psi_{F,\omega}}(x)$, if $|\cdot|_\omega$ is Archimedean, we denote by $|\cdot|_{(\psi_E\otimes\psi_F)_\omega}(x)$ the π-tensor product of the norms $|\cdot|_{\psi_{E,\omega}}(x)$ and $|\cdot|_{\psi_{F,\omega}}(x)$. Thus we obtain a metric family $\psi_E \otimes \psi_F$ on the vector bundle $E \otimes F$, called the tensor product of metric families ψ_E and ψ_F. In the case where one of the vector bundles E and F is of rank 1, we also write the tensor product metric family of ψ_E and ψ_F in an additive way as $\psi_E + \psi_F$.

Proposition 2.2.9. *Let E and F be vector bundles on X, equipped with metric families ψ_E and ψ_F respectively. We assume that E is a line bundle. If both metric families ψ_E and ψ_F are dominated* (*resp. measurable*)*, then $\psi_E + \psi_F$ is also dominated* (*resp. measurable*)*.*

Proof. Since E is of rank 1, we can identify $\mathbb{P}(E \otimes F)$ with $\mathbb{P}(F)$. Moreover, if we denote by $\pi : \mathbb{P}(F) \to X$ the structural morphism, one has $\mathcal{O}_{E\otimes F}(1) = \pi^*(E) \otimes \mathcal{O}_F(1)$, and the metric family $(\psi_E + \psi_F)^{\mathrm{FS}}$ identifies with the tensor product of $\pi^*(\psi_E)$ and ψ_F^{FS}. Hence the assertions follow from [**36**, Propositions 6.1.12 and 6.1.28]. □

Proposition 2.2.10. *Let E be a vector bundle on X. Then there exists a dominated and measurable metric family of E.*

Proof. Let L be an ample line bundle on X and φ be a dominated and measurable metric family of $L^\vee$. Then, one can find a positive integer m such that $L^m \otimes E$ is ample and generated by global sections. If $L^m \otimes E$ has a dominated and measurable metric family ψ', then the tensor product of $m\varphi$ with ψ' is a dominated and measurable metric family of E by Proposition 2.2.9, so we may assume that E is ample and generated by global sections.

Let $H^0(X,E)\otimes O_X \to E$ be the natural surjective homomorphism. Fix a basis $e_1,\ldots,e_N$ of $H^0(X,E)$ and, for each $\omega\in\Omega$ and $(a_1,\ldots,a_N)\in K_\omega^N$, we set

$$\|a_1e_1+\cdots+a_Ne_N\|_\omega=\begin{cases}\sqrt{|a_1|_\omega^2+\cdots+|a_N|_\omega^2} & \text{if } \omega\in\Omega_\infty,\\ \max\{|a_1|_\omega,\ldots,|a_N|_\omega\} & \text{if } \omega\in\Omega\setminus\Omega_\infty,\end{cases}$$

and ξ be the norm family $(\|\cdot\|_\omega)_{\omega\in\Omega}$. Let ψ be the quotient metric family of E induced by $H^0(X,E)\otimes O_X\to E$ and ξ. Let $\pi:\mathbb{P}(E)\to X$ be the projective bundle of E and $O_E(1)$ be the tautological line bundle of $\mathbb{P}(E)$. Note that the metric family ψ^{FS} of $O_E(1)$ is induced by $H^0(X,E)\otimes O_{\mathbb{P}(E)}\to O_E(1)$ and ξ, so it is dominated and measurable. Thus the assertion follows. □

2.3. Dual metric family

In this section, let X be a projective scheme over $\operatorname{Spec} K$.

Definition 2.3.1. Let E be a vector bundle on X, equipped with a metric family $\psi=(\psi_\omega)_{\omega\in\Omega}$. For any $\omega\in\Omega$ and any $x\in X_\omega^{\mathrm{an}}$, the norm $|\cdot|_{\psi_\omega}(x)$ on $E_\omega(x)$ induces a dual norm on $E_\omega(x)^\vee$, which we denote by $|\cdot|_{\psi_\omega^\vee}(x)$. It turns out that $\psi_\omega^\vee=(|\cdot|_{\psi_\omega^\vee}(x))_{x\in X_\omega^{\mathrm{an}}}$ forms a continuous metric on $E_\omega^\vee$. Hence $\psi^\vee=(\psi_\omega^\vee)_{\omega\in\Omega}$ is a metric family on $E^\vee$, called the *dual metric family of* ψ.

Proposition 2.3.2. *Let E be a vector bundle on X and ψ be a metric family of E. If ψ is dominated, then the dual metric family $\psi^\vee$ is also dominated.*

Proof. Let $\pi_E:\mathbb{P}(E)\to X$ and $\pi_{E^\vee}:\mathbb{P}(E^\vee)\to X$ be the projective bundles associated with E and $E^\vee$ respectively. We consider the fiber product $\mathbb{P}(E)\times_X\mathbb{P}(E^\vee)$ of projective bundles and denote by

$$p_1:\mathbb{P}(E)\times_X\mathbb{P}(E^\vee)\longrightarrow\mathbb{P}(E)\quad\text{and}\quad p_2:\mathbb{P}(E)\otimes_X\mathbb{P}(E^\vee)\longrightarrow\mathbb{P}(E^\vee)$$

the morphisms of projection. Let

$$O_E(1)\boxtimes O_{E^\vee}(1):=p_1^*(O_E(1))\otimes p_2^*(O_{E^\vee}(1))$$

and let

$$s\in H^0(\mathbb{P}(E)\times_X\mathbb{P}(E^\vee),O_E(1)\boxtimes O_{E^\vee}(1))$$

be the trace section of $O_E(1)\boxtimes O_{E^\vee}(1)$, which corresponds to the composition of the following universal homomorphisms

$$p_2^*(O_{E^\vee}(-1))\longrightarrow p_2^*(\pi_{E^\vee}^*(E))\cong p_1^*(\pi_E^*(E))\longrightarrow p_1^*(O_E(1)).$$

Claim 2.3.3. *Let $\psi_1=(\psi_{1,\omega})_{\omega\in\Omega}$ and $\psi_2=(\psi_{2,\omega})_{\omega\in\Omega}$ be metric families on E and $E^\vee$ respectively. We equip $O_E(1)\boxtimes O_{E^\vee}(1)$ with the metric family $\varphi=(\varphi_\omega)_{\omega\in\Omega}$ which is the tensor product of the metric families $p_1^*(\psi_1^{\mathrm{FS}})$ and $p_2^*(\psi_2^{\mathrm{FS}})$. Then, for any $\omega\in\Omega$ and $x\in X_\omega^{\mathrm{an}}$, one has*

$$\sup_{f\in E_\omega^\vee(x)\setminus\{0\}}\frac{|f|_{\psi_{1,\omega}^\vee}(x)}{|f|_{\psi_{2,\omega}}(x)}\leqslant\|s\|_{\varphi_\omega},$$

where s is the trace section of $O_E(1)\boxtimes O_{E^\vee}(1)$ defined above.

Proof. For a non-zero element f of $E_\omega^\vee(x)$, the one-dimensional $\widehat{\kappa}(x)$-vector space of $E_\omega^\vee(x)$ spanned by f determines a point $P_f\in P(E_\omega)^{\mathrm{an}}$ valued in $(\widehat{\kappa}(x),|\cdot|_x)$ which lies over $x \in X_\omega^{\mathrm{an}}$. Suppose that Q is a point of $\mathbb{P}(E_\omega^\vee)^{\mathrm{an}}$ valued in $(\widehat{\kappa}(x), |\cdot|_x)$ which lies over x. Then $s(P_f, Q)$ corresponds to the following composition of universal homomorphisms

$$\mathcal{O}_{E^\vee}(-1)(Q) \longrightarrow E_\omega(x) \longrightarrow \mathcal{O}_E(1)(P_f), \tag{2.2}$$

and $|s|_{\varphi_\omega}(P_f, Q)$ is the operator norm of this homomorphism. We pick an arbitrary non-zero element ℓ of $\mathcal{O}_{E^\vee}(-1)(Q)$. The dual element in $\mathcal{O}_E(-1)(P_f)$ of the image of ℓ by (2.2) is $f(\ell)^{-1}f$. Therefore one has

$$|s|_{\varphi_\omega}(P_f, Q) = \frac{|f(\ell)|_x}{|\ell|_{\psi_{1,\omega}}(x) \cdot |f|_{\psi_{2,\omega}}(x)} \leqslant \|s\|_{\varphi_\omega}.$$

Taking the supremum with respect to ℓ, we obtain the required inequality. □

In the above claim, if both metric families ψ_1 and ψ_2 are dominated, then the metric family φ on $\mathcal{O}_E(1) \boxtimes \mathcal{O}_{E^\vee}(1)$ is also dominated. In particular, the function

$$(\omega \in \Omega) \longmapsto \ln \|s\|_{\varphi_\omega}$$

is bounded from above by an integrable function. Then the claim shows that the function

$$(\omega \in \Omega) \longmapsto \sup_{x \in X_\omega^{\mathrm{an}}} \sup_{f \in E_\omega^\vee(x)\setminus\{0\}} \Big(\ln |f|_{\psi_{1,\omega}^\vee}(x) - \ln |f|_{\psi_{2,\omega}}(x) \Big)$$

is bounded from above by an integrable function. Therefore, the function

$$(\omega \in \Omega) \longmapsto \sup_{Q \in \mathbb{P}(E^\vee)^{\mathrm{an}}} \sup_{\substack{f \in \mathcal{O}_{E^\vee}(1)(Q) \\ f \neq 0}} \Big(\ln |f|_{\psi_{1,\omega}^{\vee,\mathrm{FS}}}(Q) - \ln |f|_{\psi_{2,\omega}^{\mathrm{FS}}}(Q) \Big)$$

is bounded from above by an integrable function. For the same reason, by interchanging the roles of E and $E^\vee$ we obtain that the function

$$(\omega \in \Omega) \longmapsto \sup_{P \in \mathbb{P}(E)^{\mathrm{an}}} \sup_{\substack{t \in \mathcal{O}_E(1)(P) \\ t \neq 0}} \Big(\ln |t|_{\psi_{2,\omega}^{\vee,\mathrm{FS}}}(P) - \ln |t|_{\psi_{1,\omega}^{\mathrm{FS}}}(P) \Big)$$

is also bounded from above by an integrable function. In particular, if we denote by $\widetilde{\varphi}$ the tensor product of the metric families $p_1^*(\psi_2^{\vee,\mathrm{PS}})$ and $p_2^*(\psi_1^{\vee,\mathrm{PS}})$, then the function

$$(\omega \in \Omega) \longmapsto \ln \|s\|_{\widetilde{\varphi}_\omega}$$

is still bounded from above by an integrable function. Hence the above claim implies that the function

$$(\omega \in \Omega) \longmapsto \sup_{x \in X_\omega^{\mathrm{an}}} \sup_{f \in E_\omega^\vee(x)\setminus\{0\}} \Big(\ln |f|_{\psi_{2,\omega}}(x) - \ln |f|_{\psi_{1,\omega}^\vee}(x) \Big)$$

is bounded from above by an integrable function. Therefore we obtain that the local distance function

$$(\omega \in \Omega) \longmapsto d_\omega(\psi_1^\vee, \psi_2)$$

is bounded from above by an integrable function. By Proposition 2.2.3, the metric family $\psi_1^\vee$ is dominated. By Proposition 2.2.10, there exists at least a dominated metric family on $E^\vee$, the assertion is thus proved. □

Definition 2.3.4. Let E be a vector bundle on X, $\psi = (\psi_\omega)_{\omega \in \Omega}$ be a metric family of E. Let K'/K be a finite extension and let $P : \operatorname{Spec} K' \to X$ be a K-morphism. Let

$$(K', (\Omega', \mathcal{A}', \nu'), \phi') = S \otimes_K K'.$$

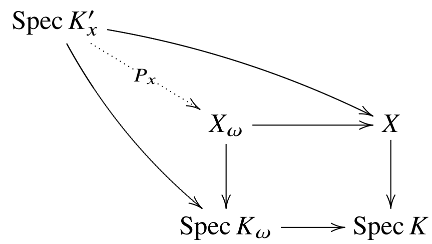
Recall that Ω' is a disjoint union

$$\Omega' = \coprod_{\omega \in \Omega} \Omega'_\omega,$$

where Ω'_ω denotes the set of all absolute values on Ω' extending $|\cdot|_\omega$. For any $\omega \in \Omega$ and any $x \in \Omega'_\omega$, we let $P_x : \operatorname{Spec} K'_x \to X_\omega$ be the morphism induced by

$$\operatorname{Spec} K'_x \longrightarrow \operatorname{Spec} K' \xrightarrow{P} X$$

and the canonical morphism $\operatorname{Spec} K'_x \to \operatorname{Spec} K_\omega$.



We denote by $\|\cdot\|_x$ the norm on

$$(E \otimes_K K') \otimes_{K'} K'_x \cong E_\omega \otimes_{K_\omega} K'_x$$

which is induced by $|\cdot|_{\psi_\omega}(P_x)$ by ε-extension of scalars if $|\cdot|_\omega$ is non-Archimedean, and by π-extension of scalars if $|\cdot|_\omega$ in Archimedean. Then, $(\|\cdot\|_x)_{x\in\Omega'}$ forms a norm family of $P^*(E)$, which we denote by $P^*(\psi)$.

Definition 2.3.5. Let Ω_0 be the set of $\omega \in \Omega$ such that the absolute value $|\cdot|_\omega$ is trivial. Let $x = (K_x, |\cdot|_x, P_x)$ be a triplet, where $(K_x, |\cdot|_x)$ is a valued extension of the trivially valued field $(K, |\cdot|_0)$, and $P_x : \operatorname{Spec} K_x \to X$ is a K-morphism. Assume that E is a vector bundle on X and $\psi = (\psi_\omega)_{\omega\in\Omega}$ is a metric family of E. Denote by E_x the K_x-vector space $P_x^*(E)$. We consider the following adelic curve

$$S_x = (K_x, (\Omega_0, \mathcal{A}_0, \nu_0), (|\cdot|_x)_{\omega\in\Omega_0}),$$

where $\mathcal{A}_0$ is the restriction of the σ-algebra $\mathcal{A}$ to Ω_0, and ν_0 is the restriction of ν to $(\Omega_0, \mathcal{A}_0)$. We denote by $x^*(\psi)$ the norm family $(|\cdot|_{\psi_\omega}(P_x^\omega))_{\omega\in\Omega_0}$ on E_x, where P_x^ω denotes the point of X_ω^{an} determined by $(P_x, |\cdot|_x)$.

Proposition 2.3.6. *Let E be a vector bundle on X and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family of E. Then the metric family ψ is measurable if and only if both of the following conditions are satisfied:*

(1) *for any finite extension K'/K and any K-morphism $P : \operatorname{Spec} K' \to X$, the norm family $P^*(\psi)$ is measurable,*
(2) *for any triplet $x = (K_x, |\cdot|_x, P_x)$, where $(K_x, |\cdot|_x)$ is a valued extension of transcendence degree* = 1 *and of rational exponent* (*see* §1.11) *of the trivially valued field $(K, |\cdot|_0)$, and $P_x : \operatorname{Spec} K_x \to X$ is a K-morphism, the norm family $x^*(\psi)$ is measurable.*

Proof. It suffices to treat the case where the field K is countable. Recall that the measurability of the metric family ψ signifies that the following two conditions are satisfied (see §1.12):

(1') for any finite extension K'/K and any K-morphism $Q : \operatorname{Spec} K' \to \mathbb{P}(E)$, the norm family $Q^*(\psi^{\mathrm{FS}})$ is measurable,
(2') for any triplet $y = (K_y, |\cdot|_y, Q_y)$, where $(K_y, |\cdot|_y)$ is a valued extension of transcendence degree = 1 and of rational exponent of the trivially valued field $(K, |\cdot|_0)$, and $Q_y : \operatorname{Spec} K_y \to \mathbb{P}(E)$ is a K-morphism, the norm family $Q_y^*(\psi^{\mathrm{FS}})$ is measurable.

Let K'/K be a finite extension. Any K-morphism $Q : \operatorname{Spec} K' \to \mathbb{P}(E)$ corresponds to a K-morphisme $P : \operatorname{Spec} K' \to X$ together with a one-dimensional quotient vector space L of $P^*(E)$, which identifies with $Q^*(O_E(1))$. Moreover, the norm family $Q^*(\psi^{\mathrm{FS}})$ identifies with the quotient norm family of $P^*(\psi)$. If the norm family $P^*(\psi)$ is measurable, by [**36**, Proposition 4.1.24], we obtain that $Q^*(\psi^{\mathrm{FS}})$ is also measurable. Conversely, if for any one-dimensional quotient vector space of $P^*(E)$, the quotient norm family of $P^*(\psi)$ on it is measurable, by passing to dual we obtain from [**36**, Proposition 4.1.24] that $P^*(\psi)^\vee$ is measurable and therefore $P^*(\psi)$ is also measurable.

Let $x = (K_x, |\cdot|_x, P_x)$ be a triplet, where $(K_x, |\cdot|_x)$ is a valued extension of transcendence degree = 1 and rational exponent of the trivially valued field $(K, |\cdot|_0)$, and $P_x : \operatorname{Spec} K_x \to X$ is a K-morphism. Note that the field K_x is countable. Similarly to the above argument, the norm family $P_x^*(\psi)$ is measurable if and only if all its quotient norm families on one-dimensional quotient subspaces are measurable. The proposition is thus proved. □

Proposition 2.3.7. *Let E be a vector bundle on X and $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family on E. If ψ is measurable, then the dual metric family $\psi^\vee$ of $E^\vee$ is also measurable.*

Proof. Let K'/K be a finite extension and $P : \operatorname{Spec} K' \to X$ be a K-morphism. If $P^*(\psi)$ is measurable, by [**36**, Proposition 4.1.24] we obtain that $P^*(\psi^\vee) = P^*(\psi)^\vee$ is measurable. Similarly, for any triplet $x = (K_x, |\cdot|_x, P_x)$, where $(K_x, |\cdot|_x)$ is a valued extension of transcendence degree $\leqslant 1$ and of rational exponent of the trivially valued field $(K, |\cdot|_0)$ and $P_x : \operatorname{Spec} K_x \to X$ is a K-morphism, if the norm family $P_x^*(\psi)$ is measurable, then $P_x^*(\psi^\vee) = P_x^*(\psi)^\vee$ is also measurable. The proposition is thus proved. □

Corollary 2.3.8. *Let E be a vector bundle on X, F be a vector subbundle of E, ψ_E be a metric family of E, and ψ_F be the restriction of ψ_E to F. If the metric family ψ_E is dominated* (*resp. measurable*)*, then the restricted metric family ψ_F is also dominated* (*resp. measurable*)*.*

Proof. The homomorphism of inclusion $F \to E$ induces by passing to dual a surjective homomorphism $E^\vee \to F^\vee$. Thus $F^\vee$ can be considered as a quotient vector bundle of $E^\vee$. Note that $\psi_F^\vee$ identifies with the quotient metric family of $\psi_E^\vee$. Hence the assertion follows from Propositions 2.3.2, 2.3.7 and 2.2.7. □

2.4. Metric families on torsion-free sheaves

In this section, we assume that the K-scheme X is geometrically integral.

Definition 2.4.1. Let E be a torsion free O_X-module and U be a non-empty Zariski open set of X such that $E|_U$ is a vector bundle. For any $\omega \in \Omega$, let ψ_ω be a continuous metric of E_ω over U_ω^{an} such that, for any $s \in H^0(X_\omega, E_\omega)$,

$$\|s\|_{\psi_\omega} := \sup\{|s|_{\psi_\omega}(x) \;:\; x \in U_\omega^{\mathrm{an}}\} < +\infty.$$

We set $\psi = (\psi_\omega)_{\omega\in\Omega}$ and $\xi_\psi = (\|\cdot\|_{\psi_\omega})_{\omega\in\Omega}$. We say that (E, U, ψ) is a *sectionally adelic torsion free O_X-module* if $(H^0(X, E), \xi_\psi)$ is an adelic vector bundle on S. By Proposition 2.2.9, if (E, ψ) is an adelic vector bundle on X, then, for any non-empty Zariski open set U of X, (E, U, ψ) is a sectionally adelic torsion free O_X-module.

Definition 2.4.2. Let E be a torsion free O_X-module and U be a non-empty Zariski open set of X such that $E|_U$ is a vector bundle. Let $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a metric family of $E|_U$. We say (E, U, ψ) is a *birationally adelic torsion free O_X-module* if it satisfies the following properties:

(1) There exist a birational morphism $\mu : X' \to X$ of geometrically integral projective schemes over K such that $\mu^{-1}(U) \to U$ is an isomorphism, an adelic vector bundle (E', ψ') on X', and an injective morphism of $\mathcal{O}_{X'}$-modules $E \to \mu_*(E')$ whose restriction to U gives an isomorphism $E|_U \to \mu_*(E')|_U \cong E'|_{\mu^{-1}(U)}$.
(2) The isomorphism $E|_U \to E'|_{\mu^{-1}(U)}$ yields an isometry

$$(E, \psi)|_U \longrightarrow (E', \psi')|_{\mu^{-1}(U)}.$$

By definition, for $s \in H^0(X, E)$ and each $\omega \in \Omega$,

$$\|s\|_{\psi_\omega} := \sup\{|s|_{\psi_\omega}(\xi) \,:\, \xi \in U_\omega^{\mathrm{an}}\}$$

belongs to $\mathbb{R}_{\geqslant 0}$. Note that $\|\cdot\|_{\psi_\omega}$ is the restriction of $\|\cdot\|_{\psi'_\omega}$ to $H^0(X, E)$ by using the injective homomorphism $H^0(X, E) \to H^0(X', E')$, so that

$$(H^0(X, E), (\|\cdot\|_{\psi_\omega})_{\omega\in\Omega})$$

is an adelic vector bundle on S, that is, a birationally adelic torsion free $\mathcal{O}_X$-module is sectionally adelic in the sense of Definition 2.4.1.

Lemma 2.4.3. *Let $\pi : X \to Y$ be a continuous map of locally compact Hausdorff spaces such that π is open and proper. Let $f : X \to \mathbb{R}$ be a continuous function on X and $\tilde{f} : Y \to \mathbb{R}$ be a function on Y given by*

$$\tilde{f}(y) = \max\{f(x) \,:\, \pi(x) = y\}.$$

Then $\tilde{f}$ is continuous on Y.

Proof. Fix $y_0 \in Y$. Since $\pi^{-1}(y_0)$ is compact, for $\varepsilon > 0$, there exist $x_1, \ldots, x_n \in \pi^{-1}(y_0)$ and open subsets $U_1, \ldots, U_n$ of X such that

$$\pi^{-1}(y_0) \subseteq U_1 \cup \cdots \cup U_n,$$

$x_i \in U_i$ for all $i \in \{1, \ldots, n\}$ and $|f(x) - f(x_i)| \leqslant \varepsilon$ for all $i \in \{1, \ldots, n\}$ and $x \in U_i$. If we set $Z = X \setminus U_1 \cup \cdots \cup U_n$, then $\pi(Z)$ is closed and $y_0 \notin \pi(Z)$. We choose an open subset W of Y such that $y_0 \in W$ and

$$W \subseteq \pi(U_1) \cap \cdots \cap \pi(U_n) \cap (Y \setminus \pi(Z)).$$

Note that $\pi^{-1}(W) \subseteq U_1 \cup \cdots \cup U_n$. Let $y \in W$ and

$$\lambda_i = \sup\{f(x) \,:\, x \in U_i \text{ and } y = \pi(x)\}.$$

Then $\tilde{f}(y) = \max\{\lambda_1, \ldots, \lambda_n\}$ and $\lambda_i - \varepsilon \leqslant f(x_i) \leqslant \lambda_i + \varepsilon$ for all $i \in \{1, \ldots, n\}$, so that

$$\tilde{f}(y) - \varepsilon \leqslant \tilde{f}(y_0) \leqslant \tilde{f}(y) + \varepsilon,$$

as required. □

Let $\pi : X \to Y$ be a generically finite morphism of geometrically integral projective schemes over $\operatorname{Spec} K$ and (M, U, ψ) be a sectionally adelic torsion free $\mathcal{O}_X$-module. Note that $\pi_*(M)$ is a torsion free $\mathcal{O}_Y$-module. The pushforward $\pi_*(\psi)$ is defined as follows: We choose a non-empty Zariski open set V of Y such that

$$\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \longrightarrow V$$

is étale and $\pi^{-1}(V) \subseteq U$. Note that $\pi_*(M)$ is locally free over V. For $y \in V_\omega^{\mathrm{an}}$ and $s \in \pi_*(M) \otimes \hat{\kappa}(y)$, $|s|_{\pi_*(\psi)_\omega}(y)$ is defined to be

$$|s|_{\pi_*(\psi_\omega)}(y) := \max\{|s|_{\psi_\omega}(x) \,:\, x \in (\pi_\omega^{\mathrm{an}})^{-1}(y)\}.$$

Since $\pi^{-1}(V)^{\mathrm{an}}_\omega \to V^{\mathrm{an}}_\omega$ is proper and open (for example, [**9**, Lemma 3.2.4]), by Lemma 2.4.3, $\pi_*(\psi)_\omega$ yields a continuous metric of $\pi_*(M)_\omega$ over V^{an}_ω. We denote $(\pi_*(\psi)_\omega)_{\omega\in\Omega}$ by $\pi_*(\psi)$. For $s \in H^0(Y, \pi_*(M)) = H^0(X, M)$, as

$$\sup\{|s|_{\pi_*(\psi)_\omega}(y) \,:\, y \in V^{\mathrm{an}}_\omega\} = \sup\{|s|_{\psi_\omega}(x) \,:\, x \in \pi^{-1}(V)^{\mathrm{an}}_\omega\},$$

one has $\|s\|_{\pi_*(\psi)_\omega} = \|s\|_{\psi_\omega} < \infty$, so that $(\pi_*(M), V, \pi_*(\psi))$ forms a sectionally adelic torsion free $\mathcal{O}_Y$-module and $(H^0(Y, \pi_*(M)), (\|\cdot\|_{\pi_*(\psi)_\omega})_{\omega\in\Omega})$ is isometric to

$$(H^0(X, M), (\|\cdot\|_{\psi_\omega})_{\omega\in\Omega}).$$

We call V an *open subscheme of definition* of $\pi_*(\psi)$.

CHAPTER 3

Volumes of normed graded linear series

This chapter deals with normed graded linear series over a trivial valued field. We first remind the geometry of adelic vector bundles on the adelic curve consisting of a sigle copy of the trivial absolute value on a field. Then we introduce in the second section the notion of normed graded algebra over such adelic curves and discuss the properties of the associated spectrum norm. The last two sections are devoted to the study of asymptotic behaviours of normed graded linear series. In this chapter, we let k be a commutative field and we denote by $|\cdot|_0$ the trivial absolute value on k. Recall that $|a|_0 = 1$ for any $a \in k^{\times}$. Moreover, $S_0 = (k, \{0\}, |\cdot|_0)$ forms an adelic curve.

3.1. Adelic vector bundle on S_0

Adelic vector bundles on S_0 are just finite-dimensional ultrametrically normed vector spaces over k. If $\overline{V} = (V, \|\cdot\|)$ is an adelic vector bundle on S_0, then the function $\|\cdot\|$ only takes finitely many values. Moreover, if the vector space V is non-zero, then one has (see [**36**, Remark 4.3.63])

$$\widehat{\mu}_{\max}(\overline{V}) = -\min_{s\in V\setminus\{0\}} \ln\|s\|, \quad \widehat{\mu}_{\min}(\overline{V}) = -\max_{s\in V} \ln\|s\|.$$

The Harder-Narasimhan $\mathbb{R}$-filtration of $\overline{V}$ is give by

$$\forall\, t\in\mathbb{R}, \quad \mathcal{F}^t(\overline{V}) = \{s \in V\;:\;\|s\| \leqslant \mathrm{e}^{-t}\}.$$

Note that

$$\begin{cases} \widehat{\deg}_+(\overline{V}) := \sup\limits_{W\subset V} \widehat{\deg}(\overline{W}) = \displaystyle\int_0^{+\infty} \dim_k(\mathcal{F}^t(\overline{V}))\,\mathrm{d}t, \\ \widehat{\deg}(\overline{V}) = -\displaystyle\int_{\mathbb{R}} t\,\mathrm{d}\dim_k(\mathcal{F}^t(\overline{V}))\,\mathrm{d}t. \end{cases}$$

3.2. Normed graded algebra

Let $V_\bullet = \bigoplus_{n\in\mathbb{N}} V_n$ be a graded k-algebra. We assume that each V_n is a finite-dimensional vector space over k. For any $n \in \mathbb{N}_{\geqslant 1}$, let $\|\cdot\|_n$ be an ultrametric norm on V_n. Then the pair $\overline{V}_\bullet = (V_\bullet, (\|\cdot\|_n)_{n\in\mathbb{N}_{\geqslant 1}})$ is called a *normed graded algebra* over $(k, |\cdot|_0)$. Let $f : \mathbb{N}_{\geqslant 1} \to \mathbb{R}_{\geqslant 0}$ be a function. If, for all $\ell \in \mathbb{N}_{\geqslant 2}$, $(n_1, \ldots, n_\ell) \in \mathbb{N}_{\geqslant 1}^{\ell}$ and $(s_1, \ldots, s_\ell) \in V_{n_1} \times \cdots \times V_{n_\ell}$, one has

$$\|s_1\cdots s_\ell\|_{n_1+\cdots+n_\ell} \leqslant \mathrm{e}^{f(n_1)+\cdots+f(n_\ell)}\|s_1\|_{n_1}\cdots\|s_\ell\|_{n_\ell}, \tag{3.1}$$

we say that $\overline{V}_\bullet$ is *f-sub-multiplicative*. In the particular case where f is the constant function taking value 0, we just say the $\overline{V}_\bullet$ is *sub-mutiplicative*. If there exist two constant C_1 and C_2 such that, for any $n \in \mathbb{N}$ and any $s \in V_n \setminus \{0\}$, one has

$$\mathrm{e}^{C_1 n} \leqslant \|s\|_n \leqslant \mathrm{e}^{C_2 n}, \tag{3.2}$$

we say that $\overline{V}_\bullet$ is *bounded*.

PROPOSITION 3.2.1. *Let $\overline{V}_\bullet$ be a normed graded algebra over $(k, |\cdot|_0)$ and $f : \mathbb{N}_{\geqslant 1} \to \mathbb{R}_{\geqslant 0}$ be a function such that*

$$\lim_{n\to+\infty} \frac{f(n)}{n} = 0.$$

Assume that $V_\bullet$ is an integral domain and that $\overline{V}_\bullet$ is f-sub-multiplicative and bounded.

(1) *For any $n \in \mathbb{N}_{\geqslant 1}$ and any $s \in V_n$, the sequence*

$$\|s^N\|_{nN}^{1/N}, \quad N \in \mathbb{N},\ N \geqslant 1$$

converges.

(2) *For any $n \in \mathbb{N}_{\geqslant t1}$, the map*

$$\|\cdot\|_{\mathrm{sp},n} : V_n \longrightarrow \mathbb{R}_{\geqslant 0}, \quad s \longmapsto \lim_{N\to+\infty} \|s^N\|_{nN}^{1/N}$$

is an ultrametric norm on V_n.

(3) *The family of norms $(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}}$ satisfies the following sub-multiplicativity condition: for any $(n, m) \in \mathbb{N}^2$ and any $(s_n, s_m) \in V_n \times V_m$,*

$$\|s_n s_m\|_{\mathrm{sp},n+m} \leqslant \|s_n\|_{\mathrm{sp},n} \cdot \|s_m\|_{\mathrm{sp},m}.$$

(4) *For any $n \in \mathbb{N}_{\geqslant 1}$ and any $s \in V_n \setminus \{0\}$, one has*

$$\|s\|_{\mathrm{sp},n} \leqslant \mathrm{e}^{f(n)} \|s\|_n. \tag{3.3}$$

PROOF. (1) It suffices to treat the case where $s \neq 0$. By (3.1), for $\ell \in \mathbb{N}_{\geqslant 2}$, and $(N_1, \ldots, N_\ell) \in \mathbb{N}_{\geqslant 1}^{\ell}$, one has

$$\ln \|s^{N_1+\cdots+N_\ell}\|_{n(N_1+\cdots+N_\ell)} \leqslant \sum_{i=1}^{\ell} \Big(\ln \|s^{N_i}\|_{nN_1} + f(nN_i) \Big).$$

Moreover, by (3.2), the sequence

$$\frac{1}{N} \ln \|s^N\|_{nN}, \quad N \in \mathbb{N},\ N \geqslant 1$$

is bounded. Therefore this sequence converges in $\mathbb{R}$ (see [**29**, Proposition 1.3.1]), which shows that the sequence

$$\|s^N\|_{nN}^{1/N}, \quad N \in \mathbb{N},\ N \geqslant 1$$

converges to a positive real number.

(2) It suffices to show that $\|\cdot\|_{\mathrm{sp},n}$ satisfies the strong triangle inequality. Let s and t be two elements of V_n. For any $N \in \mathbb{N}_{\geqslant 1}$, one has

$$(s+t)^N = \sum_{i=0}^{N} \binom{N}{i} s^i t^{N-i}$$

and hence

$$\|(s+t)^N\|_{nN} \leqslant \max_{i\in\{0,\ldots,N\}} \|s^i t^{N-i}\|_{nN}.$$

Let

$$M = \max_{j\in\mathbb{N},\, j\geqslant 1} \frac{1}{j} \max\{\ln \|s^j\|_{nj}, \ln \|t^j\|_{nj}, 0\}.$$

Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a sequence of real numbers in $[0, \frac{1}{2}]$ such that

$$\lim_{j\to+\infty} \varepsilon_j = 0, \quad \lim_{j\to+\infty} j\varepsilon_j = +\infty, \quad \lim_{j\to+\infty} (j - j\varepsilon_j) = +\infty.$$

If $i/N \leqslant \varepsilon_N$, one has

$$\frac{1}{N}\ln\|s^i t^{N-i}\|_{nN} \leqslant \varepsilon_N M + \frac{N-i}{N}\cdot\frac{1}{N-i}\ln\|t^{N-i}\|_{n(N-i)} + \frac{f(ni)}{N} + \frac{f(n(N-i))}{N}.$$

Similarly, if $(N-i)/N \leqslant \varepsilon_N$, one has

$$\frac{1}{N}\ln\|s^i t^{N-i}\|_{nN} \leqslant \frac{i}{N}\cdot\frac{1}{i}\ln\|s^i\|_{ni} + \varepsilon_N M + \frac{f(ni)}{N} + \frac{f(n(N-i))}{N}.$$

If $N\varepsilon_N < i < N - N\varepsilon_N$, one has

$$\frac{1}{N}\ln\|s^i t^{N-i}\|_{nN} \leqslant \frac{i}{N}\cdot\frac{1}{i}\ln\|s^i\|_{ni} + \frac{N-i}{N}\cdot\frac{1}{N-i}\ln\|t^{N-i}\|_{n(N-i)} + \frac{f(ni)}{N} + \frac{f(n(N-i))}{N}.$$

Taking the superior limit when $N \to +\infty$, we obtain that

$$\limsup_{N\to+\infty}\max_{i\in\{0,\ldots,N\}}\frac{1}{N}\ln\|s^i t^{N-i}\|_{nN} \leqslant \max\{\|s\|_{\mathrm{sp},n}, \|t\|_{\mathrm{sp},n}\}.$$

(3) Let $(n,m)\in\mathbb{N}^2$ and $(s_n,s_m)\in V_n\times V_m$. For any $N\in\mathbb{N}$ such that $N\geqslant 1$, one has

$$\|(s_n s_m)^N\|_{(n+m)N} \leqslant \mathrm{e}^{f(nN)+f(mN)}\|s_n^N\|_{nN}\cdot\|s_m^N\|_{mN}.$$

Taking the N-th root and letting $N\to+\infty$ we obtain

$$\|s_n s_m\|_{\mathrm{sp},n+m} \leqslant \|s_n\|_{\mathrm{sp},n}\cdot\|s_m\|_{\mathrm{sp},m}.$$

(4) For any $N\in\mathbb{N}_{\geqslant 1}$, the following inequality holds:

$$\|s^N\|_{nN} \leqslant \mathrm{e}^{Nf(n)}\|s\|_n^N.$$

Taking the N-th root and then letting $N\to+\infty$, we obtain

$$\|s\|_{\mathrm{sp},n} \leqslant \mathrm{e}^{f(n)}\|s\|_n.$$

□

3.3. Reminder on graded linear series

In this section, we let k'/k be a finitely generated extension of fields. As *graded linear series* of k'/k, we refer to a graded sub-k-algebra $V_\bullet$ of

$$k'[T] = \bigoplus_{n\in\mathbb{N}} k'T^n$$

such that $V_0 = k$. We denote by $\mathbb{N}(V_\bullet)$ the set of $n\in\mathbb{N}$ such that $V_n \neq \mathbf{0}$. If $V_\bullet$ is a graded linear series and $\mathbb{N}(V_\bullet)\neq\{0\}$, we denote by $k(V_\bullet)$ the sub-extension of k'/k generated by

$$\bigcup_{n\in\mathbb{N}(V_\bullet)\setminus\{0\}}\{f/g \mid (f,g)\in V_n\times(V_n\setminus\{0\})\}$$

over k. If $\mathbb{N}(V_\bullet)\neq\{0\}$, then we denote by $\dim(V_\bullet)$ the transcendence degree of the extension $k(V_\bullet)/k$, and call it the *Kodaira-Iitaka dimension* of $V_\bullet$. In the case where $V_n=\{0\}$ for any $n\in\mathbb{N}_{\geqslant 1}$, by convention $\dim(V_\bullet)$ is defined to be $-\infty$. If $\mathbb{N}(V_\bullet)\neq\{0\}$ and if the field $k(V_\bullet)$ coincides with k', we say that the graded linear series $V_\bullet$ is *birational*.

We say that $V_\bullet$ is *of sub-finite type* if there exists a graded linear series $W_\bullet$ of k'/k which is a k-algebra of finite type and contains $V_\bullet$ as a sub-k-algebra. By [**33**, Theorem 3.7], there exists a graded sub-k-algebra of finite type $W_\bullet$ of the polynomial ring

$$k(V_\bullet)[T] = \bigoplus_{n\in\mathbb{N}} k(V_\bullet)T^n$$

such that $k(W_\bullet) = k(V_\bullet)$, which contains $V_\bullet$ as a sub-k-algebra. In other words, $V_\bullet$ viewed as a graded linear series of $k(V_\bullet)/k$ is sub-finite.

Let $V_\bullet$ be a graded linear series of sub-finite type, and d be its Kodaria-Iitaka dimension. If $\mathbb{N}(V_\bullet) \neq \{0\}$, we define the *volume* of $V_\bullet$ as the limit (see [**33**, Theorem 6.2] for the convergence)

$$\mathrm{vol}(V_\bullet) := \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\dim_k(V_n)}{n^d/d!}.$$

Note that $V_\bullet$ satisfies the Fujita approximation property, namely, one has

$$\mathrm{vol}(V_\bullet) = \sup_{\substack{W_\bullet\subset V_\bullet\\ \dim(W_\bullet)=\dim(V_\bullet)}} \mathrm{vol}(W_\bullet),$$

where $W_\bullet$ runs over the set of all graded sub-k-algebras of finite type of $V_\bullet$ such that $\dim(W_\bullet) = \dim(V_\bullet)$.

3.4. Normed graded linear series

In this section, we fix a finitely generated extension k'/k, a graded linear series $V_\bullet$ of k'/k which is of sub-finite type, and a $f : \mathbb{N}_{\geqslant 1} \to \mathbb{R}_{\geqslant 0}$ such that

$$\lim_{n\to+\infty} \frac{f(n)}{n} = 0.$$

Let d be the Kodaira-Iitaka dimension of $V_\bullet$. We assume that $d \geqslant 0$ (namely $\mathbb{N}(V_\bullet) = \{n \in \mathbb{N} : V_n \neq \mathbf{0}\} \neq \{0\}$) and we equip the graded algebra $V_\bullet$ with a family of norms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ such that $\overline{V}_\bullet = (V_\bullet, (\|\cdot\|_n)_{n\in\mathbb{N}_{\geqslant 1}})$ forms a normed graded algebra which is f-sub-multiplicative and bounded (see §3.2). For any $n \in \mathbb{N}_{\geqslant 1}$, let $\|\cdot\|_{\mathrm{sp},n} : V_n \to \mathbb{R}_{\geqslant 0}$ be the map defined as

$$\|s\|_{\mathrm{sp},n} := \lim_{N\to+\infty} \|s^N\|_{nN}^{1/N}.$$

Then $(V_\bullet, (\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}})$ forms a normed graded algebra which is sub-multiplicative and bounded. Moreover, we denote by $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{V}_\bullet)$ the *asymptotic maximal slope* of $\overline{V}_\bullet$, which is defined as

$$\begin{aligned}\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{V}_\bullet) &= -\lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty}\ \min_{s\in V_n\setminus\{0\}} \frac{1}{n}\ln\|s\|_n \\ &= \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \widehat{\mu}_{\max}(V_n, \|\cdot\|_n).\end{aligned}$$

Note that the existence of the limit is ensured by the inequality (3.1), which implies that

$$\widehat{\mu}_{\max}(V_{n_1+\cdots+n_\ell}, \|\cdot\|_{n_1+\cdots+n_\ell}) \geqslant \sum_{i=1}^{\ell}\Big(\widehat{\mu}_{\max}(V_{n_i}, \|\cdot\|_{n_i}) - f(n_i)\Big).$$

We refer the readers to [**29**, Corollary 1.3.2] for a proof of the convergence.

Proposition 3.4.1. *The following equality holds:*

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{V}_\bullet) = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{1}{n}\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}).$$

PROOF. By Proposition 3.2.1, one has

$$\|\cdot\|_{\mathrm{sp},n} \leqslant \mathrm{e}^{f(n)}\|\cdot\|_n$$

and hence for $n \in \mathbb{N}(V_\bullet)$ the following inequality holds

$$\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}) \geqslant \widehat{\mu}_{\max}(V_n, \|\cdot\|_n) - f(n).$$

This implies

$$\lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{1}{n}\,\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}) \geqslant \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{1}{n}\,\widehat{\mu}_{\max}(V_n, \|\cdot\|_n).$$

Conversely, for any fixed $n \in \mathbb{N}(V_\bullet)$ and $s \in V_n \setminus \{0\}$ such that

$$\ln\|s\|_{\mathrm{sp},n} = -\,\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}),$$

one has

$$\begin{aligned}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V}_\bullet) &= \lim_{N\to+\infty} \frac{1}{nN}\,\widehat{\mu}_{\max}(V_{nN}, \|\cdot\|_{nN})\\ &\geqslant \lim_{N\to+\infty} \frac{-1}{nN}\ln\|s^N\|_{nN} = -\frac{1}{n}\ln\|s\|_{\mathrm{sp},n} = \frac{1}{n}\,\widehat{\mu}_{\max}(V_n, \|\cdot\|_{\mathrm{sp},n}).\end{aligned}$$

Taking the limit when $n \to +\infty$, we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{V}_\bullet) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(V_\bullet, (\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}}).$$

□

DEFINITION 3.4.2. We define the *arithmetic volume* of $\overline{V}_\bullet$ as (see §1.4 for the definition of $\widehat{\deg}_+$)

$$\widehat{\mathrm{vol}}(\overline{V}_\bullet) := \limsup_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\widehat{\deg}_+(V_n, \|\cdot\|_n)}{n^{d+1}/(d+1)!}. \tag{3.4}$$

THEOREM 3.4.3. *The superior limit in the formula* (3.4) *defining the arithmetic volume function is actually a limit. Moreover, the following equalities hold:*

$$\widehat{\mathrm{vol}}(\overline{V}_\bullet) = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\widehat{\deg}_+(V_n, \|\cdot\|_{\mathrm{sp},n})}{n^{d+1}/(d+1)!} = (d+1)\int_0^{+\infty} \mathrm{vol}(V^t_\bullet)\,\mathrm{d}t,$$

where for $t \in \mathbb{R}$,

$$V^t_\bullet := k \oplus \bigoplus_{n\in\mathbb{N},\, n\geqslant 1} \mathrm{Vect}_k(\{s \in V_n \;:\; \|s\|_{\mathrm{sp},n} \leqslant \mathrm{e}^{-nt}\}).$$

PROOF. By replacing k' by $k(V_\bullet)$, we may assume that the graded linear series $V_\bullet$ is birational. For simplifying the notation, we let M be the asymptotic maximal slope of $\overline{V}_\bullet$. Note that M is also the asymptotic maximal slope of $(V_\bullet, (\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}})$ (see Proposition 3.4.1). Moreover, since $\overline{V}_\bullet$ is bounded, there exists a constant $A \geqslant 0$ such that $\|s\|_n \leqslant \mathrm{e}^{nA}$ for any $n \in \mathbb{N}_{\geqslant 1}$ and any $s \in V_n$.

By the same argument as the proof of [**33**, Proposition 6.6], we obtain that, for any $t < M$, one has $k(V^t_\bullet) = k(V_\bullet)$. Moreover, for any $t > M$ and any $n \in \mathbb{N}_{\geqslant 1}$, one has $V^t_n = \mathbf{0}$. Therefore, combining the construction of Newton-Okounkov bodies in [**32**, Theorem 1.1] and that of the concave transform developed in [**16**, §1.3], we obtain, in a similar way as [**16**, Corollary 1.13] that

$$\widehat{\mathrm{vol}}(V_\bullet, (\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}}) = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\widehat{\deg}_+(V_n, \|\cdot\|_{\mathrm{sp},n})}{n^{d+1}/(d+1)!} = (d+1)\int_0^{+\infty} \mathrm{vol}(V_\bullet^t)\,\mathrm{d}t.$$

Moreover, by (3.3) we obtain that

$$\widehat{\deg}_+(V_n, \|\cdot\|_{\mathrm{sp},n}) \geqslant \widehat{\deg}_+(V_n, \|\cdot\|_n) - \dim_k(V_n) f(n),$$

which leads to

$$\limsup_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\widehat{\deg}_+(V_n, \|\cdot\|_n)}{n^{d+1}/(d+1)!} \leqslant \widehat{\mathrm{vol}}(V_\bullet, (\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}})$$

since $\dim_k(V_n) = O(n^d)$ when $n\in\mathbb{N}(V_\bullet)$, $n\to+\infty$.

Let ε be an element of $]0, M[$, t be an element of $[\varepsilon, M[$. Let $W_\bullet^t$ be a graded sub-k-algebra of finite type of $V_\bullet^t$, which is generated by a family of homogeneous elements $s_1,\ldots,s_\ell$ of homogeneous degrees $n_1,\ldots,n_\ell$ respectively. For any $i\in\{1,\ldots,\ell\}$, there exists $a_i\in\mathbb{N}_{\geqslant 1}$ such that the inequalities

$$\|s_i^N\|_{n_iN} \leqslant \mathrm{e}^{n_iN\varepsilon/2}\|s_i\|_{\mathrm{sp},n_i}^N \leqslant \mathrm{e}^{n_iN(\varepsilon/2-t)} \tag{3.5}$$

hold for any integer $N\geqslant a_i$. Therefore, by the inequality (3.1) we obtain that, for any $(N_1,\ldots,N_\ell)\in\mathbb{N}_{\geqslant 1}^\ell$, one has

$$\ln\|s_1^{N_1}\cdots s_\ell^{N_\ell}\|_{n_1N_1+\cdots+n_\ell N_\ell} \leqslant \sum_{i=1}^\ell \big(\ln\|s_i^{N_i}\|_{n_iN_i} + f(n_iN_i)\big).$$

By (3.5), we obtain that

$$\begin{aligned}\ln\|s_1^{N_1}\cdots s_\ell^{N_\ell}\|_{n_1N_1+\cdots+n_\ell N_\ell} &\leqslant \sum_{\substack{i\in\{1,\ldots,\ell\}\\ N_i\geqslant a_i}} n_iN_i\Big(\frac{\varepsilon}{2}-t\Big) + \sum_{\substack{i\in\{1,\ldots,\ell\}\\ N_i< a_i}} n_iN_iA \\ &\leqslant \Big(\frac{\varepsilon}{2}-t\Big)\sum_{i=1}^\ell n_i(N_i-a_i) + \sum_{i=1}^\ell n_ia_iA \\ &\leqslant \Big(\frac{\varepsilon}{2}-t\Big)\sum_{i=1}^\ell n_iN_i + \sum_{i=1}^\ell n_ia_i(A+M).\end{aligned}$$

Therefore, for $(N_1,\ldots,N_\ell)\in\mathbb{N}_{\geqslant 1}^\ell$ such that $n_1N_1+\cdots+n_\ell N_\ell$ is sufficiently large, one has

$$\|s_1^{N_1}\cdots s_\ell^{N_\ell}\|_{n_1N_1+\cdots+n_\ell N_\ell} \leqslant \mathrm{e}^{(\varepsilon-t)(n_1N_1+\cdots+n_\ell N_\ell)}.$$

In particular, for $n\in\mathbb{N}(V_\bullet)$ sufficiently large, one has

$$W_n^t \subset \mathcal{F}^{(t-\varepsilon)n}(V_n, \|\cdot\|_n),$$

which leads to

$$\liminf_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\dim_k(\mathcal{F}^{(t-\varepsilon)n}(V_n,\|\cdot\|_n))}{n^d/d!} \geqslant \mathrm{vol}(W_\bullet^t).$$

Taking the supremum when $W^t_\bullet$ varies, by the Fujita approximation property of $V^t_\bullet$ we obtain that

$$\liminf_{n\in\mathbb{N}(V_\bullet),\,n\to+\infty}\frac{\dim_k(\mathcal{F}^{(t-\varepsilon)n}(V_n,\|\cdot\|_n))}{n^d/d!}\geqslant \mathrm{vol}(V^t_\bullet). \tag{3.6}$$

Note that

$$\begin{aligned}\widehat{\deg}_+(V_n,\|\cdot\|_n)&=\int_0^{+\infty}\dim_k(\mathcal{F}^t(V_n,\|\cdot\|_n))\,\mathrm{d}t\\&=n\int_0^{+\infty}\dim_k(\mathcal{F}^{nt}(V_n,\|\cdot\|_n))\,\mathrm{d}t\\&\geqslant n\int_\varepsilon^M\dim_k(\mathcal{F}^{n(t-\varepsilon)}(V_n,\|\cdot\|_n))\,\mathrm{d}t.\end{aligned}$$

Taking the integral with respect to t, by Fatou's lemma we deduce from (3.6) that

$$\begin{aligned}&\liminf_{n\in\mathbb{N}(V_\bullet),n\to+\infty}\frac{\widehat{\deg}_+(V_n,\|\cdot\|_n)}{n^{d+1}/(d+1)!}\\&\qquad\geqslant\liminf_{n\in\mathbb{N}(V_\bullet),\,n\to+\infty}\frac{(d+1)!}{n^d}\int_\varepsilon^M\dim_k(\mathcal{F}^{n(t-\varepsilon)}(V_n,\|\cdot\|_n))\\&\qquad\qquad\geqslant(d+1)\int_\varepsilon^M\mathrm{vol}(V^t_\bullet)\,\mathrm{d}t=(d+1)\int_\varepsilon^{+\infty}\mathrm{vol}(V^t_\bullet).\end{aligned}$$

Finally, taking the supremum with respect to ε, we obtain the inequality

$$\liminf_{n\in\mathbb{N}(V_\bullet),n\to+\infty}\frac{\widehat{\deg}_+(V_n,\|\cdot\|_n)}{n^{d+1}/(d+1)!}\geqslant\widehat{\mathrm{vol}}(V_\bullet,(\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant1}}).$$

The theorem is thus proved. □

Corollary 3.4.4. *The sequences*

$$\frac{\widehat{\deg}(V_n,\|\cdot\|_n)}{n^{d+1}/(d+1)!},\quad n\in\mathbb{N}(V_\bullet)\quad\textit{and}\quad\frac{\widehat{\deg}(V_n,\|\cdot\|_{\mathrm{sp},n})}{n^{d+1}/(d+1)!},\quad n\in\mathbb{N}(V_\bullet)$$

converge to the same real number, which is equal to

$$-\int_{\mathbb{R}}t\,\mathrm{d}\,\mathrm{vol}(V^t_\bullet).$$

Proof. Let A be a positive constant such that $\|s\|_n\leqslant\mathrm{e}^{nA}$ for any $n\in\mathbb{N}_{\geqslant1}$ and any $s\in V_n$. For any $n\in\mathbb{N}_{\geqslant1}$, let $\|\cdot\|'_n=\mathrm{e}^{-nA}\|\cdot\|_n$. Then, $(V_\bullet,(\mathrm{e}^{-nA}\|\cdot\|_n)_{n\in\mathbb{N}_{\geqslant1}})$ forms a normed graded algebra over $(k,|\cdot|_0)$, which is f-sub-multiplicative and bounded. Moreover, for any $n\in\mathbb{N}_{\geqslant1}$, one has

$$\widehat{\deg}(V_n,\|\cdot\|'_n)=\widehat{\deg}_+(V_n,\|\cdot\|'_n)=nA\dim_k(V_n)+\widehat{\deg}(V_n,\|\cdot\|_n),$$

where the first equality comes from the fact that the image of $\|\cdot\|'_n$ is contained in $[0,1]$. For any $n\in\mathbb{N}$ one has

$$\|\cdot\|'_{\mathrm{sp},n}=\mathrm{e}^{-nA}\|\cdot\|_{\mathrm{sp},n}.$$

By (3.3), for any $n\in\mathbb{N}_{\geqslant1}$ and any $s\in V_n$, one has

$$\forall\,N\in\mathbb{N}_{\geqslant1},\quad\|s\|_{\mathrm{sp},n}=\|s^N\|^{1/N}_{\mathrm{sp},nN}\leqslant\mathrm{e}^{f(nN)/N}\|s^N\|^{1/N}_{nN}\leqslant\mathrm{e}^{f(nN)/N+nA}.$$

Taking the limit when $N \to +\infty$, we obtain $\|s\|_{\mathrm{sp},n} \leqslant \mathrm{e}^{nA}$ and hence $\|\cdot\|'_{\mathrm{sp},n}$ also takes value in $[0,1]$. Therefore, for any $n \in \mathbb{N}_{\geqslant 1}$, one has

$$\widehat{\deg}(V_n, \|\cdot\|'_{\mathrm{sp},n}) = \widehat{\deg}_+(V_n, \|\cdot\|'_{\mathrm{sp},n}) = nA \dim_k(V_n) + \widehat{\deg}(V_n, \|\cdot\|_{\mathrm{sp},n}),$$

Hence Theorem 3.4.3 leads to the convergence of the sequences

$$\frac{\widehat{\deg}(V_n, \|\cdot\|_n) + nA \dim_k(V_n)}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N}(V_\bullet)$$

and

$$\frac{\widehat{\deg}(V_n, \|\cdot\|_{\mathrm{sp},n}) + nA \dim_k(V_n)}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N}(V_\bullet)$$

to the same limit, which is equal to

$$\begin{aligned}(d+1)\int_0^{+\infty} \mathrm{vol}(V_\bullet^{t-A})\,\mathrm{d}t &= (d+1)\int_{-A}^{+\infty} \mathrm{vol}(V_\bullet^t)\,\mathrm{d}t \\ &= A(d+1)\,\mathrm{vol}(V_\bullet) - \int_{\mathbb{R}} t\,\mathrm{d}\,\mathrm{vol}(V_\bullet^t),\end{aligned}$$

where the last equality comes from the fact that $V_\bullet^t = V_\bullet$ when $t \leqslant -A$. By the formula

$$\lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\dim_k(V_n)}{n^d/d!} = \mathrm{vol}(V_\bullet),$$

we obtain the assertion. □

Definition 3.4.5. We define the *χ-volume* of the normed graded linear series $\overline{V}_\bullet$ as

$$\widehat{\mathrm{vol}}_\chi(\overline{V}_\bullet) = \lim_{n\in\mathbb{N}(V_\bullet),\, n\to+\infty} \frac{\widehat{\deg}(V, \|\cdot\|_n)}{n^{d+1}/(d+1)!}.$$

By Corollary 3.4.4, we obtain that $\widehat{\mathrm{vol}}_\chi(\overline{V}_\bullet) = \widehat{\mathrm{vol}}_\chi(V_\bullet, (\|\cdot\|_{\mathrm{sp},n})_{n\in\mathbb{N}_{\geqslant 1}})$.

CHAPTER 4

Arithmetic volumes over a general adelic curve

In this chapter, we use the results of the previous chapter to study the volume functions of a normed graded algebra over a general adelic curve. The main idea is to cast to the trivial valuation case by Harder-Narasimhan filtration. In the first section, we introduce the notion of graded algebra of adelic vector bundles and its casting, which is a normed graded algebra over a trivial valued field. In the second section, we show that the sequence appearing on the left-hand side of the arithmetic Hilbert-Samuel formula actually converges, by using the convergence result established in the previous chapter. In the third section, we discusses normed graded modules, which are used in the fourth section to obtain bounds of χ-volume in the case where the tensor product with a metrized torsion-free sheaf appears.

Throughout the chapter, let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be the adelic curve defined in §1.1. We let $|\cdot|_0$ be the trivial absolute value on K, and denote by $S_0 = (K, \{0\}, |\cdot|_0)$ the adelic curve consisting of a single copy of the trivial absolute value $|\cdot|_0$ on K.

4.1. Graded algebra of adelic vector bundles

In this section, we consider basic facts on graded algebras of adelic vector bundles.

Definition 4.1.1. Let $E_\bullet = \bigoplus_{n\in\mathbb{N}} E_n$ be a graded K-algebra. We assume that each vector space E_n is finite-dimensional over K. For any $n \in \mathbb{N}$, let $\xi_n = (\|\cdot\|_{n,\omega})_{\omega\in\Omega}$ be a norm family on E_n such that $\overline{E}_n = (E_n, \xi_n)$ forms an adelic vector bundle on S. We call $\overline{E}_\bullet = (\overline{E}_n)_{n\in\mathbb{N}}$ a *graded algebra of adelic vector bundles on* S. For any $n \in \mathbb{N}$ such that $n \geqslant 1$, let $(\mathcal{F}^t(\overline{E}_n))_{t\in\mathbb{R}}$ be the Harder-Narasimhan $\mathbb{R}$-filtration of $\overline{V}_n$ (see §1.6). We denote by $\|\cdot\|_n^{\mathrm{HN}}$ the norm on E_n (viewed as a vector space over $(K, |\cdot|_0)$) defined as

$$\forall\, s \in E_n, \quad \|s\|_n^{\mathrm{HN}} = \exp\big(-\sup\{t \in \mathbb{R} \,:\, s \in \mathcal{F}^t(\overline{E}_n)\}\big).$$

Then, the couple $(E_\bullet, (\|\cdot\|_n^{\mathrm{HN}})_{n\in\mathbb{N}_{\geqslant 1}})$ forms a normed graded algebra over $(K, |\cdot|_0)$ (see §3.2). Moreover, if we view $(E_n, \|\cdot\|_n^{\mathrm{HN}})$ as an adelic vector bundle on S_0, then its Harder-Narasimhan filtration coincides with that of (E_n, ξ_n). In particular, by the results recalled in §1.6, the following estimates hold:

$$0 \leqslant \widehat{\deg}(E_n, \xi_n) - \widehat{\deg}(E_n, \|\cdot\|_n^{\mathrm{HN}}) \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(E_n)\ln(\dim_K(E_n)), \tag{4.1}$$

$$0 \leqslant \widehat{\deg}_+(E_n, \xi_n) - \widehat{\deg}_+(E_n, \|\cdot\|_n^{\mathrm{HN}}) \leqslant \frac{1}{2}\nu(\Omega_\infty)\dim_K(E_n)\ln(\dim_K(E_n)). \tag{4.2}$$

Let $b = (b_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of non-negative integrable functions on $(\Omega, \mathcal{A}, \nu)$. We say that a graded algebra of adelic vector bundles $\overline{E}_\bullet$ is *b-sub-multiplicative* if for all $\omega \in \Omega$, $\ell \in \mathbb{N}_{\geqslant 2}$, $(n_1, \ldots, n_\ell) \in \mathbb{N}_{\geqslant 1}^\ell$ and $(s_1, \ldots, s_\ell) \in E_{n_1,\omega} \times \cdots \times E_{n_\ell,\omega}$, the following inequality holds

$$\|s_1 \cdots s_\ell\|_{n_1+\cdots+n_\ell,\omega} \leqslant \mathrm{e}^{b_{n_1}(\omega)+\cdots+b_{n_\ell}(\omega)}\|s_1\|_{n_1,\omega}\cdots\|s_\ell\|_{n_\ell,\omega}. \tag{4.3}$$

If for any n, b_n is the constant function taking 0 as its value, we simply say that $\overline{E}_\bullet$ is *sub-multiplicative*.

Proposition 4.1.2. *Assume that K is perfect. Let $b = (b_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of non-negative integrable functions on $(\Omega, \mathcal{A}, \nu)$, and $\overline{E}_\bullet$ be a graded algebra of adelic vector bundles on S, which is b-sub-multiplicative. Let $f : \mathbb{N}_{\geqslant 1} \to \mathbb{R}_{\geqslant 0}$ be the function defined as*

$$f(n) = \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E_n)) + \int_\Omega b_n(\omega)\,\nu(\mathrm{d}\omega).$$

Then the normed graded algebra $(E_\bullet, (\|\cdot\|_n^{\mathrm{HN}})_{n\in\mathbb{N}_{\geqslant 1}})$ is f-sub-multiplicative.

Proof. Let $\ell \in \mathbb{N}_{\geqslant 1}$ and $(n_1,\ldots,n_\ell) \in \mathbb{N}_{\geqslant 1}^\ell$. For any $i \in \{1,\ldots,\ell\}$, let F_{n_i} be a K-vector subspace of E_{n_i}. For any $\omega \in \Omega$, we consider the K_ω-linear map

$$F_{n_1,\omega}\otimes\cdots\otimes F_{n_\ell,\omega} \longrightarrow E_{n_1+\cdots+n_\ell,\omega}$$

induced by the K-algebra structure of $E_\bullet$. If we equip $F_{n_1,\omega}\otimes\cdots\otimes F_{n_\ell,\omega}$ with the ε-tensor product of the norms $\|\cdot\|_{n_1,\omega},\ldots,\|\cdot\|_{n_\ell,\omega}$ when $|\cdot|_\omega$ is non-Archimedean, and with the π-tensor product when $|\cdot|_\omega$ is Archimedean, then the operator norm of the above map is bounded from above by $\exp(b_{n_1}(\omega)+\cdots+b_{n_\ell}(\omega))$. Moreover, by [**36**, Corollary 5.6.2] (Although this result has been stated under the assumption that $\mathrm{char}(K) = 0$, this assumption is only used in the application of [**36**, Theorem 5.4.3], which actually applies to any perfect field. Moreover, the lifting of invariants from the symmetric power to the tensor power, that we have used to prove [**36**, Proposition 5.3.1], is valid in any characteristic. For details, see Remark A.3.3.), one has

$$\widehat{\mu}_{\min}(\overline{F}_{n_1}\otimes_{\varepsilon,\pi}\cdots\otimes_{\varepsilon,\pi}\overline{F}_{n_\ell}) \geqslant \sum_{i=1}^{\ell}\Big(\widehat{\mu}_{\min}(\overline{F}_{n_i}) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E_{n_i}))\Big).$$

Let $F_{n_1+\cdots+n_\ell}$ be the image of the map

$$F_{n_1}\otimes\cdots\otimes F_{n_\ell} \longrightarrow E_{n_1+\cdots+n_\ell}.$$

By [**36**, Proposition 4.3.31], we obtain that

$$\widehat{\mu}_{\min}(\overline{F}_{n_1+\cdots+n_\ell}) \geqslant \sum_{i=1}^{\ell}\Big(\widehat{\mu}_{\min}(\overline{F}_{n_i}) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E_{n_i})) - \int_\Omega b_{n_i}(\omega)\,\nu(\mathrm{d}\omega)\Big). \quad (4.4)$$

Therefore, we obtain that, for any $(t_1,\ldots,t_\ell)\in\mathbb{R}^\ell$, one has

$$\mathcal{F}^{t_1}(\overline{E}_{n_1})\cdots\mathcal{F}^{t_\ell}(\overline{E}_{n_\ell}) \subset \mathcal{F}^{t_1+\cdots+t_\ell-f(n_1)-\cdots-f(n_\ell)}(\overline{E}_{n_1+\cdots+n_\ell}),$$

which shows that the normed graded algebra $(E_\bullet, (\|\cdot\|_n^{\mathrm{HN}})_{n\in\mathbb{N}_{\geqslant 1}})$ is f-sub-multiplicative. □

Corollary-Definition 4.1.3. *Assume that the defining field K is perfect. Let $b = (b_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of non-negative integrable functions on $(\Omega,\mathcal{A},\nu)$ such that*

$$\lim_{n\to+\infty}\frac{1}{n}\int_\Omega b_n(\omega)\,\nu(\mathrm{d}\omega) = 0.$$

Let $\overline{E}_\bullet$ be a graded algebra of adelic vector bundles on S, which is b-sub-multiplicative. Denote by $\mathbb{N}(E_\bullet)$ the set of $n\in\mathbb{N}$ such that $E_n \neq \mathbf{0}$. Assume that

1. *$E_\bullet$ is isomorphic to a graded linear series of sub-finite type of a finitely generated extension of K, which is of Kodaira-Iitaka dimension $d \geqslant 0$,*
2. *there exists $C > 0$ such that, for any $n \in \mathbb{N}(E_\bullet)$,*

$$-Cn \leqslant \widehat{\mu}_{\min}(\overline{E}_n) \leqslant \widehat{\mu}_{\max}(\overline{E}_n) \leqslant Cn.$$

Then the sequences

$$\frac{\widehat{\deg}(\overline{E}_n)}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N}(E_{\bullet})$$

and

$$\frac{\widehat{\deg}_+(\overline{E}_n)}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N}(E_{\bullet})$$

converge to two real numbers $\widehat{\mathrm{vol}}_\chi(\overline{E}_{\bullet})$ *and* $\widehat{\mathrm{vol}}(\overline{E}_{\bullet})$*, which we call* χ-volume *and* volume *of* $\overline{E}_{\bullet}$*, respectively.*

Proof. These results follow from Proposition 4.1.2, Theorem 3.4.3, Corollary 3.4.4 and the comparisons (4.1), (4.2) and the convergence of the sequence

$$\frac{\dim_K(E_n)}{n^d/d!}, \quad n \in \mathbb{N}(E_{\bullet}).$$

□

Remark 4.1.4. Assume that the field K is perfect. Let $b = (b_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of non-negative integrable functions on $(\Omega, \mathcal{A}, \nu)$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\int_\Omega b_n(\omega)\,\nu(\mathrm{d}\omega) = 0.$$

Let $\overline{E}_{\bullet}$ be a graded algebra of adelic vector bundles on S, which is b-sub-multiplicative. We assume that $n_1, \ldots, n_\ell$ are elements of $\mathbb{N}(E_{\bullet}) \setminus \{0\}$ such that

$$K \oplus \bigoplus_{n\in\mathbb{N},\, n\geqslant 1} E_n$$

is generated as K-algebra by $E_{n_1} \cup \cdots \cup E_{n_\ell}$. By (4.4) we obtain that, for any $(a_1, \ldots, a_\ell) \in \mathbb{N}^\ell \setminus \{(0, \ldots, 0)\}$, the canonical image of

$$E_{n_1}^{\otimes a_1} \otimes \cdots \otimes E_{n_\ell}^{\otimes a_\ell}$$

in $E_{a_1 n_1 + \cdots + a_\ell n_\ell}$ has a minimal slope

$$\geqslant \sum_{i=1}^{\ell} a_i \bigg(\widehat{\mu}_{\min}(\overline{E}_i) - \frac{3}{2}\nu(\Omega_\infty)\ln(E_{n_i}) - \int_\Omega b_{n_i}(\omega)\,\nu(\mathrm{d}\omega)\bigg).$$

Therefore we deduce that, for any $n \in \mathbb{N}(E_{\bullet}) \setminus \{0\}$, the minimal slope of $\overline{E}_n$ is bounded from below by

$$\min_{\substack{(a_1,\ldots,a_\ell)\in\mathbb{N}^\ell\\ n=a_1n_1+\cdots+a_\ell n_\ell}} \sum_{i=1}^{\ell} a_i \bigg(\widehat{\mu}_{\min}(\overline{E}_i) - \frac{3}{2}\nu(\Omega_\infty)\ln(E_{n_i}) - \int_\Omega b_{n_i}(\omega)\,\nu(\mathrm{d}\omega)\bigg).$$

Hence there exists $C > 0$ such that $\widehat{\mu}_{\min}(\overline{E}_n) \geqslant -Cn$ holds for any $n \in \mathbb{N}(E_{\bullet})$.

4.2. Arithmetic χ-volumes of adelic line bundles

In this section, we introduce the arithmetic χ-volume of an adelic line bundle.

Theorem-Definition 4.2.1. *Let* $p : X \to \operatorname{Spec} K$ *be an integral projective scheme over* $\operatorname{Spec} K$*,* d *be the dimension of* X*, and* $\overline{L} = (L, \varphi)$ *be an adelic line bundle on* X*. We suppose that, either* K *is perfect, or* X *is geometrically integral. Assume that* L *is big and the graded* K*-algebra*

$$\bigoplus_{n\in\mathbb{N}} H^0(X, L^{\otimes n})$$

is of finite type. We denote the adelic vector bundle

$$\big(H^0(X, L^{\otimes n}), (\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega}\big)$$

over S by $p_*(\overline{L}^{\otimes n})$. Then the sequence

$$\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1 \tag{4.5}$$

*converges to a real number, which we denote by $\widehat{\mathrm{vol}}_\chi(\overline{L})$ and which we call the χ-*volume *of $\overline{L}$.*

Proof. Let K^{pc} be the perfect closure of K. Recall that, if K^{ac} denotes the algebraic closure of K, then K^{pc} is the intersection of all subfields of K^{ac} containing K which are perfect fields. Note that K^{pc}/K is a purely inseparable algebraic extension of K. Therefore, for any $\omega \in \Omega$, the absolute value $|\cdot|_\omega$ extends in a unique way to K^{pc}/K. In other words, the measure space in the adelic curve structure of $S \otimes_K K^{\mathrm{pc}}$ coincides with $(\Omega, \mathcal{A}, \nu)$.

For any $n \in \mathbb{N}$, let

$$E_n = H^0(X, L^{\otimes n}) \otimes_K K^{\mathrm{pc}} = H^0(X_{K^{\mathrm{pc}}}, L_{K^{\mathrm{pc}}}^{\otimes n}).$$

The norm family of $p_*(\overline{L}^{\otimes n})$ induces by extension of scalars a norm family on E_n, which we denote by ξ_n. By [**36**, Proposition 4.3.14], the equality

$$\widehat{\deg}(E_n, \xi_n) = \widehat{\deg}(p_*(\overline{L}^{\otimes n}))$$

holds. Moreover,

$$E_\bullet = \bigoplus_{n\in\mathbb{N}} E_n$$

is a graded K^{pc}-algebra of finite type, which is isomorphic to a graded linear series of the function field of $X_{K^{\mathrm{pc}}}$ over K^{pc}. As a graded K^{pc}-algebra of adelic vector bundles on $S \otimes_K K^{\mathrm{pc}}$, $\overline{E}_\bullet = (\overline{E}_n)_{n\in\mathbb{N}}$ is sub-multiplicative. By [**36**, Proposition 6.2.7], we obtain, following the proof of [**36**, Proposition 6.4.4], that the sequence

$$\frac{\widehat{\mu}_{\max}(\overline{E}_n)}{n}, \quad n \in \mathbb{N},\ n \geqslant 1$$

is bounded from above. Therefore the assertion follows from Corollary-Definition 4.1.3 (see also Remark 4.1.4). □

Remark 4.2.2. Under the notation and the assumption of the above theorem-definition, the following relation holds

$$\lim_{n\to+\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}))}{n\dim_K(H^0(X, L^{\otimes n}))} = \frac{\widehat{\mathrm{vol}}_\chi(\overline{L})}{(d+1)\,\mathrm{vol}(L)}.$$

4.3. Normed graded module

Let $\overline{R}_\bullet = (\overline{R}_n)_{n\in\mathbb{N}}$ be a graded algebra of adelic vector bundles on S, where $\overline{R}_n = (R_n, (\|\cdot\|_{n,\omega})_{\omega\in\Omega})$. Let $M_\bullet = \bigoplus_{n\in\mathbb{N}} M_n$ be a graded module over $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$. If each M_n is a finite-dimensional vector space over K and is equipped with a norm family $(\|\cdot\|_{n,\omega}^M)_{\omega\in\Omega}$ such that $\overline{M}_n = (M_n, (\|\cdot\|_{n,\omega}^M)_{\omega\in\Omega})$ is an adelic vector bundle on S, we say that $\overline{M}_\bullet = (\overline{M}_n)_{n\in\mathbb{N}}$ is *a graded $\overline{R}_\bullet$-module of adelic vector bundles on S*.

Assume that $\overline{R}_\bullet$ is sub-multiplicative (see Definition 4.1.1). If, for all $(n,m) \in \mathbb{N}^2$, $\omega \in \Omega$ and $(a,s) \in R_{n,\omega} \times M_{m,\omega}$, one has

$$\|as\|_{n+m,\omega}^M \leqslant \|a\|_{n,\omega} \cdot \|s\|_{m,\omega}^M,$$

we say that $\overline{M}_\bullet$ is *sub-multiplicative*.

LEMMA 4.3.1. *Let $\overline{M}_\bullet = ((M_n, \xi_{M_n}))_{n\in\mathbb{N}}$ be a graded $\overline{R}_\bullet$-module of adelic vector bundle on S. Let $Q = \bigoplus_{n=0}^{\infty} Q_n$ be a graded quotient R-module of M, that is, Q_n is a quotient vector space of M_n over K for all n and $a_\ell \cdot : M_n \to M_{n+\ell}$ induces by passing to quotient $a_\ell\cdot : Q_n \to Q_{n+\ell}$ for $a_\ell \in R_\ell$. Let ξ_{Q_n} be the quotient norm family of Q_n induced by $M_n \to Q_n$ and ξ_{M_n}. Then $\overline{Q}_\bullet = ((Q_n, \xi_{Q_n}))_{n\in\mathbb{N}}$ is a graded $\overline{R}_\bullet$-algebra.*

PROOF. Assume that ξ_{M_n} and ξ_{Q_n} are of the form $(\|\cdot\|^M_{n,\omega})_{\omega\in\Omega}$ and $(\|\cdot\|^Q_{n,\omega})_{\omega\in\Omega}$, respectively. Let $(n,n') \in \mathbb{N}^2$, $\omega \in \Omega$, $a \in R_{n,\omega}$ and $q \in Q_{n',\omega}$. For any $s \in M_{n',\omega}$ which represents the class $q \in Q_{n',\omega}$, one has

$$\|aq\|^Q_{n+n',\omega} \leqslant \|as\|^M_{n+n',\omega} \leqslant \|a\|_{n,\omega} \cdot \|s\|^M_{n',\omega}.$$

Taking the infimum with respect to s, we obtain

$$\|aq\|^Q_{n+n',\omega} \leqslant \|a\|_{n,\omega} \cdot \|q\|^Q_{n',\omega},$$

as required. □

PROPOSITION 4.3.2. *Suppose that $R_\bullet$ is a K-algebra of finite type. Let*

$$\overline{M}_\bullet = ((M_n, \xi_{M_n}))_{n\in\mathbb{N}}$$

be a graded $\overline{R}_\bullet$-module of adelic vector bundles on S, such that $M_\bullet$ is an $R_\bullet$-module of finite type. Suppose that

$$\liminf_{n\to\infty} \frac{\dim_K(M_n)}{n^d} = 0$$

for some non-negative integer d, then

$$\liminf_{n\to\infty} \frac{\widehat{\deg}(M_n, \xi_{M_n})}{n^{d+1}} \geqslant 0.$$

PROOF. Let $x_1, \ldots, x_r$ be homogeneous elements of R which generate R as K-algebra. We choose non-zero homogeneous elements $m_1, \ldots, m_\ell$ of M such that M is generated by $m_1, \ldots, m_\ell$ over R. We set $e_i = \deg(x_i)$ and $f_i = \deg(m_i)$ for $i \in \{1, \ldots, r\}$. For $\alpha = (a_1, \ldots, a_r) \in \mathbb{N}^r$, we denote $x_1^{a_1} \cdots x_r^{a_r}$ by x^α. If we set $d_n = \dim_K(M_n)$, then, for $n \geqslant \max\{f_1, \ldots, f_r\}$, we can find $\alpha_1, \ldots, \alpha_{d_n} \in \mathbb{N}^r$ and $m_{i_1}, \ldots, m_{i_{d_n}} \in \{m_1, \ldots, m_\ell\}$ such that $x^{\alpha_1} m_{i_1}, \ldots, x^{\alpha_{d_n}} m_{i_{d_n}}$ form a basis of M_n. Note that

$$\begin{aligned}
\|(x^{\alpha_1} m_{i_1}) \wedge \cdots \wedge (x^{\alpha_{d_n}} m_{i_{d_n}})\|^M_{n,\omega,\det} &\leqslant \|x^{\alpha_1} m_{i_1}\|^M_{n,\omega} \cdots \|x^{\alpha_{d_n}} m_{i_{d_n}}\|^M_{n,\omega} \\
&\leqslant \|x^{\alpha_1}\|_{n-f_{i_1},\omega} \cdots \|x^{\alpha_{d_n}}\|_{n-f_{i_{d_n}},\omega} \cdot \|m_{i_1}\|^M_{f_{i_1},\omega} \cdots \|m_{i_{d_n}}\|^M_{f_{i_{d_n}},\omega} \\
&\leqslant \max\{1, \|x_1\|_{e_1,\omega}, \ldots, \|x_r\|_{e_r,\omega}\}^{nd_n} \max\{1, \|m_1\|^M_{f_1,\omega}, \ldots, \|m_\ell\|^M_{f_\ell,\omega}\}^{d_n},
\end{aligned}$$

so that

$$\begin{aligned}
\widehat{\deg}(M_n, \xi_{M_n}) \geqslant nd_n \int_\Omega &\min\{0, -\ln\|x_1\|_{e_1,\omega}, \ldots, -\ln\|x_r\|_{e_r,\omega}\}\, \nu(\mathrm{d}\omega) \\
&+ d_n \int_\Omega \min\{0, -\ln\|m_1\|^M_{f_1,\omega}, \ldots, -\ln\|m_\ell\|^M_{f_\ell,\omega}\}\, \nu(\mathrm{d}\omega).
\end{aligned}$$

Thus the assertion follows. □

4.4. Bounds of χ-volume with auxiliary torsion free module

Let us begin with the following lemma.

Lemma 4.4.1. *Let X be an integral projective scheme over a field k, L be an invertible $\mathcal{O}_X$-module and F be a coherent $\mathcal{O}_X$-module. We assume that there exist a surjective morphism $f : X \to Y$ of integral projective schemes over k and an ample invertible $\mathcal{O}_Y$-module A such that $f^*(A) = L$. Then $R = \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})$ is a finitely generated algebra over k and $M = \bigoplus_{n=0}^{\infty} H^0(X, F \otimes L^{\otimes n})$ is a finitely generated R-module.*

Proof. By [**62**, §1.8], there exist positive integers d and n_0 such that

$$H^0(Y, A^{\otimes d}) \otimes H^0(Y, A^{\otimes n} \otimes f_*(F)) \longrightarrow H^0(Y, A^{\otimes (d+n)} \otimes f_*(F))$$

is surjective for all $n \geqslant n_0$, and hence

$$H^0(X, L^{\otimes d}) \otimes H^0(X, L^{\otimes n} \otimes F) \to H^0(X, L^{\otimes (d+n)} \otimes F)$$

is surjective for all $n \geqslant n_0$ because $f_*(L^{\otimes n}) = A^{\otimes n} \otimes f_*(\mathcal{O}_X)$, $f_*(L^{\otimes n} \otimes F) = A^{\otimes n} \otimes f_*(F)$, $\mathcal{O}_Y \subseteq f_*(\mathcal{O}_X)$. Thus, by the arguments in [**62**, §1.8], one can see the assertion. □

In the rest of the section, let $p : X \to \operatorname{Spec} K$ be a d-dimensional geometrically integral projective variety over K. Let $\overline{L} = (L, \varphi)$ be an adelic invertible $\mathcal{O}_X$-module. Let E be a torsion free $\mathcal{O}_X$-module and U be a non-empty Zariski open set of X such that $E|_U$ is a vector bundle. Let $\psi = (\psi_\omega)_{\omega \in \Omega}$ be a metric family of $E|_U$. We assume that $(L^{\otimes n} \otimes E, U, n\varphi|_U + \psi)$ is a sectionally adelic torsion free $\mathcal{O}_X$-module (see Definition 2.4.1) for all $n \in \mathbb{N}$. Note that, if the sectional algebra $\bigoplus_{n \in \mathbb{N}} H^0(X, L^{\otimes n})$ is of finite type over K (this condition is true notably when L satisfies the hypothesis of Lemma 4.4.1), by Theorem-Definition 4.2.1, the sequence

$$\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1$$

converges to a real number denoted by $\widehat{\mathrm{vol}}_\chi(\overline{L})$.

Theorem 4.4.2. *If there are a birational morphism $f : X \to Z$ of geometrically integral projective schemes over* $\operatorname{Spec} K$ *and an ample invertible $\mathcal{O}_Z$-module A such that $L = f^*(A)$, then the following inequality holds:*

$$\mathrm{rk}(E)\, \widehat{\mathrm{vol}}_\chi(\overline{L}) \leqslant \liminf_{n \to \infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!}.$$

Proof. Let r be the rank of E. Note that $p_*(\overline{L}^{\otimes n} \otimes \overline{E})$ forms an adelic vector bundle over S for any $n \in \mathbb{N}$. For a sufficiently large positive integer n_0, shrinking U if necessarily, we can find $e_1, \ldots, e_r \in H^0(X, L^{\otimes n_0} \otimes E)$ such that $e_1, \ldots, e_r$ yield a basis of $L^{\otimes n_0} \otimes E$ over U. Indeed, there is a positive integer n_0 such that

$$H^0(Z, A^{\otimes n_0} \otimes f_*(E)) \otimes \mathcal{O}_Z \longrightarrow A^{\otimes n_0} \otimes f_*(E)$$

is surjective, and hence

$$H^0(X, L^{\otimes n_0} \otimes E) \otimes \mathcal{O}_X \longrightarrow L^{\otimes n_0} \otimes E$$

is surjective on some non-empty Zariski open subset of X. Thus the assertion follows. Let $\mathcal{O}_X^{\oplus r} \to L^{\otimes n_0} \otimes E$ be the homomorphism given by

$$(a_1, \ldots, a_r) \longmapsto a_1 e_1 + \cdots + a_r e_r.$$

Let Q be the cokernel of $\mathcal{O}_X^{\oplus r} \to L^{\otimes n_0} \otimes E$. The sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow L^{\otimes n_0} \otimes E \longrightarrow Q \longrightarrow 0$$

is exact, and so is

$$0 \longrightarrow (L^{\otimes n})^{\oplus r} \longrightarrow L^{\otimes n+n_0} \otimes E \longrightarrow L^{\otimes n} \otimes Q \longrightarrow 0.$$

Thus

$$0 \longrightarrow H^0(X, L^{\otimes n})^{\oplus r} \longrightarrow H^0(X, L^{\otimes n+n_0} \otimes E) \longrightarrow H^0(X, L^{\otimes n} \otimes Q)$$

is also exact for all $n \geqslant 0$. Let Q_n be the image of

$$H^0(X, L^{\otimes n+n_0} \otimes E) \longrightarrow H^0(X, L^{\otimes n} \otimes Q).$$

We equip $H^0(X, L^{\otimes n+n_0} \otimes E)$ with the norm family

$$\xi_{(n+n_0)\varphi+\psi} = (\|\cdot\|_{(n+n_0)\varphi_\omega+\psi_\omega})_{\omega\in\Omega}.$$

Let $\xi_n^L = (\|\cdot\|_{n,\omega}^L)_{\omega\in\Omega}$ be its restricted norm family on $H^0(X, L^{\otimes n})^{\oplus r}$ induced by the injection

$$H^0(X, L^{\otimes n})^{\oplus r} \longrightarrow H^0(X, L^{\otimes n+n_0} \otimes E).$$

Let $\xi_n^Q = (\|\cdot\|_{n,\omega}^Q)_{\omega\in\Omega}$ be its quotient family on Q_n induced by the surjection

$$H^0(X, L^{\otimes n+n_0} \otimes E) \longrightarrow Q_n.$$

Then, by [**36**, Proposition 4.3.13, (4.26)],

$$\widehat{\deg}(H^0(X, L^{\otimes n})^{\oplus r}, \xi_n^L) + \widehat{\deg}(Q_n, \xi_n^Q) \leqslant \widehat{\deg}(H^0(X, L^{\otimes n+n_0} \otimes E), \xi_{(n+n_0)\varphi+\psi}).$$

Since $\dim \operatorname{Supp}(Q) < \dim X$, by Proposition 4.3.2,

$$\liminf_{n\to\infty} \frac{\widehat{\deg}(Q_n, (\|\cdot\|_{n,\omega}^Q)_{\omega\in\Omega})}{n^{d+1}} \geqslant 0.$$

Therefore, by the super-additivity of inferior limit, we obtain

$$\liminf_{n\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n})^{\oplus r}, \xi_n^L)}{n^{d+1}/(d+1)!} \leqslant \liminf_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes E))}{n^{d+1}/(d+1)!}. \tag{4.6}$$

Let us consider the homomorphism of identity

$$(H^0(X, L^{\otimes n})^{\oplus r}, (\|\cdot\|_{n\varphi_\omega}^{\oplus r})_{\omega\in\Omega}) \longrightarrow (H^0(X, L^{\otimes n})^{\oplus r}, (\|\cdot\|_{n,\omega}^L)_{\omega\in\Omega}),$$

where

$$\|(s_1, \ldots, s_r)\|_{n\varphi_\omega}^{\oplus r} = \begin{cases} \max_{i\in\{1,\ldots,r\}} \|s_i\|_{n\varphi_\omega} & \text{if } \omega \in \Omega \setminus \Omega_\infty, \\ (\|s_1\|_{n\varphi_\omega}^2 + \cdots + \|s_r\|_{n\varphi_\omega}^2)^{1/2} & \text{if } \omega \in \Omega_\infty. \end{cases}$$

If $\omega \in \Omega \setminus \Omega_\infty$, then

$$\begin{aligned} \|(s_1, \ldots, s_r)\|_{n,\omega}^L &\leqslant \|s_1 e_1 + \cdots + s_r e_r\|_{(n+n_0)\varphi_\omega+\psi_\omega} \\ &\leqslant \max_{i\in\{1,\ldots,r\}} \|s_i\|_{n\varphi_\omega} \|e_i\|_{n_0\varphi_\omega+\psi_\omega} \\ &\leqslant \|(s_1, \ldots, s_r)\|_{n\varphi_\omega}^{\oplus r} \Big(\max_{i\in\{1,\ldots,r\}} \|e_i\|_{n_0\varphi_\omega+\psi_\omega} \Big). \end{aligned}$$

Moreover, if $\omega \in \Omega_\infty$, then by Cauchy-Schwarz inequality

$$\begin{aligned}\|(s_1,\ldots,s_r)\|_{n,\omega}^{L} &\leqslant \|s_1e_1+\cdots+s_re_r\|_{(n+n_0)\varphi_\omega+\psi_\omega}\\ &\leqslant \sum\nolimits_{i=1}^r \|s_i\|_{n\varphi_\omega}\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\\ &\leqslant \Big(\sum\nolimits_{i=1}^r \|s_i\|_{n\varphi_\omega}\Big)\Big(\max_{i\in\{1,\ldots,r\}}\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\Big)\\ &\leqslant \sqrt{r}\|(s_1,\ldots,s_r)\|_{n\varphi_\omega}^{\oplus r}\max_{i\in\{1,\ldots,r\}}\|e_i\|_{n_0\varphi_\omega+\psi_\omega}.\end{aligned}$$

Therefore,

$$h(f_n)\leqslant\int_\Omega \max_{i\in\{1,\ldots,r\}}\log\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\,\nu(d\omega)+\frac{1}{2}\log(r)\operatorname{vol}(\Omega_\infty),$$

and hence, by [**36**, Proposition 4.3.18],

$$\begin{aligned}r\,\widehat{\deg}(H^0(X,L^{\otimes n}),\xi_{n\varphi}) &= \widehat{\deg}(H^0(X,L^{\otimes n})^{\oplus r},\xi_{n\varphi}^{\oplus r})\\ &\leqslant \widehat{\deg}(H^0(X,L^{\otimes n})^{\oplus r},\xi_n^{L})+\\ &\qquad rh^0(L^{\otimes n})\left(\int_\Omega \max_{i\in\{1,\ldots,r\}}\log\|e_i\|_{n_0\varphi_\omega+\psi_\omega}\,\nu(d\omega)+\frac{1}{2}\log(r)\operatorname{vol}(\Omega_\infty)\right),\end{aligned}$$

where

$$h^0(L^{\otimes n})=\dim_k H^0(X,L^{\otimes n}),\quad \xi_{n\varphi}=(\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega},\quad \xi_{n\varphi}^{\oplus r}=(\|\cdot\|_{n\varphi_\omega}^{\oplus r})_{\omega\in\Omega}.$$

Thus,

$$r\,\widehat{\operatorname{vol}}_\chi(\overline{L})\leqslant\liminf_{n\to\infty}\frac{\widehat{\deg}(H^0(X,L^{\otimes n})^{\oplus r},\xi_n^L)}{n^{d+1}/(d+1)!}.$$

Combining this inequality with (4.6), we obtain the assertion. □

COROLLARY 4.4.3. *Let $\pi: Y\to X$ be a generically finite morphism of geometrically integral projective schemes over K, $\overline{L}=(L,\varphi)$ be an adelic invertible $\mathcal{O}_X$-module and $\overline{M}=(M,\psi)$ be an adelic invertible $\mathcal{O}_Y$-module. If there are a birational morphism $p: X\to Z$ of geometrically integral projective schemes over K and an ample invertible $\mathcal{O}_Z$-module A such that $L=p^*(A)$, then*

$$\deg(\pi)\,\widehat{\operatorname{vol}}_\chi(\overline{L})\leqslant\liminf_{n\to\infty}\frac{\widehat{\deg}\big((p\circ\pi)_*(\pi^*(\overline{L})^{\otimes n}\otimes\overline{M})\big)}{n^{d+1}/(d+1)!}.$$

In particular, $\deg(\pi)\,\widehat{\operatorname{vol}}_\chi(\overline{L})\leqslant\widehat{\operatorname{vol}}_\chi(\pi^*(\overline{L}))$.

PROOF. Since $\pi^*(\overline{L}^{\otimes n})\otimes\overline{M}$ is an adelic invertible $\mathcal{O}_Y$-module, one can see that

$$(L^{\otimes n}\otimes\pi_*(M),\pi_*(n\pi^*(\varphi)+\psi))$$

is sectionally adelic for all $n\geqslant 0$ (see the last section of Chapter 2). Note that

$$\pi_*(n\pi^*(\varphi)+\psi)=n\varphi+\pi_*(\psi)\quad\text{and}\quad \operatorname{rk}(\pi_*M)=\deg(\pi).$$

Thus, by Theorem 4.4.2,

$$\deg(\pi)\,\widehat{\operatorname{vol}}_\chi(\overline{L})\leqslant\liminf_{n\to+\infty}\frac{\widehat{\deg}(H^0(X,L^{\otimes n}\otimes\pi_*(M)),(\|\cdot\|_{n\varphi_\omega+\pi_*(\psi)_\omega})_{\omega\in\Omega})}{n^{d+1}/(d+1)!}.$$

Moreover,

$$(H^0(X,L^{\otimes n}\otimes\pi_*(M)),(\|\cdot\|_{n\varphi_\omega+\pi_*(\psi)_\omega})_{\omega\in\Omega})$$

is isometric to

$$(H^0(Y, \pi^*(L^{\otimes n}) \otimes M), (\|\cdot\|_{n\pi_\omega^*(\varphi_\omega)+\psi_\omega})_{\omega\in\Omega}).$$

Thus we obtain the required inequality. □

THEOREM 4.4.4. *Let $\overline{L} = (L, \varphi)$ be an adelic invertible $\mathcal{O}_X$-module and $\overline{E} = (E, U, \psi)$ be a birationally adelic torsion free $\mathcal{O}_X$-module. We assume that there are a birational morphism $p : X \to Z$ of geometrically integral projective varieties over K and an ample invertible $\mathcal{O}_Z$-module A with $L = p^*(A)$. If either (E, ψ) is an adelic invertible $\mathcal{O}_X$-module or X is normal, then the sequence*

$$\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!}, \quad n \in \mathbb{N},\ n \geqslant 1$$

is convergent to $\operatorname{rk}(E)\,\widehat{\mathrm{vol}}_\chi(\overline{L})$.

PROOF. In view of Theorem 4.4.2, it suffices to establish the following inequality

$$\limsup_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!} \leqslant \operatorname{rk}(E)\,\widehat{\mathrm{vol}}_\chi(\overline{L}).$$

First we assume that (E, ψ) is an adelic invertible $\mathcal{O}_X$-module. Let us begin with the following claim:

CLAIM 4.4.5. *One has the following inequality:*

$$\limsup_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!} \leqslant \limsup_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes (n+n_0)}))}{n^{d+1}/(d+1)!}$$

for some positive integer n_0.

PROOF. Since L is nef and big, we can choose a positive integer n_0 and $s_0 \in H^0(X, L^{\otimes n_0} \otimes E^\vee) \setminus \{0\}$. Note that s_0 gives rise to an injective homomorphism

$$H^0(X, L^{\otimes n} \otimes E) \longrightarrow H^0(X, L^{\otimes (n+n_0)}).$$

Let $\xi_{\mathrm{sub},n} = (\|\cdot\|_{\mathrm{sub},n,\omega})_{\omega\in\Omega}$ be the restricted norm family of $H^0(X, L^{\otimes n} \otimes E)$ induced by the above injective homomorphism and

$$\xi_{(n+n_0)\varphi} = (\|\cdot\|_{(n+n_0)\varphi_\omega})_{\omega\in\Omega}.$$

In order to show Claim 4.4.5, it is sufficient to see the following two inequalities:

$$\limsup_{n\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n} \otimes E), \xi_{\mathrm{sub},n})}{n^{d+1}/(d+1)!} \leqslant \limsup_{n\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes (n+n_0)}), \xi_{(n+n_0)\varphi})}{n^{d+1}/(d+1)!}$$

and

$$\limsup_{n\to\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n} \otimes \overline{E}))}{n^{d+1}/(d+1)!} \leqslant \limsup_{n\to\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n} \otimes E), \xi_{\mathrm{sub},n})}{n^{d+1}/(d+1)!}.$$

The first inequality is a consequence of Lemma 4.3.1, Proposition 4.3.2, [**38**, Lemma 1.2.16] and [**36**, Proposition 4.3.13, (4.26)]. Let us consider the homomorphism of identity

$$f : \Big(H^0(X, L^{\otimes n} \otimes E), (\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega}\Big) \longrightarrow \Big(H^0(X, L^{\otimes n} \otimes E), \xi_{\mathrm{sub},n}\Big).$$

For $s \in H^0(X, E \otimes L^{\otimes n}) \setminus \{0\}$,

$$\begin{aligned}\frac{\|s\|_{\mathrm{sub},n,\omega}}{\|s\|_{n\varphi_\omega+\psi_\omega}} &= \frac{\|ss_0\|_{(n+n_0)\varphi_\omega}}{\|s\|_{n\varphi_\omega+\psi_\omega}} \\ &\leqslant \frac{\|s\|_{n\varphi_\omega+\psi_\omega}\|s_0\|_{n_0\varphi_\omega-\psi_\omega}}{\|s\|_{n\varphi_\omega+\psi_\omega}} = \|s_0\|_{n_0\varphi_\omega-\psi_\omega},\end{aligned}$$

so that $\|f\|_\omega \leqslant \|s_0\|_{n_0\varphi_\omega-\psi_\omega}$. Therefore, by [**36**, Proposition 4.3.18],

$$\begin{aligned}\widehat{\deg}(H^0(X, L^{\otimes n} \otimes E), (\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega}) &\leqslant \widehat{\deg}(H^0(X, L^{\otimes n} \otimes E), \xi_{\mathrm{sub},n}) \\ &\quad + \dim H^0(X, L^{\otimes n} \otimes E)\int_\Omega \log \|s_0\|_{n_0\varphi_\omega-\psi_\omega}\,\nu(d\omega).\end{aligned}$$

Thus the second inequality follows. □

By Lemma 4.4.1, Theorem-Definition 4.2.1 and the relation

$$\lim_{n\to+\infty}\frac{(n+n_0)^{d+1}}{n^{d+1}} = 1,$$

we obtain that

$$\lim_{n\to+\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes(n+n_0)}))}{n^{d+1}/(d+1)!} = \widehat{\mathrm{vol}}_\chi(\overline{L}).$$

Hence Claim 4.4.5 leads to

$$\widehat{\mathrm{vol}}_\chi(\overline{L};\overline{E}) \leqslant \widehat{\mathrm{vol}}_\chi(\overline{L}),$$

as required.

Next we assume that X is normal. We prove the assertion by induction on $r := \mathrm{rk}(E)$. Let $\mu : X' \to X$, (E', ψ') and U be a birational morphism, an adelic invertible $\mathcal{O}_{X'}$-module and a non-empty Zariski open set of X, respectively, as in Definition 2.4.2. First we suppose that $r = 1$.

Claim 4.4.6. *One has the following inequality:*

$$\limsup_{n\to\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{E}))}{n^{d+1}/(d+1)!} \leqslant \limsup_{n\to\infty}\frac{\widehat{\deg}((p\circ\mu)_*(\mu^*(\overline{L})^{\otimes n}\otimes\overline{E'}))}{n^{d+1}/(d+1)!}$$

Proof. This is a consequence of Lemma 4.3.1, Proposition 4.3.2, Lemma 4.4.1 and [**36**, (4.26) in Proposition 4.3.13]. □

By Claim 4.4.6 together with the case where (E, ψ) is an adelic invertible $\mathcal{O}_X$-module, one has

$$\limsup_{n\to\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{E}))}{n^{d+1}/(d+1)!} \leqslant \widehat{\mathrm{vol}}_\chi(\mu^*(\overline{L})).$$

On the other hand, since X is normal, one can see that $\widehat{\mathrm{vol}}_\chi(\mu^*(\overline{L})) = \widehat{\mathrm{vol}}_\chi(\overline{L})$, as desired.

In the case where $r \geqslant 2$, considering a birational morphism $X'' \to X'$ if necessarily, we may assume that there exists an exact sequence $0 \to F' \to E' \to Q' \to 0$ on X' such that F' and Q' are locally free, $\mathrm{rk}(F') = 1$ and $\mathrm{rk}(Q') = r - 1$. Let $\psi_{F'}$ be the restricted metric of F' over X' and $\psi_{Q'}$ be the quotient metric of Q' over X'. Let Q be the image of $E \to \mu_*(E') \to \mu_*(Q')$ and F be the kernel of $E \to Q$. Shrinking U if necessarily, $\psi_{Q'}$ and $\psi_{F'}$ descent to metric families ψ_Q and ψ_F of $Q|_U$ and $F|_U$. Note that $\overline{Q} = (Q, \psi_Q)$

and $\overline{F} = (F, \psi_F)$ are birationally adelic torsion free $\mathcal{O}_X$-modules by Proposition 2.2.7 and Corollary 2.3.8. Therefore, by the hypothesis of induction,

$$\begin{cases} \limsup_{n\to\infty} \dfrac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{F}))}{n^{d+1}/(d+1)!} \leqslant \widehat{\mathrm{vol}}_\chi(L,\varphi), \\ \limsup_{n\to\infty} \dfrac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{Q}))}{n^{d+1}/(d+1)!} \leqslant (r-1)\,\widehat{\mathrm{vol}}_\chi(L,\varphi). \end{cases}$$

For any $n\in\mathbb{N}$, one has an exact sequence

$$0 \to H^0(X, L^n\otimes F) \to H^0(X, L^n\otimes E) \to H^0(X, L^n\otimes Q) \to H^1(X, L^{\otimes n}\otimes F). \quad (4.7)$$

Let Q_n be the image of

$$H^0(X, L^{\otimes n}\otimes E) \longrightarrow H^0(X, L^{\otimes n}\otimes Q).$$

Let $\xi_{n,\mathrm{sub}} = (\|\cdot\|_{n,\mathrm{sub},\omega})_{\omega\in\Omega}$ be the restricted norm family of

$$\xi_{n\varphi+\psi} = (\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega}$$

on $H^0(X, L^n\otimes F)$ and $\xi_{n,\mathrm{quot}} = (\|\cdot\|_{n,\mathrm{quot},\omega})_{\omega\in\Omega}$ be the quotient norm family of $\xi_{n\varphi+\psi}$ on $H^0(X, L^n\otimes Q)$. By [**36**, (4.28)],

$$\begin{aligned} &\widehat{\deg}(H^0(X, L^n\otimes E), \xi_{n\varphi+\psi}) - \delta(H^0(X, L^n\otimes E), \xi_{n\varphi+\psi}) \\ &\quad\leqslant \Big(\widehat{\deg}(H^0(X, L^n\otimes F), \xi_{n,\mathrm{sub}}) - \delta(H^0(X, L^n\otimes F), \xi_{n,\mathrm{sub}})\Big) \\ &\quad\quad + \Big(\widehat{\deg}(H^0(X, L^n\otimes Q), \xi_{n,\mathrm{quot}}) - \delta(H^0(X, L^n\otimes Q), \xi_{n,\mathrm{quot}})\Big), \end{aligned}$$

where for any adelic vector bundle $\overline{V}$ on S, $\delta(\overline{V})$ denotes the sum $\widehat{\deg}(\overline{V}) + \widehat{\deg}(\overline{V}^\vee)$. Let $\xi_{n\varphi+\psi_Q,\mathrm{sub}} = (\|\cdot\|_{n\varphi_\omega+\psi_{Q,\omega},\mathrm{sub}})_{\omega\in\Omega}$ be the restriction of

$$\xi_{n\varphi+\psi_Q} = (\|\cdot\|_{n\varphi_\omega+\psi_{Q,\omega}})_{\omega\in\Omega}$$

to Q_n. It is easy to see that, for any $\omega\in\Omega$,

$$\|\cdot\|_{n,\mathrm{sub},\omega} = \|\cdot\|_{n\varphi_\omega+\psi_{F,\omega}}, \quad \|\cdot\|_{n,\mathrm{quot},\omega} \geqslant \|\cdot\|_{n\varphi_\omega+\psi_{Q,\omega},\mathrm{sub}}.$$

Thus, by [**36**, Proposition 4.3.18],

$$\widehat{\deg}(Q_n, \xi_{n,\mathrm{quot}}) \leqslant \widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}}),$$

so that

$$\begin{aligned} &\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{E})) - \delta(p_*(\overline{L}^{\otimes n}\otimes\overline{E})) \\ &\quad\leqslant \Big(\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{F})) - \delta(H^0(X, L^n\otimes F), \xi_{n,\mathrm{sub}})\Big) \\ &\quad\quad + \Big(\widehat{\deg}(Q_n, \xi_{n\varphi+\psi_Q,\mathrm{sub}}) - \delta(Q_n, \xi_{n,\mathrm{quot}})\Big). \end{aligned}$$

Moreover, by [**36**, Proposition 4.3.10],

$$\begin{cases} \lim_{n\to\infty} \dfrac{\delta(p_*(\overline{L}^{\otimes n}\otimes\overline{E}))}{n^{d+1}} = 0, \\ \lim_{n\to\infty} \dfrac{\delta(H^0(X, L^n\otimes F), \xi_{n,\mathrm{sub}})}{n^{d+1}} = 0, \\ \lim_{n\to\infty} \dfrac{\delta(Q_n, \xi_{n,\mathrm{quot}})}{n^{d+1}} = 0, \end{cases}$$

so that one obtains

$$\limsup_{n\to+\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{E}))}{n^{d+1}/(d+1)!}\leqslant\widehat{\mathrm{vol}}_\chi(L,\varphi)+\limsup_{n\to+\infty}\frac{\widehat{\deg}(Q_n,\xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}/(d+1)!},$$

and hence it is sufficient to show that

$$\limsup_{n\to+\infty}\frac{\widehat{\deg}(Q_n,\xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}/(d+1)!}\leqslant\limsup_{n\to+\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{Q}))}{n^{d+1}/(d+1)!}. \tag{4.8}$$

Claim 4.4.7. *If we set* $T_n = H^0(X, L^{\otimes n}\otimes Q)/Q_n$, *then*

$$\lim_{n\to+\infty}\dim_K(T_n)/n^d = 0.$$

Proof. By the Leray spectral sequence

$$E_2^{p,q}=H^p(Z,A^{\otimes n}\otimes R^qf_*(F))\Longrightarrow H^{p+q}(X,L^{\otimes n}\otimes F),$$

if n is sufficiently large, then one has an injective homomorphism

$$H^1(X,L^{\otimes n}\otimes F)\longrightarrow H^0(Z,A^{\otimes n}\otimes R^1f_*(F))$$

so that

$$\lim_{n\to+\infty}\frac{\dim_K(H^1(X,L^{\otimes n}\otimes F))}{n^d}=0$$

because $\mathrm{Supp}(R^1f_*(F))$ has Krull dimension $< d$. Thus the assertion follows by (4.7). □

By Lemma 4.4.1, $\bigoplus_{n=0}^\infty H^0(X,Q\otimes L^n)$ is finitely generated over

$$\bigoplus_{n=0}^\infty H^0(X,L^n),$$

so that $\bigoplus_{n=0}^\infty T_n$ is also finitely generated over it. Let ξ_{T_n} be the quotient norm family of $\xi_{n\varphi+\psi_Q}$ on T_n. Then by Claim 4.4.7 together with Proposition 4.3.2, we obtain that

$$\liminf_{n\to+\infty}\frac{\widehat{\deg}(T_n,\xi_{T_n})}{n^{d+1}}\geqslant 0,$$

that is, for any $\varepsilon>0$,

$$\frac{\widehat{\deg}(T_n,\xi_{T_n})}{n^{d+1}}\geqslant-\varepsilon$$

for sufficiently large n. Moreover, by [**36**, Proposition 4.3.13, (4.26)],

$$\frac{\widehat{\deg}(Q_n,\xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}}+\frac{\widehat{\deg}(T_n,\xi_{T_n})}{n^{d+1}}\leqslant\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{Q}))}{n^{d+1}},$$

so that

$$\frac{\widehat{\deg}(Q_n,\xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}}-\varepsilon\leqslant\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{Q}))}{n^{d+1}}$$

for sufficiently large n. Thus,

$$\limsup_{n\to+\infty}\frac{\widehat{\deg}(Q_n,\xi_{n\varphi+\psi_Q,\mathrm{sub}})}{n^{d+1}}-\varepsilon\leqslant\limsup_{n\to+\infty}\frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{Q}))}{n^{d+1}}.$$

Since ε is arbitrary, we obtain the inequality (4.8). □

Corollary 4.4.8. *Let (E, U, ψ) be a birational adelic torsion free $\mathcal{O}_X$-module. If X is normal and L is ample, then*

$$\lim_{n\to+\infty} \frac{\widehat{\deg}(p_*(\overline{L}^{\otimes n}\otimes\overline{E}))}{n^{d+1}/(d+1)!} = \mathrm{rk}(E)\,\widehat{\mathrm{vol}}_\chi(L,\varphi).$$

Proof. This is a consequence of Theorem 4.4.2 and Theorem 4.4.4. □

CHAPTER 5

Hilbert-Samuel property

This chapter is devoted to the proof of the arithmetic Hilbert-Samuel formula. We first show, in the first section, that the difference of the two sides of the equality does not depend on the choice of the metric family on the line bundle. Then, in the second section, we show by an explicit computation with a specific choice of metric family that the arithmetic Hilbert-Samuel formula holds for a projective space. In the third chapter, we prove the arithmetic Hilbert-Samuel formula for the trivial valuation case. The proof of the arithmetic Hilbert-Samuel formula in the general case is presented in the last two sections. We use the casting to the trivial valuation case to show that the arithmetic χ-volume is bounded from above by the arithmetic intersection number. The converse inequality is obtained by a finite projection to a projective space.

Let $f : X \to \operatorname{Spec} K$ be an integral projective scheme over $\operatorname{Spec} K$, d be the dimension of X and L be an ample invertible $\mathcal{O}_X$-module. We assume that, either the field K is perfect, or the scheme X is geometrically integral. We denote by $\mathscr{M}^+(L)$ the set of metrics families $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ such that all metrics φ_ω are semi-positive and that (L, φ) forms an adelic line bundle on X.

5.1. Definition and reduction

Definition 5.1.1. We say that $\varphi \in \mathscr{M}^+(L)$ satisfies the *Hilbert-Samuel property* if the equality

$$\widehat{\mathrm{vol}}_\chi(L, \varphi) = ((L, \varphi)^{d+1})$$

holds, namely the χ-volume coincides with the self-intersection number of (L, φ).

Remark 5.1.2. Note that Theorem-Definition 4.2.1 shows that, for any positive integer n, one has

$$\widehat{\mathrm{vol}}_\chi(L^{\otimes n}, n\varphi) = n^{d+1}\widehat{\mathrm{vol}}_\chi(L, \varphi).$$

Therefore, if φ satisfies the Hilbert-Samuel property, then for any positive integer n, the metric family $n\varphi$ also satisfies the Hilbert-Samuel property. Conversely, if there exists a positive integer n such that $n\varphi$ satisfies the Hilbert-Samuel property, then so does the metric family φ.

In order to show the Hilbert-Samuel property for all metrics families in $\mathscr{M}^+(L)$, it suffices to check the property for one arbitrary metric family in $\mathscr{M}^+(L)$.

Lemma 5.1.3. *Let E be a finite-dimensional vector space over K. If $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ and $\xi' = (\|\cdot\|'_\omega)_{\omega\in\Omega}$ are two norm families on E, then one has*

$$d_\omega(\det(\xi), \det(\xi')) \leqslant r d_\omega(\xi, \xi'). \tag{5.1}$$

In particular, if ξ is strongly dominated, so is $\det(\xi)$.

PROOF. Let r be the dimension of E over K. If η is a non-zero element of $\det(E_\omega)$, then one has

$$\ln\|\eta\|_{\omega,\det}-\ln\|\eta\|'_{\omega,\det}=\sup_{\substack{(s_1,\ldots,s_r)\in E_\omega^r\\ \eta=s_1\wedge\cdots\wedge s_r}}\ln\|s_1\wedge\cdots\wedge s_r\|_{\omega,\det}-\sum_{i=1}^r\ln\|s_i\|'_\omega$$

$$\leqslant\sup_{\substack{(s_1,\ldots,s_r)\in E_\omega^r\\ \eta=s_1\wedge\cdots\wedge s_r}}\sum_{i=1}^r\ln\|s_i\|_\omega-\ln\|s_i\|'_\omega\leqslant rd_\omega(\xi,\xi').$$

Interchanging ξ and ξ', the above inequality leads to

$$\ln\|\eta\|'_{\omega,\det}-\ln\|\eta\|_{\omega,\det}\leqslant rd_\omega(\xi,\xi').$$

Therefore, the inequality (5.1) holds. □

PROPOSITION 5.1.4. *Assume that there exists a metric family $\psi\in\mathscr{M}^+(L)$ which satisfies the Hilbert-Samuel property. Then any metric family $\varphi\in\mathscr{M}^+(L)$ satisfies the Hilbert-Samuel property.*

PROOF. For any $n\in\mathbb{N}$, let E_n be the K-vector space $H^0(X,L^{\otimes n})$ and r_n be the dimension of E_n of K. For any $\omega\in\Omega$, let $E_{n,\omega}=E_n\otimes_K K_\omega$,

$$d_{n,\omega}=\sup_{s\in E_{n,\omega}\setminus\{0\}}\Big|\ln\|s\|_{n\varphi_\omega}-\ln\|s\|_{n\psi_\omega}\Big|$$

be the distance of $\|\cdot\|_{n\varphi_\omega}$ and $\|\cdot\|_{n\psi_\omega}$, and

$$\delta_{n,\omega}=\sup_{\eta\in\det(E_{n,\omega})\setminus\{0\}}\ln\|\eta\|_{n\varphi_\omega,\det}-\ln\|\eta\|_{n\psi_\omega,\det}.$$

Note that the function $(\omega\in\Omega)\mapsto\delta_{n,\omega}$ is ν-integrable, and one has

$$\int_\Omega\delta_{n,\omega}\,\nu(\mathrm{d}\omega)=\widehat{\deg}(p_*(L^{\otimes n},n\psi))-\widehat{\deg}(p_*(L^{\otimes n},n\varphi)).$$

By Lemma 5.1.3, one has

$$|\delta_{n,\omega}|\leqslant r_nd_{n,\omega}\leqslant nr_nd_\omega(\varphi,\psi).$$

Note that the function

$$(\omega\in\Omega)\longrightarrow d_\omega(\varphi,\psi)$$

is dominated (see [**36**, Proposition 6.1.12]). Moreover, by [**18**, Theorem 1.7], one has

$$\lim_{n\to+\infty}\frac{\delta_{n,\omega}}{n^{d+1}/(d+1)!}=\sum_{j=0}^d\int_{X_\omega^{\mathrm{an}}}f_\omega(x)\,\mu_{(L_\omega,\varphi_\omega)^j(L_\omega,\psi_\omega)^{d-j}}(\mathrm{d}x),$$

where f_ω is the continuous function on X_ω^{an} such that

$$\mathrm{e}^{f_\omega(\omega)}|\cdot|_{\psi_\omega}(x)=|\cdot|_{\varphi_\omega}$$

for any $x\in X_\omega^{\mathrm{an}}$. Hence Theorem-Definition 4.2.1 and Lebesgue's dominated convergence theorem lead to (see Remark 4.2.2)

$$\widehat{\mathrm{vol}}_\chi(L,\psi)-\widehat{\mathrm{vol}}_\chi(L,\varphi)=\lim_{n\to+\infty}\frac{1}{n^{d+1}/(d+1)!}\int_\Omega\delta_{n,\omega}\,\nu(\mathrm{d}\omega)$$

$$=\sum_{j=0}^d\int_\Omega\int_{X_\omega^{\mathrm{an}}}f_\omega(x)\,\mu_{(L_\omega,\varphi_\omega)^j(L_\omega,\psi_\omega)^{d-j}}(\mathrm{d}x)\,\nu(\mathrm{d}\omega)$$

$$=((L,\psi)^{d+1})-((L,\varphi)^{d+1}).$$

The proposition is thus proved. □

DEFINITION 5.1.5. Let X be a geometrically integral projective scheme over $\operatorname{Spec} K$ and L be an ample invertible $\mathcal{O}_X$-module. If there exists a metric family $\varphi \in \mathscr{M}^+(L)$ which satisfies the Hilbert-Samuel property, or equivalently, any metric family $\varphi \in \mathscr{M}^+(L)$ satisfies the Hilbert-Samuel property (see Proposition 5.1.4), we say that the ample invertible $\mathcal{O}_X$-module L *satisfies the Hilbert-Samuel property*.

REMARK 5.1.6. The proof of Proposition 5.1.4 actually shows a more precise result: the function

$$(\varphi \in \mathscr{M}^+(L)) \longrightarrow \widehat{\mathrm{vol}}_\chi(L,\varphi) - ((L,\varphi)^{d+1})$$

is constant.

5.2. Case of a projective space

In this section, we assume that $X = \mathbb{P}^d_K$ is the projective space and $L = \mathcal{O}_{\mathbb{P}^d_K}(1)$ is the universal line bundle. We show that any metric family in $\mathscr{M}^+(L)$ satisfies the Hilbert-Samuel property. Without loss of generality (by Proposition 5.1.4), we consider a particular case as follows. Let E be a $(d+1)$-dimensional vector space over K and $(e_i)_{i=0}^d$ be a basis of E. Let $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ be the Hermitian norm family on E such that $(e_i)_{i=0}^d$ forms an orthonormal basis of E with respect to $\|\cdot\|_\omega$ for any $\omega \in \Omega$. We then identify $\mathbb{P}^d_K$ with $\mathbb{P}(E)$ and let $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be the quotient metric family on L induced by ξ. Note that, for any integer $n \in \mathbb{N}$, the vector space $H^0(X, L^{\otimes n})$ is isomorphic to the symmetric power $S^n(E)$. We denote by r_n the dimension of $S^n(E)$. One has

$$r_n = \binom{n+d}{d}.$$

DEFINITION 5.2.1. Let $\omega \in \Omega$ such that $|\cdot|_\omega$ is non-Archimedean. Let x be the point in $\mathbb{P}(E_\omega)^{\mathrm{an}}$ which consists of the generic scheme point of $\mathbb{P}(E_\omega)$ equipped with the absolute value

$$|\cdot|_x : k\big(\tfrac{e_0}{e_r}, \ldots, \tfrac{e_{r-1}}{e_r}\big) \longrightarrow \mathbb{R}_{\geqslant 0}$$

such that, for any

$$P = \sum_{\boldsymbol{a}=(a_0,\ldots,a_{r-1})\in\mathbb{N}^r} \lambda_{\boldsymbol{a}} \Big(\frac{e_0}{e_r}\Big)^{a_0} \cdots \Big(\frac{e_{r-1}}{e_r}\Big)^{a_{r-1}} \in k\big[\tfrac{e_0}{e_r}, \ldots, \tfrac{e_{r-1}}{e_r}\big],$$

one has

$$|P|_x = \max_{\boldsymbol{a}\in\mathbb{N}^r} |\lambda_{\boldsymbol{a}}|_\omega.$$

Note that the point x does not depend on the choice of the orthonormal basis $(e_j)_{j=0}^r$. In fact, the norm $\|\cdot\|$ induces a symmetric algebra norm on $K_\omega[E_\omega]$ (which is often called a *Gauss norm*) and hence defines an absolute value on the fraction field of $K_\omega[E_\omega]$. The restriction of this absolute value to the field of rational functions on $\mathbb{P}(E_\omega)$ identifies with $|\cdot|_x$. Hence x is called the *Gauss point* of $\mathbb{P}(E_\omega)^{\mathrm{an}}$.

LEMMA 5.2.2. *Let ω be an element of Ω such that $|\cdot|_\omega$ is non-Archimedean, and $n \in \mathbb{N}$. Let $\|\cdot\|_{n,\omega}$ be the ε-tensor power of $\|\cdot\|_\omega$ on the tensor power space $E_\omega^{\otimes n}$ and let $\|\cdot\|'_{n,\omega}$ be the quotient norm of $\|\cdot\|_{n,\omega}$ by the quotient homomorphism $E_\omega^{\otimes n} \to S^n(E_\omega)$. Then the norm $\|\cdot\|'_{n,\omega}$ coincides with the supremum norm $\|\cdot\|_{n\varphi_\omega}$ of the metric $n\varphi_\omega$ on $L_\omega^{\otimes n}$.*

Proof. For any $\omega \in \Omega$, we denote by E_ω the K_ω-vector space $E \otimes_K K_\omega$. By [**36**, Propositions 1.3.16 and 1.2.36], if we consider the Segre embedding $\mathbb{P}(E_\omega) \to \mathbb{P}(E_\omega^{\otimes n})$, then the metric $n\varphi_\omega$ identifies with the quotient metric induced by the norm $\|\cdot\|_{n,\omega}$. Moreover, if we denote by $\mathcal{O}_{E_\omega^{\otimes n}}(1)$ the universal invertible sheaf of $\mathbb{P}(E_\omega^{\otimes n})$ and by ψ_ω the quotient metric on this invertible sheaf induced by the norm $\|\cdot\|_{n,\omega}$. By [**36**, Proposition 2.2.22], the supremum norm $\|\cdot\|_{\psi_\omega}$ on

$$H^0(\mathbb{P}(E_\omega^{\otimes n}), \mathcal{O}_{E_\omega^{\otimes n}}(1)) = E_\omega^{\otimes n}$$

of the metric ψ_ω coincides with $\|\cdot\|_{n,\omega}$. Since $L^{\otimes n}$ is the restriction of $\mathcal{O}_{E_\omega^{\otimes n}}(1)$ to X and the restriction map

$$H^0(\mathbb{P}(E_\omega^{\otimes n}), \mathcal{O}_{E_\omega^{\otimes n}}(1)) \longrightarrow H^0(\mathbb{P}(E_\omega), L_\omega^{\otimes n})$$

identifies with the quotient homomorphism $E_\omega^{\otimes n} \to S^n(E_\omega)$. In particular, the supremum norm $\|\cdot\|_{\varphi_\omega^{\otimes n}}$ is bounded from above by the quotient norm $\|\cdot\|'_{n,\omega}$.

Let x be the Gauss point of the Berkovich analytic space $\mathbb{P}(E_\omega)^{\mathrm{an}}$ (see Definition 5.2.1). If

$$F = \sum_{\substack{I=(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} \lambda_I e_0^{a_0}\cdots e_d^{a_d}$$

is an element of $S^n(E)$, then the relation

$$F(x) = \Bigg(\sum_{\substack{I=(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} \lambda_I\Big(\frac{e_1}{e_0}\Big)^{a_1}\cdots\Big(\frac{e_d}{e_0}\Big)^{a_d}\Bigg)e_0(x)^{\otimes n}$$

holds. In particular, one has

$$\|F\|_{n\varphi_\omega} \geqslant |F|_{n\varphi_\omega}(x) = \max_{\substack{I=(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_n=d}} |\lambda_I|_\omega.$$

Since F is the image of the element

$$\widetilde{F} = \sum_{\substack{I=(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} \lambda_I e_0^{\otimes a_0}\otimes\cdots\otimes e_d^{\otimes a_d}$$

by the quotient map $E_\omega^{\otimes n} \to S^n(E_\omega)$, we obtain that

$$\|F\|_{n\varphi_\omega} \geqslant \|\widetilde{F}\|_{n,\omega} \geqslant \|F\|'_{n,\omega}.$$

Therefore the equality $\|\cdot\|_{n\varphi_\omega} = \|F\|'_{n,\omega}$ holds. □

Remark 5.2.3. As a byproduct, the proof of the above lemma shows that, for any

$$F = \sum_{\substack{I=(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} \lambda_I e_0^{a_0}\cdots e_d^{a_d} \in S^n(E_\omega),$$

one has

$$\|F\|_{n\varphi_\omega} = \max_{\substack{I=(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} |\lambda_I|_\omega.$$

In other words, the family

$$(e_0^{a_0}\cdots e_d^{a_d})_{\substack{(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}}$$

forms an orthonormal basis of $(S^d(E_\omega), \|\cdot\|_{n\varphi_\omega})$.

LEMMA 5.2.4. *For any integer $d \in \mathbb{N}$ and any any $x > 0$, let*

$$P_{d+1,x} = \{(t_0, \ldots, t_d) \in \mathbb{R}^{d+1}_{\geqslant 0} \mid t_0 + \cdots + t_d \leqslant x\},$$
$$\Delta_{d,x} = \{(t_0, \ldots, t_d) \in \mathbb{R}^{d+1}_{\geqslant 0} \mid t_0 + \cdots + t_d = x\}.$$

We denote by vol_{d+1} the Lebesgue measure on $\mathbb{R}^d$. For any affine hyperplane of $\mathbb{R}^d$, we denote by ν_d the translate of the Haar measure on the underlying hyperplane which is normalized with respect to the canonical Euclidean norm on $\mathbb{R}^{d+1}$ (namely the parallelotope spanned by an orthonormal basis has volume 1).

(1) *The volume of $P_{d+1,x}$ with respect to vol_{d+1} is $x^{d+1}/(d+1)!$.*
(2) *The volume of $\Delta_{d,x}$ with respect to ν_d is $x^d\sqrt{d+1}/d!$.*
(3) *Let μ_d be the uniform probability distribution on $\Delta_{d,x}$. One has*

$$\int_{\Delta_{d,1}} \Big(t_0 \ln(t_0) + \cdots + t_d \ln(t_d)\Big)\, \mu_d(\mathrm{d}t) = -\frac{1}{d+1}\sum_{m=1}^{d}\sum_{\ell=1}^{m}\frac{1}{\ell}.$$

PROOF. (1) We reason by induction on d. The case where $d = 0$ is trivial. In the following we assume the induction hypothesis that the lemma holds for $\mathbb{R}^d$. By Fubini's theorem, we have

$$\mathrm{vol}_{d+1}(P_{d+1,x}) = \int_0^x \mathrm{vol}_d(P_{d,x-t})\,\mathrm{d}t = \int_0^x \frac{(x-t)^d}{d!} = \frac{x^{d+1}}{(d+1)!}.$$

(2) The distance from the origin to the affine hyperplane containing $\Delta_{d,x}$ is $x/\sqrt{d+1}$. Therefore, by the equality

$$\mathrm{vol}_{d+1}(P_{d+1,x}) = \frac{1}{d+1}\cdot\frac{x}{\sqrt{d+1}}\nu_d(\Delta_{d,x}),$$

we obtain

$$\nu_d(\Delta_{d,x}) = \sqrt{d+1}\frac{x^d}{d!}.$$

(3) By Fubini's theorem, one has

$$\begin{aligned}
\int_{P_{d+1,x}} t_0\ln(t_0)\ \mathrm{vol}_{d+1}(\mathrm{d}t_0,\ldots,\mathrm{d}t_d) &= \int_0^x t\ln(t)\,\mathrm{vol}_d(P_{d,x-t})\,\mathrm{d}t\\
&= \frac{1}{d!}\int_0^x t(x-t)^d\ln(t)\,\mathrm{d}t = \frac{1}{d!}\sum_{i=0}^{d}(-1)^i\binom{d}{i}x^{d-i}\int_0^x t^{i+1}\ln(t)\,\mathrm{d}t\\
&= \frac{1}{d!}\sum_{i=0}^{d}(-1)^i\binom{d}{i}x^{d-i}\frac{1}{i+2}\left(x^{i+2}\ln(x) - \frac{1}{i+2}x^{i+2}\right)\\
&= \frac{x^{d+2}\ln(x)}{d!}\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{1}{i+2} - \frac{x^{d+2}}{d!}\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{1}{(i+2)^2}.
\end{aligned}$$

By a change of variables, we obtain

$$\int_{P_{d+1,x}} t_0\ln(t_0)\ \mathrm{vol}_{d+1}(\mathrm{d}t_0,\ldots,\mathrm{d}_{t_d}) = \frac{1}{\sqrt{d+1}}\int_0^x\int_{\Delta_{d,u}} t_0\ln(t_0)\,\nu_d(\mathrm{d}t)\,\mathrm{d}u.$$

Taking the derivative with respect to x, we obtain

$$\begin{aligned}\frac{(d+2)x^{d+1}\ln(x)+x^{d+1}}{d!}&\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{1}{i+2}\\&-\frac{(d+2)x^{d+1}}{d!}\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{1}{(i+2)^2}\\&=\frac{1}{\sqrt{d+1}}\int_{\Delta_{d,x}}t_0\ln(t_0)\,\nu_d(\mathrm{d}t)=\frac{\nu_d(\Delta_{d,x})}{\sqrt{d+1}}\int_{\Delta_{d,x}}t_0\ln(t_0)\,\mu_d(\mathrm{d}t).\end{aligned}$$

In particular, one has

$$\begin{aligned}\int_{\Delta_{d,1}}t_0\ln(t_0)\,\mu_d(\mathrm{d}t)&=\sum_{i=0}^{d}(-1)^i\binom{d}{i}\frac{1}{i+2}\Big(1-\frac{d+2}{i+2}\Big)\\&=\sum_{i=0}^{d}(-1)^i\frac{d!}{i!(d-i)!}\cdot\frac{i-d}{(i+2)^2}\\&=-\frac{1}{d+1}\sum_{i=0}^{d-1}(-1)^i\frac{(d+1)!}{(i+2)!(d-i-1)!}\cdot\frac{i+1}{i+2}.\end{aligned}$$

Therefore

$$\begin{aligned}&(d+1)\int_{\Delta_{d,1}}t_0\ln(t_0)\,\mu_d(\mathrm{d}t)-d\int_{\Delta_{d-1,1}}t_0\ln(t_0)\,\mu_{d-1}(\mathrm{d}t)\\&=-\sum_{i=0}^{d-1}(-1)^i\frac{(d+1)!}{(i+2)!(d-i-1)!}\cdot\frac{i+1}{i+2}\\&\qquad\qquad+\sum_{i=0}^{d-2}(-1)^i\frac{d!}{(i+2)!(d-i-2)!}\cdot\frac{i+1}{i+2}\\&=-\sum_{i=0}^{d-1}(-1)^i\frac{d!}{(i+2)!(d-i-1)!}\cdot\frac{i+1}{i+2}(d+1-(d-i-1))\\&=-\sum_{i=0}^{d-1}(-1)^i\frac{d!}{(i+2)!(d-i-1)!}((i+2)-1)\\&=-\sum_{i=0}^{d-1}(-1)^i\left(\binom{d}{i+1}-\frac{1}{d+1}\binom{d+1}{i+2}\right)\\&=\sum_{i=1}^{d}(-1)^i\binom{d}{i}+\frac{1}{d+1}\sum_{i=2}^{d+1}(-1)^i\binom{d+1}{i}\\&=-1+\frac{1}{d+1}(-1+(d+1))=-\frac{1}{d+1}.\end{aligned}$$

Combining with

$$2\int_{\Delta_{1,1}}t_0\ln(t_0)\,\mu_1(\mathrm{d}t)=2\int_0^1 t\ln(t)\,\mathrm{d}t=-\int_0^1 t\,\mathrm{d}t=-\frac{1}{2},$$

by induction we obtain

$$(d+1)\int_{\Delta_{d,1}} t_0 \ln(t_0)\,\mu_d(\mathrm{d}t) = -\sum_{i=1}^{d}\frac{1}{i+1} = -\sum_{\ell=2}^{d+1}\frac{1}{\ell}.$$

By symmetry of $(t_0,\ldots,t_d)$, we get

$$(d+1)\sum_{i=0}^{d}\int_{\Delta_{d,1}} t_i \ln(t_i)\,\mu_d(\mathrm{d}t) = -(d+1)\sum_{\ell=2}^{d+1}\frac{1}{\ell}.$$

Since

$$\begin{aligned}\sum_{m=1}^{d}\sum_{\ell=1}^{m}\frac{1}{\ell} &= \sum_{\ell=1}^{d}\sum_{m=\ell}^{d}\frac{1}{\ell} = \sum_{\ell=1}^{d}\frac{d+1-\ell}{\ell} = (d+1)\sum_{\ell=1}^{d}\frac{1}{\ell} - d \\ &= (d+1)\sum_{\ell=2}^{d+1}\frac{1}{\ell} + (d+1) - \frac{d+1}{d+1} - d = (d+1)\sum_{\ell=2}^{d+1}\frac{1}{\ell},\end{aligned}$$

we obtain the desired result. □

Proposition 5.2.5. *The universal invertible sheaf $\mathcal{O}_{\mathbb{P}^d_K}(1)$ satisfies the Hilbert-Samuel property.*

Proof. By Proposition 5.1.4, it suffices to prove that the particular quotient metric family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ defined in the beginning of the section satisfies the Hilbert-Samuel property. For any $n \in \mathbb{N}$, let

$$\eta_n = \bigwedge_{\substack{(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} e_0^{a_0}\cdots e_d^{a_d} \in \det(S^n(E)).$$

By Lemma 5.2.2 and [**36**, Proposition 1.2.23], for any $\omega \in \Omega$ such that $|\cdot|_\omega$ is non-Archimedean, one has

$$\|\eta_n\|_{n\varphi_\omega,\det} = 1.$$

Let ω be an element of Ω such that $|\cdot|_\omega$ is Archimedean. Similarly to Lemma 5.2.2, for each $n \in \mathbb{N}$, we let $\|\cdot\|_{n,\omega}$ be the orthogonal tensor power norm on $E_\omega^{\otimes n}$ and $\|\cdot\|'_{n,\omega}$ be its quotient norm on $S^n(E_\omega)$. Note that

$$(e_0^{a_0}\cdots e_d^{a_d})_{\substack{(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}}$$

forms an orthogonal basis of $(S^d(E_\omega), \|\cdot\|'_{n,\omega})$ and

$$\|e_0^{a_0}\cdots e_d^{a_d}\|'_{n,\omega} = \Big(\frac{a_0!\cdots a_d!}{n!}\Big)^{\frac{1}{2}}.$$

By [**36**, Proposition 1.2.25], one has

$$\|\eta_n\|'_{n,\omega,\det} = \prod_{\substack{(a_0,\ldots,a_d)\in\mathbb{N}^{d+1}\\ a_0+\cdots+a_d=n}} \Big(\frac{a_0!\cdots a_d!}{n!}\Big)^{\frac{1}{2}}.$$

In particular, using Stirling's formula one obtains

$$\begin{aligned}\lim_{n\to+\infty}\frac{\ln\|\eta_n\|'_{n,\omega,\det}}{nr_n} &= -\frac{1}{2}\int_\Delta (t_0\ln(t_0)+\cdots+t_d\ln(t_d))\,\mathrm{d}\mu \\ &= \frac{1}{2(d+1)}\sum_{m=1}^{d}\sum_{\ell=1}^{m}\frac{1}{\ell},\end{aligned}$$

where μ denotes the uniform probability measure on the simplex

$$\Delta = \{(t_0, \ldots, t_d) \in \mathbb{R}_{\geqslant 0}^{d+1} \mid t_0 + \cdots + t_d = 1\},$$

and the second equality comes from Lemma 5.2.4.

By [**13**, Lemma 4.3.6] and [**48**, Lemma 30] (see also [**65**, VIII.2.5 lemma 2]), one has

$$\sup_{s \in S^n(E_\omega)\setminus\{0\}} \left| \ln(r_n^{-\frac{1}{2}} \|s\|'_{n,\omega}) - \ln \|s\|_{\varphi_\omega^{\otimes n}} \right| = O(\ln(n)).$$

Moreover,

$$\ln(r_n^{-\frac{1}{2}}) = -\frac{1}{2} \ln r_n = O(\ln(n)).$$

Hence by Lemma 5.1.3 we obtain

$$\lim_{n\to+\infty} \frac{\ln \|\eta_n\|_{\varphi_\omega^{\otimes n},\det}}{nr_n} = \lim_{n\to+\infty} \frac{\ln \|\eta_n\|'_{n,\omega,\det}}{nr_n}.$$

The proposition is thus proved. □

5.3. Trivial valuation case

In this section, we show the Hilbert-Samuel property in the trivial valuation case. Let $v = (k, |\cdot|)$ be a trivially valued field. Let us begin with the following Lemma:

LEMMA 5.3.1. *Let X be an integral projective scheme of dimension d over* Spec k *and L be a very ample invertible $\mathcal{O}_X$-module. Let $\|\cdot\|$ be the trivial norm on $H^0(X, L)$, that is, $\|e\| = 1$ for $e \in H^0(X, L) \setminus \{0\}$. Let φ be the quotient metric of L induced by the surjective homomorphism $H^0(X, L) \otimes \mathcal{O}_X \to L$ and $\|\cdot\|$. Then we have*

$$\widehat{\mathrm{vol}}_\chi(L, \varphi) = ((L, \varphi)^{d+1})_v = 0,$$

where in the construction of $\widehat{\mathrm{vol}}_\chi(L, \varphi)$ we consider the adelic curve consisting of one copy of the trivial absolute value on k and the counting measure.

PROOF. Let $X \hookrightarrow \mathbb{P}_k^\ell$ be the embbedding given by L, where $\ell = \dim_k H^0(X, L) - 1$. We can find a positive integer n_0 such that $H^0(\mathbb{P}_k^\ell, \mathcal{O}_{\mathbb{P}_k^\ell}(n)) \to H^0(X, L^{\otimes n})$ is surjective for all $n \geqslant n_0$. In order to see $\widehat{\mathrm{vol}}_\chi(L, \varphi) = 0$, it is sufficient to show that the norm $\|\cdot\|_{n\varphi}$ is trivial for all $n \geqslant n_0$. As $H^0(\mathbb{P}_k^\ell, \mathcal{O}_{\mathbb{P}_k^\ell}(n)) = \mathrm{Sym}^n(H^0(X, L))$, one has that $\mathrm{Sym}^n(H^0(X, L)) \to H^0(X, L^{\otimes n})$ is surjective for all $n \geqslant n_0$. Let $(T_0, \ldots, T_\ell)$ be a homogeneous coordinate of $\mathbb{P}_k^\ell$. For $n \geqslant n_0$ and $s \in H^0(X, L^{\otimes n})$, if

$$s \equiv \sum_{\substack{(i_0,\ldots,i_\ell)\in\mathbb{N}^{\ell+1} \\ i_0+\cdots+i_\ell=n}} a_{i_0,\ldots,i_\ell} T_0^{i_0} \cdots T_\ell^{i_\ell}$$

modulo $\mathrm{Ker}(\mathrm{Sym}^n(H^0(X, L)) \to H^0(X, L^{\otimes n}))$, then

$$\|s\|_{n\varphi} = \sup_{x \in (X\cap U_0)^{\mathrm{an}}} \frac{\left| \sum_{\substack{(i_0,\ldots,i_\ell)\in\mathbb{N}^{\ell+1} \\ i_0+\cdots+i_\ell=n}} a_{i_0,\ldots,i_\ell} z_1^{i_1} \cdots z_\ell^{i_\ell} \right|_x}{\Big(\max\{1, |z_1|_x, \ldots, |z_\ell|_x\}\Big)^n},$$

where $z_i = T_i/T_0$ and $U_0 = \{(T_0, \ldots, T_\ell) \in \mathbb{P}^\ell_k \,:\, T_0 \neq 0\}$. Note that

$$\Big|\sum\nolimits_{\substack{(i_0,\ldots,i_\ell)\in\mathbb{N}^{\ell+1}\\ i_0+\cdots+i_\ell=n}} a_{i_0,\ldots,i_\ell} z_1^{i_1}\cdots z_\ell^{i_\ell}\Big|_x \leqslant \max\{|z_1|_x^{i_1}\cdots|z_\ell|_x^{i_\ell} \,:\, (i_0,\ldots,i_\ell)\in\mathbb{N}^{\ell+1}, i_0+\cdots+i_\ell=n\} \leqslant \Big(\max\{1,|z_1|_x,\ldots,|z_\ell|_x\}\Big)^n,$$

and hence $\|s\|_{n\varphi} \leqslant 1$. Let k^{ac} be an algebraic closure of k. We assume $s \neq 0$. We choose $\xi = (1, \xi_1, \ldots, \xi_n) \in X(k^{\mathrm{ac}})$ such that $s(\xi) \neq 0$. Then, as

$$\sum_{\substack{(i_0,\ldots,i_\ell)\in\mathbb{N}^{\ell+1}\\ i_0+\cdots+i_\ell=n}} a_{i_0,\ldots,i_\ell}\xi_1^{i_1}\cdots\xi_\ell^{i_\ell} \in k^{\mathrm{ac}}\setminus\{0\}$$

and $\xi_1, \ldots, \xi_\ell \in k^{\mathrm{ac}}$, one has

$$\Big|\sum\nolimits_{\substack{(i_0,\ldots,i_\ell)\in\mathbb{N}^{\ell+1}\\ i_0+\cdots+i_\ell=n}} a_{i_0,\ldots,i_\ell}\xi_1^{i_1}\cdots\xi_\ell^{i_\ell}\Big|_{v'} = 1 \quad\text{and}\quad \max\{1,|\xi_1|_{v'},\ldots,|\xi_\ell|_{v'}\} = 1$$

where v' is the pair of k^{ac} and its trivial absolute value. Therefore, $\|s\|_{n\varphi} = 1$.

Next let us see that $((L,\varphi)^{d+1})_v = 0$. Note that

$$H^0(\mathbb{P}^\ell_k, \mathcal{O}_{\mathbb{P}^\ell_k}(1)) = H^0(X, L) \text{ and } \mathrm{Sym}^n(H^0(\mathbb{P}^\ell_k, \mathcal{O}_{\mathbb{P}^\ell_k}(1))) = H^0(\mathbb{P}^\ell_k, \mathcal{O}_{\mathbb{P}^\ell_k}(n))$$

for $n \geqslant 1$. Let ψ be the Fubuni-Study metric of $\mathcal{O}_{\mathbb{P}^\ell_k}(1)$ induced by the surjective homomorphism $H^0(\mathbb{P}^\ell_k, \mathcal{O}_{\mathbb{P}^\ell_k}(1))) \otimes \mathcal{O}_{\mathbb{P}^\ell_k} \to \mathcal{O}_{\mathbb{P}^\ell_k}(1)$ and $\|\cdot\|$. Then $\psi|_{X^{\mathrm{an}}} = \varphi$. In the same way as before, $\|\cdot\|_{n\psi}$ on $H^0(\mathbb{P}^\ell_k, \mathcal{O}_{\mathbb{P}^\ell_k}(n))$ is trivial for $n \geqslant 1$. Therefore, the induced norm on $H^0(\check{\mathbb{P}}^\ell_k \times\cdots\times \check{\mathbb{P}}^\ell_k, \mathcal{O}_{\check{\mathbb{P}}^\ell_k}(\delta)\boxtimes\cdots\boxtimes\mathcal{O}_{\check{\mathbb{P}}^\ell_k}(\delta))$ is also trivial, where $\delta = (L^d)$. Thus the assertion follows (by [**38**, Theorem 3.9.7]). □

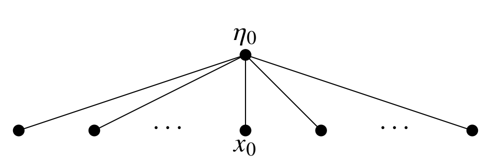
Theorem 5.3.2. *Assume that, for any $\omega \in \Omega$, $|\cdot|_\omega$ is the trivial absolute value on K. Then any ample line bundle L on X satisfies the Hilbert-Samuel property.*

Proof. By Remark 5.1.2, we may assume that L is very ample. Let E be the vector space $H^0(X, L)$. For any $\omega \in \Omega$, we denote by $\|\cdot\|_\omega$ the trivial norm on $E = E_\omega$. Let $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ and $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be the quotient metric family on L induced by ξ and the canonical closed embedding $X \to \mathbb{P}(E)$. Then, Lemma 5.3.1 implies

$$\mathrm{vol}_\chi(L,\varphi) = ((L,\varphi)^{d+1}) = 0.$$

Therefore, by Proposition 5.1.4 we obtain that the invertible sheaf L satisfies the Hilbert-Samuel property. □

Remark 5.3.3. In [**37**], an intersection product of metrized divisors has been introduced in the setting of curves over a trivially valued field $(k, |\cdot|)$. Let X be a regular projective curve over $\mathrm{Spec}\, k$. Recall that the Berkovich space X^{an} is an infinite tree



where the root point η_0 corresponds to the generic point of X together with the trivial absolute value on $\kappa(\eta)$, and each leaf x_0 corresponds to the closed point x together with the trivial absolute value on $\kappa(x)$. Moreover, each branch $]\eta_0, x_0[$ is parametrized by $]0, +\infty[$, where $t \in]0, +\infty[$ corresponds to the generic point η together with the absolute value

$$|\cdot|_{x,t} = \exp(-t \operatorname{ord}_x(\cdot)).$$

We denote by $t(\cdot) : X^{\mathrm{an}} \longrightarrow [0, +\infty]$ the parametrization map, where $t(\eta_0) = 0$ and $t(x_0) = +\infty$. Let D be a Cartier divisor on X. Recall that a Green function g of D is of the form

$$g = g_D + \varphi_g,$$

where g_D is the canonical Green function of D, which is defined as

$$g_D(\xi) = \operatorname{ord}_x(D)t(\xi),$$

and φ_g is a continuous real-valued function on X^{an} (which is hence bounded since X^{an} is compact). Then, the intersection number of two integrable metrized Cartier divisor $\overline{D}_0 = (D_0, g_0)$ and $\overline{D}_1 = (D_1, g_1)$ has been defined as

$$g_1(\eta_0)\deg(D_0) + g_0(\eta_0)\deg(D_1) - \sum_{x \in X^{(1)}} [\kappa(x) : k] \int_0^{+\infty} \varphi'_{g_0 \circ \xi_x}(t)\varphi'_{g_1 \circ \xi_x}(t)\,\mathrm{d}t, \quad (5.2)$$

where $X^{(1)}$ is the set of closed points of X, $\xi_x : [0, +\infty] \to [\eta_0, x_0]$ is the map sending $t \in [0, +\infty]$ to the point in $[\eta_0, x_0]$ of parameter t, and the function $\varphi'_{g_1 \circ \xi_x}(\cdot)$ should be considered as the right-continuous version of the Radon-Nikodym density of the function $\varphi_{g_1 \circ \xi_x}(\cdot)$ with respect to the Lebesgue measure.

Let (L_0, φ_0) and (L_1, φ_1) be integrable metrized invertible $\mathcal{O}_X$-modules. By [**37**, Remark 7.3], the above intersection number with respect to (L_0, φ_0) and (L_1, φ_1) is well-defined. To distinguish this intersection number with the intersection number defined in [**38**, Definition 3.10.1] it is denoted by $((L_0, \varphi_0) \cdot (L_1, \varphi_1))'$. Then one can see

$$((L_0, \varphi_0) \cdot (L_1, \varphi_1)) = ((L_0, \varphi_0) \cdot (L_1, \varphi_1))'. \quad (5.3)$$

Indeed, by using the linearity of $(\,\cdot\,)$ and $(\,\cdot\,)'$, we may assume that L_0 and L_1 are ample, and φ_0 and φ_1 are semipositive. Moreover, as

$$\begin{cases} ((L_0, \varphi_0) \cdot (L_1, \varphi_1)) = \dfrac{(((L_0, \varphi_0) + (L_1, \varphi_1))^2) - ((L_0, \varphi_0)^2) - ((L_1, \varphi_1)^2)}{2}, \\[2ex] ((L_0, \varphi_0) \cdot (L_1, \varphi_1))' = \dfrac{(((L_0, \varphi_0) + (L_1, \varphi_1))^2)' - ((L_0, \varphi_0)^2)' - ((L_1, \varphi_1)^2)'}{2}, \end{cases}$$

we may further assume that $(L_0, \varphi_0) = (L_1, \varphi_1)$, say (L, φ). On the one hand, by [**37**, Theorem 7.4],

$$\lim_{n \to \infty} \frac{-\ln \|s_1 \wedge \cdots \wedge s_{r_n}\|_{n\varphi, \det}}{n^2/2} = ((L, \varphi) \cdot (L, \varphi))',$$

where $\{s_1, \ldots, s_{r_n}\}$ is a basis of $H^0(X, L^{\otimes n})$. On the other hand,

$$\lim_{n \to \infty} \frac{-\ln \|s_1 \wedge \cdots \wedge s_{r_n}\|_{n\varphi, \det}}{n^2/2} = ((L, \varphi) \cdot (L, \varphi))$$

by Theorem 5.3.2 (the Hilbert-Samuel formula over a trivially valued field), as required.

5.4. Casting to the trivial valuation case

In this section, we assume that K is perfect. Let X be a projective K-scheme, d be the dimension of X, E be a finite-dimensional vector space over K, $f : X \to \mathbb{P}(E)$ be a closed embedding, and L be the restriction of the universal invertible sheaf $\mathcal{O}_E(1)$ to X. We assume that, for any positive integer n, the restriction map

$$S^n(E) = H^0(\mathbb{P}(E), \mathcal{O}_E(n)) \longrightarrow H^0(X, L^{\otimes n})$$

is surjective. We equip E with a Hermitian norm family $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ such that the couple $\overline{E} = (E, \xi)$ forms a strongly adelic vector bundle on the adelic curve S. Denote by $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ the quotient metric family on L induced by ξ and the closed embedding f.

Let $\mathcal{F} = (\mathcal{F}^t(\overline{E}))_{t\in\mathbb{R}}$ be the Harder-Narasimhan $\mathbb{R}$-filtration of $\overline{E}$. Recall that

$$\mathcal{F}^t(\overline{E}) = \sum_{\substack{0\neq F\subseteq E\\ \widehat{\mu}_{\min}(\overline{F})\geqslant t}} F$$

(cf. [**36**, Corollary 4.3.4]). Note that this $\mathbb{R}$-filtration actually defines an ultrametric norm $\|\cdot\|_0$ on E, where we consider the trivial absolute value $|\cdot|_0$ on the field K. More precisely, for any $s \in E$, one has

$$\|s\|_0 = \exp(-\sup\{t \in \mathbb{R} \,:\, s \in \mathcal{F}^t(\overline{E})\})$$

(cf. [**36**, Remark 1.1.40]). Denote by φ_0 the quotient metric on L induced by $\|\cdot\|_0$. If we consider the adelic curve S_0 consisting of a single copy of the trivial absolute value on K, then (L, φ_0) becomes an adelic line bundle on X.

Proposition 5.4.1. *The following inequality holds:*

$$((L,\varphi)^{d+1}) \geqslant ((L,\varphi_0)^{d+1}) - \nu(\Omega_\infty)\big((d+1)\delta\ln(r) + \ln(\delta!)\big), \tag{5.4}$$

where r denotes the dimension of E over K and δ is the degree of X with respect to the line bundle L, that is, $\delta = (L^d)$.

Proof. For any $\omega \in \Omega$, let $\|\cdot\|_{\omega,*}$ be the dual norm on $E_\omega^\vee$ and let $\|\cdot\|_{\omega,*,\delta}$ be the δ-th symmetric power of the norm $\|\cdot\|_{\omega,*}$, that is the quotient norm of the ε-tensor power (resp. orthogonal tensor power) of $\|\cdot\|_{\omega,*}$ by the canonical quotient map if $|\cdot|_\omega$ is non-Archimedean (resp. Archimedean). Let $\|\cdot\|'_{\omega,*}$ be the ε-tensor product (resp. orthogonal tensor product) of $d+1$ copies of the norm $\|\cdot\|_{\omega,*,\delta}$ if $|\cdot|_\omega$ is non-Archimedean (resp. Archimedean). By [**36**, Proposition 1.2.36], this norm also identifies with the quotient of the tensor power of $\|\cdot\|_{\omega,*}$ by the quotient map

$$p_\omega : E_\omega^{\vee\otimes\delta(d+1)} \cong (E_\omega^{\vee\otimes\delta})^{\otimes(d+1)} \longrightarrow S^\delta(E_\omega^\vee)^{\otimes(d+1)}. \tag{5.5}$$

We denote by ξ' the norm family $(\|\cdot\|'_{\omega,*})_{\omega\in\Omega}$. It turns out that

$$(S^\delta(E^\vee)^{\otimes(d+1)}, \xi')$$

forms an adelic vector bundle on S. Moreover, if we let $R \in S^\delta(E^\vee)^{\otimes(d+1)}$ be a resultant of X with respect to $d+1$ copies of the closed embedding $f : X \to \mathbb{P}(E)$, then the following inequality holds:

$$((L,\varphi)^{d+1}) \geqslant -\widehat{\deg}_{\xi'}(R) - \frac{1}{2}\nu(\Omega_\infty)(d+1)\ln\binom{r+\delta-1}{\delta}, \tag{5.6}$$

where r is the dimension of E over K. This is a consequence of [**38**, Theorem 3.9.7] and [**13**, Corollary 1.4.3, formula (1.4.10) and Lemma 4.3.6]. Note that in the case where $\Omega_\infty = \varnothing$, the equality

$$((L,\varphi)^{d+1}) = -\widehat{\deg}_{\xi'}(R) \tag{5.7}$$

holds.

We now consider the trivial absolute value $|\cdot|_0$ on K and we let ξ'_0 be the ultrametric norm on $S^\delta(E_\omega^\vee)^{\otimes(d+1)}$ defined as the quotient norm of the ε-tensor power of $\|\cdot\|_{0,*}$ by the quotient map

$$p : E^{\vee\otimes\delta(d+1)} \cong (E^{\vee\otimes\delta})^{\otimes(d+1)} \longrightarrow S^\delta(E^\vee)^{\otimes(d+1)}.$$

Similarly to (5.7), the following equality holds:

$$((L,\varphi_0)^{d+1}) = -\widehat{\deg}_{\xi'_0}(R). \tag{5.8}$$

Note that the dual norm $\|\cdot\|_{0,*}$ corresponds to the Harder-Narasimhan $\mathbb{R}$-filtration of the dual adelic vector bundle $\overline{E}^\vee = (E^\vee,\xi^\vee)$, where $\xi^\vee = (\|\cdot\|_{\omega,*})_{\omega\in\Omega}$ (see the proof of [**36**, Proposition 4.3.41]). Therefore, if we denote by Ψ the one-dimensional vector sub-space of $S^\delta(E^\vee)^{\otimes(d+1)}$ spanned by the resultants of X with respect to $d+1$ copies of $f : X \to \mathbb{P}(E)$, then the dual statement of [**36**, Theorem 5.6.1] (see Remark A.3.3) leads to

$$\widehat{\deg}(\Psi,\xi') \leqslant \widehat{\deg}(\Psi,\xi'_0) + \frac{1}{2}\nu(\Omega_\infty)\delta(d+1)\ln(r) + \nu(\Omega_\infty)\ln(\delta!),$$

or equivalently

$$-\widehat{\deg}_{\xi'}(R) \geqslant -\widehat{\deg}_{\xi'_0}(R) - \frac{1}{2}\nu(\Omega_\infty)\delta(d+1)\ln(r) - \nu(\Omega_\infty)\ln(\delta!). \tag{5.9}$$

In the case where Ω_∞ is empty, we use Theorem A.3.5 to determine the Harder-Narasimhan $\mathbb{R}$-filtration of $S^\delta(\overline{E}^\vee)$ and apply the dual statement to the tensor product of $d+1$ copies of $S^\delta(\overline{E}^\vee)$. In the case where Ω_∞ is not empty, we uses the anti-symmetrization map (see Remark A.2.6) to identify $S^\delta(E^\vee)$ with a vector subspace of $E^{\vee\otimes\delta}$ and apply the dual statement to $\delta(d+1)$ copies of $\overline{E}^\vee$. Note that the anti-symmetrization map sym′ has height $\leqslant \nu(\Omega_\infty)\ln(\delta!)$ (see Propositions A.2.4 and A.2.5). By (5.6), (5.8) and (5.9), we obtain

$$\begin{aligned}((L,\varphi)^{d+1}) &\geqslant ((L,\varphi_0)^{d+1}) - \frac{1}{2}\nu(\Omega_\infty)(d+1)\ln\binom{r+\delta-1}{\delta}\\ &\qquad - \frac{1}{2}\nu(\Omega_\infty)\delta(d+1)\ln(r) - \nu(\Omega_\infty)\ln(\delta!)\\ &\geqslant ((L,\varphi_0)^{d+1}) - \nu(\Omega_\infty)\delta(d+1)\ln(r) - \nu(\Omega_\infty)\ln(\delta!),\end{aligned}$$

by using the inequality

$$\binom{r+\delta-1}{\delta} \leqslant r^\delta.$$

The proposition is thus proved. □

5.5. Arithmetic Hilbert-Samuel theorem

The purpose of this section is to prove the following theorem.

THEOREM 5.5.1. *Let X be an integral projective K-scheme, d be the dimension of X and L be an ample invertible O_X-module. We assume that, either K is perfect, or X is geometrically integral. Then for any metric family $\varphi \in \mathscr{M}^+(L)$, the following equality holds*

$$\widehat{\mathrm{vol}}_\chi(L,\varphi) = ((L,\varphi)^{d+1}). \tag{5.10}$$

Proof. **Step 1:** *We first prove the inequality* $\widehat{\mathrm{vol}}_\chi(L,\varphi) \leqslant ((L,\varphi)^{d+1})$.

Let K' be the perfect closure of K. Note that each absolute value $|\cdot|_\omega$, $\omega\in\Omega$, extends in a unique way to K', so that the underlying measure space of $S\otimes_K K'$ identifies with $(\Omega,\mathcal{A},\nu)$. Let $X' = X\times_{\operatorname{Spec} K}\operatorname{Spec} K'$, $L' = L\otimes_K K'$, and φ' be the extension of φ to L'. Let $(s_1,\ldots,s_N)$ be a basis of $H^0(X,L^{\otimes n})$. Note that, for any $\omega\in\Omega$, the norm $\|\cdot\|_{n\varphi'_\omega}$ is an extension of $\|\cdot\|_{n\varphi_\omega}$ (cf. [**36**, Proposition 2.1.19]). Therefore, by [**36**, Proposition 1.1.66] or Appendix A.2.6, one has

$$\|s_1\wedge\cdots\wedge s_N\|_{n\varphi_\omega,\det} \geqslant \|s_1\wedge\cdots\wedge s_N\|_{n\varphi'_\omega,\det},$$

so that $\widehat{\mathrm{vol}}_\chi(L,\varphi)\leqslant\widehat{\mathrm{vol}}_\chi(L',\varphi')$. Moreover, by [**38**, Theorem 4.3.6],

$$((L,\varphi)^{d+1}) = ((L',\varphi')^{d+1}).$$

Thus, if the assertion of Step 1 holds for K', then one has

$$\widehat{\mathrm{vol}}_\chi(L,\varphi)\leqslant\widehat{\mathrm{vol}}_\chi(L',\varphi')\leqslant((L',\varphi')^{d+1}) = ((L,\varphi)^{d+1}).$$

Therefore we may assume that K is perfect.

By taking a tensor power of L we may assume that L is very ample and the canonical K-linear map

$$S^n(H^0(X,L))\longrightarrow H^0(X,L^{\otimes n}) \tag{5.11}$$

is surjective for any integer $n\geqslant 1$. Moreover, by Remark 5.1.6, the difference

$$\widehat{\mathrm{vol}}_\chi(L,\varphi) - ((L,\varphi)^{d+1})$$

does not depend on the choice of the metric family φ. Therefore, we may assume that φ identifies with the quotient metric family induced by the norm family $\xi_1 = (\|\cdot\|_{\varphi_\omega})_{\omega\in\Omega}$. By [**36**, Proposition 2.2.22 (2)], for any positive integer n, the metric $n\varphi$ identifies with the quotient metric family induced by the norm family $\xi_n = (\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega}$. Moreover, by changing metrics we may also assume that the minimal slope of $(H^0(X,L),\xi_1)$ is non-negative. Since the K-linear map (5.11) is surjective, by [**36**, Proposition 6.3.25], we obtain that the minimal slope of $(H^0(X,L^{\otimes n}),\xi_n)$ is non-negative for any positive integer n. By [**36**, Theorem 4.1.26], there exists a Hermitian norm family $\xi'_n = (\|\cdot\|'_{n,\omega})$ of $H^0(X,L^{\otimes n})$ such that $\|\cdot\|_{n,\omega} = \|\cdot\|_{n\varphi_\omega}$ when $|\cdot|_\omega$ is non-Archimedean and

$$\|\cdot\|'_{n,\omega}\leqslant\|\cdot\|_{n\varphi_\omega}\leqslant(2r_n)^{1/2}\|\cdot\|'_{n,\omega} \tag{5.12}$$

when $|\cdot|_\omega$ is Archimedean, where r_n denotes the dimension of $H^0(X,L^{\otimes n})$. Note that

$$\Big|\widehat{\deg}(H^0(X,L^{\otimes n}),\xi_n) - \widehat{\deg}(H^0(X,L^{\otimes n}),\xi'_n)\Big|\leqslant\frac{1}{2}\nu(\Omega_\infty)r_n\ln(2r_n),$$

so that

$$\widehat{\mathrm{vol}}_\chi(L,\varphi) = \lim_{n\to+\infty}\frac{\widehat{\deg}(H^0(X,L^{\otimes n}),\xi'_n)}{n^{d+1}/(d+1)!}. \tag{5.13}$$

For any positive integer n, let $\|\cdot\|_n$ be the ultrametric norm on $H^0(X,L^{\otimes n})$ corresponding to the Harder-Narasimhan $\mathbb{R}$-filtration of $(H^0(X,L^{\otimes n}),\xi'_n)$, where we consider the trivial absolute value $|\cdot|_0$ on K. Let $\widetilde{\varphi}_n$ be the continuous metric on L (where we still consider the trivial absolute value on K) such that $n\widetilde{\varphi}_n$ identifies with the quotient metric on $L^{\otimes n}$ induced by $\|\cdot\|_n$. By [**36**, Proposition 2.2.22 (2)], one has $\|\cdot\|_{n\widetilde{\varphi}_n} = \|\cdot\|_n$ on $H^0(X,L^{\otimes n})$ and hence

$$\widehat{\deg}(H^0(X,L^{\otimes n}),\|\cdot\|_{n\widetilde{\varphi}_n}) = \widehat{\deg}(H^0(X,L^{\otimes n}),\|\cdot\|_n) = \widehat{\deg}(H^0(X,L^{\otimes n}),\xi'_n). \tag{5.14}$$

By Proposition 5.4.1 and the second inequality of (5.12) we obtain that

$$\begin{aligned}&((nL,n\varphi)^{d+1})+\frac{1}{2}\nu(\Omega_\infty)(d+1)n^d(L^d)\ln(2r_n)\\&\geqslant((nL,n\widetilde{\varphi}_n)^{d+1})-\nu(\Omega_\infty)n^d(L^d)\Big((d+1)\ln(r_n)+\ln(n^d(L^d))\Big),\end{aligned}\tag{5.15}$$

where we consider X as an arithmetic variety over the adelic curve S (resp. as an arithmetic variety over the adelic curve consisting of a single copy of the trivial absolute value on K) in the computation of the arithmetic intersection number on the left-hand side (resp. right-hand). Moreover, by Theorem 5.3.2, the following equality holds:

$$\widehat{\mathrm{vol}}_\chi(L,\widetilde{\varphi}_n)=((L,\widetilde{\varphi}_n)^{d+1}).\tag{5.16}$$

By [**31**, Corollary 5.2] (see also the proof of Theorem 7.3 of *loc. cit.*), there exists a positive constant C such that, for any positive integer n, one has

$$\widehat{\deg}(H^0(X,L^{\otimes n}),\|\cdot\|_{n\widetilde{\varphi}_n})\leqslant\frac{\widehat{\mathrm{vol}}_\chi(nL,n\widetilde{\varphi}_n)}{(d+1)!}+Cn^d.$$

The constant C can be taken in the form an invariant of the graded linear series

$$\bigoplus_{m\in\mathbb{N}}H^0(X,L^{\otimes m})$$

multiplied by

$$\sup_{m\in\mathbb{N},\,m\geqslant1}\frac{\widehat{\mu}_{\max}(H^0(X,L^{\otimes m}),\xi'_m)}{m}.$$

By (5.14), (5.15) and (5.16), we deduce that

$$\begin{aligned}\widehat{\deg}(H^0(X,L^{\otimes n}),\xi'_n)\leqslant{}&\frac{n^{d+1}}{(d+1)!}((L,\varphi)^{d+1})+Cn^d\\&+\frac{1}{2}\nu(\Omega_\infty)(d+1)n^d(L^d)\ln(2r_n^3)+\nu(\Omega_\infty)n^d(L^d)\ln(n^d(L^d)).\end{aligned}$$

Dividing the two sides of the inequality by $n^{d+1}/(d+1)!$ and then taking the limit when $n\longrightarrow+\infty$, by (5.13) we obtain

$$\widehat{\mathrm{vol}}_\chi(L,\varphi)\leqslant((L,\varphi)^{d+1}).$$

Step 2: *the converse inequality* $\widehat{\mathrm{vol}}_\chi(L,\varphi)\geqslant((L,\varphi)^{d+1})$.

By replacing L by a tensor power, we may assume that L is very ample. Moreover, by the normalization of Noether (cf. [**38**, Proposition 1.7.4]), we may also assume that there is a finite K-morphism $\pi:X\longrightarrow\mathbb{P}^d_K$ such that $L\cong\pi^*(\mathcal{O}_{\mathbb{P}^d_K}(1))$. By Remark 5.1.6, we may further assume that there exists an element $\psi=(\psi_\omega)_{\omega\in\Omega}$ of $\mathscr{M}^+(\mathcal{O}_{\mathbb{P}^d_K}(1))$ such that φ equals the pull-back of ψ by π. Then, by Corollary 4.4.3, Proposition 5.2.5 and [**38**, Theorem 4.4.9], one has

$$\widehat{\mathrm{vol}}_\chi(L,\varphi)\geqslant\deg(\pi)\widehat{\mathrm{vol}}_\chi(\mathcal{O}_{\mathbb{P}^d_K}(1),\psi)=\deg(\pi)((\mathcal{O}_{\mathbb{P}^d_K}(1),\psi)^{d+1})=((L,\varphi)^{d+1}),$$

as required. □

Corollary 5.5.2. *Let X be a d-dimensional geometrically integral projective scheme over* Spec K, *$\overline{L}=(L,\varphi)$ be an adelic line bundle on X and $\overline{E}=(E,U,\psi)$ be a birational* *adelic torsion free $\mathcal{O}_X$-module. Assume that L is ample and the metrics in φ are semi-positive. Moreover we suppose that either (E, ψ) is an adelic invertible $\mathcal{O}_X$-module or X is normal. Then one has*

$$\lim_{n\to+\infty} \frac{\widehat{\deg}(H^0(X, L^{\otimes n} \otimes E), (\|\cdot\|_{n\varphi_\omega+\psi_\omega})_{\omega\in\Omega})}{n^{d+1}/(d+1)!} = \mathrm{rk}(E)(\overline{L}^{d+1}).$$

Proof. This is a consequence of Theorem 5.5.1 together with Theorem 4.4.4. □

CHAPTER 6

Relative ampleness and nefness

The aim of this chapter is to discuss strong relative positivity conditions on adelic line bundles. In the first two sections, we introduce a numerical invariant, asymptotic minimal slope, to measure the relative positivity. In the third section, we define the relative ampleness of an adelic line bundle and discuss its properties. In particular, we establish a lower bound for the arithmetic intersection number in terms of asymptotic minimal slopes. In the fourth section, we extend by continuity the function of asymptotic minimal slope to the cone of relatively nef adelic line bundles and generalize the lower bound for the arithmetic intersection number in this setting. In the fifth section, we prove a generalized Hodge index theorem which gives a bigness criterion of relatively nef adelic line bundles in terms of the positivity of the arithmetic self-intersection number. In the sixth section we prove the non-decreasing property of the asymptotic minimal slope by the pull-back by a projective morphism. This property is useful to provide lower bounds of the asymptotic minimal slope. In the seventh section we compare the asymptotic minimal slope of a generically big and relatively nef adelic line bundle to normalized height of the arithmetic variety with respect to the adelic line bundle, by using the arithmetic Hilbert-Samuel formula.

Throughout the chapter, we assume that the underlying field K of the adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ is perfect.

6.1. Convergence of minimal slopes

Lemma 6.1.1. *Let k be a field, X and Y be projective k-schemes and $g : Y \to X$ be a projective k-morphism such that $g_*(\mathcal{O}_Y) = \mathcal{O}_X$. Let L be an ample line bundle on Y and M be an ample line bundle on X. Then there exists $N \in \mathbb{N}_{\geqslant 1}$ such that, for any $(n, m) \in \mathbb{N}^2$ satisfying $\min\{n, m\} \geqslant N$, the k-linear map*

$$H^0(Y, L^{\otimes n}) \otimes_K H^0(X, M^{\otimes m}) = H^0(Y, L^{\otimes n}) \otimes_K H^0(Y, g^*(M^{\otimes m})) \\ \longrightarrow H^0(Y, L^{\otimes n} \otimes g^*(M)^{\otimes m})$$

defined by multiplication of sections is surjective.

Proof. Consider the graphe

$$\Gamma_g : Y \longrightarrow Y \times_k X$$

of the morphism $g : Y \to X$. It is a closed immersion since g is separated. Denote by I the ideal sheaf of the image of Γ_g. Let $p : Y \times_k X \to Y$ and $q : Y \times_k X \to X$ be the two projections, and $A = p^*(L) \otimes q^*(M)$. Since M and L are both ample, the line bundle A on $Y \times_k X$ is ample. Moreover, one has $\Gamma_g^*(A) = L \otimes g^*(M)$. The short exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{Y \times_K X} \longrightarrow \mathcal{O}_{Y \times_k X}/I \longrightarrow 0$$

induces, by tensor product with the invertible sheaf $p^*(L^{\otimes n})\otimes q^*(M^{\otimes m})$ and then by taking cohomology groups on $Y\times_k X$, an exact sequence of K-vector spaces

$$\begin{aligned}H^0(Y,L^{\otimes n})\otimes_k H^0(Y,g^*(M)^{\otimes m}) &\longrightarrow H^0(Y,L^{\otimes n}\otimes g^*(M)^{\otimes m})\\ &\longrightarrow H^1(Y\times_k X, I\otimes p^*(L^{\otimes n})\otimes q^*(M^{\otimes m})).\end{aligned}$$

By [**56**, Example 1.4.4], the line bundles $p^*(L)$ and $q^*(M)$ are nef. By Fujita's vanishing theorem (cf. [**46**, Theorem 5.1]), there exists $N\in\mathbb{N}_{\geqslant 1}$ such that, for any $(n,m)\in\mathbb{N}^2$ such that $\min\{n,m\}>N$, one has

$$\begin{aligned}&H^1(Y\times_k X, I\otimes p^*(L^{\otimes n})\otimes q^*(M^{\otimes m}))\\ &= H^1(Y\times_k X, I\otimes A^{\otimes N}\otimes p^*(L^{\otimes(n-N)})\otimes q^*(M^{\otimes(m-N)}))=\mathbf{0}.\end{aligned}$$

Therefore the assertion follows. □

Lemma 6.1.2. *Let $(k,|\cdot|)$ be a field equipped with a complete absolute value. Let X be a projective scheme over k, L be a semi-ample line bundle on X and φ be a semi-positive metric of L. Then, for any projective K-morphism $g:Y\to X$, $g^*(\varphi)$ is also semi-positive.*

Proof. Replacing L by a tensor power, we may assume that L is generated by global sections, and that there exists a sequence of quotient metric families $(\varphi_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty}\frac{1}{n}d(n\varphi,\varphi_n)=0.$$

Note that for each $n\in\mathbb{N}$, the pull-back $g^*(\varphi_n)$ is still a quotient metric, and one has

$$d(ng^*(\varphi),g^*(\varphi_n))\leqslant d(n\varphi,\varphi_n).$$

Therefore we obtain that $g^*(\varphi)$ is semi-positive. □

In the remaining of the section, we let $f:X\to\operatorname{Spec}K$ be a *non-empty and reduced* projective scheme over $\operatorname{Spec}K$. Since the base field K is supposed to be perfect, the K-scheme X is geometrically reduced.

Proposition 6.1.3. *Let $\overline{L}=(L,\varphi)$ be an adelic line bundle on X such that L is ample. Then the sequence*

$$\frac{\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n}))}{n},\quad n\in\mathbb{N},\ n\geqslant 1 \tag{6.1}$$

converges in $\mathbb{R}$.

Proof. For any $n\in\mathbb{N}_{\geqslant 1}$ let $\overline{E}_n=(E_n,\xi_n)$ be the adelic vector bundle $f_*(\overline{L}^{\otimes n})$. Since L is ample, by Lemma 6.1.1 there exists $N\in\mathbb{N}_{\geqslant 1}$ such that, for any $(n,m)\in\mathbb{N}^2_{\geqslant N}$, the map

$$E_n\otimes_K E_m\longrightarrow E_{n+m},\quad s\otimes t\longmapsto st$$

is surjective. Moreover, if we equip $E_n\otimes E_m$ with the ε,π-tensor product of the norm families ξ_n and ξ_m, the above map has height $\leqslant 0$. By [**36**, Proposition 4.3.31], one has

$$\widehat{\mu}_{\min}(\overline{E}_{n+m})\geqslant\widehat{\mu}_{\min}(\overline{E}_n\otimes_{\varepsilon,\pi}\overline{E}_m).$$

Moreover, since the field K is assumed to be perfect, by [**36**, Corollary 5.6.2] (see also Remark A.3.3), one has

$$\begin{aligned}\widehat{\mu}_{\min}(\overline{E}_n\otimes_{\varepsilon,\pi}\overline{E}_m)&\geqslant\widehat{\mu}_{\min}(\overline{E}_n)+\widehat{\mu}_{\min}(\overline{E}_m)\\ &\quad-\frac{3}{2}\nu(\Omega_\infty)(\ln(\dim_K(E_n))+\ln(\dim_K(E_m))).\end{aligned}$$

Note that

$$\ln(\dim_K(E_n)) = O(\ln(n)),$$

and, by [**36**, Propositions 6.4.4 and 6.2.7], there exists a constant $C > 0$ such that

$$\widehat{\mu}_{\min}(\overline{E}_n) \leqslant \widehat{\mu}_{\max}(\overline{E}_n) \leqslant Cn.$$

Therefore, by [**29**, Corollary 3.6], we obtain the convergence of the sequence (6.1). □

6.2. Asymptotic minimal slope

DEFINITION 6.2.1. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. If L is ample, we define the *asymptotic minimal slope of* $\overline{L}$ as

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) := \lim_{n\to+\infty} \frac{\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n}))}{n}.$$

By definition, for any $m \in \mathbb{N}$ such that $m \geqslant 1$, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes m}) = m\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}). \tag{6.2}$$

PROPOSITION 6.2.2. *Let $\overline{L} = (L, \varphi)$ and $\overline{M} = (M, \psi)$ be adelic line bundles on X such that L and M are ample. Then one has*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L} \otimes \overline{M}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{M}). \tag{6.3}$$

PROOF. By Lemma 6.1.1, for sufficiently large natural number n, the K-linear map

$$H^0(X, L^{\otimes n}) \otimes_K H^0(X, M^{\otimes n}) \longrightarrow H^0(X, (L \otimes M)^{\otimes n}), \quad s \otimes t \longmapsto st$$

is surjective. Moreover, for any $\omega \in \Omega$, the following inequality holds:

$$\forall\,(s,t) \in H^0(X_\omega, L_\omega^{\otimes n}) \times H^0(X_\omega, M_\omega^{\otimes n}), \quad \|st\|_{n(\varphi_\omega+\psi_\omega)} \leqslant \|s\|_{n\varphi_\omega} \cdot \|t\|_{n\psi_\omega}.$$

Therefore, if we equip $H^0(X, L^{\otimes n}) \otimes_K H^0(X, M^{\otimes n})$ with the ε, π-tensor product norm family, then the above K-linear map has height $\leqslant 0$. Hence, by [**36**, Proposition 4.3.31 and Corollary 5.6.2] (see also Remark A.3.3), we obtain

$$\begin{aligned}\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n} \otimes \overline{M}^{\otimes n})) &\geqslant \widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n})) + \widehat{\mu}_{\min}(f_*(\overline{M}^{\otimes n})) \\ &\quad - \frac{3}{2}\nu(\Omega_\infty)\Big(\ln(\dim_K(H^0(X, L^{\otimes n}))) + \ln(\dim_K(H^0(X, M^{\otimes n})))\Big).\end{aligned}$$

We divide the two sides of the inequality by n and then take the limit when $n \to +\infty$, using

$$\lim_{n\to+\infty} \frac{1}{n} \ln(\dim_K(H^0(X, L^{\otimes n}))) = \lim_{n\to+\infty} \frac{1}{n} \ln(\dim_K(H^0(X, M^{\otimes n}))) = 0.$$

we obtain the inequality (6.3). □

PROPOSITION 6.2.3. *Let L be an ample line bundle on X and φ_1 and φ_2 be metric families on L such that (L, φ_1) and (L, φ_2) are both adelic line bundles. Then the following inequality holds:*

$$\Big|\widehat{\mu}^{\mathrm{asy}}_{\min}(L, \varphi_1) - \widehat{\mu}^{\mathrm{asy}}_{\min}(L, \varphi_2)\Big| \leqslant d(\varphi_1, \varphi_2). \tag{6.4}$$

PROOF. For any $n \in \mathbb{N}$, the identity maps

$$f_*(L^{\otimes n}, n\varphi_1) \longrightarrow f_*(L^{\otimes n}, n\varphi_2)$$

and

$$f_*(L^{\otimes n}, n\varphi_2) \longrightarrow f_*(L^{\otimes n}, n\varphi_1)$$

have heights $\leqslant d(n\varphi_1, n\varphi_2) = n\, d(\varphi_1, \varphi_2)$. By [**36**, Proposition 4.3.31], we obtain that

$$\Big|\widehat{\mu}_{\min}(f_*(L^{\otimes n}, n\varphi_1)) - \widehat{\mu}_{\min}(f_*(L^{\otimes n}, n\varphi_2))\Big| \leqslant n\, d(\varphi_1, \varphi_2).$$

Dividing the two sides of the inequality by n and then taking the limit when $n \to +\infty$, we obtain (6.4). □

Assume that X is integral. Let $\overline{L}$ be an adelic line bundle on X such that L is ample. Note that

$$\dim_K(H^0(X, L^{\otimes n})) = \frac{(L^d)}{d!}n^d + o(n^d), \quad n \to +\infty.$$

Therefore, one has

$$\lim_{n\to+\infty} \frac{\widehat{\mu}(f_*(\overline{L}^{\otimes n}))}{n} = \frac{\widehat{\mathrm{vol}}_\chi(\overline{L})}{(d+1)(L^d)}. \tag{6.5}$$

We denote by $\widehat{\mu}^{\mathrm{asy}}(\overline{L})$ the value $\dfrac{\widehat{\mathrm{vol}}_\chi(\overline{L})}{(d+1)(L^d)}$ and call it the *asymptotic slope* of $\overline{L}$. By Theorem 5.5.1, if $\overline{L} = (L, \varphi)$ is an adelic line bundle on X such that L is ample and φ is semi-positive, that is, $\overline{L}$ is relatively ample in the sense of Definition 6.3.1 below, then $\widehat{\mathrm{vol}}_\chi(\overline{L}) = (\overline{L}^{d+1})_S$ and hence

$$\widehat{\mu}^{\mathrm{asy}}(\overline{L}) = \frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)}. \tag{6.6}$$

Remark 6.2.4. Let $\overline{L}$ be an adelic line bundle on X such that L is ample. By definition the following inequality holds:

$$\widehat{\mu}^{\mathrm{asy}}(\overline{L}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}). \tag{6.7}$$

6.3. Relative ampleness and lower bound of intersection number

Definition 6.3.1. Let (L, φ) be an adelic line bundle on X. We say (L, φ) is *relatively ample* if L is ample and φ is semi-positive. By [**36**, Proposition 2.3.5], if $\overline{L}$ and $\overline{M}$ are relatively ample adelic line bundle, then the tensor product $\overline{L} \otimes \overline{M}$ is relatively ample.

Theorem 6.3.2. *Let $\overline{L}_i = (L_i, \varphi_i)$ be a family of relatively ample adelic line bundles on X, where $i \in \{0, \ldots, d\}$. For any $i \in \{0, \ldots, d\}$, let*

$$\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d).$$

Then the following inequality holds:

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \sum_{i=0}^{d} \delta_i\, \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i). \tag{6.8}$$

Proof. Without loss of generality, we may assume that $L_0, \ldots, L_d$ are very ample. For any $n \in \mathbb{N}_{\geqslant 1}$ and any $i \in \{0, \ldots, d\}$, we denote by $E_{i,n}$ the K-vector space $H^0(X, L_i^{\otimes n})$, and set $r_{i,n} = \dim_K(E_{i,n}) - 1$. We denote by $\xi_{n\varphi_i}$ the norm family $(\|\cdot\|_{n\varphi_{i,\omega}})_{\omega\in\Omega}$ on $E_{i,n}$, and let $\xi_{i,n}$ be a Hermitian norm family on $E_{i,n}$ such that $(E_{i,n}, \xi_{i,n})$ forms an adelic vector bundle and that

$$d_\omega(\xi_{i,n}, \xi_{n\varphi_i}) \leqslant \frac{1}{2}\,\mathbb{1}_{\Omega_\infty}(\omega)\ln(r_{i,n} + 2).$$

The existence of such a Hermitian norm family is ensured by [**36**, Theorem 4.1.26]. Let $\varphi_i^{(n)}$ be the metric family on L_i such that $n\varphi_i^{(n)}$ identifies with the quotient metric family induced by the closed embedding $X \to \mathbb{P}(E_{i,n})$ and the norm family $\xi_{i,n}$. Since

$$\lim_{n\to+\infty} \frac{1}{n}\ln(r_{i,n}+2) = 0$$

and the metric families φ_i are semi-positive, by [**38**, Proposition 3.3.12], we obtain that

$$\lim_{n\to+\infty} d(\varphi_i^{(n)},\varphi_i) = \lim_{n\to+\infty}\int_\Omega d_\omega(\varphi_i^{(n)},\varphi_i)\,\nu(\mathrm{d}\omega) = 0.$$

For any $n\in\mathbb{N}_{\geqslant 1}$, let R_n be the one-dimensional vector space of

$$S^{n^d\delta_0}(E_{0,n}^\vee)\otimes_k\cdots\otimes_k S^{n^d\delta_d}(E_{d,n}^\vee) \tag{6.9}$$

spanned by any resultant of the closed embeddings $X \to \mathbb{P}(E_{i,n})$. We equip each $S^{n^d\delta_0}(E_{i,n}^\vee)$ with the orthogonal symmetric power norm family of $\xi_{i,n}^\vee$, and the tensor product space (6.9) with the orthogonal tensor product norm family. By [**38**, Remark 4.2.14] and [**13**, Corollary 1.4.3 and Lemma 4.3.8], we obtain that

$$\begin{aligned}((L_0,\varphi_0^{(n)})\cdots(L_d,\varphi_d^{(n)}))_S &\geqslant -\frac{1}{n^{d+1}}\Bigg(\widehat{\deg}(\overline{R}_n)+\nu(\Omega_\infty)\sum_{i=0}^d \ln\binom{r_{i,n}+n^d\delta_i}{n^d\delta_i}\Bigg)\\ &\geqslant -\frac{1}{n^{d+1}}\Big(\widehat{\deg}(\overline{R}_n)+\nu(\Omega_\infty)\sum_{i=0}^d n^d\delta_i\ln(r_{i,n}+1)\Big),\end{aligned} \tag{6.10}$$

where the second inequality comes from

$$\forall\,(a,b)\in\mathbb{N}_{\geqslant 1}^2,\quad \binom{a+b}{b}\leqslant (a+1)^b.$$

Note that

$$\widehat{\deg}(\overline{R}_n)\leqslant \widehat{\mu}_{\max}(S^{n^d\delta_0}(E_{0,n}^\vee,\xi_{0,n}^\vee)\otimes\cdots\otimes S^{n^d\delta_d}(E_{d,n}^\vee,\xi_{d,n}^\vee)). \tag{6.11}$$

In the case where K is of characteristic 0, by Remark A.2.6 and [**36**, Proposition 4.3.31], we obtain

$$\begin{aligned}&\widehat{\mu}_{\max}\big(S^{n^d\delta_0}(E_{0,n}^\vee,\xi_{0,n}^\vee)\otimes\cdots\otimes S^{n^d\delta_d}(E_{d,n}^\vee,\xi_{d,n}^\vee)\big)\\ &\leqslant \widehat{\mu}_{\max}\big((E_{0,n}^\vee,\xi_{0,n}^\vee)^{\otimes n^d\delta_0}\otimes\cdots\otimes(E_{d,n}^\vee,\xi_{d,n}^\vee)^{\otimes n^d\delta_d}\big)+\nu(\Omega_\infty)\sum_{i=0}^d n^d\delta_i\ln(n^d\delta_i).\end{aligned} \tag{6.12}$$

By [**36**, Corollaries 4.3.27 and 5.6.2], we have

$$\begin{aligned}&\widehat{\mu}_{\max}\big((E_{0,n}^\vee,\xi_{0,n}^\vee)^{\otimes n^d\delta_0}\otimes\cdots\otimes(E_{d,n}^\vee,\xi_{d,n}^\vee)^{\otimes n^d\delta_d}\big)\\ &\leqslant \sum_{i=0}^d n^d\delta_i\Big(\widehat{\mu}_{\max}(E_{i,n}^\vee,\xi_{i,n}^\vee)+\frac{1}{2}\nu(\Omega_\infty)\ln(r_{i,n}+1)\Big)\\ &=\sum_{i=0}^d n^d\delta_i\Big(-\widehat{\mu}_{\min}(E_{i,n},\xi_{i,n})+\frac{1}{2}\nu(\Omega_\infty)\ln(r_{i,n}+1)\Big)\end{aligned} \tag{6.13}$$

Combining (6.10), (6.11), (6.12) and (6.13), we obtain

$$\begin{aligned}((L_0,\varphi_0^{(n)})\cdots(L_d,\varphi_d^{(n)}))_S \geqslant \sum_{i=0}^d \delta_i \frac{\widehat{\mu}_{\min}(E_{i,n},\xi_{i,n})}{n}\\ -\frac{3}{2}\nu(\Omega_\infty)\sum_{i=0}^d \frac{\delta_i}{n}\ln(r_{i,n}+1)-\nu(\Omega_\infty)\sum_{i=0}^d\frac{\delta_i}{n}\ln(n^d\delta_i).\end{aligned}\tag{6.14}$$

In the case where K is of positive characteristic, by Corollary A.3.2 and Theorem A.3.5, we obtain

$$\widehat{\mu}_{\max}\big(S^{n^d\delta_0}(E_{0,n}^\vee,\xi_{0,n}^\vee)\otimes\cdots\otimes S^{n^d\delta_d}(E_{d,n}^\vee,\xi_{d,n}^\vee)\big)\leqslant\sum_{i=0}^d n^d\delta_i\,\widehat{\mu}_{\max}(E_{i,n}^\vee,\xi_{i,n}^\vee).$$

Hence the inequality (6.14) still holds in this case. Since $r_{i,n}=O(n^d)$, taking the limit when n goes to the infinity, we obtain the inequality (6.8). □

6.4. Relative nefness and continuous extension of $\widehat{\mu}_{\min}^{\mathrm{asy}}$

Proposition 6.4.1. *Let $\overline{L}$ and $\overline{A}$ be adelic line bundle on X. Assume that L is nef and A is ample. Then the sequence*

$$\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}),\quad n\in\mathbb{N}_{\geqslant 1}\tag{6.15}$$

converges in $\mathbb{R}\cup\{-\infty\}$, and the limit does not depend on the choice of $\overline{A}$. In particular, in the case where L is ample, the following equality holds:

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A})=\widehat{\mu}_{\min}(\overline{L}).\tag{6.16}$$

Proof. Let p be a positive integer. By Proposition 6.2.2, for any $\ell\in\mathbb{N}_{\geqslant 1}$ and any $r\in\{1,\ldots,p\}$, one has

$$\begin{aligned}\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes p}\otimes\overline{A})&=\frac{1}{\ell+1}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes(\ell+1)p}\otimes\overline{A}^{\otimes(\ell+1)})\\ &\geqslant\frac{1}{\ell+1}\Big(\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes(\ell p+r)}\otimes\overline{A})+\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes(p-r)}\otimes\overline{A})+(\ell-1)\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A})\Big).\end{aligned}$$

Taking the limit superior when $\ell p+r\to+\infty$, we obtain

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes p}\otimes\overline{A})\geqslant p\limsup_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})+\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}),$$

which leads to

$$\liminf_{p\to+\infty}\frac{1}{p}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes p}\otimes\overline{A})\geqslant\limsup_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}).$$

Therefore the sequence (6.15) converges in $[-\infty,+\infty]$. Moreover, still by Proposition 6.2.2, for any $p\in\mathbb{N}_{\geqslant 1}$, one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}\otimes\overline{A})=\frac{1}{p}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes p}\otimes\overline{A}^{\otimes p})\geqslant\frac{1}{p}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes p}\otimes\overline{A})+\frac{p-1}{p}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}),$$

which shows that

$$\lim_{p\to+\infty}\frac{1}{p}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes p}\otimes\overline{A})\leqslant\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}\otimes\overline{A})-\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A})<+\infty.$$

To prove the second assertion, we first show that the limit of the sequence does not depend on the choice of the metric family on $\overline{A}$. For this purpose, we consider two metric families φ_1 et φ_2 on A such that both (A,φ_1) and (A,φ_2) are adelic line bundles on X. By Proposition 6.2.3, for any $n\in\mathbb{N}$ one has

$$\Big|\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes(A,\varphi_1))-\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes(A,\varphi_1))\Big|\leqslant d(\varphi_1,\varphi_2),$$

so that

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes(A,\varphi_1))=\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes(A,\varphi_2)).\tag{6.17}$$

We then show that, for any $p\in\mathbb{N}_{\geqslant 2}$, the following inequality holds:

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})=\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}^{\otimes p}).\tag{6.18}$$

In fact, by (6.2), for any $n\in\mathbb{N}_{\geqslant 1}$ one has

$$\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})=\frac{1}{np}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes np}\otimes\overline{A}^{\otimes p}).$$

Taking the limit when $n\to+\infty$, we obtain the equality (6.18).

Note that if $\overline{B}$ is another adelic line bundle such that B is ample, then the following inequality holds:

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})\leqslant\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}\otimes\overline{B}).\tag{6.19}$$

In fact, by Proposition 6.2.2, for any $n\in\mathbb{N}_{\geqslant 1}$, one has

$$\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}\otimes\overline{B})\geqslant\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})+\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{B}).$$

Taking the limit when $n\to+\infty$, we obtain (6.19).

Finally, we show that, if $\overline{B}$ is an arbitrary adelic line bundle such that B is ample, then the equality

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})=\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{B})\tag{6.20}$$

holds. In fact, there exists $p\in\mathbb{N}_{\geqslant 1}$ such that $N=B^{\otimes p}\otimes A^{\vee}$ is ample. We equip it with an arbitrary metric family such that $\overline{N}$ forms an adelic line bundle. By (6.19) we obtain

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})\leqslant\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}\otimes\overline{N}).$$

Since $A\otimes N$ is isomorphic to $B^{\otimes p}$, by (6.17) and (6.18) we obtain

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}\otimes\overline{N})=\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{B}).$$

Therefore, we deduce

$$\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})\leqslant\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{B}).$$

Interchanging the roles of $\overline{A}$ and $\overline{B}$ we obtain the converse inequality.

To obtain the equality (6.16), it suffices to apply the equality (6.20) in the particular case where $\overline{A}=\overline{L}$. The proposition is thus proved. □

Definition 6.4.2. Let $\overline{L}$ be an adelic line bundle on X such that L is nef, we define

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}):=\lim_{n\to+\infty}\frac{1}{n}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}),$$

where $\overline{A}$ is an arbitrary adelic line bundle such that A is ample. The element $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L})$ of $\mathbb{R}\cup\{-\infty\}$ is called *asymptotic minimal slope* of $\overline{L}$.

Remark 6.4.3. It is an interesting question to ask when the asymptotic minimal slope is a real number. As we will show in Theorem 6.6.6, the asymptotic minimal slope does not decrease if we replace the adelic line bundle by its pullback by a projective morphism. In particular, if L is the pullback of an ample line bundle by a projective morphism, then $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \in \mathbb{R}$. Luo [**59**] gives a partial answer in the function field case.

Proposition 6.4.4. *Let $\overline{L}$ and $\overline{M}$ be adelic line bundles on X such that L and M are nef. One has*

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L} \otimes \overline{M}) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) + \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{M}). \tag{6.21}$$

Moreover, one has

$$\lim_{n \to +\infty} \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) = \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \tag{6.22}$$

provided that $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{M}) > -\infty$.

Proof. Let $\overline{A}$ be an adelic line bundle on X such that A is ample. For any $n \in \mathbb{N}_{\geqslant 1}$, by Proposition 6.2.2 one has

$$\frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}^{\otimes n} \otimes \overline{A}^{\otimes 2}) \geqslant \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{A}) + \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{M}^{\otimes n} \otimes \overline{A}).$$

Taking the limit when $n \to +\infty$, we obtain the inequality (6.21).

By (6.21), we obtain, for any positive integer n, the inequality

$$\frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) + \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{M}).$$

Since $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{M}) \in \mathbb{R}$, taking the limit inferior when $n \to +\infty$, we obtain

$$\liminf_{n \to +\infty} \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}).$$

Pick an adelic line bundle $\overline{A}$ on X such that A is ample. Since $A \otimes M$ is ample, one has

$$\lim_{n \to +\infty} \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M} \otimes \overline{A}) = \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}). \tag{6.23}$$

Moreover, by (6.21) one has

$$\frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M} \otimes \overline{A}) \geqslant \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) + \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}).$$

Taking the limit superior, by (6.23) we obtain

$$\limsup_{n \to +\infty} \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) \leqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}).$$

Hence the equality (6.22) holds. □

Definition 6.4.5. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. We say that $\overline{L}$ is *relatively nef* if there exists a relatively ample adelic line bundle $\overline{A}$ on X and a positive integer N such that, for any $n \in \mathbb{N}_{\geqslant N}$, the tensor product $\overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample. Note that a relatively nef adelic line bundle is integrable in the sense of §1.15.

Proposition 6.4.6. *Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X such that L is semi-ample and φ is semi-positive. Then, for any adelic line bundle $\overline{A} = (A, \psi)$ on X which is relatively ample and any $n \in \mathbb{N}$, the tensor product $\overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample. In particular, $\overline{L}$ is relatively nef.*

Proof. Since L is semi-ample, we obtain that, for any $n \in \mathbb{N}$, $L^{\otimes n} \otimes A$ is ample. Moreover, by [**36**, Proposition 2.3.5], $n\varphi + \psi$ is semi-positive. Hence $\overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample. □

Proposition 6.4.7. *Let $\overline{L}$ and $\overline{M}$ be adelic line bundles on X which are relatively nef. Then the tensor product $\overline{L} \otimes \overline{M}$ is also relatively nef.*

Proof. Let $\overline{A}$ and $\overline{B}$ be relatively ample adelic line bundles on X, and N be a positive integer such that $\overline{L}^{\otimes n} \otimes \overline{A}$ and $\overline{M}^{\otimes n} \otimes \overline{B}$ are relatively ample for any integer $n \geqslant N$. We then obtain that $(\overline{L} \otimes \overline{M})^{\otimes n} \otimes (\overline{A} \otimes \overline{B})$ is relatively ample. Therefore $\overline{L} \otimes \overline{M}$ is relatively nef. □

Proposition 6.4.8. *Let $\overline{L}_0, \ldots, \overline{L}_d$ be a family of relatively nef adelic line bundles on X. For any $i \in \{0, \ldots, d\}$, let*

$$\delta_i = (L_0 \cdots L_{i-1} L_{i+1} \cdots L_d).$$

Assume that $\delta_i > 0$ for those $i \in \{0, \ldots, d\}$ such that $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i) = -\infty$. Then the following inequality holds:

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \sum_{i=0}^{d} \delta_i \, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i). \tag{6.24}$$

Proof. If there is $i \in \{0, \ldots, d\}$ such that $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i) = -\infty$, then the assertion is obvious, so that we may assume that $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i) > -\infty$ for all $i \in \{0, \ldots, d\}$.

Let $\overline{A}_i$ be a relatively ample adelic line bundle on X such that $\overline{L}_i^{\otimes n} \otimes \overline{A}_i$ is relatively ample for sufficiently large positive integer n. For any $i \in \{0, \ldots, d\}$ and any positive integer n, let

$$\overline{L}_{i,n} = \overline{L}_i^{\otimes n} \otimes \overline{A}_i,$$
$$\delta_{i,n} = (L_{0,n} \cdots L_{i-1,n} L_{i+1,n} \cdots L_{d,n}).$$

By the multi-linearity of intersection product, we obtain that

$$\lim_{n \to +\infty} \frac{\delta_{i,n}}{n^d} = \delta_i, \quad \lim_{n \to +\infty} \frac{(\overline{L}_{0,n} \cdots \overline{L}_{d,n})_S}{n^{d+1}} = (\overline{L}_0 \cdots \overline{L}_d)_S.$$

Note that Theorem 6.3.2 leads to

$$(\overline{L}_{0,n} \cdots \overline{L}_{d,n})_S \geqslant \sum_{i=0}^{d} \delta_{i,n} \, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_{i,n})$$

for sufficiently large positive integer n. Dividing the two sides by n^{d+1} and then taking the limit when $n \longrightarrow +\infty$, we obtain the inequality (6.24). □

6.5. Generalized Hodge index theorem

Theorem 6.5.1 (Generalized Hodge index theorem). *Let (L, φ) be a relatively nef adelic invertible O_X-module. Then one has*

$$\widehat{\mathrm{vol}}(L, \varphi) \geqslant ((L, \varphi)^{d+1}). \tag{6.25}$$

Proof. Let (A, ψ) be a relatively ample adelic invertible O_X-module and $n_0 \in \mathbb{N}$ such that $(L, \varphi)^{\otimes n} \otimes (A, \psi)$ is relatively ample for $n \in \mathbb{N}_{\geqslant n_0}$. Then, by Theorem 5.5.1,

$$\widehat{\mathrm{vol}}((L, \varphi)^{\otimes n} \otimes (A, \psi)) \geqslant \widehat{\mathrm{vol}}_\chi((L, \varphi)^{\otimes n} \otimes (A, \psi)) \geqslant (((L, \varphi)^{\otimes n} \otimes (A, \psi))^{d+1})_S$$

for $n \geqslant n_0$, and hence by [**36**, Theorem 6.4.14 and Theorem 6.4.24],

$$\begin{aligned}\widehat{\mathrm{vol}}(L,\varphi) &= \lim_{n\to+\infty}\frac{1}{n^{d+1}}\widehat{\mathrm{vol}}((L,\varphi)^{\otimes n}\otimes(A,\psi))\\ &\geqslant \lim_{n\to+\infty}\frac{1}{n^{d+1}}(((L,\varphi)^{\otimes n}\otimes(A,\psi))^{d+1})_S = ((L,\varphi)^{d+1}),\end{aligned}$$

as desired. □

Corollary 6.5.2. *Let (L,φ) be a relatively nef adelic invertible O_X-module. If $((L,\varphi)^{d+1}) > 0$, then L is big.*

Proof. By Corollary 6.5.1, $\widehat{\mathrm{vol}}(L,\varphi) > 0$. Let (A,ψ) be a relatively ample adelic invertible O_X-module. By the continuity of $\widehat{\mathrm{vol}}$ (see [**36**, Theorem 6.4.24]), there is a positive integer n such that $\widehat{\mathrm{vol}}((L,\varphi)^{\otimes n}\otimes(A,\psi)^\vee) > 0$, so that, for some positive integer m, $H^0(X,(L^{\otimes n}\otimes A^\vee)^{\otimes m}) \neq \{0\}$. Therefore L is big. □

6.6. Pull-back by a projective morphism

Lemma 6.6.1. *If $\overline{L} = (L,\varphi)$ is a relatively nef adelic line bundle on X and if $g : Y \to X$ is projective morphism from a reduced K-scheme Y to X, then the pull-back $g^*(\overline{L})$ is a relatively nef adelic line bundle on Y.*

Proof. Let $\overline{A} = (A,\psi)$ be a relatively ample line bundle on X and N be a positive integer such that $\overline{L}^{\otimes n}\otimes\overline{A} = (L^{\otimes n}\otimes A, n\varphi+\psi)$ is relatively ample for any $n \in \mathbb{N}_{\geqslant N}$. Note that $L^{\otimes n}\otimes A$ is ample and hence $g^*(L)^{\otimes n}\otimes g^*(A)$ is semi-ample. Moreover, by Lemma 6.1.2, $n\varphi+\psi$ is semi-positive. We choose an arbitrary relatively ample adelic line bundle $\overline{B}$ on Y. By Proposition 6.4.6, we obtain that $g^*(\overline{L})^{\otimes n}\otimes(g^*(\overline{A})\otimes\overline{B})$ is relatively ample for any $n \in \mathbb{N}_{\geqslant N}$. Thus the assertion follows. □

Proposition 6.6.2. *Let $\overline{L} = (L,\varphi)$ be an adelic line bundle on X such that L is nef. For any non-empty and reduced closed subscheme Y of X, the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}|_Y) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}). \tag{6.26}$$

Proof. We first consider the case where L is ample. Clearly the restriction of L to Y is ample, and there exists $n_0 \in \mathbb{N}$ such that the restriction map

$$\pi_n : H^0(X,L^{\otimes n}) \longrightarrow H^0(Y,L|_Y^{\otimes n})$$

is surjective for any $n \in \mathbb{N}_{\geqslant n_0}$. Moreover, if we denote by φ_ω^Y the restriction of the metric φ_ω to $L_\omega|_{Y_\omega}$, then, for any $s \in H^0(X_\omega, L_\omega^{\otimes n})$, the inequality

$$\|s\|_{n\varphi_\omega} \geqslant \|\pi_{n,\omega}(s)\|_{n\varphi_\omega^Y}$$

holds, so that, by [**36**, Proposition 4.3.31], we obtain

$$\widehat{\mu}_{\min}(H^0(Y,L|_Y^{\otimes n}),(\|\cdot\|_{n\varphi_\omega^Y})_{\omega\in\Omega}) \geqslant \widehat{\mu}_{\min}(H^0(X,L^{\otimes n}),(\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega})$$

for any $n \in \mathbb{N}_{\geqslant n_0}$. Dividing the two sides of the inequality by n and taking the limit when $n \to +\infty$, we obtain the inequality (6.26).

In general, let $\overline{A}$ be an adelic line bundle on X such that A is ample. By the above argument, one has $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}|_Y) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) > -\infty$. Since L is nef, $L|_Y$ is also nef (see [**56**, Example 1.4.4]) and therefore $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}|_Y)$ is well defined. By (6.22) and the above case, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}|_Y) = \lim_{n\to+\infty}\frac{1}{n}\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}|_Y^{\otimes n}\otimes\overline{A}|_Y) \geqslant \lim_{n\to+\infty}\frac{1}{n}\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{A}) = \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}),$$

as required. □

PROPOSITION 6.6.3. *Let Y be a reduced and non-empty closed subscheme of X and r be the dimension of Y. Let $\overline{L}_0, \ldots, \overline{L}_r$ be a family of relatively nef adelic line bundles on X. For any $i \in \{0, \ldots, r\}$, let*

$$\delta_i = (L_0|_Y \cdots L_{i-1}|_Y L_{i+1}|_Y \cdots L_r|_Y).$$

Assume that, for any $i \in \{0, \ldots, r\}$, $\delta_i > 0$ once $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i|_Y) = -\infty$. Then the following inequality holds:

$$(\overline{L}_0|_Y \cdots \overline{L}_r|_Y)_S \geqslant \sum_{i=0}^{r} \delta_i \ \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}_i|_Y). \tag{6.27}$$

PROOF. This is a consequence of Proposition 6.4.8 and Lemma 6.6.1. □

PROPOSITION 6.6.4. *Let (E, ξ) be an adelic vector bundle on S, L be a quotient line bundle of $f^*(E)$ and φ be the quotient metric family induced by ξ. Then the adelic line bundle (L, φ) is relatively nef. Moreover, the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \geqslant \widehat{\mu}_{\min}(\overline{E}) - \frac{3}{2}\nu(\Omega_\infty) \ln(\dim_K(E)) \tag{6.28}$$

PROOF. By [**57**, Propositions 6.1.8 and 6.1.2], $f^*(E)$ is a nef vector bundle on X and hence L is a nef line bundle. Moreover, since quotient metrics are semi-positive (see [**36**, Remark 2.3.1]), the adelic line bundle $\overline{L}$ is relatively nef.

In the following, we prove the inequality (6.28). Let p be an integer and $\overline{A}$ be a relatively ample adelic line bundle on X. Then $\overline{L}^{\otimes p} \otimes \overline{A}$ is relatively ample. Let $Y = \mathbb{P}(f^*(E)^{\otimes p})$ and $g : Y \to X$ be the structural morphism. The quotient homomorphism $f^*(E) \to L$ induces by taking the tensor product a surjective homomorphism $f^*(E)^{\otimes p} \to L^{\otimes p}$, which corresponds to a section $s : X \to Y$ such that $s^*(O_Y(1)) \cong L^{\otimes p}$. Hence

$$s^*(O_Y(1) \otimes g^*(A)) \cong L^{\otimes p} \otimes A.$$

By Proposition 6.6.2, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p} \otimes \overline{A}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{O_Y(1)} \otimes g^*(\overline{A})), \tag{6.29}$$

where we consider Fubini-Study metric fiber by fiber on $O_Y(1)$. Note that, for any integer $n \in \mathbb{N}_{\geqslant 1}$, by the adjunction formula one has

$$H^0(Y, O_Y(n) \otimes g^*(A)^{\otimes n}) = H^0(X, S^n(f^*(E)^{\otimes p}) \otimes A^{\otimes n}) = S^n(E^{\otimes p}) \otimes H^0(X, A^{\otimes n}).$$

Moreover, the projection map

$$E^{\otimes np} \otimes H^0(X, A^{\otimes n}) \longrightarrow S^n(E^{\otimes p}) \otimes H^0(X, A^{\otimes n})$$

has height $\leqslant 0$, where we consider the ε, π-tensor product norm family on the left hand side of the arrow, and the adelic vector bundle structure of $(fg)_*(\overline{O_Y(n)} \otimes g^*(\overline{A})^{\otimes n})$ on the right hand side. By [**36**, Corollary 5.6.2] (see also Remark A.3.3), we obtain

$$\begin{aligned}\frac{\widehat{\mu}_{\min}\big(g_*(\overline{O_Y(n)} \otimes g^*(\overline{A}^{\otimes n}))}{n} \geqslant \frac{1}{n}\bigg(& np\, \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(f_*(\overline{A}^{\otimes n})) \\ & - \frac{3}{2}\nu(\Omega_\infty) \ln\big(\dim_K(E)^{np} \cdot \dim_K(H^0(X, A^{\otimes n}))\big)\bigg).\end{aligned}$$

Taking the limit when $n \to +\infty$, we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{O}_Y(1) \otimes g^*(\overline{A})) \geqslant p\, \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) - \frac{3}{2}\nu(\Omega_\infty) p \ln(\dim_K(E)).$$

Combining this inequality with (6.29), we obtain

$$\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p}\otimes\overline{A})\geqslant\widehat{\mu}_{\min}(\overline{E})+\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A})-\frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E)).$$

Thus, due to Definition 6.4.2, we obtain

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})=\lim_{p\to\infty}\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes p}\otimes\overline{A})\geqslant\widehat{\mu}_{\min}(\overline{E})-\frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E)),$$

as required. □

Proposition 6.6.5. *Let $g:Y\to X$ be a projective morphism of K-schemes, which is surjective and such that $g_*(\mathcal{O}_Y)=\mathcal{O}_X$. Let $\overline{L}$ be an adelic line bundle on X such that L is nef. Then the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}))\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

Proof. By [**56**, Example 1.4.4], the line bundle $g^*(L)$ is nef, and hence $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}))$ is well defined. We first consider the case where L is ample. Let p be a positive integer and $\overline{A}$ be an adelic line bundle on Y such that A is ample. By Lemma 6.1.1, for sufficiently positive integer n, the K-linear map

$$H^0(Y,A^{\otimes n})\otimes H^0(X,L^{\otimes pn})\longrightarrow H^0(Y,A^{\otimes n}\otimes g^*(L)^{\otimes pn})$$

is surjective. Moreover, if we equip the left hand side of the arrow with the ε,π-tensor product norm family of those of $(fg)_*(\overline{A}^{\otimes n})$ and $f_*(\overline{L}^{\otimes pn})$, then the K-linear map has height $\leqslant 0$. Therefore, by [**36**, Corollary 5.6.2] we obtain

$$\begin{aligned}\widehat{\mu}_{\min}((fg)_*(\overline{A}^{\otimes n}\otimes g^*(\overline{L})^{\otimes pn}))&\geqslant\widehat{\mu}_{\min}((fg)_*(\overline{A}^{\otimes n}))+\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes pn}))\\&\quad-\frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(H^0(Y,A^{\otimes n}))\dim_K(H^0(X,L^{\otimes pn}))).\end{aligned}$$

Dividing the two sides of the inequality by pn and then taking the limit when $n\to+\infty$, we obtain

$$\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}\otimes g^*(\overline{L})^{\otimes p})\geqslant\frac{\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A})}{p}+\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}),$$

which leads to

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}))=\lim_{p\to+\infty}\frac{1}{p}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}\otimes g^*(\overline{L})^{\otimes p})\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

We now consider the general case. Let $\overline{B}$ be an adelic line bundle on X such that B is ample. By the above argument we obtain that $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{B}))\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{B})>-\infty$ and, for any positive integer n,

$$\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}^{\otimes n})\otimes g^*(\overline{B}))\geqslant\frac{1}{n}\,\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}^{\otimes n}\otimes\overline{B}).$$

Taking the limit when $n\to+\infty$, by (6.22) we obtain $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}))\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$. □

Theorem 6.6.6. *Let $g:Y\to X$ be a projective morphism of K-schemes. We assume that Y is non-empty and reduced. For any adelic line bundle $\overline{L}$ on X such that L is nef, one has $\widehat{\mu}^{\mathrm{asy}}_{\min}(g^*(\overline{L}))\geqslant\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L})$.*

Proof. The projective morphism g can be written as the composition of a closed immersion from Y into a projective bundle on X and the projection from the projective bundle to X. Hence the inequality follows from Propositions 6.6.5 and 6.6.2. □

6.7. Comparison with the normalized height

The following height estimate can be deduced from Theorem 6.4.8. Here we provide an alternative proof in the particular case where X is integral by using the arithmetic Hilbert-Samuel formula.

Proposition 6.7.1. *Let $\overline{L}$ be a relatively nef adelic line bundle on X such that $(L^d) > 0$. Then the following inequality holds*

$$\frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)} \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}). \tag{6.30}$$

Proof. We assume that X is integral. In the case where $\overline{L}$ is relatively ample, it is a consequence of Theorem 5.5.1 and Remark 6.2.4.

We now consider the general case where $\overline{L}$ is only relatively nef. Let $\overline{A}$ be a relatively ample adelic line bundle and N be a positive integer such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample for any $n \in \mathbb{N}_{\geqslant N}$. For any $n \in \mathbb{N}_{\geqslant N}$, the adelic line bundle $\overline{L}_n = \overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample. Hence the particular case of the proposition proved above shows that

$$\forall\, n \in \mathbb{N}_{\geqslant N}, \quad \frac{(\overline{L}_n^{d+1})_S}{(d+1)(\overline{L}_n^d)} \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_n).$$

Moreover, by the multi-linearity of intersection product, one has

$$\lim_{n\to+\infty} \frac{(\overline{L}_n^{d+1})_S}{n^{d+1}} = (\overline{L}^{d+1})_S, \quad \lim_{n\to+\infty} \frac{(L_n^d)}{n^d} = (L^d).$$

Therefore, one obtains

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) = \lim_{n\to+\infty} \frac{1}{n}\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_n) \leqslant \lim_{n\to+\infty} \frac{(\overline{L}_n^{d+1})_S}{n(d+1)(L_n^d)} = \frac{(\overline{L}^{d+1})}{(d+1)(L^d)}.$$

□

Corollary 6.7.2. *Let $\overline{L}$ be a relatively nef adelic line bundle on X. For any non-empty and reduced closed subscheme Y of X, the following inequality holds:*

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \leqslant \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)}. \tag{6.31}$$

In particular, for any closed point x, one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \leqslant h_{\overline{L}}(x),$$

where $h_{\overline{L}}(x)$ denotes

$$\frac{(\overline{L}|_x)_S}{[K(x):K]} = \widehat{\deg}(x^*(\overline{L})).$$

Proof. By Lemma 6.6.1, the restriction of $\overline{L}$ to Y is relatively nef. By Proposition 6.7.1 one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_Y) \leqslant \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)}.$$

By Proposition 6.6.2, we obtain (6.31). □

CHAPTER 7

Global adelic space of an arithmetic variety

This chapter is devoted to the construction of global adelic space of an arithmetic variety. This construction will be useful further in the study of equidistribution of closed subvarieties. In the first section we establish a link between metric family on the trivial invertible sheaf and family of continuous functions on local analytifications. In the second section we prove the measurability of partial derivatives, which will be useful in the proof of Bogomolov conjecture over an adelic curve with Archimedean places. In the third section, we interpret the arithmetic χ-volume by concave transform on the Newton-Okounkov body and show its convexity with respect to choices of metric families. In the fourth section, we prove the Gâteaux differentiability of the arithmetic χ-volume. In the fifth section, we prove the measurability of fiber integrals of a measurable family of continuous functions. In the last two sections, we construct the global adelic space of an arithmetic variety, which is a measure space fibered over the adelic curve, admitting the fiber integrals as the disintegration with respect to the base measure.

Throughout this chapter, we fix an adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ such that the underlying field K is countable and perfect. Let X be an integral projective scheme over $\operatorname{Spec} K$ and d be the dimension of X.

7.1. Function associated with a metric family

Definition 7.1.1. For any invertible $\mathcal{O}_X$-module L, we denote by $\mathscr{M}(L)$ the set of metric families φ on L such that (L, φ) forms an adelic line bundle on X. If L_1 and L_2 are two invertible $\mathcal{O}_X$-modules, and $(\varphi_1, \varphi_2) \in \mathscr{M}(L_1) \times \mathscr{M}(L_2)$, we denote by $\varphi_1 + \varphi_2$ the tensor product of the metric families φ_1 and φ_2, which is an element of $\mathscr{M}(L_1 \otimes L_2)$.

Definition 7.1.2. Let U be a non-empty Zariski open set of X. Let

$$\mathscr{C}^0(U) := \prod_{\omega \in \Omega} C^0(U_\omega^{\mathrm{an}})$$

and $f = (f_\omega)_{\omega \in \Omega} \in \mathscr{C}^0(U)$. We say that f is *measurable* if the following conditions are satisfied (see [**36**, Definition 6.1.27]):

(a) For any closed point P of U, the function from $\Omega_{K(P)}$ to $\mathbb{R}$ sending $v \in \Omega_{K(P)}$ to $f(x_{P,v})$ is $\mathcal{A}_{K(P)}$-measurable, where $x_{P,v}$ denotes the point of X_ω^{an} represented by $(P_v, |\cdot|_v)$, P_v being the point of $X_\omega(K(P)_v)$ extending P.

(b) Let X^{an} be the Berkovich analytification of X with respect to the trivial absolute value on K. For any $x \in X^{\mathrm{an}}$, whose underlying scheme point is of dimension 1, and such that the absolute value (in the structure of x) of the residue field has a rational exponent (see §1.11), the function $(\omega \in \Omega_0) \mapsto f_\omega(x)$ is $\mathcal{A}|_{\Omega_0}$-measurable, where Ω_0 denotes the set of $\omega \in \Omega$ such that $|\cdot|_\omega$ is trivial.

Under the assumption $U = X$, the family f is said to be *dominated* if there exists an integrable function g on $(\Omega, \mathcal{A})$ such that $\sup_{x \in X_\omega^{\mathrm{an}}} |f_\omega|(x) \leqslant g(\omega)$ for all $\omega \in \Omega$. Here we set

$$\begin{cases} \mathscr{C}^0_{\mathrm{m}}(U) := \{f \in \mathscr{C}^0(U) \mid f \text{ is measurable}\}, \\ \mathscr{C}^0_{\mathrm{d}}(X) := \{f \in \mathscr{C}^0(X) \mid f \text{ is dominated}\}, \\ \mathscr{C}^0_{\mathrm{a}}(X) := \{f \in \mathscr{C}^0(X) \mid f \text{ is measurable and dominated}\}. \end{cases}$$

Sometimes $\mathscr{C}^0(U)$, $\mathscr{C}^0_{\mathrm{m}}(U)$, $\mathscr{C}^0_{\mathrm{d}}(X)$ and $\mathscr{C}^0_{\mathrm{a}}(X)$ are denoted by

$$\mathscr{C}^0(U;\Omega), \quad \mathscr{C}^0_{\mathrm{m}}(U;\Omega), \quad \mathscr{C}^0_{\mathrm{d}}(X;\Omega) \quad \text{and} \quad \mathscr{C}^0_{\mathrm{a}}(X;\Omega),$$

respectively, to emphasize the parameter space Ω.

For $f \in \mathscr{C}^0(X)$, a metric family of $\mathcal{O}_X$ given by $(e^{-f_\omega}|\cdot|_\omega)_{\omega\in\Omega}$ is denoted by e^{-f}. Note that a map given by $f \mapsto (\mathcal{O}_X, \mathrm{e}^{-f})$ yields a bijection between $\mathscr{C}^0(X)$ and the set of all metric families of $\mathcal{O}_X$. In the situation where we use the additive notation to represent a tensor product metric family, by abuse of notation we also denote the metric family e^{-f} by f in order to facilite comprehensions.

Proposition 7.1.3. *For $f \in \mathscr{C}^0(X)$, we have the following equivalence:*

$$\begin{cases} f \in \mathscr{C}^0_{\mathrm{m}}(X) \Longleftrightarrow (\mathcal{O}_X, \mathrm{e}^{-f}) \textit{ is measurable}, \\ f \in \mathscr{C}^0_{\mathrm{d}}(X) \Longleftrightarrow (\mathcal{O}_X, \mathrm{e}^{-f}) \textit{ is dominated}, \\ f \in \mathscr{C}^0_{\mathrm{a}}(X) \Longleftrightarrow (\mathcal{O}_X, \mathrm{e}^{-f}) \textit{ is an adelic line bundle}. \end{cases}$$

Proof. The first equivalence is obvious. The second is a consequence of [**36**, Proposition 6.1.12] and the fact that the zero metric on $\mathcal{O}_X$ is dominated. The third follows from the first and second. □

By abuse of notation, we often identify $\mathscr{C}^0_{\mathrm{a}}(X)$ with $\mathscr{M}(\mathcal{O}_X)$. Let Ω' be a measurable subset of Ω (i.e. $\Omega' \in \mathcal{A}$), and $S' = (K, (\Omega', \mathcal{A}', \nu'), \phi')$ be the restriction of S to Ω'. Note that S' is also an adelic curve. We consider a natural correspondence $\mathscr{C}^0(X;\Omega') \to \mathscr{C}^0(X;\Omega)$ given by the following way: if $f = (f_\omega)_{\omega\in\Omega'}$, then $\tilde{f} = (\tilde{f}_\omega)_{\omega\in\Omega}$ is defined to be

$$\tilde{f}_\omega := \begin{cases} f_\omega & \text{if } \omega \in \Omega', \\ 0 & \text{otherwise}. \end{cases}$$

Then, as a corollary of the above proposition, we have the following.

Corollary 7.1.4. *The above correspondence yields $\mathscr{C}^0_{\mathrm{a}}(X;\Omega') \subseteq \mathscr{C}^0_{\mathrm{a}}(X;\Omega)$.*

7.2. Measurability of partial derivatives

We assume that K is algebraically closed and $\Omega = \Omega_\infty$. We fix a root $\sqrt{-1}$ of the equation $T^2 + 1 = 0$ in K, and a family $(\iota_\omega)_{\omega\in\Omega_\infty}$ of embeddings $K \to \mathbb{C}$ which satisfies the following conditions (c.f. [**38**, Lemma 4.2.5]):

(i) for any $\omega \in \Omega_\infty$, $\iota_\omega(\sqrt{-1}) = i$, where $i \in \mathbb{C}$ denotes the usual imaginary unit,
(ii) for any $\omega \in \Omega_\infty$, $|\cdot|_\omega = |\iota_\omega(\cdot)|$,
(iii) for any $a \in K$, the function $(\omega \in \Omega_\infty) \mapsto \iota_\omega(a)$ is measurable.

Let $\pi : X \to \mathbb{P}^d_K$ be a finite projection and V be an affine Zariski open set of $\mathbb{P}^d_K$ such that if we set $U = \pi^{-1}(V)$, then U is smooth over K and $\pi : U \to V$ is étale. Let

$$\mathscr{C}^\infty(U) := \prod_{\omega\in\Omega} C^\infty(U^{\mathrm{an}}_\omega) \quad \text{and} \quad \mathscr{C}^\infty_{\mathrm{m}}(U) := \mathscr{C}^\infty(U) \cap \mathscr{C}^o_{\mathrm{m}}(U).$$

Let x be a closed point of U. For any $\omega \in \Omega$, we denote by x_ω the unique point of X_ω^{an} whose underly scheme point of X_ω lies over x. Let $(z_j)_{j=1}^d$ be a coordinate of V. Note $\pi^*(z_1), \ldots, \pi^*(z_d)$ yields a local étale coordinate of U around x. By abuse of notation, $\pi^*(z_1), \ldots, \pi^*(z_d)$ are denoted by $z_1, \ldots, z_d$.

Proposition 7.2.1. *If $f = (f_\omega) \in \mathscr{C}_{\mathrm{m}}^\infty(U)$, then the function given by*

$$(\omega \in \Omega) \longmapsto \frac{\partial^2(f_\omega)}{\partial z_j \partial \bar{z}_\ell}(x_\omega)$$

is $\mathcal{A}$-measurable.

Proof. We may assume that $\pi(x) = (0, \ldots, 0)$. As V is a Zariski open set of $\mathbb{A}_K^d$, we can find a non-negative integer e and a non-zero polynomial

$$h = \sum_{\substack{i_1, \ldots, i_d \in \mathbb{Z}_{\geqslant 0}, \\ i_1 + \cdots + i_d \leqslant e}} a_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d} \quad (a_{i_1, \ldots, i_d} \in K)$$

such that $h(0, \ldots, 0) \neq 0$ and

$$V' := \{(x_1, \ldots, x_d) \in K^d \mid h(x_1, \ldots, x_n) \neq 0\} \subseteq V.$$

If we set

$$h_\omega = \sum_{i_1, \ldots, i_d \in \mathbb{Z}_{\geqslant 0}} \iota_\omega(a_{i_1, \ldots, i_d}) X_1^{i_1} \cdots X_d^{i_d}$$

for each $\omega \in \Omega$, then ${V'_\omega}^{\mathrm{an}} = \{(\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d \mid h_\omega(\zeta_1, \ldots, \zeta_n) \neq 0\}$. For a polynomial

$$g = \sum_{\substack{i_1, \ldots, i_d \in \mathbb{Z}_{\geqslant 0}, \\ i_1 + \cdots + i_d \leqslant e}} b_{i_1, \ldots, i_d} X_1^{i_1} \cdots X_d^{i_d} \in \mathbb{C}[X_1, \ldots, X_d],$$

we define $\rho(g)$ to be

$$\rho(g) := \inf\{(|\zeta_1|^2 + \cdots + |\zeta_d|^2)^{1/2} \mid g(\zeta_1, \ldots, \zeta_d) = 0\}.$$

Note that $\rho(g) = 0$ if and only if $g(0, \ldots, 0) = 0$, and ρ is continuous with respect to the coefficients $(b_{i_1, \ldots, i_d})$, so if we set $r_\omega = \rho(h_\omega)$, then $r_\omega > 0$ and the function given by $(\omega \in \Omega) \mapsto r_\omega$ is $\mathcal{A}$-measurable. Moreover,

$$W_\omega = \{(\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d \mid |\zeta_1|^2 + \cdots + |\zeta_d|^2 < r_\omega^2\} \subseteq {V'_\omega}^{\mathrm{an}}.$$

As W_ω is simply connected and $\pi_\omega^{-1}(W_\omega)$ is étale over W_ω, $\pi_\omega^{-1}(W_\omega)$ is a disjoint union of connected open sets. Let T_ω be the connected component of $\pi_\omega^{-1}(W_\omega)$ such that $x_\omega \in T_\omega$. Then $\pi_\omega : T_\omega \to W_\omega$ is an isomorphism.

Let n be a positive integer and $A_n := \{\omega \in \Omega \mid r_\omega \geqslant 1/n\} \in \mathcal{A}$. Let

$$(p_1 + \sqrt{-1}q_1, \ldots, p_d + \sqrt{-1}q_d) \in \mathbb{Q}(\sqrt{-1})^d$$

such that $(p_1^2 + q_1^2) + \cdots + (p_d^2 + q_d^2) \leqslant 1/n^2$. Then, for each $\omega \in A_n$, we can find $y_\omega \in T_\omega$ such that $z_\omega(y_\omega) = (p_1 + iq_1, \ldots, p_d + iq_d)$. Further, since $y_\omega \in T_\omega$ for all $\omega \in A_n$, there exists $y \in U$ such that $z(y) = (p_1 + \sqrt{-1}q_1, \ldots, p_d + \sqrt{-1}q_d)$ and y is the image of y_ω by $X_\omega \to X$ for all $\omega \in A_n$. Therefore, for any $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon < 1/(\sqrt{2}n)$, we can find $y_{1,\varepsilon}, y_{2,\varepsilon}, y_{3,\varepsilon} \in U'(K)$ such that

$$z(y_{1,\varepsilon}) = \varepsilon e_j, \ z(y_{2,\varepsilon}) = \sqrt{-1}\varepsilon e_\ell, \ z(y_{3,\varepsilon}) = \varepsilon e_j + \sqrt{-1}\varepsilon e_\ell$$

and

$$\lim_{\varepsilon \to 0} (y_{1,\varepsilon})_\omega = x_\omega, \quad \lim_{\varepsilon \to 0} (y_{2,\varepsilon})_\omega = x_\omega \quad \text{and} \quad \lim_{\varepsilon \to 0} (y_{3,\varepsilon})_\omega = x_\omega.$$

Thus one obtains

$$\frac{\partial^2(f_\omega)}{\partial z_j \partial \overline{z}_\ell}(x_\omega) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\Big[f_\omega((y_{3,\varepsilon})_\omega) - f_\omega((y_{1,\varepsilon})_\omega) - f_\omega((y_{2,\varepsilon})_\omega) + f_\omega(x_\omega)\Big]$$

for $\omega \in A_n$. Note that, for any rational point y of U, the function $(\omega \in \Omega) \mapsto f_\omega(y_\omega)$ is $\mathcal{A}$-measurable. Therefore, if we set

$$b_n(\omega) = \begin{cases} \dfrac{\partial^2(f_\omega)}{\partial z_j \partial \overline{z}_\ell}(x_\omega) & \omega \in A_n, \\ 0 & \text{otherwise,} \end{cases}$$

then b_n is $\mathcal{A}$-measurable on Ω, so the assertion is proved because

$$\lim_{n\to\infty} b_n(\omega) = \frac{\partial^2(f_\omega)}{\partial z_j \partial \overline{z}_\ell}(x_\omega).$$

□

7.3. Relative volume and χ-volume

If L is an invertible O_X-module, we denote by

$$V_\bullet(L) = \bigoplus_{n\in\mathbb{N}} H^0(X, L^{\otimes n})$$

the total graded linear series of L. Recall that the *volume* of L is defined as

$$\mathrm{vol}(L) = \limsup_{n\to+\infty} \frac{\dim_K(H^0(X, L^{\otimes n}))}{n^d/d!}.$$

The invertible O_X-module L is said to be *big* if $\mathrm{vol}(L) > 0$.

If L is a big invertible O_X-module and $V_\bullet$ is a graded linear series of L (namely a graded sub-K-algebra of $V_\bullet(L)$), we denote by $\Delta(V_\bullet)$ the Newton-Okounkov body of $V_\bullet$. We refer to [**53, 58**] for the construction of Newton-Okounkov body under the assumption that the function field of X admits a $\mathbb{Z}^d$-valuation of one-dimensional leaves over K, see also [**32, 33**] for the arithmetic construction which applies to the general case. Recall that $\Delta(V_\bullet)$ is a closed convex subset of $\mathbb{R}^d$, whose Lebesgue measure is equal to

$$\frac{1}{d!}\,\mathrm{vol}(V_\bullet) := \limsup_{n\to+\infty} \frac{\dim_K(V_n)}{n^d}.$$

In the case where $V_\bullet$ is the total graded linear series of L, the Newton-Okounkov body is denoted by $\Delta(L)$. Recall that, if L and L' are big invertible O_X-modules, and $V_\bullet$, $V'_\bullet$ and $W_\bullet$ are respectively graded linear series of L, L' and $L\otimes L'$, such that

$$\forall\, n \in \mathbb{N}, \quad V_n \cdot V'_n \subseteq W_n,$$

then one has

$$\Delta(V_\bullet) + \Delta(V'_\bullet) := \{x + y \;:\; (x, y) \in \Delta(V_\bullet) + \Delta(V'_\bullet)\} \subseteq \Delta(W_\bullet).$$

Let (L, φ) be an adelic line bundle on X. For any $n \in \mathbb{N}$, we denote by $\xi_{n\varphi}$ the norm family $(\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega}$ on $V_n(L) = H^0(X, L^{\otimes n})$, so that $(V_n(L), \xi_{n\varphi})$ forms an adelic vector bundle on S. Let $\mathcal{F}$ be the Harder-Narasimhan $\mathbb{R}$-filtration of this adelic line bundle. Recall that

$$\forall\, t \in \mathbb{R}, \quad \mathcal{F}^t(V_n(L), \xi_{n\varphi}) = \sum_{\substack{\mathbf{0}\neq F\subseteq V_n(L)\\ \widehat{\mu}_{\min}(\overline{F})\geqslant t}} F,$$

where F runs over the set of non-zero vector subspaces of $V_n(L)$ such that the minimal slope of F equipped with restricted norm family of $\xi_{n\varphi}$ is not less than t. Note that this $\mathbb{R}$-filtration determines an ultrametric norm on $V_n(L)$ (where we consider the trivial absolute value on K), which we denote by $\|\cdot\|_{n\varphi}$. Denote by $\|\cdot\|_{n\varphi,\mathrm{sp}}$ the corresponding spectral norm, which is defined as

$$\|s\|_{n\varphi,\mathrm{sp}} := \lim_{N\to+\infty} \|s^N\|_{nN\varphi}^{\frac{1}{N}}.$$

For any $t \in \mathbb{R}$, let

$$V_n^{\varphi,t}(L) := \{s \in V_n(L) \;:\; \|s\|_{n\varphi,\mathrm{sp}} \leqslant \mathrm{e}^{-nt}\}.$$

Note that $V_\bullet^{\varphi,t}(L)$ is a graded linear series of L.

Definition 7.3.1. Let (L, φ) be an adelic line bundle on X such that L is big. We call *concave transform* of the metric family φ the function $G_\varphi : \Delta(L) \to \mathbb{R}$ defined as follows:

$$G_\varphi(x) = \sup\{t \in \mathbb{R} \;:\; x \in \Delta(V_\bullet^{\varphi,t}(L))\}.$$

This function is concave. Moreover, for any $x \in \Delta(L)$, one has

$$\liminf_{n\to+\infty} \frac{\widehat{\mu}_{\min}(V_n(L), \xi_{n\varphi})}{n} \leqslant G_\varphi(x) \leqslant \limsup_{n\to+\infty} \frac{\widehat{\mu}_{\max}(V_n(L), \xi_{n\varphi})}{n}.$$

For any positive integer n, one has $\Delta(L^{\otimes n}) = n\Delta(L)$ and

$$G_{n\varphi}(nx) = nG_\varphi(x). \tag{7.1}$$

In the case where

$$\liminf_{n\to+\infty} \frac{\widehat{\mu}_{\min}(V_n(L), \xi_{n\varphi})}{n} > -\infty,$$

the following equality holds:

$$\int_{\Delta(L)} G_\varphi(x)\,\mathrm{d}x = \frac{1}{(d+1)!}\widehat{\mathrm{vol}}_\chi(L, \varphi). \tag{7.2}$$

Definition 7.3.2. We say that an invertible $\mathcal{O}_X$-module L is *slope-bounded* if there exists a metric family $\varphi \in \mathscr{M}(L)$ such that

$$\liminf_{n\to+\infty} \frac{\widehat{\mu}_{\min}(V_n(L), \xi_{n\varphi})}{n} > -\infty. \tag{7.3}$$

Note that, for any element $\psi \in \mathscr{M}(L)$, one has

$$\forall\, n \in \mathbb{N},\quad \left|\widehat{\mu}_{\min}(V_n(L), \xi_{n\varphi}) - \widehat{\mu}_{\min}(V_n(L), \xi_{n\psi})\right| \leqslant n \int_\Omega \sup_{x\in X_\omega^{\mathrm{an}}} |\varphi_\omega - \psi_\omega|(x)\,\nu(\mathrm{d}\omega)$$

and therefore

$$\liminf_{n\to+\infty} \frac{\widehat{\mu}_{\min}(V_n(L), \xi_{n\psi})}{n} > -\infty$$

if (7.3) is true.

Example 7.3.3. Let L be a big invertible $\mathcal{O}_X$-module. Assume that the graded linear series $V_\bullet(L) := \bigoplus_{n\in\mathbb{N}} V_n(L)$ is of finite type over K. Then the invertible $\mathcal{O}_X$-module L is slope-bounded. In particular, semiample and big invertible $\mathcal{O}_X$-modules are slope-bounded (cf. Remark 7.3.4 below).

REMARK 7.3.4. Let L be a semiample invertible O_X-module. Then $V_\bullet(L)$ is of finite type over K. Indeed, there exist a surjective morphism $f : X \to Y$ of projective integral schemes over K, an ample invertible O_Y-module A and a positive integer a such that $L^{\otimes a} = f^*(A)$. Thus, by Lemma 4.4.1, $R = V_\bullet(L^{\otimes a})$ is of finite type over K and

$$M_i = \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes i} \otimes L^{\otimes na})$$

is finitely generated over R for every $0 \leqslant i < a$. Therefore, $V_\bullet(L) = M_0 \oplus \cdots \oplus M_{a-1}$ is also finitely generated over R, and hence the assertion follows.

PROPOSITION 7.3.5. *Let L_1 and L_2 be big invertible O_X-modules. If (φ_1, φ_2) is an element of $\mathscr{M}(L_1) \times \mathscr{M}(L_2)$, then the following inequality holds:*

$$\forall\, (x, y) \in \Delta(L_1) \times \Delta(L_2), \quad G_{\varphi_1+\varphi_2}(x + y) \geqslant G_{\varphi_1}(x) + G_{\varphi_2}(y).$$

PROOF. Let $(t_1, t_2) \in \mathbb{R}^2$, $n \in \mathbb{N}_{\geqslant 1}$ and $(s_1, s_2) \in V_n^{\varphi_1,t_1}(L_1) \times V_n^{\varphi_2,t_2}(L_2)$. By definition, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}_{\geqslant 1}$ such that

$$\forall\, i \in \{1, 2\}, \quad s_i^N \in \mathcal{F}^{nN(t_i-\varepsilon)}(V_{nN}(L_i), \xi_{nN\varphi_i}).$$

By [**36**, Corollary 5.6.2] (see also Remark A.3.3), we obtain that

$$\begin{aligned} - \ln \|(s_1 s_2)^N\|_{nN(\varphi_1+\varphi_2)} &\geqslant nN(t_1 + t_2 - 2\varepsilon) \\ &\quad - \frac{3}{2}\nu(\Omega_\infty)\Big(\ln \dim_K(V_{nN}(L_1)) + \ln \dim_K(V_{nN}(L_2))\Big). \end{aligned}$$

Dividing the two sides of the equality by N and taking the limit when $N \to +\infty$, we obtain

$$- \ln \|s_1 s_2\|_{n(\varphi_1+\varphi_2),\mathrm{sp}} \geqslant n(t_1 + t_2 - 2\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$- \ln \|s_1 s_2\|_{n(\varphi_1+\varphi_2),\mathrm{sp}} \geqslant n(t_1 + t_2).$$

Therefore, one has

$$V_n^{\varphi_1,t_1}(L_1) \cdot V_n^{\varphi_2,t_2}(L_2) \subseteq V_n^{\varphi_1+\varphi_2,t_1+t_2}(L_1 \otimes L_2),$$

which implies

$$\Delta(V_\bullet^{\varphi_1,t_1}(L_1)) + \Delta(V_\bullet^{\varphi_2,t_2}(L_2)) \subseteq \Delta(V_\bullet^{\varphi_1+\varphi_2,t_1+t_2}(L_1 \otimes L_2)).$$

Let (x, y) be an element of $\Delta(V_\bullet(L_1)) \times \Delta(V_\bullet(L_2))$. For any $\varepsilon > 0$ and any $(t_1, t_2) \in \mathbb{R}^2$ such that $t_1 \leqslant G_{\varphi_1}(x) - \varepsilon$ and $t_2 \leqslant G_{\varphi_2}(y) - \varepsilon$, one has

$$(x, y) \in \Delta(V_\bullet^{\varphi_1,t_1}(L_1)) \times \Delta(V_\bullet^{\varphi_2,t_2}(L_2))$$

and hence $x + y \in \Delta(V_\bullet^{\varphi_1+\varphi_2,t_1+t_2}(L_1 \otimes L_2))$. We thus obtain

$$G_{\varphi_1+\varphi_2}(x + y) \geqslant t_1 + t_2 - 2\varepsilon.$$

Since t_1, t_2 and ε are arbitrary, we deduce

$$G_{\varphi_1+\varphi_2}(x + y) \geqslant G_{\varphi_1}(x) + G_{\varphi_2}(x).$$

□

COROLLARY 7.3.6. *Let L be a slope-bounded invertible O_X-module, φ_1 and φ_2 be elements of $\mathscr{M}(L)$, and $\delta \in [0, 1]$. Then the following inequality holds*

$$\widehat{\mathrm{vol}}_\chi(L, \delta\varphi_1 + (1 - \delta)\varphi_2) \geqslant \delta\, \widehat{\mathrm{vol}}_\chi(L, \varphi_1) + (1 - \delta)\, \widehat{\mathrm{vol}}_\chi(L, \varphi_2). \tag{7.4}$$

In other words, the function from $\mathscr{M}(L)$ to $\mathbb{R}$ sending φ to $\widehat{\mathrm{vol}}_\chi(L, \varphi)$ is concave.

Proof. We first treat the case where δ is a rational number. Let k and N be positive integers such that $N > k$. By (7.1), (7.2) and Proposition 7.3.5, we obtain

$$\frac{\widehat{\mathrm{vol}}_\chi(L^{\otimes N}, k\varphi_1 + (N-k)\varphi_2)}{\mathrm{vol}(L^{\otimes N})} \geqslant \frac{\widehat{\mathrm{vol}}_\chi(L^{\otimes k}, k\varphi_1)}{\mathrm{vol}(L^{\otimes k})} + \frac{\widehat{\mathrm{vol}}_\chi(L^{\otimes (N-k)}, (N-k)\varphi_2)}{\mathrm{vol}(L^{\otimes (N-k)})},$$

or equivalently,

$$N\,\widehat{\mathrm{vol}}_\chi(L, \tfrac{k}{N}\varphi_1 + \tfrac{N-k}{N}\varphi_2) \geqslant k\,\widehat{\mathrm{vol}}_\chi(L,\varphi_1) + (N-k)\,\widehat{\mathrm{vol}}_\chi(L,\varphi_2).$$

Therefore the inequality (7.4) holds in the case where δ is rational. The general case follows from the rational case together with the following estimate

$$\forall\,(\varphi,\psi)\in\mathscr{M}(L)^2,$$
$$\Big|\widehat{\mathrm{vol}}(L,\varphi) - \widehat{\mathrm{vol}}(L,\psi)\Big| \leqslant (d+1)\,\mathrm{vol}(L)\int_\Omega \sup_{x\in X_\omega^{\mathrm{an}}} |(\varphi-\psi)(x)|\,\nu(\mathrm{d}\omega).$$

□

7.4. Gâteaux differentiability

We show that the χ-volume function $\widehat{\mathrm{vol}}_\chi(\cdot)$ is Gâteaux differentiable along any direction defined by metric families on the open cone $\widehat{\mathrm{Pic}}_A(X)$ of adelic line bundles $\overline{L}$ on X such that L is semi-ample and big.

Definition 7.4.1. Let L be a big invertible $\mathcal{O}_X$-module. Let φ and ψ be two elements of $\mathscr{M}(L)$. For any $\omega\in\Omega$, we denote by $\mathrm{vol}(L_\omega,\varphi_\omega,\psi_\omega)$ the relative volume of L_ω with respect to the metric pair $(\varphi_\omega,\psi_\omega)$, which is defined as

$$-\lim_{n\to+\infty}\frac{(d+1)!}{n^{d+1}}\ln\frac{\|\cdot\|_{n\varphi_\omega,\det}}{\|\cdot\|_{n\psi_\omega,\det}}.$$

We refer to [**34**, Theorem 4.5] for the convergence of the sequence defining the relative volume.

Proposition 7.4.2. *Let L be a semi-ample and big invertible $\mathcal{O}_X$-module and $(\varphi,\psi)\in\mathscr{M}(L)^2$. The following equality holds:*

$$\widehat{\mathrm{vol}}_\chi(L,\varphi) - \widehat{\mathrm{vol}}_\chi(L,\psi) = \int_\Omega \mathrm{vol}(L_\omega,\varphi_\omega,\psi_\omega)\,\nu(\mathrm{d}\omega).$$

Proof. For any positive integer n, let α_n be a non-zero element of $\det H^0(X, L^{\otimes n})$. By definition

$$\widehat{\mathrm{vol}}_\chi(L,\varphi) - \widehat{\mathrm{vol}}_\chi(L,\psi) = -\lim_{n\to+\infty}\frac{(d+1)!}{n^{d+1}}\int_{\omega\in\Omega}\ln\frac{\|\alpha_n\|_{n\varphi_\omega,\det}}{\|\alpha_n\|_{n\psi_\omega,\det}}\,\nu(\mathrm{d}\omega).$$

Note that

$$\frac{1}{n\dim_K(H^0(X,L^{\otimes n}))}\left|\ln\frac{\|\cdot\|_{n\varphi_\omega,\det}}{\|\cdot\|_{n\psi_\omega,\det}}\right| \leqslant \sup_{x\in X_\omega^{\mathrm{an}}}|\varphi_\omega-\psi_\omega|(x).$$

By dominated convergence theorem we obtain

$$\begin{aligned}\widehat{\mathrm{vol}}_\chi(L,\varphi) - \widehat{\mathrm{vol}}_\chi(L,\psi) &= -\int_{\omega\in\Omega}\lim_{n\to+\infty}\frac{(d+1)!}{n^{d+1}}\ln\frac{\|\cdot\|_{n\varphi_\omega,\det}}{\|\cdot\|_{n\psi_\omega,\det}}\,\nu(\mathrm{d}\omega)\\ &= \int_\Omega \mathrm{vol}(L_\omega,\varphi_\omega,\psi_\omega)\,\nu(\mathrm{d}\omega).\end{aligned}$$

□

PROPOSITION 7.4.3. *Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. We assume that L is semi-ample and big and that φ is semi-positive. The function $\widehat{\mathrm{vol}}_\chi(\cdot)$ on $\widehat{\mathrm{Pic}}_A(X)$ is Gâteaux differentiable at $\overline{L}$ along the directions of $\mathscr{M}(O_X)$. Moreover, for any $f \in \mathscr{M}(O_X)$, the function*

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} f_\omega \, c_1(L_\omega, \varphi_\omega)^d$$

is ν-integrable, and one has

$$\lim_{t\to 0} \frac{\widehat{\mathrm{vol}}_\chi(\overline{L}(tf)) - \widehat{\mathrm{vol}}_\chi(\overline{L})}{t} = (d+1)\int_\Omega \nu(\mathrm{d}\omega) \int_{X_\omega^{\mathrm{an}}} f_\omega \, c_1(L_\omega, \varphi_\omega)^d.$$

PROOF. By [**19**, Theorem 1.2] and [**10**, Theorem B], for any $\omega \in \Omega$, one has

$$\lim_{t\to 0} \frac{\mathrm{vol}(L_\omega, \varphi_\omega + tf_\omega, \varphi_\omega)}{t} = (d+1)\int_{X_\omega^{\mathrm{an}}} f_\omega \, c_1(L_\omega, \varphi_\omega)^d.$$

Note that

$$\left|\frac{\mathrm{vol}(L_\omega, \varphi_\omega + tf_\omega, \varphi_\omega)}{t}\right| \leqslant (d+1)\deg_L(X) \sup_{x \in X_\omega^{\mathrm{an}}} |f_\omega|(x).$$

Since the function

$$(\omega \in \Omega) \longmapsto \sup_{x \in X_\omega^{\mathrm{an}}} |f_\omega|(x)$$

is integrable, by Lebesgue's dominated convergence theorem we obtain, by using Proposition 7.4.2, that

$$\lim_{t\to 0} \frac{\widehat{\mathrm{vol}}_\chi(\overline{L}(tf)) - \widehat{\mathrm{vol}}_\chi(\overline{L})}{t} = (d+1)\int_\Omega \nu(\mathrm{d}\omega) \int_{X_\omega^{\mathrm{an}}} f_\omega \, c_1(L_\omega, \varphi_\omega)^d.$$

□

REMARK 7.4.4. We conjecture that any big invertible O_X-module L is slope-bounded. If this is true, then for any metric family $\varphi \in \mathscr{M}(L)$, the χ-volume $\widehat{\mathrm{vol}}(L, \varphi)$ takes real values. Hence the results of Propositions 7.4.2 and 7.4.3 hold without semi-amplitude assumption on L. Correspondingly, we conjecture that Theorem 8.11.2 also holds when L is only nef and big. Actually Luo [**59**] give a generalization of Theorem 8.11.2 under the assumption $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \in \mathbb{R}$.

COROLLARY 7.4.5. *Let $(M_1, \psi_1), \ldots, (M_d, \psi_d)$ be relatively nef adelic line bundles. For any $f \in \mathscr{C}_{\mathrm{a}}^0(X)$, the function*

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} f_\omega \, c_1(M_{1,\omega}, \psi_{1,\omega}) \cdots c_1(M_{d,\omega}, \psi_{d,\omega}) \tag{7.5}$$

is ν-integrable.

PROOF. By the multi-linearity of Monge-Ampère measure, we may assume without loss of generality that all adelic line bundles (M_i, ψ_i) are equal to the same one (M, ψ) (c.f. [**38**, Proposition 1.1.4]). Let (L, φ) be a relatively ample adelic line bundle on X. By Proposition 7.4.3, for any $n \in \mathbb{N}_{\geqslant 1}$, the function

$$(\omega \in \Omega) \longmapsto \frac{1}{n^d} \int_{X_\omega^{\mathrm{an}}} f_\omega \, c_1(M_\omega^{\otimes n} \otimes L_\omega, n\psi_\omega + \varphi_\omega)^d$$

is $\mathcal{A}$-measurable. Passing to limit when $n \to +\infty$, we obtain the $\mathcal{A}$-measurability of the function

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} f_\omega \, c_1(M_\omega, \psi_\omega)^d.$$

Finally, since $f \in \mathscr{C}^0_{\mathrm{a}}(X)$, by [**36**, Proposition 6.1.12], we obtain that there exist two ν-integrable functions A_1 and A_2 on Ω such that

$$\forall\, x \in X_\omega^{\mathrm{an}}, \quad A_1(\omega) \leqslant f_\omega(x) \leqslant A_2(\omega).$$

Since each $c_1(M_\omega, \psi_\omega)^d$ has measure $c_1(M)^d$, the function

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} f_\omega \, c_1(M_\omega, \psi_\omega)^d$$

is ν-integrable. □

7.5. Measurability of fiber integrals

Definition 7.5.1. Let Ω' be an element of $\mathcal{A}$. As *Borel measure family* on X over Ω', we refer to a family $\eta = (\eta_\omega)_{\omega \in \Omega'}$, where each η_ω is a Borel measure on X_ω, such that, for any $f = (f_\omega)_{\omega \in \Omega'} \in \mathscr{C}^0_{\mathrm{a}}(X; \Omega')$, the function

$$(\omega \in \Omega') \longmapsto \int_{X_\omega^{\mathrm{an}}} f_\omega(x) \, \eta_\omega(\mathrm{d}x)$$

is $\mathcal{A}|_{\Omega'}$-measurable and integrable with respect to the restriction of the measure ν to Ω'. We denote by $\eta(f)$ the integral

$$\int_{\Omega'} \int_{X_\omega^{\mathrm{an}}} f_\omega(x) \, \eta_\omega(\mathrm{d}x) \, \nu(\mathrm{d}\omega).$$

Remark 7.5.2. Let $\eta = (\eta_\omega)_{\omega \in \Omega'}$ be a Borel measure family on X over Ω'. For any ν-integrable function $A : \Omega \to \mathbb{R}$ which vanishes on $\Omega \setminus \Omega'$, we consider the family $f_A = (f_{A,\omega})_{\omega \in \Omega}$, where $f_{A,\omega}$ denotes the constant function on X_ω^{an} taking value $A(\omega)$. We then obtain that the function

$$(\omega \in \Omega') \longmapsto A(\omega)\eta_\omega(X_\omega^{\mathrm{an}}) = \int_{x \in X_\omega^{\mathrm{an}}} f_{A,\omega}(x) \, \eta_\omega(\mathrm{d}x)$$

is ν-integrable. This observation shows that the function

$$(\omega \in \Omega') \longmapsto \eta_\omega(X_\omega^{\mathrm{an}})$$

is essentially bounded, namely there exists $C > 0$ such that

$$\{\omega \in \Omega' \,:\, \eta_\omega(X_\omega^{\mathrm{an}}) > C\}$$

is a zero measure set.

Example 7.5.3. Let $\overline{L} = (L, \varphi)$ be a relatively ample adelic line bundle on X, namely L is an ample invertible O_X-module and $\varphi = (\varphi_\omega)_{\omega \in \Omega}$ is a measurable and dominated family of semi-positive metrics. Let Y be a reduced closed subscheme. Denote by $\delta_{\overline{L}, Y, \Omega'} = (\delta_{\overline{L}, Y, \omega})_{\omega \in \Omega'}$ the Borel measure family on X over Ω' defined as follows: for any $\omega \in \Omega$, and any positive Borel function f_ω on X_ω^{an},

$$\int_{X_\omega^{\mathrm{an}}} f_\omega(x) \, \delta_{\overline{L}, Y, \omega}(\mathrm{d}x) := \frac{1}{\deg_L(Y)} \int_{Y_\omega^{\mathrm{an}}} f_\omega(y) \, c_1(L_\omega|_{Y_\omega}, \varphi_\omega|_{Y_\omega})^{\dim(Y)}(\mathrm{d}y).$$

This is a Borel probability measure on X_ω^{an}, which is supported on Y_ω^{an}. In the case where Y is a closed point, the measure $\delta_{\overline{L},Y,\omega}$ is given by the weighted average on some points of X_ω^{an}. We refer to Proposition 7.4.3 for the integrability of the function

$$(\omega\in\Omega)\longmapsto\int_{X_\omega^{\mathrm{an}}} f_\omega(x)\,\delta_{\overline{L},Y,\omega}(\mathrm{d}x)$$

when $f\in\mathscr{C}^0_{\mathrm{a}}(X)$ (so that the restriction of f to Y belongs to $\mathscr{C}^0_{\mathrm{a}}(Y)$). In the case where S is proper and $\Omega'=\Omega$, one can also interpret the expression $\delta_{\overline{L},Y,\Omega}$ in terms of the arithmetic intersection theory. For any $f\in\mathscr{C}^0_{\mathrm{a}}(X)$ and any $t\in\mathbb{R}$, we denote by $\overline{L}(tf)$ the adelic line bundle $(L,\varphi+\mathrm{e}^{-tf})$. Then the following equality holds:

$$\delta_{\overline{L},Y,\Omega'}(f)=\lim_{t\to 0}\frac{(\overline{L}(tf)|_Y^{\dim(Y)+1})_S-(\overline{L}^{\dim(Y)+1}|_Y)_S}{(\dim(Y)+1)\deg_L(Y)t},$$

provided that $(\mathcal{O}_X,\mathrm{e}^{-f})$ formes an integrable adelic line bundle.

Example 7.5.4. Let $(M_1,\psi_1),\ldots,(M_d,\psi_d)$ be relatively nef adelic line bundles on X. For any $\omega\in\Omega$, let

$$\eta_\omega=c_1(M_{1,\omega},\psi_{1,\omega})\cdots c_1(M_{d,\omega},\psi_{d,\omega}).$$

By Corollary 7.4.5, $(\eta_\omega)_{\omega\in\Omega}$ forms a Borel measure family on X over Ω.

Proposition 7.5.5. *Let Ω' be an element of $\mathcal{A}$ and $\eta=(\eta_\omega)_{\omega\in\Omega'}$ be a Borel measure family on X over Ω'. Let $\overline{M}=(M,\psi)$ be an adelic line bundle on X and s be a non-zero global section of M. Then the function*

$$(\omega\in\Omega')\longmapsto\int_{X_\omega^{\mathrm{an}}}(-\ln|s|_{\psi_\omega}(x))\,\eta_\omega(\mathrm{d}x)$$

is $\mathcal{A}$-measurable and is bounded from below by an integrable function.

Proof. Let Ω_0 be the set of $\omega\in\Omega$ such that $|\cdot|_\omega$ is trivial. Then

$$\Omega\setminus\Omega_0=\bigcup_{a\in K\setminus\{0\}}\{\omega\in\Omega\;:\;|a|_\omega\neq 1\}$$

is σ-finite (namely a countable union of elements of $\mathcal{A}$ which have a finite measure). Moreover, by [**36**, Proposition 6.1.12], a comparison with the trivial metric family over Ω_0 shows that the function

$$(\omega\in\Omega_0)\longmapsto\sup_{x\in X_\omega^{\mathrm{an}}}\big|\ln|s|_{\psi_\omega}(x)\big|$$

is integrable. Therefore the set

$$\Omega_{0,s}=\{\omega\in\Omega_0\;:\;|s|_{\psi_\omega}\text{ is not identically }1\}$$

is σ-finite. We may then choose a non-negative ν-integrable function A on Ω such that $A(\omega)>0$ for any $\omega\in(\Omega\setminus\Omega_0)\cup\Omega_{0,s}$. In fact, if we write $(\Omega\setminus\Omega_0)\cup\Omega_{0,s}$ as a countable union $\bigcup_{n\in\mathbb{N}}B_n$, where each B_n is an element of finite measure in $\mathcal{A}$, then, with arbitrary choices of positive real numbers b_n such that $b_n\nu(B_n)\leqslant 2^{-n}$, $n\in\mathbb{N}$, the following function is ν-integrable

$$\sum_{n\in\mathbb{N}}b_n\,\mathbb{1}_{B_n}$$

and vanishes nowhere on $(\Omega\setminus\Omega_0)\cup\Omega_{0,s}$.

For any $t>0$ and any $\omega\in\Omega$, let $f_{t,\omega}:X_\omega^{\mathrm{an}}\to\mathbb{R}$ be the function defined as follows:

$$f_{t,\omega}(x):=\min\{-\ln|s|_{\psi_\omega}(x),tA(\omega)\}.$$

This is a continuous function on X_ω^{an}, which yields a continuous metric e^{-f_ω} on O_{X_ω}. Moreover, the function $f_{t,\omega}$ is bounded from below by

$$\min\{-\ln\|s\|_{\psi_\omega}, tA(\omega)\}$$

and bounded from above by $tA(\omega)$. By Proposition [**36**, Proposition 6.2.12], the function

$$g_s:\Omega\to\mathbb{R},\quad (\omega\in\Omega)\longmapsto -\ln\|s\|_{\psi_\omega}$$

is integrable. Therefore, the family $f_t=(f_{t,\omega})_{\omega\in\Omega}$ belongs to $\mathscr{C}^0_{\mathrm{a}}(X)$ and hence the function

$$(\omega\in\Omega')\longmapsto\int_{X_\omega^{\mathrm{an}}} f_{t,\omega}(x)\,\eta_\omega(\mathrm{d}x)$$

is $\mathcal{A}$-measurable. Passing to limit when $t\to+\infty$, we obtain the measurability of the function

$$(\omega\in\Omega')\longmapsto\int_{X_\omega^{\mathrm{an}}}(-\ln|s|_{\psi_\omega}(x))\,\eta_\omega(\mathrm{d}x).$$

Finally, by definition, for any $\omega\in\Omega'$, one has

$$\int_{X_\omega^{\mathrm{an}}}(-\ln|s|_{\psi_\omega}(x))\,\eta_\omega(\mathrm{d}x)\geqslant -\ln\|s\|_{\psi_\omega}\eta_\omega(X_\omega^{\mathrm{an}}).$$

Since the function $(\omega\in\Omega')\mapsto\eta_\omega(X_\omega^{\mathrm{an}})$ is essentially bounded, the second assertion is true. □

Proposition 7.5.6. *Let U be a non-empty Zariski open subset of X, and $f=(f_\omega)_{\omega\in\Omega}$ be a measurable family, where each f_ω is a continuous function on U_ω^{an}, such that there exists a ν-integrable function $g:\Omega\to\mathbb{R}$ satisfying*

$$\forall\,\omega\in\Omega,\ \forall\,x\in X_\omega^{\mathrm{an}},\quad f_\omega(x)\geqslant g(\omega).$$

Let Ω' be a σ-finite element of $\mathcal{A}$ and $(\eta_\omega)_{\omega\in\Omega}$ be a Borel measure family on X over Ω'. Assume that, there exist an adelic line bundle (M,ψ) and a non-zero section $s\in H^0(X,M)$ such that the non-vanishing locus of s is contained in U and that

$$\int_{X_\omega^{\mathrm{an}}}(-\ln|s|_{\psi_\omega}(x))\,\eta_\omega(\mathrm{d}x)<+\infty\quad \nu\text{-almost everywhere on }\Omega'.$$

Then the function

$$(\omega\in\Omega)\longmapsto\int_{U_\omega^{\mathrm{an}}} f_\omega(x)\,\eta_\omega(\mathrm{d}x)$$

is $\mathcal{A}$-measurable.

Proof. We choose a non-negative ν-integrable function A on Ω such that $A(\omega)>0$ for any $\omega\in\Omega'$. This is possible since Ω' is σ-finite. Moreover, without loss of generality, we may assume (by Remark 7.5.2 and the condition of the proposition on $(\eta_\omega)_{\omega\in\Omega}$) that

$$\int_{X_\omega^{\mathrm{an}}}(-\ln|s|_{\psi_\omega}(x))\,\eta_\omega(\mathrm{d}x)\in\mathbb{R}$$

for any $\omega\in\Omega'$.

For any $t>0$ and any $\omega\in\Omega$, let $f_{t,\omega}:X_\omega^{\mathrm{an}}\to\mathbb{R}$ be the function defined as

$$f_{t,\omega}(x):=\min\{f_\omega(x)-\ln|s|_{\psi_\omega}(x), tA(\omega)\},$$

where by convention $f_{t,\omega}(x)=tA(\omega)$ when $s(x)=0$. Since $f_\omega(x)$ is continuous and $-\ln|s|_{\psi_\omega}(x)$ tends to $+\infty$ when x tends to some point $x_0\in X_\omega^{\mathrm{an}}$ such that $s(x_0)=0$, we obtain that the function $f_{t,\omega}$ is continuous on X_ω^{an}. Moreover, the function $f_{t,\omega}$ is bounded from above by $tA(\omega)$ and bounded from below by

$$\min\{g(\omega) - \ln \|s\|_{\psi_\omega}, tA(\omega)\}.$$

Therefore the family $(f_{t,\omega})_{\omega\in\Omega}$ belongs to $\mathscr{C}_{\mathrm{a}}^0(X)$. Hence by Proposition 7.5.5 we obtain the measurability of the function

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} (f_\omega(x) - \ln |s|_{\psi_\omega}(x))\, \eta_\omega(\mathrm{d}x).$$

□

Remark 7.5.7. Let (L, φ) be a relatively nef adelic line bundle. For any $\omega \in \Omega$, let $\eta_\omega = c_1(L_\omega, \varphi_\omega)^d$. Then, for any adelic line bundle (M, ψ) and any non-zero section $s \in H^0(X, M)$, one has

$$\forall\, \omega \in \Omega, \quad \int_{X_\omega^{\mathrm{an}}} (-\ln |s|_{\psi_\omega}(x))\, \eta_\omega(\mathrm{d}x) \in \mathbb{R}.$$

We refer to [**27**] for the non-Archimedean case.

7.6. Global adelic space

Recall that we fix a adelic curve

$$S = (K, (\Omega, \mathcal{A}, \nu), \phi)$$

such that K is countable and perfect. If X is an integral projective K-scheme, we denote by $K(X)$ the field of rational functions on X. This is also a countable field.

Let X be a projective scheme over K and Ω' be an element of $\mathcal{A}$. We denote by $X_{\Omega'}^{\mathrm{an}}$ the disjoint union $\coprod_{\omega\in\Omega'} X_\omega^{\mathrm{an}}$. Denote by $\pi : X_{\Omega'}^{\mathrm{an}} \to \Omega'$ the map sending the elements of X_ω^{an} to ω. For any Zariski open subset U of X, let $U_{\Omega'}^{\mathrm{an}}$ be the disjoint union $\coprod_{\omega\in\Omega'} U_\omega^{\mathrm{an}}$.

Definition 7.6.1. We equip $X_{\Omega'}^{\mathrm{an}}$ with the smallest σ-algebra $\mathcal{B}_{X,\Omega'}$ which satisfies the following conditions:

(1) the map $\pi : X_{\Omega'}^{\mathrm{an}} \to \Omega'$ is measurable,
(2) for any Zariski open subset U of X, the set $U_{\Omega'}^{\mathrm{an}}$ belongs to $\mathcal{B}_{X,\Omega'}$,
(3) for any adelic line bundle (L, φ) on X and any section s of L on some Zariski open subset U of X, the function

$$U_{\Omega'}^{\mathrm{an}} \longrightarrow \mathbb{R}, \quad (x \in U_\omega^{\mathrm{an}}) \longmapsto |s|_{\varphi_\omega}(x)$$

is $\mathcal{B}_{X,\Omega'}$-measurable.

Remark 7.6.2. The above third condition can be replaced by the following (3)':

(3)' For any adelic line bundle (L, φ) on X and any global section s of L, the function

$$X_{\Omega'}^{\mathrm{an}} \longrightarrow \mathbb{R}, \quad (x \in X_\omega^{\mathrm{an}}) \longmapsto |s|_{\varphi_\omega}(x)$$

is $\mathcal{B}_{X,\Omega'}$-measurable.

In fact, suppose that U is a Zariski open subset of X and s is a section of L over U. We an auxiliary adelic vector bundle (M, ψ) such that M is very ample and global sections $t_1, \ldots, t_n$ of M such that the non-vanishing loci $D(t_1), \ldots, D(t_n)$ of $t_1, \ldots, t_n$ form an open cover of U. Then there exists an integer $\ell \geqslant 1$ such that $t_1^\ell s, \ldots, t_n^\ell s$ extend to global sections of $L \otimes M^{\otimes \ell}$. Locally on $D(t_i)_{\Omega'}^{\mathrm{an}}$ we can then write the function $(x \in X_\omega^{\mathrm{an}}) \mapsto |s|_{\varphi_\omega}$ as the quotient of $(x \in X_\omega^{\mathrm{an}}) \mapsto |t_i^\ell s|_{\ell\psi_\omega+\varphi_\omega}(x)$ by $(x \in X_\omega^{\mathrm{an}}) \mapsto |t_i|_{\psi_\omega}^\ell(x)$.

Remark 7.6.3. Let U be a Zariski open subset of X. We consider the trivial metric family on $\mathcal{O}_X$. Then the point (3) in the above definition shows that, for any regular function f on U, the function $|f|_{\Omega'}: U^{\mathrm{an}}_{\Omega'} \to \mathbb{R}_{\geqslant 0}$, which sends $x \in U^{\mathrm{an}}_{\omega}$ to $|f|_{\omega}(x)$, is $\mathcal{B}_{X,\Omega'}|_{U^{\mathrm{an}}_{\Omega'}}$-measurable.

Assume that the scheme X is integral. Let q be a rational function on X and U be the maximal open subscheme over which the rational function q is defined. We consider the function $|q|_{\Omega'}$ on $X^{\mathrm{an}}_{\Omega'}$ sending $x \in X^{\mathrm{an}}_{\omega}$ to $|q|_{\omega}(x)$. Note that, on the Zariski open subset U, the rational function q coincides with a regular function b on U. Moreover, the following equality holds

$$|q|_{\Omega'}(x) = \begin{cases} |b|_{\Omega'}(x), & \text{if } x \in U^{\mathrm{an}}_{\Omega'}, \\ +\infty, & \text{if } x \in X^{\mathrm{an}}_{\Omega'} \setminus U^{\mathrm{an}}_{\Omega'}. \end{cases}$$

In particular, the function $|q|_{\Omega'}$ is $\mathcal{B}_{X,\Omega'}$-measurable.

Proposition 7.6.4. *Let $f: X^{\mathrm{an}}_{\Omega'} \to \mathbb{R}_{\geqslant 0}$ be a $\mathcal{B}_{X,\Omega'}$-measurable function.*

(1) *For any $\omega \in \Omega'$, $f|_{X^{\mathrm{an}}_{\omega}}$ is a Borel measurable function.*

(2) *Let $\eta = (\eta_\omega)_{\omega \in \Omega'}$ be a Borel measure family on X over Ω'. Then the function*

$$(\omega \in \Omega') \longmapsto \int_{X^{\mathrm{an}}_{\omega}} f(x)\, \eta_\omega(\mathrm{d}x)$$

is $\mathcal{A}|_{\Omega'}$-measurable.

Proof. Let $\mathcal{H}_\eta$ be the set of bounded functions $f: X^{\mathrm{an}}_{\Omega'} \to \mathbb{R}$ which satisfies the condition predicted in the proposition, namely $f|_{X^{\mathrm{an}}_{\omega}}$ is a Borel function for any $\omega \in \Omega'$, and the function

$$(\omega \in \Omega') \longmapsto \int_{X^{\mathrm{an}}_{\omega}} f(x)\, \eta_\omega(\mathrm{d}x)$$

is $\mathcal{A}|_{\Omega'}$-measurable. Note that $\mathcal{H}_\eta$ is a λ-family, namely

(i) the constant function 1 belongs to $\mathcal{H}_\eta$;

(ii) if f and g are two elements of $\mathcal{H}_\eta$, and a and b are non-negative numbers, then $af + bg \in \mathcal{H}_\eta$;

(iii) if f and g are two elements of $\mathcal{H}_\eta$ such that $f \leqslant g$, then $g - f \in \mathcal{H}_\eta$;

(iv) if $(f_n)_{n\in\mathbb{N}}$ is an increasing and uniformly bounded sequence of functions in $\mathcal{H}_\eta$, then the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{H}_\eta$.

Let C be set of functions $X^{\mathrm{an}}_{\Omega'} \to \mathbb{R}$ of the form $1\!\!1_{\pi^{-1}(A)}|b|_{\varphi,\Omega'}$, where A is an element of $\mathcal{A}$ contained in Ω' and b is a global section of some adelic vector bundle (L,φ). Moreover, the σ-algebra $\mathcal{B}_{X,\Omega'}$ is equal to the σ-algebra $\sigma(C)$ generated by C. By Proposition 7.5.6, the family C is contained in $\mathcal{H}_\eta$. If A' is another element of $\mathcal{A}$ contained in Ω', b' is a global section of some adelic line bundle (L',φ'), then one has

$$(1\!\!1_{\pi^{-1}(A)}|b|_{\varphi,\Omega'})(1\!\!1_{\pi^{-1}(A')}|b'|_{\varphi',\Omega'}) = 1\!\!1_{\pi^{-1}(A\cap A')}|bb'|_{\varphi+\varphi',\Omega'},$$

where wen consider bb' as a global section of the adelic line bundle $(L \otimes L', \varphi \otimes \varphi')$. The function family C is hence stable by multiplication. By monotone class theorem (see for example [**36**, Theorem A.1.3]), $\mathcal{H}_\eta$ contains all bounded $\sigma(C)$-measurable functions. Since any $\mathcal{B}_{X,\Omega'}$-measurable function can be written as a limit of bounded $\mathcal{B}_{X,\Omega'}$-measurable functions, the proposition is thus proved. □

Definition 7.6.5. Let Ω' be an element of $\mathcal{A}$ and η be a Borel measure family on X over Ω'. Proposition 7.6.4 shows that, the map

$$(B \in \mathcal{B}_{X,\Omega'}) \longmapsto \int_{\omega\in\Omega'} \nu(\mathrm{d}\omega) \int_{X^{\mathrm{an}}_{\omega}} 1\!\!1_B(x)\, \eta_\omega(\mathrm{d}x)$$

defines a measure on the measurable space $(X^{\mathrm{an}}_{\Omega'}, \mathcal{B}_{X,\Omega'})$. We denote by $\eta_{\Omega'}$ this measure. By abuse of notation, it is often denoted by η. For any non-negative $\mathcal{B}_{X,\Omega'}$-measurable function $f : X^{\mathrm{an}}_{\Omega'} \to \mathbb{R}_{\geqslant 0}$, one has

$$\int_{X^{\mathrm{an}}_{\Omega'}} f(x)\,\eta_{\Omega'}(\mathrm{d}x) = \int_{\Omega'} \nu(\mathrm{d}\omega) \int_{X^{\mathrm{an}}_{\omega}} f(x)\,\eta_{\omega}(\mathrm{d}x).$$

Remark 7.6.6. If there exists an adelic line bundle (M, ψ) such that M is ample and that, for any $n \in \mathbb{N}_{\geqslant 1}$ and any non-zero section $s \in H^0(X, M^{\otimes n})$, one has

$$\forall\, \omega \in \Omega, \quad \int_{X^{\mathrm{an}}_{\omega}} (-\ln |s|_{\psi_\omega}(x))\,\eta_\omega(\mathrm{d}x) \in \mathbb{R},$$

then, viewed as a measure on $(X_{\Omega'}, \mathcal{B}_{X,\Omega'})$, η is uniquely determined by the integrals of functions in $\mathscr{C}^0_{\mathrm{a}}(X; \Omega')$. In other words, if $\eta' = (\eta'_\omega)_{\omega\in\Omega}$ is another Borel measure family on X over Ω' such that

$$\int_{\omega\in\Omega'} \nu(\mathrm{d}\omega) \int_{X^{\mathrm{an}}_{\omega}} f_\omega(x)\,\eta_\omega(\mathrm{d}x) = \int_{\omega\in\Omega'} \nu(\mathrm{d}\omega) \int_{X^{\mathrm{an}}_{\omega}} f_\omega(x)\,\eta'_\omega(\mathrm{d}x),$$

then, as measures on $(X^{\mathrm{an}}_{\Omega'}, \mathcal{B}_{X,\Omega'})$, one has $\eta_{\Omega'} = \eta'_{\Omega'}$. This follows from the proofs of Propositions 7.5.5 and 7.5.6.

Definition 7.6.7. We equip $X^{\mathrm{an}}_{\Omega'}$ the smallest σ-algebra $\mathcal{B}'_{X,\Omega'}$ which satisfies the following conditions:

(1) the map $\pi : X^{\mathrm{an}}_{\Omega'} \to \Omega'$ is measurable,
(2) for any Zariski open subset U of X, the set $U^{\mathrm{an}}_{\Omega'}$ belongs to $\mathcal{B}'_{X,\Omega'}$,
(3) for any Zariski open subset U of X and any regular function b on U, the function $|b|_{\Omega'}$ on $U^{\mathrm{an}}_{\Omega'}$ defined as

$$\forall\, \omega \in \Omega',\ \forall\, x \in U^{\mathrm{an}}_{\omega}, \quad |b|_{\Omega'}(x) := |b|_\omega(x)$$

is $\mathcal{B}'_{X,\Omega'}$-measurable.

Obviously $\mathcal{B}'_{X,\Omega'} \subseteq \mathcal{B}_{X,\Omega'}$ (see Remark 7.6.3).

Proposition 7.6.8. *Let L be a very ample invertible $\mathcal{O}_X$-module and φ be a family of semipositive metrics of L such that (L, φ) is measurable. Then, for $s \in H^0(X, L)$, the function $|s|_{\varphi,\Omega'} : X^{\mathrm{an}}_{\Omega'} \to \mathbb{R}$ sending $x \in X^{\mathrm{an}}_{\omega}$ to $|s|_{\varphi_\omega}(x)$ is $\mathcal{B}'_{X,\Omega'}$-measurable.*

Proof. We begin with the case where φ is a quotient metric family. Let $E = H^0(X, L)$, ξ be a norm family on E such that (E, ξ) forms a measurable vector bundle on S, and $u : X \to \mathbb{P}(E)$ be the canonical projective K-morphism. Note that $u^*(\mathcal{O}_E(1)) \cong L$. Assume that the metric family φ is induced by ξ and the morphism u. If $s = 0$, then the assertion is obvious, so we may assume that $s \neq 0$. Let U be a Zariski open set of X given by $\{\xi \in X \mid s(\xi) \neq 0\}$. For $t \in E \setminus \{0\}$, we choose $\lambda \in K(X)^\times$ such that $t = \lambda s$. Note that λ is regular on U. Let us consider $|\lambda|^{-1}_{\Omega'}(x) \cdot \|t\|_\omega$ on U^{an}_{Ω}. For a point x with $|\lambda|_{\Omega'}(x) = 0$ (i.e. $t(x) = 0$), the value of $|\lambda|^{-1}_{\Omega'}(x) \cdot \|t\|_\omega$ is defined to be ∞. Thus,

$$\inf_{\substack{t\in E\setminus\{0\},\ \lambda\in K(X)^\times \\ t=\lambda s}} |\lambda|^{-1}_{\Omega'}(x) \cdot \|t\|_\omega = \inf_{\substack{t\in E\setminus\{0\},\ \lambda\in K(X)^\times \\ t(x)\neq 0,\ t=\lambda s}} |\lambda|^{-1}_{\Omega'}(x) \cdot \|t\|_\omega,$$

and hence

$$|s|_{\varphi,\Omega'}(x) = \inf_{\substack{t\in E\setminus\{0\},\ \lambda\in K(X)^\times \\ t=\lambda s}} |\lambda|^{-1}_{\Omega'}(x) \cdot \|t\|_\omega$$

for all $\omega \in \Omega$ and $x \in U_\omega^{\mathrm{an}}$. Therefore $|s|_{\varphi,\Omega'}$ is $\mathcal{B}'_{X,\Omega'}$-measurable because $|\lambda|_{\Omega'}^{-1}(x) \cdot \|t\|_\omega$ is $\mathcal{B}'_{X,\Omega'}$-measurable and E is countable.

Let $(\varphi_n)_{n\in\mathbb{N}}$ is a family of metric families on L. We assume that (L, φ_n) forms a measurable line bundle on X, and that, for any $\omega \in \Omega'$,

$$\lim_{n\to+\infty} d_\omega(\varphi_n, \varphi) = 0.$$

If for any $n \in \mathbb{N}$, the measurable line bundle (L, φ_n) verifies the assertion of the proposition, then so does (L, φ). Therefore the proposition follows. □

Remark 7.6.9. The above proposition implies that the σ-algebra $\mathcal{B}_{X,\Omega'}$ is spanned by functions of the form $|b|_{\Omega'} f_{\Omega'}$, where b is a regular function on some Zariski open subset U of X and f is an element of $\mathscr{C}_{\mathrm{a}}^0(X)$. In fact, for any adelic line bundle (L, φ) such that L is ample, one can always find an element $f \in \mathscr{C}_{\mathrm{a}}^0(X)$ such that $\varphi + f$ is semi-positive.

7.7. Determination of fiber integral by global adelic measure

In this section, we let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve such that K is countable, and Ω' be an element of $\mathcal{A}$. We have seen in §7.6 that any Borel measure family $\eta = (\eta_\omega)_{\omega\in\Omega'}$ on X over Ω' determines a measure $\eta_{\Omega'}$ on the measurable space $(X_{\Omega'}^{\mathrm{an}}, \mathcal{B}_{X,\Omega'})$, such that, for any non-negative $\mathcal{B}_{X,\Omega'}$-measurable function f on $X_{\Omega'}^{\mathrm{an}}$, one has

$$\int_{X_{\Omega'}^{\mathrm{an}}} f(x)\,\eta_{\Omega'}(\mathrm{d}x) = \int_{\Omega'} \nu(\mathrm{d}\omega) \int_{X_\omega^{\mathrm{an}}} f(x)\,\eta_\omega(\mathrm{d}x).$$

We show that the fiber measures η_ω are almost everywhere determined by the global measure $\eta_{\Omega'}$.

Proposition 7.7.1. *Let $\eta = (\eta_\omega)_{\omega\in\Omega'}$ and $\tau = (\tau_\omega)_{\omega\in\Omega'}$ be Borel measure family on X over Ω'. If $\eta_{\Omega'} = \tau_{\Omega'}$, then there exists a measurable subset Ω'' of Ω' such that $\nu(\Omega'') = 0$ and that $\eta_\omega = \tau_\omega$ for any $\omega \in \Omega' \setminus \Omega''$.*

Proof. We first show that, for any non-negative $\mathcal{B}_{X,\Omega'}$-measurable function f on $X_{\Omega'}^{\mathrm{an}}$, the set Ω'_f of $\omega \in \Omega'$ such that

$$\int_{X_\omega^{\mathrm{an}}} f(x)\,\eta_\omega(\mathrm{d}x) \neq \int_{X_\omega^{\mathrm{an}}} f(x)\,\tau_\omega(\mathrm{d}x)$$

has measure 0 with respect to ν. Let A be the set of $\omega \in \Omega'$ such that

$$\int_{X_\omega^{\mathrm{an}}} f(x)\,\eta_\omega(\mathrm{d}x) > \int_{X_\omega^{\mathrm{an}}} f(x)\,\tau_\omega(\mathrm{d}x).$$

By Proposition 7.6.4, we obtain that A is a measurable set. Moreover, by the equality

$$\int_{X_{\Omega'}^{\mathrm{an}}} \mathbb{1}_A(\pi(x)) f(x)\,\eta_{\Omega'}(\mathrm{d}x) = \int_{X_{\Omega'}^{\mathrm{an}}} \mathbb{1}_A(\pi(x)) f(x)\,\tau_{\Omega'}(\mathrm{d}x)$$

we obtain that $\nu(A) = 0$, where $\pi : X_{\Omega'}^{\mathrm{an}} \to \Omega'$ sends the elements of X_ω^{an} to ω. Similarly, the set of $\omega \in \Omega'$ such that

$$\int_{X_\omega^{\mathrm{an}}} f(x)\,\eta_\omega(\mathrm{d}x) < \int_{X_\omega^{\mathrm{an}}} f(x)\,\tau_\omega(\mathrm{d}x)$$

is also $\mathcal{A}$-measurable and has measure 0 with respect to ν.

We now pick an auxiliary adelic line bundle $\overline{L} = (L, \varphi)$ such that L is ample and φ is semi-positive. We let

$$\Theta = \bigcup_{n \in \mathbb{N}} H^0(X, L^{\otimes n}).$$

Since K is countable and each linear series $H^0(X, L^{\otimes n})$ is a finite-dimensional vector space over K, the set Θ is countable. For any $s \in \Theta$, we consider the function $f_s : X^{\mathrm{an}}_{\Omega'} \to \mathbb{R}_{\geqslant 0}$ which sends $x \in X^{\mathrm{an}}_{\omega}$ to $|s|_{\varphi_\omega}(x)$. This function is $\mathcal{B}_{X,\Omega'}$-measurable. We let

$$\Omega'' := \bigcup_{s \in \Theta} \Omega'_{f_s}.$$

By the above argument, we obtain that Ω'' belongs to $\mathcal{A}$ and $\nu(\Omega'') = 0$.

Let ω be an element of $\Omega' \setminus \Omega''$ and denote by $\mathcal{H}_\omega$ the set of positive and bounded Borel function g_ω on X^{an}_ω such that

$$\int_{X^{\mathrm{an}}_\omega} g_\omega(x)\, \eta_\omega(\mathrm{d}x) = \int_{X^{\mathrm{an}}_\omega} g_\omega(x)\, \tau_\omega(\mathrm{d}x).$$

Clearly $\mathcal{H}_\omega$ is a λ-family. Let C_ω be the set of functions of the form $(x \in X^{\mathrm{an}}_\omega) \longmapsto |s|_{\varphi_\omega}(x)$, where s is an element of Θ. Clearly the family C_ω is stable by multiplication. Therefore, by monotone class theorem (see for example [**36**, Theorem A.1.3]), $\mathcal{H}_\omega$ contains all bounded $\sigma(C_\omega)$-measurable functions. Finally, for any $n \in \mathbb{N}$, $H^0(X, L^{\otimes n})$ is dense in $H^0(X_\omega, L^{\otimes n}_\omega) \cong H^0(X, L^{\otimes n}) \otimes_K K_\omega$. For any $s \in H^0(X_\omega, L^{\otimes n}_\omega)$, there exists a sequence $(s_\ell)_{\ell \in \mathbb{N}}$ of elements in $H^0(X, L^{\otimes n})$ such that

$$\|s_\ell - s\|_{n\varphi_\omega} = \sup_{x \in X^{\mathrm{an}}_\omega} |s_\ell - s|_{\varphi_\omega}(x)$$

converges to 0. Therefore we deduce that $\mathcal{H}_\omega$ actually contains all bounded positive Borel functions on X^{an}_ω, which means that $\eta_\omega = \tau_\omega$. □

Lemma 7.7.2. *Let $(K, |\cdot|)$ be a trivially valued field, X be an geometrically integral projective scheme of degree d over* Spec K*, L be an ample invertible $\mathcal{O}_X$-module, and φ and φ' be two continuous metrics on L. Assume that the equality $c_1(L, \varphi)^d = c_1(L, \varphi')^d$ holds. Then there exists a constant $\lambda \in \mathbb{R}$ such that $\varphi' = e^\lambda \varphi$.*

Proof. By definition there exists a continuous function λ on X^{an} such that $\varphi' = e^{\lambda(\cdot)}\varphi$. It suffices to prove that the function $\lambda(\cdot)$ is constant.

Let u be a real number such that $0 < u < 1$. For $f(T) = \sum_{n=0}^{\infty} a_i T^i \in K[[T]]$, we define $|f(T)|$ to be

$$|f(T)| = \sup_{i \in \mathbb{Z}_{\geqslant 0}} |a_i| u^i,$$

which extends to a non-trivial absolute value on $K((T))$. Let $K' = K((T))$ and $\pi : X^{\mathrm{an}}_{K'} \to X^{\mathrm{an}}$ be the projection map. By Proposition [**38**, Proposition 3.9.9], one has

$$\begin{cases} \Big((0, \lambda \circ \pi) \cdot \big(L_{K'}, \varphi_{K'}\big)^d\Big) = ((0, \lambda) \cdot (L, \varphi)^d), \\ \Big((0, \lambda \circ \pi) \cdot \big(L_{K'}, \varphi'_{K'}\big)^d\Big) = ((0, \lambda) \cdot (L, \varphi')^d), \end{cases}$$

and hence

$$\begin{aligned} \int_{X^{\mathrm{an}}_{K'}} \lambda(\pi(x))\, c_1\big(L_{K'}, \varphi_{K'}\big)^d &= \int_{X^{\mathrm{an}}} \lambda(x)\, c_1(L, \varphi)^d \\ = \int_{X^{\mathrm{an}}} \lambda(x)\, c_1(L, \varphi')^d &= \int_{X^{\mathrm{an}}_{K'}} \lambda(\pi(x))\, c_1\big(L_{K'}, \varphi'_{K'}\big)^d. \end{aligned}$$

By the same method of the proof of [**76**, Corollary 2.2], we obtain that $\lambda(\pi(\cdot))$ is a constant function, and hence $\lambda(\cdot)$ is a constant function since π is surjective. □

PROPOSITION 7.7.3. *Let L and L' be invertible O_X-modules, and $\varphi = (\varphi_\omega)_{\omega\in\Omega'}$ and $\varphi' = (\varphi'_\omega)_{\omega\in\Omega'}$ be semi-positive metric families of L and L' over Ω', respectively, such that (L,φ) and (L',φ') forms adelic line bundles over Ω'. If $\delta_{(L,\varphi),X,\Omega'} = \delta_{(L',\varphi'),X,\Omega'}$ on $X^{\mathrm{an}}_{\Omega'}$, then there exists $\Omega'' \in \mathcal{A}$ such that $\Omega'' \subseteq \Omega'$, $\nu(\Omega'') = 0$ and $c_1(L_\omega,\varphi_\omega)^{\dim X} = c_1(L'_\omega,\varphi'_\omega)^{\dim X}$ for all $\omega \in \Omega' \setminus \Omega''$. Moreover, in the case where $L = L'$ and L is ample, then there exists an integrable function $\lambda : \Omega' \to \mathbb{R}$ such that $\varphi'_\omega = e^{\lambda(\omega)}\varphi_\omega$ for any $\omega \in \Omega' \setminus \Omega''$*

PROOF. The first statement is a consequence of Proposition 7.7.1. For the second statement, let $f = (f_\omega)_{\omega\in\Omega'} \in \mathscr{C}^0_{\mathrm{a}}(X;\Omega')$ such that $\varphi'_\omega = e^{f_\omega}\varphi_\omega$ for all $\omega \in \Omega'$. By the uniqueness part of Calabi-Yau theorem (see [**76**, Corollary 2.2] and the lemma above, see also [**23, 54, 12**]), f_ω is a constant $c(\omega)$ on $\Omega' \setminus \Omega''$. Thus, if we set

$$\lambda(\omega) = \begin{cases} c(\omega) & \text{if } \omega \in \Omega' \setminus \Omega'', \\ 0 & \text{otherwise,} \end{cases}$$

then the second assertion follows. □

CHAPTER 8

Generically big and pseudo-effective adelic line bundles

The purpose of this chapter is to study weak relative positivity conditions of adelic line bundle. In the first section, we first extend the arithmetic intersection product in allowing the appearance of one non-integrable adelic line bundle. In the second and third sections, we introduce a numerical invariant, the asymptotic maximal slope, to measure the weak relative positivity of an adelic line bundle. In the fourth section, we show that the asymptotic maximal slope does not decrease by the pull-back by a surjective projective morphism. In the fifth section, we prove a relative version of Fujita's approximation theorem for the asymptotic maximal slope of an adelic line bundle by the asymptotic minimal slope of relatively nef adelic line subbundles. In the sixth section, we proved a strong lower bound of the arithmetic intersection product with the appearance of the asymptotic maximal slope of one adelic line bundle instead of its asymptotic minimal slope. In the seventh section, we discuss asymptotic first minimum, which is similar, but in general not equal, to the asymptotic maximal slope. In the eighth section, we compare the asymptotic maximal slope to the normalized height. In the ninth section, we introduce the condition of strong Minkowskianness for adelic line bundles. Under this condition the adelic line bundles behave similarly to the classic number field case. In the tenth section, we study the successive minima of the normalized height function and discuss its link with sectional invariants such as the asymptotic maximal and minimal slopes. In the eleventh and last section, we prove an equidistribution theorem for a generic sequence of integral closed subscheme.

In this chapter, we fix an adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ such that, either $(\Omega, \mathcal{A})$ is discrete, or K is countable. We assume in addition that K is perfect. Let X be a projective scheme over $\operatorname{Spec} K$ and d be the dimension of X.

8.1. Extension of arithmetic intersection product

In this section, we extends the construction of intersection product allowing the appearance of one non-integrable adelic line bundle. Let $\overline{L}_0 = (L_0, \varphi_0), \ldots, \overline{L}_d = (L_d, \varphi_d)$ be adelic line bundles on X. We assume that $\overline{L}_1, \ldots, \overline{L}_d$ are integrable.

Proposition 8.1.1. *Assume that the invertible $\mathcal{O}_X$-module L_0 admits a global section s which forms a regular meromorphic section of L_0 on X. Then the function*

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} \ln |s|_{\varphi_{0,\omega}} c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega})$$

is ν-integrable.

Proof. By the multi-linearity of the local mixed Monge-Ampère measure

$$c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega})$$

with respect to $\overline{L}_1, \ldots, \overline{L}_d$, we may assume without loss of generality that the adelic line bundle $\overline{L}_1, \ldots, \overline{L}_d$ are relatively ample. By Proposition 7.5.5, we obtain that the function

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} \ln |s|_{\varphi_{0,\omega}} c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega})$$

is $\mathcal{A}$-measurable and bounded from above by a ν-integrable function. Let $\overline{A} = (A, \psi)$ be a relatively ample adelic line bundle on X such that $L_0^\vee \otimes A$ admits a global section t. By Proposition 7.5.5 again, the function

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} (-\ln |t|_{\psi_\omega - \varphi_{0,\omega}})\, c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega})$$

is bounded from below by a ν-integrable function. Since $\overline{A}$ is relatively ample and hence integrable, by [**38**, Theorem 4.2.12] we obtain that the function

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} \ln |st|_{\psi}\, c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega})$$

is ν-integrable, which shows that the function

$$(\omega \in \Omega) \longmapsto \int_{X_\omega^{\mathrm{an}}} \ln |s|_{\varphi_{0,\omega}} c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega})$$

is bounded from below by a ν-integrable function. The proposition is thus proved. □

Definition 8.1.2. Let $\overline{L}_1 = (L_1, \varphi_1), \ldots, \overline{L}_d = (L_d, \varphi_d)$ be integrable adelic line bundles. Let $\overline{L}_0 = (L_0, \varphi_0)$ be an adelic line bundle. If L_0 admits a non-zero global section s which forms a regular meromorphic section of L_0 over X and

$$\mathrm{div}(s) = \sum_{j=1}^n a_j Z_j$$

is the decomposition of $\mathrm{div}(s)$ into linear combination of prime divisors, we define the *arithmetic intersection number* of $\overline{L}_0, \ldots, \overline{L}_d$ as

$$\sum_{j=1}^n a_j (\overline{L}_1|_{Z_j} \cdots \overline{L}_d|_{Z_j})_S - \int_\Omega \nu(\mathrm{d}\omega) \int_{X_\omega^{\mathrm{an}}} \ln |s|_{\varphi_{0,\omega}}\, c_1(L_{1,\omega}, \varphi_{1,\omega}) \cdots c_1(L_{d,\omega}, \varphi_{d,\omega}),$$

denoted by $(\overline{L}_0 \cdots \overline{L}_d)_S$. In the general case, we write $\overline{L}_0$ in the form $\overline{M}_0 \otimes \overline{N}_0^\vee$, where $\overline{M}_0$ and $\overline{N}_0$ are adelic line bundles such that M_0 and N_0 are ample. Then we define the arithmetic intersection number of $\overline{L}_0, \ldots, \overline{L}_d$ as

$$(\overline{M}_0 \cdot \overline{L}_1 \cdots \overline{L}_d)_S - (\overline{N}_0 \cdot \overline{L}_1 \cdots \overline{L}_d)_S.$$

Note that, for fixed integral adelic line bundle $\overline{L}_1, \cdots, \overline{L}_d$, the map

$$(\overline{L}_0 \in \widehat{\mathrm{Pic}}(X)) \longmapsto (\overline{L}_0 \cdot \overline{L}_1 \cdots \overline{L}_d)_S$$

defines a linear form on $\widehat{\mathrm{Pic}}(X)$.

Proposition 8.1.3. *Let $\overline{L}_0 = (L_0, \varphi_0), \ldots, \overline{L}_d = (L_d, \varphi_d)$ be a family of adelic line bundles on X. For any $i \in \{0, \ldots, d\}$, let*

$$\delta_i = (L_0 \cdots L_{i-1} \cdot L_{i+1} \cdots L_d).$$

Assume that $\overline{L}_1,\ldots,\overline{L}_d$ are relatively nef, L_0 admits a global section s which is a regular meromorphic section, and, for any $i\in\{1,\ldots,d\}$, $\delta_i>0$ once $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i)=-\infty$. Then the following inequality holds:

$$\begin{aligned}(\overline{L}_0\cdots\overline{L}_d)_S\geqslant{}&\sum_{i=1}^d\delta_i\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i)\\&-\int_\Omega\int_{X_\omega^{\mathrm{an}}}\ln\|s\|_{\varphi_{0,\omega}}c_1(L_{1,\omega},\varphi_{1,\omega})\cdots c_1(L_{d,\omega},\varphi_{d,\omega})\,\nu(\mathrm{d}\omega).\end{aligned}\tag{8.1}$$

Proof. If there exists $i\in\{1,\ldots,d\}$ such that $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i)=-\infty$, then the inequality (8.1) is trivial. Therefore, we can assume that $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i)\in\mathbb{R}$ for any $i\in\{1,\ldots,n\}$. Let $\mathrm{div}(s)=a_1Z_1+\cdots+a_nZ_n$ be the decomposition of $\mathrm{div}(s)$ as linear combination of prime divisors, where $a_1,\ldots,a_n$ are non-negative integers since s is a global section. By proposition 6.6.2, for any $i\in\{1,\ldots,d\}$ and and any $j\in\{1,\ldots,n\}$, one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i|_{Z_j})\geqslant\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i).\tag{8.2}$$

By [**38**, Proposition 4.4.4], one has

$$\begin{aligned}(\overline{L}_0\cdots\overline{L}_d)_S={}&\sum_{j=1}^n a_j(\overline{L}_1|_{Z_j}\cdots\overline{L}_d|_{Z_j})_S\\&-\int_\Omega\int_{X_\omega^{\mathrm{an}}}\ln|s|_{\varphi_{0,\omega}}(x)\,c_1(L_{1,\omega},\varphi_{1,\omega})\cdots c_1(L_{d,\omega},\varphi_{d,\omega})(\mathrm{d}x)\,\nu(\mathrm{d}\omega).\end{aligned}$$

By Proposition 6.6.3, one has

$$\sum_{j=1}^n a_j(\overline{L}_1|_{Z_j}\cdots\overline{L}_d|_{Z_j})_S\geqslant\sum_{j=1}^n a_j\sum_{i=1}^d\delta_{i,j}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i|_{Z_j})\geqslant\sum_{j=1}^n a_j\sum_{i=1}^d\delta_{i,j}\,\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i),$$

where

$$\delta_{i,j}:=(L_1|_{Z_j}\cdots L_{i-1}|_{Z_j}\cdot L_{i+1}|_{Z_j}\cdots L_d|_{Z_j}),$$

and the second inequality comes from (8.2). Note that, for any $i\in\{1,\ldots,d\}$, one has

$$\sum_{j=1}^n a_j\delta_{i,j}=\delta_i.$$

Hence we obtain the desired inequality. □

8.2. Convergence of maximal slopes

Proposition 8.2.1. *Let X be an integral projective scheme over* Spec K, *and $\overline{L}=(L,\varphi)$ and $\overline{M}=(M,\psi)$ be adelic line bundles on X such that $H^0(X,L)$ and $H^0(X,M)$ are non-zero. Then the following inequality holds:*

$$\widehat{\mu}_{\max}(f_*(\overline{L}\otimes\overline{M}))\geqslant\widehat{\mu}_{\max}(f_*(\overline{L}))+\widehat{\mu}_{\max}(f_*(\overline{M}))-\frac{3}{2}\nu(\Omega_\infty)(\ln(h^0(L)\cdot h^0(M))),$$

where $h^0(L)=\dim_K(H^0(X,L))$ and $h^0(M)=\dim_K(H^0(X,M))$.

Proof. By [**36**, Theorem 4.3.58], there exist non-zero vector subspaces E and F of $H^0(X,L)$ and $H^0(X,M)$, respectively, such that

$$\widehat{\mu}_{\min}(\overline{E})=\widehat{\mu}_{\max}(f_*(\overline{L})),\quad\widehat{\mu}_{\min}(\overline{F})=\widehat{\mu}_{\max}(f_*(\overline{M})),$$

where we consider restricted norm families on E and F. Since X is integral, the map

$$E \otimes_K F \longrightarrow H^0(X, L \otimes M), \quad s \otimes t \longmapsto st$$

is non-zero. Moreover, for any $\omega \in \Omega$, one has

$$\forall\,(s,t) \in E_\omega \times F_\omega, \quad \|st\|_{\varphi_\omega+\psi_\omega} \leqslant \|s\|_{\varphi_\omega} \cdot \|t\|_{\psi_\omega}.$$

Therefore, the height of the above K-linear map is $\leqslant 0$ if we consider the ε, π-tensor product norm family on $E \otimes_K F$. By [**36**, Theorem 4.3.31 and Corollary 5.6.2], we obtain

$$\begin{aligned}
&\widehat{\mu}_{\max}(f_*(\overline{L} \otimes \overline{M}))\\
&\geqslant \widehat{\mu}_{\min}(\overline{E} \otimes_{\varepsilon,\pi} \overline{F}) \geqslant \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(\overline{F}) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E)\cdot\dim_K(F))\\
&= \widehat{\mu}_{\max}(f_*(\overline{L})) + \widehat{\mu}_{\max}(f_*(\overline{M})) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E)\cdot\dim_K(F))\\
&\geqslant \widehat{\mu}_{\max}(f_*(\overline{L})) + \widehat{\mu}_{\max}(f_*(\overline{M})) - \frac{3}{2}\nu(\Omega_\infty)(\ln(h^0(L)\cdot h^0(M))),
\end{aligned}$$

as required. □

Corollary 8.2.2. *Let $\overline{L}$ be an adelic line bundle on X such that $H^0(X, L^{\otimes n})$ is non-zero for sufficiently large natural number n. The sequence*

$$\frac{1}{n}\,\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n})), \quad n \in \mathbb{N}_{\geqslant 1}$$

converges in $\mathbb{R}$.

Proof. The convergence of the sequence follows from Proposition 8.2.1, using the same argument as in the proof of Proposition 6.1.3. □

8.3. Asymptotic maximal slope

In this section, we let $f : X \to \operatorname{Spec} K$ be an integral projective K-scheme.

Definition 8.3.1. Let $\overline{L}$ be an adelic line bundle on X such that L is big. We define

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) := \lim_{n\to+\infty} \frac{\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n}))}{n}.$$

By definition, for any $p \in \mathbb{N}_{\geqslant 1}$, the following equality holds:

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes p}) = p\,\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}).$$

Proposition 8.3.2. *Let $\overline{L}$ and $\overline{M}$ be adelic line bundles on X such that L and M are both big. One has*

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L} \otimes \overline{M}) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{M}). \tag{8.3}$$

Proof. For any $n \in \mathbb{N}_{\geqslant 1}$, let $a_n = \dim_K(H^0(X, L^{\otimes n}))$ and $b_n = \dim_K(X, M^{\otimes n})$. One has

$$\ln(a_n) = O(\ln(n)), \quad \ln(b_n) = O(\ln(n)), \quad n \to +\infty.$$

By Proposition 8.2.1, for sufficiently large n, one has

$$\frac{\widehat{\mu}_{\max}(f_*((\overline{L}\otimes\overline{M})^{\otimes n}))}{n} \geqslant \frac{\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n}))}{n} + \frac{\widehat{\mu}_{\max}(f_*(\overline{M}^{\otimes n}))}{n} - \frac{3}{2}\nu(\Omega_\infty)\frac{\ln(a_n b_n)}{n}.$$

Taking the limit when $n \to +\infty$, we obtain the inequality (8.3). □

PROPOSITION 8.3.3. *Let $\overline{L}$ and $\overline{A}$ be adelic line bundle on X. We assume that L is pseudo-effective and A is big. Then the sequence*

$$\frac{1}{n}\,\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}),\quad n\in\mathbb{N}_{\geqslant 1}$$

converges in $\mathbb{R}\cup\{-\infty\}$. Moreover, its limit does not depend on the choice of $\overline{A}$. In particular, in the case where L is big, the following equality holds:

$$\lim_{n\rightarrow+\infty}\frac{1}{n}\,\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})=\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}).\tag{8.4}$$

PROOF. The proof relies on the super-additivity of the function $\widehat{\mu}_{\max}^{\mathrm{asy}}(\cdot)$ (see Proposition 8.3.2) and follows the same strategy as that of Proposition 6.4.1. We omit the details. □

DEFINITION 8.3.4. Let $\overline{L}$ be an adelic line bundle on X such that L is pseudo-effective. We define $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})$ as the limit

$$\lim_{n\rightarrow+\infty}\frac{1}{n}\,\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A}),$$

where $\overline{A}$ is an arbitrary adelic line bundle on X such that A is big. The element $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})$ of $\mathbb{R}\cup\{-\infty\}$ is called the *asymptotic maximal slope* of $\overline{L}$.

PROPOSITION 8.3.5. *Let $\overline{L}$ and $\overline{M}$ be adelic line bundles on X such that L and M are pseudo-effective. Then the following inequality holds:*

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}\otimes\overline{M})\geqslant\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})+\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{M}).$$

PROOF. Let $\overline{A}$ be an adelic line bundle on X such that A is big. For any $n\in\mathbb{N}$,

$$(L\otimes M)^{\otimes n}\otimes A^{\otimes 2}=(L^{\otimes n}\otimes A)\otimes(M^{\otimes n}\otimes A)$$

is big. Moreover, by Proposition 8.3.2, one has

$$\frac{1}{n}\,\widehat{\mu}_{\max}^{\mathrm{asy}}\big((\overline{L}\otimes\overline{M})^{\otimes n}\otimes\overline{A}^{\otimes 2}\big)\geqslant\frac{1}{n}\,\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n}\otimes\overline{A})+\frac{1}{n}\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{M}^{\otimes n}\otimes\overline{A}).$$

Taking the limit when $n\rightarrow+\infty$, we obtain

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}\otimes\overline{M})\geqslant\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})+\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{M}).$$

□

8.4. Pullback by a surjective projective morphism

Let X and Y be integral projective K-schemes and $g:Y\rightarrow X$ be a surjective projective morphism.

LEMMA 8.4.1. *Let L be an invertible $\mathcal{O}_X$-module. If L is pseudo-effective, then the pullback $g^*(L)$ is also pseudo-effective.*

PROOF. Let A be a big invertible $\mathcal{O}_X$-module and B be a big invertible $\mathcal{O}_Y$-module. For any positive integer p, the invertible $\mathcal{O}_X$-module $L^{\otimes p}\otimes A$ is big and hence $g^*(L^{\otimes p}\otimes A)$ is pseudo-effective since it has a tensor power which is effective. Similarly, $g^*(A)$ is also pseudo-effective. Thus we obtain that $g^*(A)\otimes B$ and $g^*(L)^{\otimes p}\otimes g^*(A)\otimes B$ are big. In particular, $g^*(L)$ is pseudo-effective. □

PROPOSITION 8.4.2. *Let $\overline{L}$ be an adelic line bundle on X such that L is pseudo-effective. Then the following inequality holds:*

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(g^*(\overline{L}))\geqslant\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}).$$

Proof. We have seen in Lemma 8.4.1 that the invertible O_X-module L is pseudo-effective, so that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ is well defined. We choose an adelic line bundle $\overline{A}$ on X such that A is big.

We first assume that L is big. Let n and p be positive integers. We consider the K-linear map

$$H^0(X, g_*(A^{\otimes n})) \otimes H^0(X, L^{\otimes np}) \longrightarrow H^0(X, g_*(A^{\otimes n}) \otimes L^{\otimes np}) = H^0(Y, A^{\otimes n} \otimes g^*(L^{\otimes np}))$$

induced by multiplication of sections. Let E be the destabilizing vector subspace of $(fg)_*(\overline{A}^{\otimes n})$ and let F be the destabilizing vector subspace of $f_*(\overline{L}^{\otimes np})$. By [**36**, Proposition 4.3.31], one has

$$\widehat{\mu}_{\min}(\overline{E} \otimes_{\varepsilon,\pi} \overline{F}) \leqslant \widehat{\mu}_{\max}((fg)_*(\overline{A}^{\otimes n} \otimes g^*(\overline{L}^{\otimes np}))).$$

by [**36**, Corollary 5.6.2 and Remark 4.3.48] (see also Remark A.3.3), one deduces

$$\begin{aligned}
&\widehat{\mu}_{\max}((fg)_*(\overline{A}^{\otimes n} \otimes g^*(\overline{L}^{\otimes np})))\\
&\geqslant \widehat{\mu}_{\min}(\overline{E}) + \widehat{\mu}_{\min}(\overline{F}) - \frac{3}{2}\nu(\Omega_\infty)(\ln(\dim_K(E)) + \ln(\dim_K(F)))\\
&\geqslant \widehat{\mu}_{\max}((fg)_*(\overline{A}^{\otimes n})) + \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes np})) - 2\nu(\Omega_\infty)\ln\dim_K(H^0(Y, A^{\otimes n}))\\
&\quad - 2\nu(\Omega_\infty)\ln\dim_K(H^0(X, L^{\otimes np})).
\end{aligned}$$

If we divide the two sides by np, taking the limit when $n \to +\infty$, we obtain

$$\frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{A} \otimes g^*(\overline{L}^{\otimes p})) \geqslant \frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{A}) + \widehat{\mu}_{\max}(\overline{L}).$$

Taking the limit when $p \to +\infty$, we obtain $\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$, as required.

We then consider the general case where L is only assumed to be pseudo-effective. Let $\overline{B}$ be an adelic line bundle on X such that B is big. Note that, for any positive integer p, $L^{\otimes p} \otimes B$ is big. Hence, by the particular case of the proposition shown above, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})^{\otimes p} \otimes g^*(\overline{B})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes p} \otimes B).$$

Therefore, by Proposition 8.3.5, we obtain

$$\frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})^{\otimes p} \otimes g^*(\overline{B}) \otimes \overline{A}) \geqslant \frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes p} \otimes \overline{B}) + \frac{1}{p}\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{A}).$$

Taking the limit when $p \to +\infty$, we obtain $\widehat{\mu}^{\mathrm{asy}}_{\max}(g^*(\overline{L})) \geqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$. □

Remark 8.4.3. Let $\overline{L}$ be an adelic line bundle on X. Assume that L is the pull-back of a big line bundle by a surjective projective morphism. Then Proposition 8.4.2 shows that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) \in \mathbb{R}$.

8.5. Relative Fujita approximation

Let $f : X \to \operatorname{Spec} K$ be an integral projective K-scheme, $K(X)$ be the field of rational functions on X, and $\mathscr{M}_X$ be the sheaf of meromorphic functions on X.

Definition 8.5.1. Let L be a big line bundle on X. Note that $\mathscr{M}_X \otimes_{O_X} L$ is isomorphic to the trivial invertible $\mathscr{M}_X$-module. In particular, if s and t are two global sections of L such that $s \neq 0$, then there exists a unique rational function $\lambda \in K(X)$ such that $t = \lambda s$. We denote by t/s this rational function. If E is a K-vector subspace of $H^0(X, L)$. We denote by $K(E)$ the sub-extension of $K(X)/K$ generated by elements of the form t/s, where t and s are non-zero sections in $K(E)$. We say that E is *birational* if $K(E) = K(X)$. Moreover L is said to be *birational* if $K(H^0(X, L)) = K(X)$.

Remark 8.5.2. Let L and M be line bundles on X, E be a vector subspace of $H^0(X, L)$, s be a non-zero global section of M and

$$F = \{ts \mid t \in E\} \subseteq H^0(X, L \otimes M).$$

Then by definition one has $K(F) = K(E)$. In particular, if E is birational, so is F; if L is birational, so is $L \otimes M$.

Proposition 8.5.3. *Let L be a big line bundle on X. For sufficiently positive integer p, the line bundle $L^{\otimes p}$ is birational.*

Proof. Since L is big, there exist a positive integer q, an ample line bundle A and an effective line bundle M on X such that $L^{\otimes q} \cong A \otimes M$. By replacing q by a multiple, we may assume that the graded K-algebra

$$\bigoplus_{n \in \mathbb{N}} H^0(X, A^{\otimes n})$$

is generated by $H^0(X, A)$ and that $L^{\otimes (q+1)}$ is effective. For any $a \in \mathbb{N}_{\geqslant 1}$, one has

$$X = \mathrm{Proj}\Big(\bigoplus_{n \in \mathbb{N}} H^0(X, A^{\otimes an})\Big),$$

which implies that $A^{\otimes a}$ is birational and hence $L^{\otimes aq}$ is birational. Moreover, since $L^{\otimes (q+1)}$ is effective, for any $b \in \mathbb{N}_{\geqslant 1}$, the line bundle $L^{\otimes b(q+1)}$ is also effective. Therefore, for any $(a, b) \in \mathbb{N}_{\geqslant 1}^2$, the line bundle $L^{\otimes aq + b(q+1)}$ is birational. Since q and $q + 1$ are coprime, we obtain that $L^{\otimes p}$ is birational for sufficiently large $p \in \mathbb{N}$. □

Definition 8.5.4. Let $\overline{L} = (L, \varphi)$ be an adelic line bundle on X. If $s \in H^0(X, L)$ is a non-zero global section such that $\|s\|_{\varphi_\omega} \leqslant 1$ for any $\omega \in \Omega$, we say that the global section s is *effective*. We say that $\overline{L}$ is *effective* if it admits at least an effective global section.

Lemma 8.5.5. *Let $\overline{L}$ be an adelic line bundle such that L is big. For any $t < \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})$ and any $N \in \mathbb{N}_{\geqslant 1}$, there exists an integer $p \geqslant N$ and a vector subspace E of $H^0(X, L^{\otimes p})$ such that $K(E) = K(X)$ and $\widehat{\mu}_{\min}(\overline{E}) > pt$.*

Proof. By replacing $\overline{L}$ by one of its tensor powers, we may assume without loss of generality that L is birational. For any $n \in \mathbb{N}$, let $r_n = \dim_K(H^0(X, L^{\otimes n}))$. Since $t < \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})$, for sufficiently large $n \in \mathbb{N}$, one has

$$\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n})) > (n+1)t - \widehat{\mu}_{\min}(f_*(\overline{L})) + \frac{3}{2}\nu(\Omega_\infty) \ln\big(r_n \cdot r_1\big).$$

Let F be a vector subspace of $H^0(X, L^{\otimes n})$ such that

$$\widehat{\mu}_{\min}(\overline{F}) = \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n})).$$

The existence of F is ensured by [**36**, Theorem 4.3.58]. Let E be the image of $F \otimes_K H^0(X, L)$ by the K-linear map

$$H^0(X, L^{\otimes n}) \otimes H^0(X, L) \longrightarrow H^0(X, L^{\otimes n+1}), \quad s \otimes t \longmapsto st.$$

Since L is birational and F is non-zero, we obtain that E is birational. By [**36**, Corollary 5.6.2], one has

$$\begin{aligned}&\widehat{\mu}_{\min}(\overline{E}) \geqslant \widehat{\mu}_{\min}(\overline{F} \otimes_{\varepsilon,\pi} f_*(\overline{L}))\\ \geqslant\ &\widehat{\mu}_{\min}(\overline{F}) + \widehat{\mu}_{\min}(f_*(\overline{L})) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(F)\cdot r_1)\\ =\ &\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n})) + \widehat{\mu}_{\min}(f_*(\overline{L})) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(F)\cdot r_1) > (n+1)t.\end{aligned}$$

□

THEOREM 8.5.6 (Relative Fujita approximation). *Let $\overline{L}$ be an adelic line bundle on X such that L is big. For any real number $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$, there exist a positive integer p, a birational projective K-morphism $g : X' \to X$, a relatively nef adelic line bundle $\overline{A}$ and an effective adelic line bundle $\overline{M}$ on X' such that A is big, $g^*(\overline{L}^{\otimes p})$ is isomorphic to $\overline{A} \otimes \overline{M}$ and $\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{A}) \geqslant pt$.*

PROOF. Let p be a positive integer p and V be a birational vector subspace of $H^0(X, L^{\otimes p})$ such that

$$\widehat{\mu}_{\min}(\overline{V}) = \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes p})) \geqslant pt + \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(H^0(X, L^{\otimes p}))).$$

Let $g : X' \to X$ be the blow-up of L along the base locus of V, namely

$$X' = \mathrm{Proj}\left(\mathrm{Im}\left(\bigoplus_{n\in\mathbb{N}} S^n(f^*(V)) \longrightarrow \bigoplus_{n\in\mathbb{N}} L^{\otimes np}\right)\right).$$

Denote by E the exceptional divisor and by s_E the global section of $\mathcal{O}_X(E)$ which trivializes $\mathcal{O}_X(E)$ outside of the exceptional divisor. One has

$$\mathcal{O}_{X'}(1) \cong g^*(L^{\otimes p}) \otimes \mathcal{O}_X(-E).$$

Moreover, the canonical surjective homomorphism

$$g^*(f^*(V)) \longrightarrow \mathcal{O}_{X'}(1) \tag{8.5}$$

induces a K-morphism $i : X' \to \mathbb{P}(V)$ such that $i^*(\mathcal{O}_V(1)) = \mathcal{O}_{X'}(1)$, where $\mathcal{O}_V(1)$ denotes the universal invertible sheaf on $\mathbb{P}(V)$. Since V is birational, the line bundle $\mathcal{O}_{X'}(1)$ is big.

We equip V with the induced norm family of $(\|\cdot\|_{p\varphi_\omega})_{\omega\in\Omega}$ and $\mathcal{O}_{X'}(1)$ with the quotient metric family $\varphi' = (\varphi'_\omega)_{\omega\in\Omega}$ induced by $(\|\cdot\|_{p\varphi_\omega})_{\omega\in\Omega}$ and the surjective homomorphism (8.5). We identify $\mathcal{O}_X(E)$ with $g^*(L^{\otimes p}) \otimes \mathcal{O}_{X'}(1)^\vee$ and equip it with the tensor product metric family. Then the section s_E is effective. Moreover, by Proposition 6.6.4, the adelic line bundle $\overline{\mathcal{O}_{X'}(1)}$ is relatively nef, and the following inequality holds

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{\mathcal{O}_{X'}(1)}) \geqslant \widehat{\mu}_{\min}(\overline{V}) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(V)) \geqslant pt,$$

as required. □

REMARK 8.5.7. Let $\overline{L}$ be an adelic line bundle on X such that L is big. Let $\overline{B}$ be a relatively ample adelic line bundle. There exists a positive integer N such that $L^{\otimes m} \otimes B^\vee$ is big for any $m \in \mathbb{N}_{\geqslant N}$. Let t be a real number such that $t < \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$. There exists $m \in \mathbb{N}_{\geqslant N}$ such that

$$mt - \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{B}) < (m-N)\,\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}) + \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes N} \otimes \overline{B}^\vee) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}^{\otimes m} \otimes \overline{B}^\vee),$$

where the second inequality comes from Proposition 8.3.5. If we apply Theorem 8.5.6 to $\overline{L}^{\otimes m} \otimes \overline{B}^{\vee}$, we obtain the existence of a positive integer p, a birational projective K-morphism $g : X' \to X$, a relatively nef adelic line bundle $\overline{A}$ and an effective adelic line bundle $\overline{M}$ on X' such that A is big, $g^*(\overline{L}^{\otimes mp} \otimes \overline{B}^{\vee\otimes p})$ is isomorphic to $\overline{A} \otimes \overline{M}$ and

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) \geqslant p(mt - \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{B})). \tag{8.6}$$

Let $\overline{N} = \overline{A} \otimes g^*(\overline{B})^{\otimes p}$. This is a relatively ample line bundle, and one has

$$\overline{N} \otimes \overline{M} \cong \overline{A} \otimes \overline{M} \otimes g^*(\overline{B})^{\otimes p} \cong g^*(\overline{L}^{\otimes mp}).$$

Moreover, one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{N}) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) + p\, \widehat{\mu}_{\min}^{\mathrm{asy}}(g^*(\overline{B})) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) + p\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{B}),$$

where the first inequality comes from Proposition 6.4.4, and the second comes from Theorem 6.6.6. By (8.6), we obtain

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{N}) \geqslant pmt.$$

Therefore, in Theorem 8.5.6, the adelic line bundle $\overline{A}$ can be taken to be relatively ample.

8.6. Lower bound of intersection product

THEOREM 8.6.1. *Let X be an integral projective K-scheme, and $\overline{L}_0, \ldots, \overline{L}_d$ be adelic line bundles on X. For any $i \in \{0, \ldots, d\}$, let*

$$\delta_i = (L_0 \cdots L_{i-1} \cdot L_{i+1} \cdots L_d).$$

Suppose that

(1) *$\overline{L}_1, \ldots, \overline{L}_d$ are relatively nef and L_0 is pseudo-effective.*
(2) *if $\delta_0 = 0$, then $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_0) > -\infty$,*
(3) *for any $i \in \{1, \ldots, d\}$, if $\delta_i = 0$, then $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i) > -\infty$.*

Then the following inequality holds:

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_0) + \sum_{i=1}^{d} \delta_i\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i). \tag{8.7}$$

PROOF. If the set

$$\{\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_0), \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_1), \ldots, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_d)\}$$

contains $-\infty$, then the inequality (8.7) is trivial. So we may assume without loss of generality that

$$\{\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_0), \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_1), \ldots, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_d)\} \subseteq \mathbb{R}.$$

Let $\overline{M}$ be an adelic line bundle on X such that M is big. For any $n \in \mathbb{N}_{\geqslant 1}$, let

$$\overline{L}_{0,n} = \overline{L}_0^{\otimes n} \otimes \overline{M}.$$

For any $i \in \{1, \ldots, n\}$, let

$$\delta'_i = (ML_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_d)$$
$$\delta_{i,n} = (L_{0,n}L_1 \cdots L_{i-1} \cdot L_{i+1} \cdots L_d) = n\delta_i + \delta'_{i,n}.$$

By Theorem 8.5.6 (see also Remark 8.5.7), for any real number $t < \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_{0,n})$, there exists a positive integer p, a birational projective morphism $g : X' \to X$, a relatively ample adelic line bundle $\overline{A}$ and an effective adelic line bundle E on X' such that

$$g^*(\overline{L}_{0,n}^{\otimes p}) = \overline{A} \otimes \overline{E}, \quad \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) \geqslant pt.$$

By Theorem 6.6.6 for any $i \in \{1, \ldots, d\}$, one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(g^*(\overline{L}_i)) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i).$$

Therefore, by Proposition 8.1.3 and Proposition 6.4.8, we obtain

$$(\overline{E} \cdot g^*(\overline{L}_1) \cdots g^*(\overline{L}_d))_S \geqslant \sum_{i=1}^{d} (E \cdot L_1 \cdots L_{i-1} \cdot L_i \cdots L_d)\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i),$$

$$(\overline{A} \cdot g^*(\overline{L}_1) \cdots g^*(\overline{L}_d))_S \geqslant \delta_0\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) + \sum_{i=1}^{d} (A \cdot L_1 \cdots L_{i-1} \cdot L_i \cdots L_d)\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i).$$

Taking the sum, we obtain

$$(\overline{L}_{0,n}^{\otimes p} \cdot \overline{L}_1 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) + \sum_{i=1}^{d} p\delta_{i,n}\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i) \geqslant \delta_0 pt + \sum_{i=1}^{d} p\delta_{i,n}\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i).$$

Since t is arbitrary, we deduce

$$(\overline{L}_{0,n}^{\otimes p} \cdot \overline{L}_1 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_{0,n}) + \sum_{i=1}^{d} \delta_{i,n}\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i).$$

Dividing the two sides by n and then taking the limit when $n \to +\infty$, we obtain

$$(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant \delta_0\, \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_0) + \sum_{i=1}^{d} \delta_i\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}_i).$$

□

8.7. Convergence of the first minimum

In this section, we let $f : X \to \operatorname{Spec} K$ be an integral projective scheme over $\operatorname{Spec} K$.

Definition 8.7.1. Let $\overline{E} = (E, (\|\cdot\|_\omega)_{\omega \in \Omega})$ be an adelic vector bundle on S. For any non-zero element s in E, let

$$\widehat{\deg}(s) := -\int_{\Omega} \ln \|s\|_\omega \, \nu(\mathrm{d}\omega).$$

If E is non zero, we define

$$\lambda_{\max}(\overline{E}) := \sup_{s \in E \setminus \{0\}} \widehat{\deg}(s).$$

Clearly one has

$$\lambda_{\max}(\overline{E}) \leqslant \widehat{\mu}_{\max}(\overline{E}). \tag{8.8}$$

Proposition 8.7.2. *Let $\overline{L} = (L, \varphi)$ and $\overline{M} = (M, \psi)$ be adelic line bundles on X such that both $H^0(X, L)$ and $H^0(X, M)$ are non-zero. Then the following inequality holds:*

$$\lambda_{\max}(f_*(\overline{L} \otimes \overline{M})) \geqslant \lambda_{\max}(f_*(\overline{L})) + \lambda_{\max}(f_*(\overline{M})).$$

Proof. Let s and t be respectively non-zero elements of $H^0(X, L)$ and $H^0(X, M)$. For any $\omega \in \Omega$, one has

$$\|st\|_{\varphi_\omega + \psi_\omega} \leqslant \|s\|_{\varphi_\omega} \cdot \|t\|_{\psi_\omega},$$

which leads to

$$\lambda_{\max}(f_*(\overline{L} \otimes \overline{M})) \geqslant \widehat{\deg}(st) \geqslant \widehat{\deg}(s) + \widehat{\deg}(t).$$

Taking the supremum with respect to s and t, we obtain the required inequality. □

Let $\overline{L}$ be an adelic line bundle on X such that L is big. Similarly to Corollary 8.2.2, the sequence

$$\frac{1}{n}\lambda_{\max}(f_*(\overline{L}^{\otimes n})), \quad n \in \mathbb{N}_{\geqslant 1}$$

converges to a real number, which we denote by $\lambda_{\max}^{\mathrm{asy}}(\overline{L})$ and called the *asymptotic first minimum* of $\overline{L}$. By definition, for any $p \in \mathbb{N}_{\geqslant 1}$ one has

$$\lambda_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes p}) = p\lambda_{\max}^{\mathrm{asy}}(\overline{L}).$$

Proposition 8.7.2 also implies that, if $\overline{L}$ and $\overline{M}$ are adelic line bundles on X such that both L and M are big, one has

$$\lambda_{\max}^{\mathrm{asy}}(\overline{L} \otimes \overline{M}) \geqslant \lambda_{\max}^{\mathrm{asy}}(\overline{L}) + \lambda_{\max}^{\mathrm{asy}}(\overline{M}). \tag{8.9}$$

Similarly to Proposition 8.3.3, this inequality allows to extend continuously the function $\lambda_{\max}^{\mathrm{asy}}(\cdot)$ to the cone of adelic line bundles $\overline{L}$ such that L is pseudo-effective: if $\overline{L}$ is an adelic line bundle on X such that L is pseudo-effective, then, for any adelic line bundle $\overline{A}$ on X, the sequence

$$\frac{1}{n}\lambda_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{A}), \quad n \in \mathbb{N}_{\geqslant 1} \tag{8.10}$$

converges in $\mathbb{R} \cup \{-\infty\}$ and its limit does not depend on the choice of $\overline{A}$. For the proof of this statement one can following the strategy of the proof of Proposition 6.4.1 in using the inequality 8.9 and the fact that, if A is a big line bundle and B is a line bundle on X, then there exists a positive integer p such that $B^\vee \otimes A^{\otimes p}$ is big. We denote the limit of the sequence (8.10) by $\lambda_{\max}^{\mathrm{asy}}(\overline{L})$. By (8.8) we obtain that

$$\lambda_{\max}^{\mathrm{asy}}(\overline{L}) \leqslant \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) \tag{8.11}$$

for any adelic line bundle $\overline{L}$ such that L is pseudo-effective.

8.8. Height inequalities

Proposition 8.8.1. *Let $f : X \to \operatorname{Spec} K$ be an integral projective scheme over* Spec K *and $\overline{L}$ be an adelic line bundle on X which is relatively nef and such that $(L^d) > 0$. Then the following inequality holds:*

$$\widehat{\mu}^{\mathrm{asy}}(\overline{L}) = \frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)} \leqslant \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}). \tag{8.12}$$

Proof. We first consider the case where L is relatively ample. As in the proof of Proposition 6.7.1, one has

$$\frac{(\overline{L}^{d+1})_S}{(d+1)(L^d)} = \lim_{n\to+\infty}\frac{\widehat{\mu}(f_*(\overline{L}^{\otimes n}))}{n} \leqslant \lim_{n\to+\infty}\frac{\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes n}))}{n} = \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}).$$

We now consider the general case. Let $\overline{A}$ be a relatively ample adelic line bundle on X such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample for sufficiently large positive integer n. For any $n \in \mathbb{N}_{\geqslant 1}$, let $\overline{L}_n = \overline{L}^{\otimes n} \otimes \overline{A}$. The particular case of the proposition proved above shows that

$$\frac{(\overline{L}_n^{d+1})_S}{(d+1)(L_n^d)} \leqslant \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}_n)$$

if n is sufficiently large. Taking the limit when $n \to +\infty$, by the relations

$$\lim_{n\to+\infty}\frac{(\overline{L}_n^{d+1})_S}{n^{d+1}} = (\overline{L}^{d+1})_S, \quad \lim_{n\to+\infty}\frac{(L_n^d)}{n^d} = (L^d)$$

and Proposition 8.3.3 we obtain the desired result. □

REMARK 8.8.2. Combining Propositions 8.8.1 and 6.7.1, we obtain that, if $\overline{L}$ is relatively nef and if $(L^d) > 0$, then the inequality $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \leqslant \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L})$ is true. This inequality also holds for relatively nef adelic line bundle $\overline{L}$ with $(L^d) = 0$. It suffices to choose an auxiliary relatively ample adelic line bundle $\overline{M}$ and deduce the inequality from

$$\frac{1}{n}\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) \leqslant \frac{1}{n}\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M})$$

by taking the limit when $n \to +\infty$.

THEOREM 8.8.3. *Let X be a non-empty and reduced projective K-scheme and Θ_X be the set of all integral closed subschemes of X. Let $\overline{L} = (L, \varphi)$ be a relatively ample adelic line bundle on X. Then the following equality holds:*

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) = \inf_{Y \in \Theta_X} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y) = \inf_{Y \in \Theta_X} \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y) + 1)\deg_L(Y)}. \tag{8.13}$$

PROOF. For any $Y \in \Theta_X$ and any $n \in \mathbb{N}$, let $V_{Y,n}(L)$ be the image of the restriction map

$$H^0(X, L^{\otimes n}) \longrightarrow H^0(Y, L|_Y^{\otimes n}).$$

We equip $V_{Y,n}(L)$ with the quotient norm family $\xi_n^Y = (\|\cdot\|_{n\varphi_\omega,\mathrm{quot}}^Y)_{\omega\in\Omega}$ induced by $\xi_{n\varphi} = (\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega}$ to obtain adelic vector bundle on S.

CLAIM 8.8.4. *For any $Y \in \Theta_X$, the following equality holds*

$$\lim_{n\to+\infty} \frac{\widehat{\mu}_{\max}(V_{Y,n}(L), \xi_n^Y)}{n} = \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y).$$

PROOF. Since L is ample, there exists $N \in \mathbb{N}_{\geqslant 1}$ such that, for any $n \in \mathbb{N}_{\geqslant N}$, one has $V_{Y,n}(L) = H^0(Y, L|_Y^{\otimes n})$. We denote by φ^Y the restriction of the metric family φ to $L|_Y$. By definition, for any $n \in \mathbb{N}_{\geqslant N}$, one has

$$\|\cdot\|_{n\varphi_\omega,\mathrm{quot}}^Y \geqslant \|\cdot\|_{n\varphi_\omega^Y}.$$

Moreover, by [**44**, Theorem 1.3] (for the trivial valuation case we apply the method of [**35**, Theorem 4.1] to reduce to non-trivial valuation case), one has

$$\lim_{n\to+\infty} \frac{1}{n} d_\omega(\xi_n^Y, \xi_{n\varphi^Y}) = 0,$$

where $\xi_{n\varphi^Y} = (\|\cdot\|_{n\varphi_\omega^Y})_{\omega\in\Omega}$.

By [**36**, Proposition 2.2.22 (5)], the function

$$(\omega \in \Omega) \longmapsto \frac{1}{n} d_\omega(\xi_n^Y, \xi_{n\varphi^Y})$$

is dominated. Therefore, by Lebesgue's dominated convergence theorem we obtain

$$\lim_{n\to+\infty} \frac{1}{n} d(\xi_n^Y, \xi_{n\varphi^Y}) = \lim_{n\to+\infty} \frac{1}{n} \int_\Omega d_\omega(\xi_n^Y, \xi_{n\varphi^Y})\, \nu(\mathrm{d}\omega). \tag{8.14}$$

Finally, by [**36**, Proposition 4.3.31], one has

$$\left|\frac{\widehat{\mu}_{\max}(V_{Y,n}(L), \xi_n^Y)}{n} - \frac{\widehat{\mu}_{\max}(V_{Y,n}(L), \xi_{n\varphi^Y})}{n}\right| \leqslant \frac{1}{n} d(\xi_n^Y, \xi_{n\varphi^Y}).$$

Passing to limit when $n \to +\infty$, we obtain the desired equality. □

Combining Claim 8.8.4 with [**36**, Theorem 7.2.4], we obtain

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \geqslant \inf_{Y\in\Theta_X} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y).$$

By Proposition 8.8.1, for any $Y \in \Theta_X$, one has

$$\frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)} \leqslant \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y).$$

Finally, by Corollary 6.7.2, for any $Y \in \Theta_X$, one has

$$\frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)} \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}).$$

Thus (8.13) is proved. □

Corollary 8.8.5. *Let $\overline{L}$ be a relatively ample adelic line bundle on X. If we assume that $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) > \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L})$, then one has*

$$\begin{aligned}\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) &= \inf_{Y\in\Theta_X\setminus\{X\}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y)\\ &= \inf_{Y\in\Theta_X\setminus\{X\}} \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)} = \inf_{Y\in\Theta_X\setminus\{X\}} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_Y).\end{aligned}$$

Proof. The first equality comes from (8.13) and the hypothesis $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) > \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L})$. By Propositions 8.8.1 and 6.7.1, we obtain

$$\inf_{Y\in\Theta_X\setminus\{X\}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y) \geqslant \inf_{Y\in\Theta_X\setminus\{X\}} \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)} \geqslant \inf_{Y\in\Theta_X\setminus\{X\}} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_Y).$$

Moreover, by Proposition 6.6.2, for any $Y \in \Theta_X$ one has $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_Y) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L})$. The assertion is thus proved. □

Proposition 8.8.6. *Let X be an integral projective K-scheme and Θ_X be the set of all integral closed subschemes of X. Let $\overline{L} = (L, \varphi)$ be a relatively ample adelic line bundle on X. Then the following inequality holds:*

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{1}{d+1}\lambda_{\max}^{\mathrm{asy}}(\overline{L}) + \frac{d}{d+1}\inf_{Y\in\Theta_X\setminus\{X\}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y). \tag{8.15}$$

In particular, if

$$\lambda_{\max}^{\mathrm{asy}}(\overline{L}) \geqslant \frac{(\overline{L}^{(d+1)})}{(d+1)(L^d)},$$

then the inequality

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \inf_{Y\in\Theta_X\setminus\{X\}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y) \tag{8.16}$$

holds.

Proof. The case where $d = 0$ is trivial. In the following, we suppose that $d \geqslant 1$. By replacing $\overline{L}$ by a tensor power, we may assume that

$$V_\bullet(L) = \bigoplus_{n\in\mathbb{N}} H^0(X, L^{\otimes n})$$

is generated as K-algebra by $V_1(L)$. For any $n \in \mathbb{N}$, we let $h^0(L^{\otimes n})$ be the dimension of $H^0(X, L^{\otimes n})$ over K. Let s be a non-zero global section of L and $I_\bullet$ be the homogeneous ideal of $V_\bullet(L)$ generated by s. Then one can find a sequence

$$I_\bullet = I_{0,\bullet} \subsetneq I_{1,\bullet} \subsetneq \ldots \subsetneq I_{r,\bullet} = V_\bullet(L)$$

of homogeneous ideals of $R_\bullet$ and non-zero homogeneous prime ideals $P_{i,\bullet}$, $i \in \{1, \ldots, r\}$, of $V_\bullet(L)$ such that

$$\forall\, i \in \{1, \ldots, r\}, \quad P_{i,\bullet} \cdot I_{i,\bullet} \subset I_{i-1,\bullet}.$$

We denote by Y_i the integral closed subscheme of X defined by the homogeneous ideal $P_{i,\bullet}$.

Consider the following sequence

$$\begin{array}{ccccccccc}
V_0(L) & \xrightarrow{\cdot s} & I_{0,1} & \hookrightarrow \cdots \hookrightarrow & I_{i,1} & \hookrightarrow \cdots \hookrightarrow & I_{r,1} = V_1(L) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \xrightarrow{\cdot s} & I_{0,j} & \hookrightarrow \cdots \hookrightarrow & I_{i,j} & \hookrightarrow \cdots \hookrightarrow & I_{r,j} = V_j(L) \\
 & \xrightarrow{\cdot s} & I_{0,j+1} & \hookrightarrow \cdots \hookrightarrow & I_{i,j+1} & \hookrightarrow \cdots \hookrightarrow & I_{r,j+1} = V_{j+1}(L) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \xrightarrow{\cdot s} & I_{0,n} & \hookrightarrow \cdots \hookrightarrow & I_{i,n} & \hookrightarrow \cdots \hookrightarrow & I_{r,n} = V_n(L)
\end{array}$$

By [**36**, Proposition 4.3.13], one has

$$\widehat{\deg}(f_*(\overline{L}^{\otimes n})) \geqslant \sum_{j=1}^{n} \sum_{i=1}^{r} \widehat{\deg}(\overline{I_{i,j}/I_{i-1,j}}) + \widehat{\deg}(s) \sum_{k=0}^{n-1} h^0(L^{\otimes k}). \tag{8.17}$$

By [**36**, Proposition 7.1.6] and (8.14), one has

$$\liminf_{m \to +\infty} \frac{\widehat{\mu}_{\min}(\overline{I_{i,m}/I_{i-1,m}})}{m} \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_{Y_i}). \tag{8.18}$$

Moreover, by the asymptotic Riemann-Roch formula, one has

$$h^0(L^{\otimes k}) = \frac{(L^d)}{d!} k^d + O(k^{d-1}),$$

which leads to

$$\lim_{n \to +\infty} \frac{1}{n h^0(L^{\otimes n})} \sum_{j=0}^{n-1} h^0(L^{\otimes j}) = \frac{1}{d+1}.$$

For any integers n and m such that $1 \leqslant m \leqslant n$, we deduce from (8.17) that

$$\begin{aligned}
\widehat{\deg}(f_*(\overline{L}^{\otimes n})) \geqslant & \sum_{j=1}^{m} \sum_{i=1}^{r} \widehat{\deg}(\overline{I_{i,j}/I_{i-1,j}}) \\
& + \min_{i \in \{1,\ldots,r\}} \inf_{\ell \in \mathbb{N}_{\geqslant m}} \frac{\widehat{\mu}_{\min}(\overline{I_{i,\ell}/I_{i-1,\ell}})}{\ell} \sum_{j=m+1}^{n} j(h^0(L^{\otimes j}) - h^0(L^{\otimes (j-1)})) \\
& + \widehat{\deg}(s) \sum_{k=0}^{n-1} h^0(L^{\otimes k}).
\end{aligned}$$

Dividing the two sides by $nh^0(L^{\otimes n})$ and taking the limit when $n \to +\infty$, we obtain

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{d}{d+1} \min_{i \in \{1,\ldots,r\}} \inf_{\ell \in \mathbb{N}_{\geqslant m}} \frac{\widehat{\mu}_{\min}(\overline{I_{i,\ell}/I_{i-1,\ell}})}{\ell} + \frac{1}{d+1} \widehat{\deg}(s).$$

Since m is arbitrary, taking the limit when $m \to +\infty$, by (8.18) we obtain

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{d}{d+1} \min_{i\in\{1,\ldots,r\}} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_{Y_i}) + \frac{1}{d+1}\widehat{\deg}(s).$$

By Theorem 8.8.3, for any $i \in \{1,\ldots,r\}$, one has

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}|_{Y_i}) = \inf_{Z\in\Theta_{Y_i}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Z) \geqslant \inf_{Z\in\Theta_X\setminus\{X\}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Z).$$

Since s is arbitrary, we obtain

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{1}{d+1}\widehat{\lambda}_{\max}(f_*(\overline{L})) + \frac{d}{d+1} \inf_{\substack{Y\in\Theta_X\\ Y\neq X}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y).$$

Finally, replacing $\overline{L}$ by $\overline{L}^{\otimes p}$ for $p \in \mathbb{N}_{\geqslant 1}$, we obtain

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \frac{1}{p(d+1)}\widehat{\lambda}_{\max}(f_*(\overline{L}^{\otimes p})) + \frac{d}{d+1} \inf_{\substack{Y\in\Theta_X\\ Y\neq X}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y).$$

Taking the limit when $p \to +\infty$, we obtain the inequality (8.15). □

Proposition 8.8.7. *Let X be an integral projective scheme over* Spec K *and $\overline{L}$ be an adelic line bundle on X such that L is big. Then the following inequality holds:*

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) \leqslant \sup_{\substack{Y\in\Theta_X\\ Y\neq X}} \inf_{x\in(X\setminus Y)^{(0)}} h_{\overline{L}}(x),$$

where $(X\setminus Y)^{(0)}$ denotes the set of closed points of $X\setminus Y$, and

$$h_{\overline{L}}(x) = \frac{(\overline{L}|_x)_S}{[K(x):K]} = \widehat{\deg}(x^*(\overline{L})).$$

Proof. See [**36**, Proposition 6.4.4]. □

8.9. Minkowskian adelic line bundles

Definition 8.9.1. Let X be a reduced projective K-scheme and $\overline{L}$ be an adelic line bundle on X. We say that $\overline{L}$ is *Minkowskian* if the inequality below holds:

$$\lambda_{\max}^{\mathrm{asy}}(\overline{L}) \geqslant \widehat{\mu}^{\mathrm{asy}}(\overline{L}) = \frac{(\overline{L}^{\dim(X)+1})_S}{(\dim(X)+1)\deg_L(X)}.$$

Moreover, $\overline{L}$ is said to be *strongly Minkowskian* if for any integral closed sub-scheme Y of X, the restricted adelic line bundle $\overline{L}|_Y$ is Minkowskian. Note that the strongly Minkowskian condition is satisfied in the following cases:

(1) S is the adelic curve associated with a number field, and the metrics of $\overline{L}$ over non-Archimedean places are almost everywhere induced by a common integral model defined over the ring of algebraic integers in the number field;
(2) S is the adelic curve associated with a regular projective curve over a field, and the metrics of $\overline{L}$ are induced by an integral model of L over the base curve;
(3) S is the adelic curve of a single copy of the trivial absolute value.

The case (1) comes from the classic Minkowski theory of Euclidean lattices. The case (2) is a consequence of Riemann-Roch theorem on curves. The case (3) follows from [**36**, Remark 4.3.63].

Corollary 8.9.2. *Let X be an integral projective K-scheme and $\overline{L}$ be a relatively ample adelic line bundle on X. Assume that $\overline{L}$ is strongly Minkowskian. Then the following inequality holds:*

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \inf_{x\in X^{(0)}} h_{\overline{L}}(x), \tag{8.19}$$

where $X^{(0)}$ denotes the set of closed points of X. Moreover, one has

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) = \inf_{x\in X^{(0)}} h_{\overline{L}}(x). \tag{8.20}$$

Proof. We reason by induction on the dimension d of X. The case where $d = 0$ is trivial. Assume that $d \geqslant 1$ and that the result is true for any integral projective K-scheme of dimension $< d$. By Proposition 8.8.6 one has

$$\frac{(\overline{L}^{(d+1)})_S}{(d+1)(L^d)} \geqslant \inf_{Y\in\Theta_X\setminus\{X\}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Y) \geqslant \inf_{Y\in\Theta_X\setminus\{X\}} \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)},$$

where the second inequality comes from Proposition 8.8.1. For any $Y \in \Theta_X$ such that $Y \neq X$, one has $\dim(Y) < \dim(X)$. Hence the induction hypothesis leads to

$$\frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)} \geqslant \inf_{x\in Y^{(0)}} h_{\overline{L}}(x) \geqslant \inf_{x\in X^{(0)}} h_{\overline{L}}(x). \tag{8.21}$$

The inequality (8.19) is thus proved.

By Corollary 6.7.2, the inequality

$$\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) \leqslant \inf_{x\in X^{(0)}} h_{\overline{L}}(x)$$

holds. Conversely, by Theorem 8.8.3 and the inequality (8.21), one has

$$\begin{aligned}\widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}) &= \inf_{Y\in\Theta_X} \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)\deg_L(Y)}\\ &\geqslant \inf_{Y\in\Theta_X}\inf_{x\in Y^{(0)}} h_{\overline{L}}(x) = \inf_{x\in X^{(0)}} h_{\overline{L}}(x).\end{aligned}$$

□

Lemma 8.9.3. *Let $\pi : X \to Y$ be a generically finite and surjective morphism of d-dimensional projective integral schemes over K. Let $\overline{M}$ be a relatively ample adelic line bundle on Y. Then we have the following:*

(1) $\widehat{\mu}^{\mathrm{asy}}(\pi^*(\overline{M})) = \widehat{\mu}^{\mathrm{asy}}(\overline{M})$.
(2) $\lambda^{\mathrm{asy}}_{\max}(\pi^*(\overline{M})) \geqslant \lambda^{\mathrm{asy}}_{\max}(\overline{M})$.
(3) *If $\overline{M}$ is Minkowskian, then $\pi^*(\overline{M})$ is also Minkowskian.*

Proof. (1) By the Hilbert-Samual formula,

$$\widehat{\mu}^{\mathrm{asy}}(\overline{M}) = \frac{(\overline{M}^{d+1})_S}{(M^d)(\dim Y+1)} \quad\text{and}\quad \widehat{\mu}^{\mathrm{asy}}(\pi^*(\overline{M})) = \frac{(\pi^*(\overline{M})^{d+1})_S}{(\pi^*(M)^d)(\dim X+1)},$$

and hence the assertion follows because

$$(\pi^*(\overline{M})^{d+1})_S = (\deg\pi)(\overline{M}^{d+1})_S \quad\text{and}\quad (\pi^*(M)^d) = (\deg\pi)(M^d).$$

(2) is obvious because $\widehat{\deg}(s) = \widehat{\deg}(\pi^*(s))$ for $s \in H^0(Y, M)\setminus\{0\}$. Moreover, (3) is a consequence (1) and (2). □

PROPOSITION 8.9.4. *Let $\pi : X \to Y$ be a finite morphism of projective integral schemes over K. Let $\overline{M}$ be an adelic line bundle on Y such that M is ample and $\overline{M}$ is semi-positive. If $\overline{M}$ is strongly Minkowskian, then $\pi^*(\overline{M})$ is also strongly Minkowskian.*

PROOF. Let Z be a subvariety of X. Then $\pi|_Z : Z \to \pi(Z)$ is a finite and surjective morphism, and hence, by Lemma 8.9.3, $\pi^*(\overline{M})|_Z = (\pi|_Z)^*(\overline{M}|_{\pi(Z)})$ is Minkowskian, as required. □

REMARK 8.9.5. Let L be a very ample line bundle on X. Then there exists a finite and surjective morphism $\pi : X \to \mathbb{P}^d_K$ such that $\pi^*(\mathcal{O}_{\mathbb{P}^d_K}(1)) \simeq L$. Let $\{s_0, \ldots, s_d\}$ be a basis of $H^0(\mathbb{P}^d_K, \mathcal{O}_{\mathbb{P}^d_K}(1))$. For $\omega \in \Omega$, let $\|\cdot\|_\omega$ be the norm on $H^0(\mathbb{P}^d_{K_\omega}, \mathcal{O}_{\mathbb{P}^d_{K_\omega}}(1)) = H^0(\mathbb{P}^d_K, \mathcal{O}_{\mathbb{P}^d_K}(1)) \otimes K_\omega$ given by

$$\|a_0 s_1 + \cdots + a_d s_d\|_\omega = \begin{cases} \max\{|a_0|_\omega, \ldots, |a_d|_\omega\} & \text{if } \omega \in \Omega_{\mathrm{fin}}, \\ \sqrt{|a_0|^2_\omega + \cdots + |a_d|^2_\omega} & \text{if } \omega \in \Omega_\infty, \end{cases}$$

where $a_0, \ldots, a_d \in K_\omega$. Let ψ_ω be the Fubini-Study metric of $\mathcal{O}_{\mathbb{P}^d_{K_\omega}}(1)$ induced by $\|\cdot\|_\omega$. Then it is not difficult to see that $(\mathcal{O}_{\mathbb{P}^d_K}(1), \psi = (\psi_\omega)_{\omega\in\Omega})$ is semipositive and Minkowskian, so that, by Lemma 8.9.3, $(L, \pi^*(\psi))$ is Minkowskian.

8.10. Successive minima

Let X be a reduce projective scheme over $\operatorname{Spec} K$ and $\overline{L}$ be a relatively ample adelic line bundle on X. For any $i \in \{1, \ldots, d+1\}$, let

$$e_i(\overline{L}) = \sup_{\substack{Y \subseteq X \text{ closed} \\ \operatorname{codim}(Y) \geqslant i}} \inf_{\substack{Z \in \Theta_X \\ Z \not\subseteq Y}} \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_Z).$$

By definition, the following inequalities hold:

$$e_1(\overline{L}) \geqslant \ldots \geqslant e_{d+1}(\overline{L}).$$

Moreover, by Theorem 8.8.3, one has

$$e_{d+1}(\overline{L}) = \widehat{\mu}^{\mathrm{asy}}_{\min}(\overline{L}).$$

PROPOSITION 8.10.1. *Assume that the scheme X is integral. For any relatively ample adelic line bundle $\overline{L}$ on X, the equality $e_1(\overline{L}) = \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ holds.*

PROOF. If Y is a closed subscheme of codimension 1 of X, then $X \not\subseteq Y$. Therefore, the inequality $e_1(\overline{L}) \leqslant \widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L})$ holds. In the following, we show the converse inequality. Let t be a real number such that $t > e_1(\overline{L})$. By definition, there exists a family $(Z_i)_{i\in I}$ of integral closed subschemes of Y such that $\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}|_{Z_i}) \leqslant t$ for any $i \in I$ and that the generic points of Z_i form a Zariski dense family in X.

Let m be a positive integer and E_m be a vector subspace of $H^0(X, L^{\otimes m})$ such that

$$\widehat{\mu}_{\min}(\overline{E}_m) = \widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes m})). \tag{8.22}$$

For any positive integer n, let $F_{m,n}$ be the image of $E_m^{\otimes n}$ by the multiplication map

$$H^0(X, L^{\otimes m})^{\otimes n} \longrightarrow H^0(X, L^{\otimes mn}).$$

By [**36**, Proposition 4.3.31 and Corollary 5.6.2] (see also Remark A.3.3), one has

$$\widehat{\mu}_{\min}(\overline{F}_{m,n}) \geqslant n\Big(\widehat{\mu}_{\min}(\overline{E}_m) - \frac{3}{2}\nu(\Omega_\infty)\ln(\dim_K(E_m))\Big). \tag{8.23}$$

Moreover, there exists $i \in I$ such that the generic point of Z_i does not belong to the base locus of E_m (namely the closed subscheme of X defined by the ideal sheaf $\mathrm{Im}(E_m \otimes L^{\vee \otimes m} \to \mathcal{O}_X)$). Therefore the image of $F_{m,n}$ by the restriction map

$$H^0(X, L^{\otimes mn}) \longrightarrow H^0(Z_i, L^{\otimes mn}|_{Z_i})$$

is non-zero. By [**36**, Proposition 4.3.31], one has

$$\widehat{\mu}_{\min}(\overline{F}_{m,n}) \leqslant \widehat{\mu}_{\max}((f|_{Z_i})_*(\overline{L}|_{Z_i}^{\otimes mn})).$$

Combining this inequality with (8.22) and (8.23), we obtain

$$\frac{1}{m}\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes m})) \leqslant \frac{1}{mn}\widehat{\mu}_{\max}((f|_{Z_i})_*(\overline{L}|_{Z_i}^{\otimes mn})) + \frac{3}{2m}\nu(\Omega_\infty)\ln(\dim_K(E_m)).$$

Taking the limit when $n \to +\infty$, we obtain

$$\frac{1}{m}\widehat{\mu}_{\max}(f_*(\overline{L}^{\otimes m})) \leqslant t + \frac{3}{2m}\nu(\Omega_\infty)\ln(\dim_K(E_m)).$$

Taking the limit when $m \to +\infty$, we obtain $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) \leqslant t$. Since $t > e_1(\overline{L})$ is arbitrary, we get $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) \leqslant e_1(\overline{L})$, as required. □

Remark 8.10.2. Let $\overline{L}$ be a relatively ample adelic line bundle on X. For any $t \in \mathbb{R}$ and any positive integer n, we let $V_n^t(\overline{L})$ be the vector subspace of $H^0(X, L^{\otimes n})$ generated by non-zero vector subspaces of minimal slope $\geqslant nt$ and $r_n(t)$ be the dimension of the base locus of $V_n^t(\overline{L})$. For $t \in \mathbb{R}$, let

$$r(t) = \liminf_{n \to +\infty} r_n(t).$$

By using the method used in the proof of Proposition 8.10.1, we can show that, for any $i \in \{1, \ldots, d+1\}$

$$\sup\{t \in \mathbb{R} \mid r(t) \leqslant i\} \leqslant e_i(\overline{L}).$$

It is a natural question to ask if the equality holds. Moreover, we expect that the following inequality is true:

$$(d+1)\widehat{\mu}^{\mathrm{asy}}(\overline{L}) \geqslant \sum_{i=1}^{d+1} e_i(\overline{L}). \tag{8.24}$$

For any $i \in \{1, \ldots, d+1\}$, one has

$$e_i(\overline{L}) = \sup_{\substack{Y \subseteq X \text{ closed} \\ \mathrm{codim}(Y) \geqslant i}} \inf_{\substack{Z \in \Theta_X \\ Z \not\subseteq Y}} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Z) \leqslant \sup_{\substack{Y \subseteq X \text{ closed} \\ \mathrm{codim}(Y) \geqslant i}} \inf_{x \in (X \setminus Y)^{(0)}} h_{\overline{L}}(x),$$

where $(X \setminus Y)^{(0)}$ denotes the set of closed points of X outside of Y. In the case where S is the adelic curve consisting of places of a number field, by [**8**, Theorem 1.5], for any integral closed subscheme Z of X, one has

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Z) = \sup_{\substack{W \in \Theta_Z \\ W \neq Z}} \inf_{x \in (Z \setminus W)^{(0)}} h_{\overline{L}}(x).$$

If Z is not contained in Y, then

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Z) \geqslant \inf_{x \in (Z \setminus Y)^{(0)}} h_{\overline{L}}(x) \geqslant \inf_{x \in (X \setminus Y)^{(0)}} h_{\overline{L}}(x).$$

Therefore, in this case $e_i(\overline{L})$ identifies with the i-th minimum of the height function $h_{\overline{L}}$ in the sense of Zhang. In particular, the inequality (8.24) follows from [**81**, Theorem 5.2].

8.11. Equidistribution theorem

Throughout this section, we assume that S is proper.

Definition 8.11.1. Let X be a reduced projective scheme over $\operatorname{Spec} K$ and $\overline{L}$ be a relatively nef adelic line bundle on X. For any integral closed subscheme Y of X such that $L|_Y$ is big, we define the *normalized height* of Y with respect to $\overline{L}$ as

$$h_{\overline{L}}(Y) := \frac{(\overline{L}|_Y^{\dim(Y)+1})_S}{(\dim(Y)+1)(L|_Y^{\dim(Y)})}.$$

Theorem 8.11.2. *Let X be an integral projective scheme over* $\operatorname{Spec} K$ *and $\overline{L}$ be an adelic line bundle on X. We assume that L is big and semi-ample, and that φ is semi-positive. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of integral closed subschemes of X. Assume that,*

(1) *the sequence $(Y_n)_{n\in\mathbb{N}}$ is generic, namely, for any strict closed subscheme Z of X, the set $\{n \in \mathbb{N} : Y_n \subseteq Z\}$ is finite,*
(2) *for any $n \in \mathbb{N}$, $L|_{Y_n}$ is big,*
(3) *the sequence $(Y_n)_{n\in\mathbb{N}}$ is small, namely the sequence $(h_{\overline{L}}(Y_n))_{n\in\mathbb{N}}$ converges to $h_{\overline{L}}(X)$.*

Let Ω' be a measurable subset of Ω (i.e. $\Omega' \in \mathcal{A}$). Then the sequence $(\delta_{\overline{L},Y_n,\Omega'})_{n\in\mathbb{N}}$ converges weakly to $\delta_{\overline{L},X,\Omega'}$. In other words, for any $f \in \mathscr{C}^0_{\mathrm{a}}(X)$, one has

$$\lim_{n\to+\infty} \delta_{\overline{L},Y_n,\Omega'}(f) = \delta_{\overline{L},X,\Omega'}(f). \tag{8.25}$$

Proof. First we assume that $\Omega' = \Omega$. For any $f \in \mathscr{C}^0_{\mathrm{a}}(X))$, let

$$\Psi(f) = \liminf_{n\to+\infty} \frac{\widehat{\mathrm{vol}}_\chi(\overline{L}(f)|_{Y_n})}{(\dim(Y_n)+1)\deg_L(Y_n))}.$$

By Corollary 7.3.6, this is a concave function on $\mathscr{C}^0_{\mathrm{a}}(X)$. Since the sequence $(Y_n)_{n\in\mathbb{N}}$ is generic, $\Psi(f)$ is bounded from below by (see Proposition 8.10.1)

$$\widehat{\mu}^{\mathrm{asy}}_{\max}(\overline{L}(f)) \geqslant \frac{\widehat{\mathrm{vol}}_\chi(\overline{L}(f))}{(d+1)\deg_L(X)}.$$

Moreover, the hypothesis of the theorem implies that the equality

$$\Psi(0) = \frac{\widehat{\mathrm{vol}}_\chi(\overline{L}(f))}{(d+1)\deg_L(X)}. \tag{8.26}$$

By Proposition 7.4.3, the function $f \mapsto \widehat{\mathrm{vol}}_\chi(\overline{L}(f))$ is Gâteaux differentiable at $f=0$ and its differential is given by the linear form

$$f \longmapsto (d+1)\deg_L(X)\delta_{\overline{L},X}(f).$$

Since Ψ is a concave function, there exists a linear form $\ell : \mathscr{C}^0_{\mathrm{a}}(X) \to \mathbb{R}$ such that $\ell(f) + \Psi(0) \geqslant \Psi(f)$ for any $f \in \mathscr{C}^0_{\mathrm{a}(X)}$. We then deduce, by the equality (8.26), that $\ell(f) \geqslant \delta_{\overline{L},X}(f)$ for any $f \in \mathscr{C}^0_{\mathrm{a}}(X)$, which leads to $\ell = \delta_{\overline{L},X}(f)$. Thus $\delta_{\overline{L},X}(\cdot)$ is the unique linear form on $\mathscr{C}^0_{\mathrm{a}}(X)$ such that $\delta_{\overline{L},X}(f) + \Psi(0) \geqslant \Psi(f)$ for any $f \in \mathscr{C}^0_{\mathrm{a}}(X)$.

By the condition (2) of the theorem, one has

$$\Psi(0) = \lim_{n\to+\infty} \frac{\widehat{\mathrm{vol}}_\chi(\overline{L}|_{Y_n})}{(\dim(Y_n)+1)\deg_L(Y_n))}$$

and hence

$$\Psi(f) - \Psi(0) = \liminf_{n\to+\infty} \frac{\widehat{\mathrm{vol}}_\chi(\overline{L}(f)|_{Y_n}) - \widehat{\mathrm{vol}}_\chi(\overline{L}|_{Y_n})}{(\dim(Y_n)+1)\deg_L(Y_n)}.$$

Since the function

$$(f \in \mathscr{C}^0_{\mathrm{a}}(X)) \to \widehat{\mathrm{vol}}_\chi(\overline{L}(f)|_{Y_n})$$

is concave and Gâteaux differentiable at $f = 0$ with differential $\delta_{\overline{L},Y_n}(\cdot)$, we obtain

$$\Psi(f) - \Psi(0) \leqslant \liminf_{n\to+\infty} \delta_{\overline{L},Y_n}(f).$$

Applying this inequality to tf with $t > 0$, and taking the limit when $t \to 0$, we obtain

$$\delta_{\overline{L},X}(f) \leqslant \liminf_{n\to+\infty} \delta_{\overline{L},Y_n}(f).$$

Replacing f by $-f$, we obtain

$$\delta_{\overline{L},X}(f) \geqslant \limsup_{n\to+\infty} \delta_{\overline{L},Y_n}(f).$$

Therefore,

$$\delta_{\overline{L},X}(f) = \lim_{n\to+\infty} \delta_{\overline{L},Y_n}(f).$$

The general case is a consequence of the previous case and Corollary 7.1.4. □

Remark 8.11.3. Let L be a big invertible $\mathcal{O}_X$-module and let Z be its augmented base locus. Note that Z is a proper closed subset of X. If $(Y_n)_{n\in\mathbb{N}}$ is a generic sequence of integral closed subschemes of X, the set $\{n \in \mathbb{N} : Y_n \subseteq Z\}$ is finite. Therefore, for sufficiently large $n \in \mathbb{N}$, the restriction of L to Y_n is big. This is a consequence of [**11**, Theorem 1.5] on the positivity of restricted volumes.

Remark 8.11.4. Note that

$$\begin{cases} \delta_{\overline{L},Y_n,\Omega'}(f) = \displaystyle\int_{\Omega'} \nu(\mathrm{d}\omega) \int_{X^{\mathrm{an}}_\omega} f_\omega(x)\, \delta_{\overline{L},Y_n,\omega}(\mathrm{d}x), \\ \delta_{\overline{L},X,\Omega'}(f) = \displaystyle\int_{\Omega'} \nu(\mathrm{d}\omega) \int_{X^{\mathrm{an}}_\omega} f_\omega(x)\, \delta_{\overline{L},X,\omega}(\mathrm{d}x), \end{cases}$$

so if $\nu(\Omega') = 0$, then $\delta_{\overline{L},Y_n,\Omega'}(f) = \delta_{\overline{L},X,\Omega'}(f) = 0$. Thus Theorem 8.11.2 has a meaning in the case where $\nu(\Omega') > 0$.

Remark 8.11.5. By Theorem 8.11.2, we can recover the equidistribution theorem over an arithmetic function field including a number field case (cf. [**60**, Theorem 6.1]). Theorem 8.11.2 also gives a partial answer for [**77**, Conjecture 5.4.1].

CHAPTER 9

Global positivity conditions

In this chapter, we study global positivity conditions, notably ampleness, nefness, bigness and pseudo-effectivity. In the first and second sections, we define these positivity conditions and discuss the links with the corresponding relative positivity conditions and the positivity of asymptotic sectional invariants. The third section is a reminder on the construction of the canonical metric family of an arithmetic dynamical system. Then in the fourth section, we prove a theorem of Bogomolov conjecture type in the framework of Abelian varieties over an adelic curve with Archimedean places. In the fifth section, we discuss arithmetic dynamical systems.

As in the previous chapter, we fix an adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ such that, either $(\Omega, \mathcal{A})$ is discrete, or K is countable. We assume in addition that K is perfect and $\nu(\Omega) \not\subseteq \{0, +\infty\}$.

9.1. Ampleness and nefness

In this section, we let X be a non-empty and reduced projective scheme over $\operatorname{Spec} K$, and let d be the dimension of X.

Definition 9.1.1. We say that an adelic line bundle $\overline{L}$ on X is *ample* if it is relatively ample and if there exists $\varepsilon > 0$ such that the inequality

$$h_{\overline{L}}(Y) \geqslant \varepsilon \deg_L(Y)(\dim(Y) + 1)$$

holds for any integral closed subscheme Y of X.

Proposition 9.1.2. *Let $\overline{L}$ be an adelic line bundle which is relatively ample. Then the following statements are equivalent:*

(1) *$\overline{L}$ is ample,*
(2) $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) > 0$,
(3) *there exists $\varepsilon > 0$ such that, for any integral closed subscheme Y of X, one has $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y) > \varepsilon$.*

Proof. This is a consequence of Theorem 8.8.3. □

Proposition 9.1.3. *If $\overline{L}_0, \ldots, \overline{L}_d$ are ample adelic line bundles on X, then the inequality*

$$(\overline{L}_0 \cdots \overline{L}_d)_S > 0$$

holds.

Proof. This is a consequence of Theorem 6.3.2 and Proposition 9.1.2. □

Proposition 9.1.4. *Let $\overline{L}$ be an adelic line bundle which is relatively ample and strongly Minkowskian. Then the following conditions are equivalent:*

(1) *$\overline{L}$ is ample,*

(2) *there exists* $\varepsilon > 0$ *such that, for any closed point* x *of* X*, one has* $h_{\overline{L}}(x) > \varepsilon$.

Proof. This is a consequence of Corollary 8.9.2. □

Definition 9.1.5. We say that an adelic line bundle $\overline{L}$ on X is *nef* if there exists an ample adelic line bundle $\overline{A}$ and a positive integer N such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is ample for any $n \in \mathbb{N}_{\geqslant N}$.

Proposition 9.1.6. *Let* $\overline{L}$ *be an adelic line bundle on* X*. The following conditions are equivalent:*

(1) $\overline{L}$ *is nef,*
(2) $\overline{L}$ *is relatively nef and* $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \geqslant 0$.

Proof. Assume that $\overline{L}$ is nef. By definition, it is relatively nef. Let $\overline{A}$ be an ample adelic line bundle and N be a positive integer such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is ample for any $n \in \mathbb{N}_{\geqslant N}$. Then one has $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{A}) > 0$, which leads to

$$\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) = \lim_{n \to +\infty} \frac{1}{n} \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{A}) \geqslant 0.$$

Conversely, we assume that $\overline{L}$ is relatively nef and $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) \geqslant 0$. Since $\overline{L}$ is relatively nef, there exists a relatively ample line bundle $\overline{A}$ and a positive integer N such that $\overline{L}^{\otimes n} \otimes \overline{A}$ is relatively ample for any $n \in \mathbb{N}_{\geqslant N}$. By dilating the metrics of $\overline{A}$, we may assume that $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) > 0$. Then, by Proposition 6.4.4 we obtain that

$$\forall\, n \in \mathbb{N}_{\geqslant N}, \quad \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{A}) \geqslant n\, \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) + \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) \geqslant \widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{A}) > 0.$$

Therefore $\overline{L}^{\otimes n} \otimes \overline{A}$ is ample. □

Proposition 9.1.7. (1) *If* $\overline{L}_0, \ldots, \overline{L}_d$ *are nef adelic line bundles on* X*, then the inequality* $(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant 0$ *holds.*

(2) *If* $\overline{L}$ *is a nef adelic line bundle on* X *and if* $g : Y \to X$ *is a projective* K*-morphism, then the pullback* $g^*(\overline{L})$ *is nef.*
(3) *If* $\overline{L}$ *is a nef adelic line bundle on* X*, for any integral closed subscheme* Y *of* X*, one has* $(\overline{L}|_Y^{\dim(Y)+1})_S \geqslant 0$.
(4) *If* $\overline{L}$ *is a relatively ample adelic line bundle on* X *such that* $h_{\overline{L}}(Y) \geqslant 0$ *for any integral closed subscheme* Y *of* X*, then* $\overline{L}$ *is nef.*
(5) *If* $\overline{L}$ *is a relatively ample adelic line bundle on* X *such that* $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}|_Y) \geqslant 0$ *for any integral closed subscheme* Y *of* X*, then* $\overline{L}$ *is nef.*

Proof. The first statement is a consequence of Proposition 6.4.8 and Proposition 9.1.6. The second statement follows from Lemma 6.6.1, Theorem 6.6.6 and Proposition 9.1.6. The third statement is a consequence of the first and the second ones. The last two statements are consequences of Theorem 8.8.3 and Proposition 9.1.6. □

9.2. Bigness and pseudo-effectivity

In this section, we let X be an integral projective K-scheme $f : X \to \operatorname{Spec} K$ and let d be its dimension.

DEFINITION 9.2.1. Let $\overline{L}$ be an adelic line bundle on X. We define the *arithmetic volume* of $\overline{L}$ as

$$\widehat{\mathrm{vol}}(\overline{L}) := \limsup_{n\to+\infty} \frac{\widehat{\deg}_+(f_*(\overline{L}^{\otimes n}))}{n^{d+1}/(d+1)!}.$$

If $\widehat{\mathrm{vol}}(\overline{L}) > 0$, we say that $\overline{L}$ is *big*. It has been shown in [**36**, Proposition 6.4.18] that $\overline{L}$ is big if and only if L is big and $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) > 0$.

PROPOSITION 9.2.2. *An ample adelic line bundle is big.*

PROOF. Let $\overline{L}$ be an ample adelic line bundle on X. Then one has $\widehat{\mu}_{\min}^{\mathrm{asy}}(\overline{L}) > 0$, namely for sufficiently large positive integer n one has $\widehat{\mu}_{\min}(f_*(\overline{L}^{\otimes n})) > 0$. By [**36**, Proposition 4.3.13], for such n one has

$$\widehat{\deg}(f_*(\overline{L}^{\otimes n})) = \widehat{\deg}_+(f_*(\overline{L}^{\otimes n})),$$

which leads to, by Theorem 5.5.1,

$$\widehat{\mathrm{vol}}(\overline{L}) = (\overline{L}^{d+1})_S > 0,$$

where the inequality comes from Proposition 9.1.3. Hence $\overline{L}$ is big. □

REMARK 9.2.3. We expect that a variant of the method in the proof of Theorem 8.5.6 leads to an arithmetic version of Fujita's approximation theorem for big adelic line bundles, which generalizes the results of [**28, 75**].

PROPOSITION 9.2.4. *Let $\overline{L}_0, \ldots, \overline{L}_d$ be adelic line bundles on X. If $\overline{L}_0$ is big and $\overline{L}_1, \ldots, \overline{L}_d$ are ample, then*

$$(\overline{L}_0 \cdots \overline{L}_d)_S > 0.$$

PROOF. This is a consequence of Theorem 8.6.1. □

DEFINITION 9.2.5. Let $\overline{L}$ be an adelic line bundle on X. We say $\overline{L}$ is *pseudo-effective* if there exist a big adelic line bundle $\overline{M}$ on X and a positive integer n_0 such that $\overline{L}^{\otimes n} \otimes \overline{M}$ is big for any $n \in \mathbb{N}_{\geqslant n_0}$.

PROPOSITION 9.2.6. *Let $\overline{L}$ be an adelic line bundle on X. The following assertions are equivalent:*

(1) *$\overline{L}$ is pseudo-effective,*
(2) *L is pseudo-effective and $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) \geqslant 0$.*

PROOF. Assume that $\overline{L}$ is pseudo-effective. Let $\overline{M}$ be a big adelic line bundle and n_0 be a positive integer such that $\overline{L}^{\otimes n} \otimes \overline{M}$ is big for any integer $n \geqslant n_0$. In particular, $L^{\otimes n} \otimes M$ is big for any integer $n \geqslant n_0$. Hence L is pseudo-effective. Moreover, for $n \geqslant n_0$, one has $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) > 0$, which implies that

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) = \lim_{n\to+\infty} \frac{1}{n} \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) \geqslant 0.$$

Conversely, assume that L is pseudo-effective and $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) \geqslant 0$. Let $\overline{M}$ be a big adelic line bundle on X. Since L is pseudo-effective, for any positive integer n, $L^{\otimes n} \otimes M$ is big. Moreover, by Proposition 8.3.5 one has

$$\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes n} \otimes \overline{M}) \geqslant n\, \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) + \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{M}) > 0.$$

Hence $\overline{L}^{\otimes n} \otimes \overline{M}$ is big for any $n \in \mathbb{N}$, which shows that $\overline{L}$ is pseudo-effective. □

PROPOSITION 9.2.7. (1) *Let $\overline{L}_0, \ldots, \overline{L}_d$ be adelic line bundles on X. Assume that $\overline{L}_0$ is pseudo-effective and that $\overline{L}_1, \ldots, \overline{L}_d$ are nef, then the inequality $(\overline{L}_0 \cdots \overline{L}_d)_S \geqslant 0$ holds.*
(2) *If $\overline{L}$ is a pseudo-effective adelic line bundle on X and if $g : Y \to X$ is a surjective and projective morphism, then the pullback $g^*(\overline{L})$ is also pseudo-effective.*
(3) *If $\overline{L}$ is nef, then it is pseudo-effective.*

PROOF. The first statement is a consequence of Theorem 8.6.1; the second one is a consequence of Proposition 8.4.2.

(3) Since $\overline{L}$ is nef, we obtain that L is nef, and hence is pseudo-effective. Let $\overline{A}$ be an ample adelic line bundle. For any positive integer p, by Proposition 8.3.5 one has

$$\frac{1}{p}\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}^{\otimes p} \otimes \overline{A}) \geqslant \widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) + \frac{1}{p}\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{A}).$$

Taking the limit when $p \to +\infty$, we obtain $\widehat{\mu}_{\max}^{\mathrm{asy}}(\overline{L}) \geqslant 0$. By Proposition 9.2.6, $\overline{L}$ is pseudo-effective. □

9.3. Canonical compactification

In this section, we recall several basic facts on the canonical compactification of an algebraic dynamical system.

9.3.1. Local canonical compactification. Let $(k, |\cdot|)$ be a field equipped with a complete absolute value. Let X be a geometrically integral projective scheme over k. In particular, $H^0(X, \mathcal{O}_X) = k$. Let $f : X \to X$ be a surjective endomorphism of X over k and L be a semiample line bundle on X. We assume that there exists an isomorphism $\alpha : f^*(L) \simeq L^{\otimes d}$ for some integer $d \geqslant 2$. It is well-known that there exists a unique semi-positive metric $\varphi_{f,\alpha}$ of L^{an} such that α induces an isometry $f^*(L, \varphi_{f,\alpha}) \simeq (L, \varphi_{f,\alpha})^{\otimes d}$ (cf. [**36**, Proposition 2.5.11]), that is, $|\cdot|_{f^*(\varphi_{f,\alpha})} = |\alpha(\cdot)|_{d\varphi_{f,\alpha}}$. The metric $\varphi_{f,\alpha}$ is called the (local) canonical compactification of L. Let us begin with the following lemma.

LEMMA 9.3.1. *Let λ be a continuous function on X^{an}, $a \in \mathbb{R}$ and $b \in \mathbb{R}_{>1}$. If $f^*(\lambda) = b\lambda + a$, then λ is a constant and $\lambda = -a/(b-1)$.*

PROOF. By our assumption,

$$\max_{x \in X^{\mathrm{an}}} \{\lambda(x)\} = b \max_{x \in X^{\mathrm{an}}} \{\lambda(x)\} + a \quad \text{and} \quad \min_{x \in X^{\mathrm{an}}} \{\lambda(x)\} = b \min_{x \in X^{\mathrm{an}}} \{\lambda(x)\} + a,$$

that is,

$$(b-1) \max_{x \in X^{\mathrm{an}}} \{\lambda(x)\} = (b-1) \min_{x \in X^{\mathrm{an}}} \{\lambda(x)\} = -a,$$

and hence the assertion follows. □

PROPOSITION 9.3.2. *If we change the isomorphism α by $c\alpha$ ($c \in k^\times$), then*

$$|\cdot|_{\varphi_{f,c\alpha}} = |c|^{-1/(d-1)}|\cdot|_{\varphi_{f,\alpha}}.$$

PROOF. Indeed, we can find a continuous function λ on X^{an} such that $|\cdot|_{\varphi_{f,c\alpha}} = \exp(\lambda)|\cdot|_{\varphi_{f,\alpha}}$. Thus,

$$\begin{aligned}\exp(f^*(\lambda))|\cdot|_{f^*(\varphi_{f,\alpha})} = |\cdot|_{f^*(\varphi_{f,c\alpha})} = |(c\alpha)(\cdot)|_{d\varphi_{f,c\alpha}} &= |c| \exp(d\lambda)|\alpha(\cdot)|_{d\varphi_{f,\alpha}} \\ &= \exp(d\lambda + \log|c|)|\cdot|_{f^*(\varphi_{f,\alpha})},\end{aligned}$$

and hence $f^*(\lambda) = d\lambda + \log|c|$. Therefore, by Lemma 9.3.1, λ is constant and $\lambda = -\log|c|/(d-1)$, as required. □

Let $g : X \to X$ be another surjective endomorphism of X such that there exists an isomorphism $\beta : g^*(L) \simeq L^{\otimes e}$ for some integer $e \geqslant 2$. We assume that $f \circ g = g \circ f$. Let us consider the following homomorphisms

$$\begin{cases} g^*(f^*(L)) \xrightarrow[g^*(\alpha)]{\sim} g^*(L^{\otimes d}) \xrightarrow[\beta^{\otimes d}]{\sim} L^{\otimes de}, \\ g^*(f^*(L)) = f^*(g^*(L)) \xrightarrow[f^*(\beta)]{\sim} f^*(L^{\otimes e}) \xrightarrow[\alpha^{\otimes e}]{\sim} L^{\otimes de}. \end{cases}$$

Then there exists $r \in k^\times$ such that $\beta^{\otimes d} \circ g^*(\alpha) = r \cdot \alpha^{\otimes e} \circ f^*(\beta)$. In the case where $r = 1$, we say that (f, α) is compatible with (g, β).

Proposition 9.3.3. *One has* $|\cdot|_{\varphi_{g,\beta}} = |r|^{-1/(d-1)(e-1)} |\cdot|_{\varphi_{f,\alpha}}$. *In particular, if* (f, α) *is compatible with* (g, β)*, then* $\varphi_{f,\alpha} = \varphi_{g,\beta}$.

Proof. We can find a continuous function λ on X^{an} such that

$$|\cdot|_{g^*(\varphi_{f,\alpha})} = \exp(\lambda) |\beta(\cdot)|_{e\varphi_{f,\alpha}}.$$

Thus

$$\begin{aligned} |\cdot|_{f^*(g^*(\varphi_{f,\alpha}))} &= \exp(f^*(\lambda)) |f^*(\beta)(\cdot)|_{e f^*(\varphi_{f,\alpha})} \\ &= \exp(f^*(\lambda)) |\alpha^{\otimes e}(f^*(\beta)(\cdot))|_{de\varphi_{f,\alpha}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\cdot|_{g^*(f^*(\varphi_{f,\alpha}))} = |g^*(\alpha)(\cdot)|_{g^*(d\varphi_{f,\alpha})} &= \exp(\lambda)^d |\beta^{\otimes d}(g^*(\alpha)(\cdot))|_{de\varphi_{f,\alpha}} \\ &= \exp(\lambda)^d |r\alpha^{\otimes e}(f^*(\beta)(\cdot))|_{de\varphi_{f,\alpha}} \\ &= \exp(\lambda)^d |r| |\alpha^{\otimes e}(f^*(\beta)(\cdot))|_{de\varphi_{f,\alpha}}, \end{aligned}$$

so $f^*(\lambda) = d\lambda + \log|r|$. Therefore, by Lemma 9.3.1, λ is a constant function and $\lambda = -\log|r|/(d-1)$.

Here we set $|\cdot|_{\varphi_{g,\beta}} = \exp(\mu) |\cdot|_{\varphi_{f,\alpha}}$ for some continuous function μ on X^{an}. Then, as $|\cdot|_{g^*(\varphi_{g,\beta})} = |\beta(\cdot)|_{e\varphi_{g,\beta}}$, one has

$$\begin{aligned} \exp(g^*(\mu)) |\cdot|_{g^*(\varphi_{f,\alpha})} = |\cdot|_{g^*(\varphi_{g,\beta})} = |\beta(\cdot)|_{e\varphi_{g,\beta}} &= \exp(e\mu) |\beta(\cdot)|_{e\varphi_{f,\alpha}} \\ &= \exp(e\mu - \lambda) |\cdot|_{g^*(\varphi_{f,\alpha})}, \end{aligned}$$

that is, $g^*(\mu) = e\mu - \lambda$. Therefore, by Lemma 9.3.1, μ is a constant and $\mu = \lambda/(e-1)$, and hence

$$\mu = -\log|r|/(d-1)(e-1),$$

as required. ◻

Remark 9.3.4. If $\beta^{\otimes d} \circ g^*(\alpha) = r \cdot \alpha^{\otimes e} \circ f^*(\beta)$ $(r \in k^\times)$, then, for $c \in k^\times$,

$$(c\beta)^{\otimes d} \circ g^*(\alpha) = c^d \cdot \beta^{\otimes d} \circ g^*(\alpha) = (c^d r) \cdot \alpha^{\otimes e} \circ f^*(\beta) = (c^{d-1} r) \cdot \alpha^{\otimes e} \circ f^*(c\beta).$$

Thus if there exists $c \in k$ such that $c^{d-1} r = 1$, then (f, α) is compatible with $(g, c\beta)$. In particular, if k is algebraically closed, then, for (f, α), we can find β such that (f, α) is compatible with (g, β).

9.3.2. Global canonical compactification. Let K be a field equipped with an adelic structure $((\Omega, \mathcal{A}, \nu), \phi)$. We assume that $\nu(\mathcal{A}) \not\subseteq \{0, \infty\}$. Let X be a geometrically integral projective variety over K and $f : X \to X$ be a surjective endomorphism of X over K. Let L be a semiample line bundle on X such that there exists an isomorphism $\alpha : f^*(L) \xrightarrow{\sim} L^{\otimes d}$ for some integer $d \geqslant 2$. For each $\omega \in \Omega$, let φ_ω be the local compactification of L_ω, that is, the isomorphism $\alpha_\omega : f_\omega^*(L_\omega) \xrightarrow{\sim} L_\omega^{\otimes d}$ induces an isometry $f_\omega^*(L_\omega, \varphi_\omega) \simeq (L_\omega, \varphi_\omega)^{\otimes d}$.

Proposition 9.3.5. *If we set $\varphi = (\varphi_\omega)_{\omega \in \Omega}$, then (L, φ) is a nef adelic line bundle on X. The family φ is called the global compactification of L.*

Proof. Let $\varphi_0 = (\varphi_{0,\omega})_{\omega \in \Omega}$ be a metric family of L such that (L, φ_0) is a relatively nef adelic line bundle. By the assumption "$\nu(\mathcal{A}) \not\subseteq \{0, \infty\}$", we can find a non-negative measurable function ϑ on Ω such that

$$\hat{\mu}_{\min}^{\mathrm{asy}}(L, \varphi_0) + \int_\Omega \vartheta(\omega)\nu(\mathrm{d}\omega) \geqslant 0.$$

If we set $\varphi_{0,\omega}^{\vartheta} = \exp(-\vartheta(\omega))\varphi_{0,\omega}$ and $\varphi_0^{\vartheta} = (\varphi_{0,\omega}^{\vartheta})_{\omega \in \Omega}$, then it is easy to see that

$$\hat{\mu}_{\min}^{\mathrm{asy}}(L, \varphi_0^{\vartheta}) = \hat{\mu}_{\min}^{\mathrm{asy}}(L, \varphi_0) + \int_\Omega \vartheta(\omega)\nu(\mathrm{d}\omega) \geqslant 0,$$

so $(L, \varphi_0^{\vartheta})$ is nef by Proposition 9.1.6. Thus we may assume that (L, φ_0) is nef.

For each $\omega \in \Omega$, one can find a continuous function λ_ω on X_ω^{an} such that

$$|\cdot|_{f^*(\varphi_{0,\omega})} = |\alpha_\omega(\cdot)|_{d\varphi_{0,\omega}} \exp(\lambda_\omega).$$

Note that $(\mathcal{O}_X, (\exp(\lambda_\omega)|\cdot|_\omega)_{\omega \in \Omega})$ is an adelic line bundle. Let

$$h_{0,\omega} = 0 \quad \text{and} \quad h_{n,\omega} = \sum_{i=0}^{n-1} \frac{1}{d^{i+1}} (f_\omega^i)^*(\lambda_\omega) \quad (n \geqslant 1),$$

and $|\cdot|_{\varphi_{n,\omega}} = |\cdot|_{\varphi_{0,\omega}} \exp(h_{n,\omega})$. Then, in the same way as [**36**, Proposition 2.5.11], one can see that $\{h_{n,\omega}\}_{n=0}^{\infty}$ converges uniformly to a continuous function $h_{\infty,\omega}$ on X_ω^{an} and $\alpha_\omega : f_\omega^*(L_\omega) \xrightarrow{\sim} L_\omega^{\otimes d}$ induces an isometry

$$f_\omega^*(L_\omega, (\varphi_{n-1,\omega})_{\omega \in \Omega}) \simeq (L_\omega, (\varphi_{n,\omega})\omega \in \Omega)^{\otimes d}.$$

In particular, if we set $|\cdot|_{\varphi_{\infty,\omega}} = |\cdot|_{\varphi_{0,\omega}} \exp(h_{\infty,\omega})$, then α_ω yields an isometry

$$f_\omega^*(L_\omega, (\varphi_{\infty,\omega})_{\omega \in \Omega}) \simeq (L_\omega, (\varphi_{\infty,\omega})_{\omega \in \Omega})^{\otimes d},$$

and hence $\varphi_{\infty,\omega}$ is the local canonical compactification of L_ω. Thus, by the uniqueness of the local canonical compactfication, we have $\varphi_{\infty,\omega} = \varphi_\omega$ for all $\omega \in \Omega$. Let $\varphi_n = (\varphi_{n,\omega})_{\omega \in \Omega}$. By [**36**, Proposition 6.1.29], (L, φ) is measurable because (L, φ_n) is measurable for all n. Moreover, in the same way as [**36**, Proposition 2.5.11], we obtains

$$d_\omega(\varphi, \varphi_0) \leqslant \frac{\|\lambda_\omega\|_{\sup}}{d-1},$$

so $(\omega \in \Omega) \mapsto d_\omega(\varphi, \varphi_{g_0})$ is dominated. Thus (L, φ) is dominated by [**36**, Proposition 6.1.12]. Further, as $f^*(L, \varphi_{n-1}) \simeq (L, \varphi_n)^{\otimes d}$, we can see that (L, φ_n) is nef for all n. Therefore, (L, φ) is also nef, as required. □

Remark 9.3.6. We assume that the adelic structure is proper.

(1) Let $\alpha : f^*(L) \to L^{\otimes d}$ be the isomorphism. If we change the isomorphism α by $c\alpha$ ($c \in K^\times$), then, by Proposition 9.3.2,

$$|\cdot|_{\varphi_{f,c\alpha,\omega}} = |c|_\omega^{-1/(d-1)} |\cdot|_{\varphi_{f,\alpha,\omega}}$$

for all $\omega \in \Omega$. Thus, by the product formula, one has $h_{(L,\varphi_{f,c\alpha})} = h_{(L,\varphi_{f,\alpha})}$.

(2) Let $g : X \to X$ be another surjective endomorphism of X over K. We assume that $f \circ g = g \circ f$ and there exists an isomorphism $\beta : g^*(L) \to L^{\otimes e}$ for some integer $e \geqslant 2$. Then, by Proposition 9.3.3, there exists $r \in K^\times$ such that

$$|\cdot|_{\varphi_{f,\alpha,\omega}} = |r|_\omega^{-1/(d-1)(e-1)} |\cdot|_{\varphi_{g,\beta,\omega}}.$$

for all $\omega \in \Omega$. Therefore, by the product formula, one has $h_{(L,\varphi_{f,\alpha})} = h_{(L,\varphi_{g,\beta})}$.

9.4. Bogomolov's conjecture over a countable field of characteristic zero

Throughout this section, we assume that S is proper and that K is algebraically closed, countable and of characteristic 0. We assume in addition that $\nu(\Omega_\infty) > 0$ and $\nu(\mathcal{A}) \notin \{0, 1\}$.

Let X be a projective integral scheme over K and $\overline{L}$ be an adelic line bundle on X. The *essential minimum* $\hat{\mu}_{\mathrm{ess}}(\overline{L})$ of $\overline{L}$ is defined to be

$$\hat{\mu}_{\mathrm{ess}}(\overline{L}) := \sup_{Z \subsetneq X} \inf_{x \in (X \setminus Z)(K)} h_{\overline{L}}(x)$$

where Z runs over the set of all proper closed subsets of X.

Let A be an abelian variety over $\operatorname{Spec} K$. For any integer n, let $[n] : A \to A$ be the morphism of multiplication by n. Let L be a symmetric ample invertible $\mathcal{O}_A$-module. If we fix an isomorphism $[2]^*(L) \simeq L^{\otimes 4}$, then, for each $\omega \in \Omega$, we can assign the local canonical compactification φ_ω (with respect to $[2]$) to L_ω. If we set $\varphi = (\varphi_\omega)_{\omega \in \Omega}$, then, by Proposition 9.3.5, (L, φ) is a nef adelic line bundle on X. Since $[2]$ and $[n]$ are commutative and K is algebraically closed, by Remark 9.3.4, we can find a suitable isomorphism $[n]^*(L) \simeq L^{\otimes n^2}$ such that the local canonical compatification of L_ω with respect to $[n]$ coincides with φ_ω.

Theorem 9.4.1. *Let X be an integral subscheme of A such that the stabilizer of X is trivial. If* $\dim X > 0$*, then the essential minimum $\hat{\mu}_{\mathrm{ess}}(\overline{L}|_X)$ of X is strictly positive.*

Proof. There exists an integer $m \geqslant 1$ such that the morphism

$$f : X^m \longrightarrow A^{m-1}, \quad (x_1, \ldots, x_m) \longmapsto (x_2 - x_1, \ldots, x_m - x_{m-1})$$

is birational onto its image but not finite. We assume by contradiction that there exists a generic sequence $(x_n)_{n \in \mathbb{N}}$ in $X(K)$ such that $h_{\overline{L}}(x_n)$ converges to 0 when $n \to +\infty$. This sequence permits us to construct a generic sequence $(y_n)_{n \in \mathbb{N}}$ in X^m such that $h_{\overline{L}^{\boxtimes m}}(y_n)$ converges to 0 when $n \to +\infty$. Moreover, since Néron-Tate height is a quadratic form, we also deduce that $h_{\overline{L}^{\boxtimes (m-1)}}(f(y_n))$ converges to 0 when $n \to +\infty$, that is,

$$\lim_{n \to \infty} h_{f^*(\overline{L}^{\boxtimes (m-1)})}(y_n) = 0.$$

Applying the equidistribution theorem (c.f. Theorem 8.11.2) to the sequence $(y_n)_{n \in \mathbb{N}}$, we deduce that the sequences of measures

$$\delta_{\overline{L}^{\boxtimes m}, y_n, \Omega_\infty} \quad \text{and} \quad \delta_{f^*(\overline{L}^{\boxtimes (m-1)}), y_n, \Omega_\infty}, \quad n \in \mathbb{N}$$

converge weakly to $\delta_{\overline{L}^{\boxtimes m}, X^m, \Omega_\infty}$ and $\delta_{f^*(\overline{L}^{\boxtimes (m-1)}), X^m, \Omega_\infty}$, respectively, and hence

$$\delta_{\overline{L}^{\boxtimes m}, X^m, \Omega_\infty} = \delta_{f^*(\overline{L}^{\boxtimes (m-1)}), X^m, \Omega_\infty}$$

holds as measures (see Remark 7.6.6). Therefore, by Proposition 7.7.3, there exists $\Omega' \subseteq \Omega_\infty$ such that $\nu(\Omega_\infty \setminus \Omega') = 0$ and that

$$c_1((L_\omega, \varphi_\omega)^{\boxtimes m})^{m \dim X} = f_\omega^*(c_1((L_\omega, \varphi_\omega)^{\boxtimes m-1}))^{m \dim X}$$

on $(X^m_{\mathrm{reg}})^{\mathrm{an}}_\omega$ for any $\omega \in \Omega'$, where X_{reg} is the regular locus of X. The above equation leads to a contradiction because $c_1((L_\omega, \varphi_\omega)^{\boxtimes m})^{m \dim X}$ is positive and

$$f^*_\omega(c_1((L_\omega, \varphi_\omega)^{\boxtimes m-1}))^{m \dim X}$$

vanishes at the diagonal points of $(X^m_{\mathrm{reg}})^{\mathrm{an}}_\omega$. □

As a consequence of the above theorem, we have the following answer of Bogomolov's conjecture for K.

Corollary 9.4.2. *Let X be an integral subscheme of A. If $\hat{\mu}_{\mathrm{ess}}(\overline{L}|_X) = 0$, then X is a translation of an abelian subvariety by a closed point of height* 0.

Proof. In the same argument as [**62**, the last paragraph in the proof of Theorem 9.20], we may assume that the stabilizer of X is trivial. Thus, by Theorem 9.4.1, one has $\dim X = 0$, so we set $X = \{x\}$. Thus $\hat{\mu}_{\mathrm{ess}}(\overline{L}|_X) = h_{\overline{L}}(x) = 0$, as required. □

Remark 9.4.3. Assume that any finitely generated subfield of K has Northcott's property (cf. [**38**, Theorem 2.7.18]). Then any closed point of height 0 with respect to the Néron-Tate height on the abelian variety A is a torsion point. Indeed, we choose a subfield K' of K such that A, L and x are defined over K' and K' is finitely generated over $\mathbb{Q}$. Then, by Northcott's property, $\{nx \mid n \in \mathbb{Z}\}$ is a finite group because $h_{\overline{L}}(nx) = 0$ for all $n \in \mathbb{Z}$, so x is a torsion point.

Remark 9.4.4. The geometric analogue of Bogomolov's conjecture for Abelian varieties over function fields has been proved by a series of works of Gubler [**50**], Yamaki [**69, 70, 71, 72, 73**], Gao-Habbegger [**47**], Cantat-Gao-Habegger-Xie [**24**] and Xie-Yuan [**68**]. It is an interesting question to investigate the condition (on the polarized Abelian variety (A, L)) under which the result of Theorem 9.4.1 holds without the assumption $\nu(\Omega_\infty) > 0$.

9.5. Dynamical systems over a countable field

Throughout this section, we assume that S is proper, $\nu(\mathcal{A}) \not\subseteq \{0, +\infty\}$, and that K is algebraically closed and countable.

Let X be a projective integral scheme over K and L be an ample line bundle on X. Let $\mathrm{End}(X; L)$ be the set of all endomorphisms $f : X \to X$ such that $f^*(L) \simeq L^{\otimes d}$ for some $d \in \mathbb{Z}_{>1}$. For each $f \in \mathrm{End}(X; L)$, we denote the set of all preperiodic closed points by $\mathrm{Prep}(f)$, that is,

$$\mathrm{Prep}(f) := \{x \in X(K) \mid f^m(x) = f^n(x) \text{ for some } m, n \in \mathbb{Z}_{\geqslant 1} \text{ with } m \neq n\}.$$

Moreover, let $\varphi_f = \{\varphi_{f,\omega}\}_{\omega\in\Omega}$ denote the global canonical compactification of L. Note that, for each $\omega \in \Omega$, $\varphi_{f,\omega}$ is semipositive and

$$f^*_\omega(L_\omega, \varphi_{f,\omega}) \simeq (L_\omega, \varphi_{f,\omega})^{\otimes d}.$$

It is easy to see that

$$\mathrm{Prep}(f) \subseteq \{x \in X(K) \mid h_{(L,\varphi_f)}(x) = 0\}. \tag{9.1}$$

Indeed, as $h_{(L,\varphi_f)}(f^m(x)) = h_{(L,\varphi_f)}(f^n(x))$ for some positive integers m, n with $m \neq n$, one has

$$d^m h_{(L,\varphi_f)}(x) = h_{(L,\varphi_f)}(f^m(x)) = h_{(L,\varphi_f)}(f^n(x)) = d^n h_{(L,\varphi_f)}(x),$$

as required. Moreover, if S has Northcott's property for any finitely generated subfield of K, then

$$\mathrm{Prep}(f) = \{x \in X(K) \mid h_{(L,\varphi_f)}(x) = 0\} \tag{9.2}$$

because $\{f^n(x) \mid n \in \mathbb{Z}_{\geqslant 0}\}$ is finite if $h_{(L,\varphi_f)}(x) = 0$. The main theorem of this section is following.

THEOREM 9.5.1. *For $f, g \in \mathrm{End}(X; L)$, the following are equivalent:*

(1) $h_{(L,\varphi_f)} = h_{(L,\varphi_g)}$.
(2) $\{x \in X(K) \mid h_{(L,\varphi_f)}(x) = 0\} = \{x \in X(K) \mid h_{(L,\varphi_g)}(x) = 0\}$.
(3) $\{x \in X(K) \mid h_{(L,\varphi_f)}(x) = h_{(L,\varphi_g)}(x) = 0\}$ *is Zariski dense in $X(K)$.*

PROOF. "(1) $\Longrightarrow$ (2)" and "(2) $\Longrightarrow$ (3)" are obvious because $\mathrm{Prep}(f)$ and $\mathrm{Prep}(g)$ are Zariki dense in $X(K)$. We assume (3). Choose a generic sequence $\{x_n\}_{n\in\mathbb{N}}$ in

$$\{x \in X(K) \mid h_{(L,\varphi_f)}(x) = h_{(L,\varphi_g)}(x) = 0\}.$$

Since the sequence is small with respect to both $h_{(L,\varphi_f)}$ and $h_{(L,\varphi_g)}$, by the equidistibution theorem (cf. Theorem 8.11.2), one has

$$\delta_{(L,\varphi_f),X,\Omega} = \delta_{(L,\varphi_g),X,\Omega'},$$

and hence, by Proposition 7.7.3, there exist $\Omega' \in \mathcal{A}$ and an integrable function ℓ on Ω such that $\nu(\Omega \setminus \Omega') = 0$ and $\varphi_{g,\omega} = \exp(\ell(\omega))\varphi_{f,\omega}$ for all $\omega \in \Omega'$. Thus one can see

$$h_{(L,\varphi_f)} = h_{(L,\varphi_g)} + c$$

for some constant c. On the other hand, $c = 0$ because the set

$$\{x \in X(K) \mid h_{(L,\varphi_f)}(x) = h_{(L,\varphi_g)}(x) = 0\}$$

is not empty. Thus one has (1). ☐

REMARK 9.5.2. By the above theorem, we can recover [**78**, Theorem 1.3].

APPENDIX A

Appendix

A.1. Tensorial semi-stability

We recall some constructions and facts of multi-linear algebra and classical invariant theory. Then we prove a lifting theorem for invariants in a symmetric power of a tensor product under the action of the product of special linear groups. In subsections A.1.1–A.1.6, we fix a commutative ring k with unit.

A.1.1. Symmetric power. Let V be a free k-module of finite rank and δ be a natural number. We denote by $V^{\otimes\delta}$ the δ-th tensor power of the k-module V. Note that the symmetric group $\mathfrak{S}_\delta$ acts k-linearly on $V^{\otimes\delta}$ by permuting the tensor factors. The quotient k-module of $V^{\otimes\delta}$ by this action of $\mathfrak{S}_\delta$ is denoted by $S^\delta(V)$. The class of $x_1 \otimes \cdots \otimes x_\delta$ in $S^\delta(V)$ is denoted by $x_1 \cdots x_\delta$. If $\boldsymbol{e} = (e_i)_{i=1}^d$ is a basis of V over k, then

$$\boldsymbol{e}^{\boldsymbol{a}} := \prod_{i\in\{1,\ldots,d\}} e_i^{a_i}, \quad \boldsymbol{a} = (a_i)_{i=1}^d \in \mathbb{N}^d,\ |\boldsymbol{a}| := a_1 + \cdots + a_d = \delta$$

form a basis of $S^\delta(V)$ over k. In particular, $S^\delta(V)$ is a free k-module. More generally, if V is decomposed into a direct sum

$$V = V^{(1)} \oplus \cdots \oplus V^{(r)}$$

of free sub-k-modules, then the k-linear map

$$\bigoplus_{\substack{\boldsymbol{b}=(b_1,\ldots,b_r)\in\mathbb{N}^r \\ |\boldsymbol{b}|=b_1+\cdots+b_r=\delta}} S^{b_1}(V^{(1)}) \otimes \cdots \otimes S^{b_r}(V^{(r)}) \longrightarrow S^\delta(V), \tag{A.1}$$

which sends

$$x_1 \otimes \cdots \otimes x_r \in S^{b_1}(V^{(1)}) \otimes \cdots \otimes S^{b_r}(V^{(r)})$$

to $x_1 \cdots x_r \in S^\delta(V)$, is an isomorphism.

We call $S^\delta(V)$ the *δ-th symmetric power* of the free k-module V. Note that S^δ defines a functor from the category of free k-modules of finite rank to itself. Moreover, it preserves the extension of scalars, namely, for any commutative k-algebra A, one has

$$S^\delta(V \otimes_k A) \cong S^\delta(V) \otimes_k A.$$

The graded k-algebra structure of the tensor algebra

$$T(V) := \bigoplus_{\delta\in\mathbb{N}} V^{\otimes\delta}$$

induces by passing to quotient a graded k-algebra structure on

$$\mathrm{Sym}(V) := \bigoplus_{\delta\in\mathbb{N}} S^\delta(V).$$

This k-algebra is commutative and finitely generated, and it is isomorphic to the polynomial ring $k[X_1, \ldots, X_r]$ over k, where r is the rank of V.

A.1.2. Exterior power. Let V be a free k-module. We denote by

$$\Lambda(V) = \bigoplus_{\delta\in\mathbb{N}} \Lambda^{\delta}(V)$$

the quotient graded k-algebra of $T(V)$ by the two-sided ideal generated by elements of the form $x \otimes x$, $x \in V$. If $x_1, \ldots, x_\delta$ are elements of V, the image of $x_1 \otimes \cdots \otimes x_\delta$ in $\Lambda^\delta(V)$ is denoted by $x_1 \wedge \cdots \wedge x_\delta$. Note that, if $(e_i)_{i=1}^d$ is a basis of V over k, then

$$e_{i_1} \wedge \cdots \wedge e_{i_\delta}, \quad 1 \leqslant i_1 < \ldots < i_\delta \leqslant d$$

form a basis of $\Lambda^\delta(V)$ over k. The k-module $\Lambda^\delta(V)$ is called the *δ-th exterior power* of V. Note that Λ^δ also defines a functor from the category of free k-modules of finite rank to itself, and preserves extensions of scalars.

If δ is a natural number, we denote by $\iota_\delta : \Lambda^\delta(V) \to V^{\otimes\delta}$ the anti-symmetrization map which sends $x_1 \wedge \cdots \wedge x_\delta \in \Lambda^\delta(V)$ to

$$\sum_{\sigma\in\mathfrak{S}_\delta} \operatorname{sgn}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(\delta)}.$$

This is an injective $\mathrm{GL}(V)$-linear map. It is however *not* a section of the projection $E^{\otimes\delta} \to \Lambda^\delta(V)$.

A.1.3. Schur functor. We denote by $\mathbb{N}_{\geqslant 1}$ the set of positive integers and by $\mathbb{N}^{\oplus\infty}$ the set of sequences $\lambda = (\lambda_i)_{i\in\mathbb{N}_{\geqslant 1}}$ of natural numbers indexed by $\mathbb{N}_{\geqslant 1}$ such that $\lambda_i = 0$ except finitely many i. For any $\lambda = (\lambda_i)_{i\in\mathbb{N}_{\geqslant 1}} \in \mathbb{N}^{\oplus\infty}$, we denote by $|\lambda|$ the sum

$$\sum_{i\in\mathbb{N}_{\geqslant 1}} \lambda_i,$$

called the *weight* of λ. If n is a natural number and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, by abuse of notation we denote by λ the sequence

$$(\lambda_1, \ldots, \lambda_n, 0, \ldots, 0, \ldots) \in \mathbb{N}^{\oplus\infty}.$$

We call *partition* any sequence $\lambda = (\lambda_i)_{i\in\mathbb{N}_{\geqslant 1}} \in \mathbb{N}^{\oplus\infty}$ such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots.$$

The value $\sup\{i \in \mathbb{N}_{\geqslant 1} : \lambda_i > 0\}$ (with the convention $\sup\varnothing = 0$) is called the *length* of the partition λ. For any $\delta \in \mathbb{N}$, we denote by $\mathscr{P}_\delta$ the set of partitions of weight δ.

If $\lambda = (\lambda_i)_{i\in\mathbb{N}_{\geqslant 1}}$ is a sequence in $\mathbb{N}^{\oplus\infty}$, we denote by $\widetilde{\lambda} = (\widetilde{\lambda}_n)_{n\in\mathbb{N}_{\geqslant 1}}$ the sequence defined as

$$\widetilde{\lambda}_n = \sum_{i\in\mathbb{N}_{\geqslant 1},\, \lambda_i \geqslant n} 1.$$

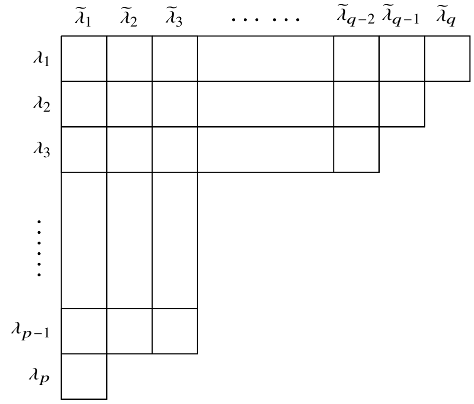
We call $\widetilde{\lambda}$ the *transpose* of λ. Clearly one has

$$\widetilde{\lambda}_1 \geqslant \widetilde{\lambda}_2 \geqslant \ldots$$

and hence $\widetilde{\lambda}$ is a partition. Moreover, the following equalities hold:

$$\sum_{n\in\mathbb{N}_{\geqslant 1}} \widetilde{\lambda}_n = \sum_{\substack{(i,n)\in\mathbb{N}_{\geqslant 1}^2 \\ \lambda_i > n}} 1 = \sum_{i\in\mathbb{N}_{\geqslant 1}} \sum_{\substack{n\in\mathbb{N}_{\geqslant 1} \\ n\leqslant\lambda_i}} 1 = \sum_{i\in\mathbb{N}_{\geqslant 1}} \lambda_i.$$

Note that the double transpose $\widetilde{\widetilde{\lambda}}$ is equal to the sequence λ sorted in the decreasing order. The following graph illustrates the transpose of a partition.



Let V be a free k-module of finite rank. For any

$$\lambda = (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, \ldots) \in \mathbb{N}^{\oplus\infty},$$

we let

$$\begin{aligned} V^{\otimes\lambda} &:= V^{\otimes\lambda_1} \otimes \cdots \otimes V^{\otimes\lambda_p} \\ \Lambda^{\lambda}(V) &:= \Lambda^{\lambda_1}(V) \otimes \cdots \otimes \Lambda^{\lambda_p}(V), \\ S^{\lambda}(V) &:= S^{\lambda_1}(V) \otimes \cdots \otimes S^{\lambda_p}(V). \end{aligned}$$

By the isomorphism (A.1), we can identify $S^{\lambda}(V)$ with a direct summand of $S^{\delta}(V^{\oplus p})$. Moreover, if λ and μ are two elements of $\mathbb{N}^{\oplus\infty}$, one has a commutative diagram of canonical $\mathrm{SL}(V)$-linear maps

$$\begin{array}{ccc} V^{\otimes\lambda} \otimes V^{\otimes\mu} & \xrightarrow{\simeq} & V^{\otimes(\lambda+\mu)} \\ \big\downarrow & & \big\downarrow \\ S^{\lambda}(V) \otimes S^{\mu}(V) & \longrightarrow & S^{\lambda+\mu}(V) \end{array}$$

If $\lambda = (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, \ldots)$ is a partition if its transpose is of the form

$$\widetilde{\lambda} = (\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_q, 0, \ldots, 0, \ldots)$$

with $\widetilde{\lambda}_q > 0$, we denote by $L^{\lambda}(V)$ the image of the following composed map

$$\Lambda^{\lambda}(V) = \Lambda^{\lambda_1}(V) \otimes \cdots \otimes \Lambda^{\lambda_p}(V) \xrightarrow{\alpha_\lambda} V^{\otimes|\lambda|} \xrightarrow{\beta_\lambda} S^{\widetilde{\lambda}_1}(V) \otimes \cdots \otimes S^{\widetilde{\lambda}_q}(V) = S^{\widetilde{\lambda}}(V),$$

where α_λ is induced by the anti-symmetrization maps $\Lambda^{\lambda_i}(V) \to V^{\otimes\lambda_i}$, and β_λ sends $x_1 \otimes \cdots \otimes x_{|\lambda|}$ to

$$\begin{aligned} &(x_1 x_{\lambda_1+1} x_{\lambda_1+\lambda_2+1} \cdots x_{\lambda_1+\cdots+\lambda_{\widetilde{\lambda}_1-1}+1}) \\ &\qquad \otimes (x_2 x_{\lambda_1+2} x_{\lambda_1+\lambda_2+2} \cdots x_{\lambda_1+\cdots+\lambda_{\widetilde{\lambda}_2-1}+2}) \\ &\qquad\qquad \otimes \cdots \otimes (x_{\lambda_1} x_{\lambda_1+\lambda_1} x_{\lambda_1+\lambda_2+\lambda_1} \cdots x_{\lambda_1+\cdots+\lambda_{\widetilde{\lambda}_q-1}+\lambda_1}). \end{aligned}$$

The following are two fundamental examples for partitions of $\delta \in \mathbb{N}$

$$L^{(\delta)}(V) = \Lambda^{\delta}(V), \quad L^{(1,\ldots,1)}(V) = S^{\delta}(V).$$

It can be shown that $L^{\lambda}(V)$ is a free k-module of finite rank, and L^{λ} defines a functor from the category of free k-modules of finite rank to itself (see [**3**, §II.2]). It is called the *Schur functor* with respect to λ.

A.1.4. First fundamental theorem of classical invariant theory. In this subsection, we assume that k is a field (of any characteristic). We recall the first fundamental theorem of classical invariant theory in a form proved by De Concini and Procesi. We refer to [**41**, Theorem 3.3] for proof, see also [**42**, Theorem 2.1].

Theorem A.1.1. *Let V be a finite-dimensional vector space over k, r be the dimension of V over k, and p be a positive integer. Let $V_1, \ldots, V_p$ be p identical copies of V. We consider the canonical action of the special linear group* $\mathrm{SL}(V)$ *on the symmetric algebra* $\mathrm{Sym}(V_1 \oplus \cdots \oplus V_p)$. *Then the invariant sub-k-algebra* $\mathrm{Sym}(V_1 \oplus \cdots \oplus V_p)^{\mathrm{SL}(V)}$ *is generated by one-dimensional k-vector subspaces of the form*

$$\mathrm{Im}\Big(\det(V) = \Lambda^r(V) \longrightarrow V_{i_1} \otimes \cdots \otimes V_{i_r}\Big)$$

in identifying $V_{i_1} \otimes \cdots \otimes V_{i_r}$ with a direct factor of $S^r(V_1 \oplus \cdots \oplus V_p)$ via the isomorphism (A.1)*, where $i_1, \ldots, i_r$ are distinct elements of $\{1, \ldots, p\}$, and in the above formula we consider the anti-symmetrization map defined in §A.1.2.*

Remark A.1.2. Let λ be a partition, p be the length of λ, and δ be the weight of λ. We identify $S^\lambda(V)$ with a vector subspace of $S^\delta(V^{\oplus p})$. Theorem A.1.1 shows that one can lift the invariant vectors of $S^\delta(V^{\oplus p})$ to tensor powers. More precisely, if $S^\lambda(V)^{\mathrm{SL}(V)}$ is not zero, then δ should be divisible by r, and $S^\lambda(V)^{\mathrm{SL}(V)}$ identifies with the image of

$$\bigoplus_{\substack{(\mu_1,\ldots,\mu_{\delta/r})\in\mathscr{D}_r^{\delta/r}\\ \mu_1+\cdots+\mu_{\delta/r}=\lambda}} \det(V)^{\otimes(\delta/r)} \longrightarrow S^\lambda(V).$$

where $\mathscr{D}_r$ denotes the set of sequences in $\mathbb{N}^{\oplus\infty}$ of weight r and with coordinates in $\{0,1\}$, and for any $(\mu_1, \ldots, \mu_{\delta/r}) \in \mathscr{D}_r^{\delta/r}$ such that $\mu_1 + \cdots + \mu_{\delta/r} = \lambda$, we consider the composed map

$$\det(V)^{\otimes(\delta/r)} \longrightarrow V^{\otimes\mu_1} \otimes \cdots \otimes V^{\otimes\mu_{\delta/r}} \longrightarrow V^{\otimes\lambda} \longrightarrow S^\lambda(V),$$

where the first arrow is induced by anti-symmetrization maps.

A.1.5. Cauchy decomposition. In this subsection, we consider two free k-modules of finite rank V and W. The symmetric algebra $\mathrm{Sym}(V \otimes W)$ is naturally equipped with a structure of graded $\mathrm{GL}(V) \times \mathrm{GL}(W)$-module. In the case where k contains $\mathbb{Q}$, then it is known that $\mathrm{Sym}(V \otimes W)$ is isomorphic as $\mathrm{GL}(V) \times \mathrm{GL}(W)$-module to a direct sum

$$\bigoplus_\lambda L^\lambda(V) \otimes L^\lambda(W),$$

where λ runs over the set of all partitions. In general, such decomposition is not always possible.

We equip the set $\mathbb{N}^{\oplus\infty}$ with the lexicographic order. This is a total order. For any $\delta \in \mathbb{N}$, the $\mathrm{GL}(V) \times \mathrm{GL}(W)$-module $S^\delta(V \otimes W)$ admits a decreasing filtration indexed by $\mathscr{P}_\delta$ such that the sub-quotient indexed by λ is isomorphic to $L^\lambda(V) \otimes L^\lambda(W)$. In particular, $S^\delta(V \otimes W)$ admits a sub-$\mathrm{GL}(V) \times \mathrm{GL}(W)$-module which is isomorphic to

$$L^{(\delta)}(V) \otimes L^{(\delta)}(W) = \Lambda^\delta(V) \otimes \Lambda^\delta(W)$$

This result is called the *Cauchy decomposition formula* for symmetric power. We refer the readers to [**3**, Theorem III.1.4] for more details.

A.1.6. Case of several modules. In this subsection, we apply Cauchy decomposition formula to several k-modules. We first illustrate the case of three modules. Let V_1, V_2 and V_3 be three free k-modules of finite rank and δ be a natural number. By Cauchy decomposition formula, the symmetric power $S^{\delta}(V_1\otimes V_2\otimes V_3)$ admits a decreasing filtration of sub-$\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)\times\mathrm{GL}(V_3)$-modules indexed by

$$\mathscr{P}_{\delta}=\{\lambda\in\mathbb{N}^{\oplus\infty}\;:\;|\lambda|=\delta\}$$

such that the subquotient indexed by λ of the filtration is isomorphic to

$$L^{\lambda}(V_1)\otimes L^{\lambda}(V_2\otimes V_3).$$

Let $\widetilde{\lambda}=(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_q,0,\ldots,0,\ldots)$ be the transpose of λ. By definition $L^{\lambda}(V_2\otimes V_3)$ is a sub-$\mathrm{GL}(V_2)\times\mathrm{GL}(V_3)$-module of

$$S^{\widetilde{\lambda}_1}(V_2\otimes V_3)\otimes\cdots\otimes S^{\widetilde{\lambda}_q}(V_2\otimes V_3). \tag{A.2}$$

We now apply Cauchy decomposition formula to each of the tensor powers

$$S^{\widetilde{\lambda}_i}(V_2\otimes V_3).$$

By passing to the tensor product of filtrations, we obtain a decreasing filtration of (A.2) indexed by $\mathscr{P}_{\widetilde{\lambda}_1}\times\cdots\times\mathscr{P}_{\widetilde{\lambda}_q}$ (equipped with the lexicographic order), such that the subquotient indexed by

$$(\mu_1,\ldots,\mu_q)\in\mathscr{P}_{\widetilde{\lambda}_1}\times\cdots\times\mathscr{P}_{\widetilde{\lambda}_q}$$

is isomorphic to

$$L^{\mu_1}(V_2)\otimes\cdots\otimes L^{\mu_q}(V_2)\otimes L^{\mu_1}(V_3)\otimes\cdots\otimes L^{\mu_q}(V_3). \tag{A.3}$$

By combining all non-zero coordinates of $\widetilde{\mu}_1,\ldots,\widetilde{\mu}_q$ into a single partition, we obtain a partition η and can identify (A.3) with a sub-$\mathrm{GL}(V_2)\times\mathrm{GL}(V_3)$-module of $S^{\eta}(V_2)\otimes S^{\eta}(V_3)$. This filtration induces by restriction a decreasing filtration on $L^{\lambda}(V_2\otimes V_3)$. The sub-quotient of the latter indexed by $(\mu_1,\ldots,\mu_q)$ identifies with a sub-$\mathrm{GL}(V_2)\times\mathrm{GL}(V_3)$-module of (A.3). By induction we obtain the following result.

Proposition A.1.3. *Let $d\in\mathbb{N}_{\geqslant 2}$ and $(V_i)_{i=1}^{d}$ be a family of free k-modules of finite rank and δ be a natural number. Let V be the tensor product $V_1\otimes\cdots\otimes V_d$ and $G=\mathrm{SL}(V_1)\times\cdots\times\mathrm{SL}(V_d)$. There exist a finite totally ordered set $\Theta_{\delta,d}$, a map*

$$h=(h_1,\ldots,h_d):\Theta_{\delta,d}\longrightarrow\mathscr{P}_{\delta}^{d},$$

and a decreasing $\Theta_{\delta,d}$-filtration of $S^{\delta}(V)$ such that the subquotient indexed by $a\in\Theta_{\delta,d}$ is isomorphic to a sub-G-module of

$$S^{h_1(a)}(V_1)\otimes\cdots\otimes S^{h_d(a)}(V_d).$$

Proof. We reason by induction on d. The case where $d=2$ comes from Cauchy decomposition formula. Assume that $d\geqslant 3$ and that the proposition has been proved for $d-1$ free k-modules of finite rank. We apply the induction hypothesis to $V_1,\ldots,V_{d-2}$ and $V_{d-1}\otimes V_d$ to obtain a finite totally ordered set $\Theta_{\delta,d-1}$ and a map

$$h=(h_1,\ldots,h_{d-1}):\Theta_{\delta,d-1}\longrightarrow\mathscr{P}_{\delta}^{d-1}$$

together with a decreasing $\Theta_{\delta,d-1}$-filtration of $S^{\delta}(V)$ such that the subquotient indexed by $a\in\Theta_{\delta,d-1}$ is isomorphic to a sub-G-module of

$$S^{h_1(a)}(V_1)\otimes\cdots\otimes S^{h_{d-2}(a)}(V_{d-2})\otimes S^{h_{d-1}(a)}(V_{d-1}\otimes V_d).$$

Assume that $h_{d-1}(a)$ is of the form $(\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, \ldots)$. We apply Cauchy decomposition formula to $S^{\lambda_i}(V_{d-1} \otimes V_d)$ to obtain a decreasing filtration of

$$S^{h_{d-1}(a)}(V_{d-1} \otimes V_d)$$

indexed by $\mathscr{P}_{\lambda_1} \times \cdots \times \mathscr{P}_{\lambda_p}$ such that the sub-quotient indexed by

$$(\mu_1, \ldots, \mu_p) \in \mathscr{P}_{\lambda_1} \times \cdots \times \mathscr{P}_{\lambda_p}$$

is isomorphic to $L^{\mu_1}(V_{d-1}) \otimes \cdots \otimes L^{\mu_p}(V_{d-1}) \otimes L^{\mu_1}(V_d) \otimes \cdots \otimes L^{\mu_p}(V_d)$, which identifies a sub-$\mathrm{GL}(V_{d-1}) \times \mathrm{GL}(V_d)$-module of some $S^{\eta}(V_{d-1}) \otimes S^{\eta}(V_d)$, where η is a partition of weight δ. In this way we obtain a refinement of the $\Theta_{\delta,d-1}$-filtration of $S^{\delta}(V)$ which satisfies the required property. The proposition is thus proved. $\square$

Theorem A.1.4. *We keep the notation and the assumptions of Proposition A.1.3, and assume in addition that k is a field. For any $i \in \{1, \ldots, d\}$, let r_i be the dimension of V_i over k. If the space $S^{\delta}(V)^G$ of G-invariant vectors in $S^{\delta}(V)$ is non-zero, then δ is divisible by $\mathrm{lcm}(r_1, \ldots, r_d)$. Moreover, $S^{\delta}(V)^G$ identifies with the image of the following k-linear map*

$$\bigoplus_{(\sigma_1,\ldots,\sigma_d)\in\mathfrak{S}_{\delta}^d} \det(V_1)^{\otimes(\delta/r_1)} \otimes \cdots \otimes \det(V_d)^{\otimes(\delta/r_d)} \longrightarrow S^{\delta}(V), \qquad \text{(A.4)}$$

where for each $(\sigma_1, \ldots, \sigma_d) \in \mathfrak{S}_{\delta}^d$, we consider the composed map

$$\begin{array}{ccc} \det(V_1)^{\otimes(\delta/r_1)} \otimes \cdots \otimes \det(V_d)^{\otimes(\delta/r_d)} & \longrightarrow & V_1^{\otimes\delta} \otimes \cdots \otimes V_d^{\otimes\delta} \\ & & \Big\downarrow {\scriptstyle \sigma_1\otimes\cdots\otimes\sigma_d} \\ & & V_1^{\otimes\delta} \otimes \cdots \otimes V_d^{\otimes\delta} \cong V^{\otimes\delta} \longrightarrow S^{\delta}(V) \end{array}$$

where the first arrow is induced by the anti-symmetrization map.

Proof. By Proposition A.1.3, there exists a finite totally ordered set $\Theta_{\delta,d}$, a map

$$h = (h_1, \ldots, h_d) : \Theta_{\delta,d} \longrightarrow \mathscr{P}_{\delta}^d,$$

and a decreasing $\Theta_{\delta,d}$-filtration $\mathcal{F}$ of $S^{\delta}(V)$ such that the subquotient indexed by $a \in \Theta_{\delta,d}$ is isomorphic to a sub-G-module of

$$S^{h_1(a)}(V_1) \otimes \cdots \otimes S^{h_d(a)}(V_d).$$

Let s be a non-zero element of $S^{\delta}(V)^G$, a be the greatest element of $\Theta_{\delta,d}$ such that $s \in \mathcal{F}^a(S^{\delta}(V))$. Let $\mathrm{sq}^a(S^{\delta}(V))$ be the subquotient of the filtration $\mathcal{F}$ at a. By definition, the canonical image of s in $\mathrm{sq}^a(S^{\delta}(V))$ is a non-zero element of $\mathrm{sq}^a(S^{\delta}(V))^G$, which is contained in

$$\big(S^{h_1(a)}(V_1) \otimes \cdots \otimes S^{h_d(a)}(V_d)\big)^G = S^{h_1(a)}(V_1)^{\mathrm{SL}(V_1)} \otimes \cdots \otimes S^{h_1(a)}(V_d)^{\mathrm{SL}(V_d)}.$$

By Remark A.1.2, there exists an element s' in the image of (A.4) such that $s - s'$ belongs to

$$\bigcup_{b\in\Theta_{\delta,d},\, b>a} \mathcal{F}^b(S^{\delta}(V)).$$

Iterating this procedure we obtain that s actually belongs to the image of (A.4). The assertion is thus proved. $\square$

A.2. Symmetric power norm

Throughout the section, we let k be a field and V be a finite-dimensional vector space over k. We assume that the field k is equipped with an absolute value $|\cdot|$ such that k is complete with respect to the topology induced by $|\cdot|$. We also assume that the vector space V is equipped with a norm $\|\cdot\|$, which is either ultrametric (when $|\cdot|$ is non-Archimedean) or induced by an inner product (when $|\cdot|$ is Archimedean). We denote by $\|\cdot\|_*$ the dual norm of $\|\cdot\|$ on the dual vector space $V^\vee$. Recall that the norm $\|\cdot\|_*$ is defined as

$$\forall\, f \in V^\vee, \quad \|f\|_* = \sup_{x \in V\setminus\{0\}} \frac{|f(x)|}{\|x\|}.$$

It is also ultrametric or induced by an inner product.

A.2.1. Orthogonal basis. Let α be an element of $]0,1]$. We say that a basis $(e_i)_{i=1}^d$ of V is *α-orthogonal* if the following inequality holds:

$$\forall\,(\lambda_1,\ldots,\lambda_d) \in k^n, \quad \|\lambda_1 e_1 + \cdots + \lambda_d e_d\| \geqslant \alpha \max_{i\in\{1,\ldots,d\}} |\lambda_i|\cdot\|e_i\|.$$

A 1-orthogonal basis is also called an *orthogonal basis*. It is not hard to check that, in the case where $|\cdot|$ is Archimedean (and $\|\cdot\|$ is induced by an inner product $\langle\,,\rangle$), the orthogonality is equivalent to the usual definition (cf. [**36**, Proposition 1.2.3]): a basis $(e_i)_{i=1}^d$ is 1-orthogonal if and only if

$$\forall\,(i,j) \in \{1,\ldots,d\}^2, \quad \text{if } i \neq j \text{ then } \langle e_i, e_j\rangle = 0.$$

Assume that $|\cdot|$ is non-Archimedean. Let $\boldsymbol{e} = (e_i)_{i=1}^d$ be a basis of V and let $\|\cdot\|_{\boldsymbol{e}}$ be the norm of V defined as

$$\forall\,(\lambda_1,\ldots,\lambda_d) \in k^d, \quad \|\lambda_1 e_1 + \cdots + \lambda_d e_d\|_{\boldsymbol{e}} = \max_{i\in\{1,\ldots,d\}} |\lambda_i|\cdot\|e_i\|.$$

Note that $\|\cdot\|_{\boldsymbol{e}}$ is an ultrametric norm of V, and one has $\|\cdot\| \leqslant \|\cdot\|_{\boldsymbol{e}}$ (since the norm $\|\cdot\|$ is ultrametric). Moreover, $\boldsymbol{e}$ is an orthogonal basis of $(V, \|\cdot\|_{\boldsymbol{e}})$. For any $\alpha \in\,]0,1]$ the basis $(e_i)_{i=1}^d$ is α-orthogonal with respect to $\|\cdot\|$ if and only if

$$d(\|\cdot\|, \|\cdot\|_{\boldsymbol{e}}) := \sup_{x\in V\setminus\{0\}} \Big| \ln\|x\| - \ln\|x\|_{\boldsymbol{e}} \Big| \leqslant |\ln(\alpha)|.$$

By the ultrametric Gram-Schimdt procedure (see for example [**36**, Proposition 1.2.30]), for any $\alpha \in\,]0,1[$, the ultrametrically normed vector space $(V, \|\cdot\|)$ admits an α-orthogonal basis. Therefore, there exists a a sequence of ultrametric norms $(\|\cdot\|_{n\in\mathbb{N}})$ such that $(V, \|\cdot\|_n)$ admits an orthogonal basis for any n, and that

$$\lim_{n\to+\infty} d(\|\cdot\|, \|\cdot\|_n) = 0.$$

A.2.2. Direct sum. Let $(V_i, \|\cdot\|_i)$, $i \in \{1,\ldots,\delta\}$ be a family of finite-dimensional normed vector space over k. We assume that, for any $i \in \{1,\ldots,\delta\}$, the norm $\|\cdot\|_i$ is either ultrametric or induced by an inner product. In the case where $|\cdot|$ is non-Archimedean, we equip $V_1 \oplus \cdots \oplus V_\delta$ with the *ultrametric direct sum norm*, defined as

$$\forall\,(x_1,\ldots,x_\delta) \in V_1 \oplus \cdots \oplus V_\delta, \quad \|(x_1,\ldots,x_\delta)\| = \max_{i\in\{1,\ldots,\delta\}} \|x_i\|_i.$$

In the case where $|\cdot|$ is Archimedean, we equip $V_1 \oplus \cdots \oplus V_\delta$ with the *orthogonal direct sum norm*, namely

$$\|(x_1,\ldots,x_\delta)\|^2 = \sum_{i=1}^{\delta} \|x_i\|_i^2.$$

A.2.3. Symmetric power norm. Let δ be a natural number and let

$$\pi : V^{\otimes\delta} \longrightarrow S^{\delta}(V)$$

be the surjective k-linear map which sends $x_1 \otimes \cdots \otimes x_\delta \in V^{\otimes\delta}$ to $x_1 \cdots x_\delta$. We equip $V^{\otimes\delta}$ with the ε-tensor product norm or the orthogonal tensor product norm according to whether $|\cdot|$ is non-Archimedean or Archimedean, respectively. We then equip $S^{\delta}(V)$ with the quotient norm.

Proposition A.2.1. *Assume that the absolute value $|\cdot|$ is non-Archimedean. Let $\alpha \in\,]0,1]$ and let $\boldsymbol{e} = (e_i)_{i=1}^d$ be an α-orthogonal basis of V. Then the elements*

$$\boldsymbol{e}^{\boldsymbol{a}}, \quad \boldsymbol{a} \in \mathbb{N}^d,\ |\boldsymbol{a}| = \delta$$

form an α^{δ}-orthogonal basis of $S^{\delta}(V)$. Moreover, for any $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$ such that $|\boldsymbol{a}| = \delta$, one has

$$\alpha^{\delta} \prod_{i=1}^{d} \|e_i\|^{a_i} \leqslant \|\boldsymbol{e}^{\boldsymbol{a}}\| \leqslant \prod_{i=1}^{d} \|e_i\|^{a_i}. \tag{A.5}$$

Proof. Denote by $f : \{1, \ldots, d\}^{\delta} \to \mathbb{N}^d$ the map which sends $(b_1, \ldots, b_\delta)$ to the vector

$$\Big(\operatorname{card}\big(\{j \in \{1, \ldots, \delta\} \mid b_j = i\}\big)\Big)_{i=1}^{d}.$$

Let $\pi : V^{\otimes\delta} \to S^{\delta}(V)$ be the projection map. For any

$$b = (b_1, \ldots, b_\delta) \in \{1, \ldots, d\}^{\delta},$$

denote by e_b the split tensor $e_{b_1} \otimes \cdots \otimes e_{b_\delta} \in V^{\otimes\delta}$.

For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ such that $|\boldsymbol{a}| = \delta$, one has

$$\|\boldsymbol{e}^{\boldsymbol{a}}\| = \inf\left\{\left\|\sum_{b \in f^{-1}(\{\boldsymbol{a}\})} \lambda_b e_b\right\| \;:\; \sum_{b \in f^{-1}(\{\boldsymbol{a}\})} \lambda_b = 1,\ \right\}.$$

Hence (see [**36**, Remark 1.1.56])

$$\|\boldsymbol{e}^{\boldsymbol{a}}\| \leqslant \|e_1\|^{a_1} \cdots \|e_n\|^{a_n}.$$

Since $(e_i)_{i=1}^d$ is an α-orthogonal basis, $(e_b)_{b \in \{1,\ldots,d\}^{\delta}}$ is an α^{δ}-orthogonal basis of $V^{\otimes\delta}$ (see [**36**, Proposition 1.2.19]). For any $(\lambda_b)_{b \in f^{-1}(\{\boldsymbol{a}\})} \in k^{f^{-1}(\{\boldsymbol{a}\})}$ such that

$$\sum_{b \in f^{-1}(\{\boldsymbol{a}\})} \lambda_b = 1,$$

one has

$$\|e_1\|^{i_1} \cdots \|e_d\|^{i_d} \leqslant \|e_1\|^{i_1} \cdots \|e_d\|^{i_d} \max_{b \in f^{-1}(\{\boldsymbol{a}\})} |\lambda_b| \leqslant \alpha^{-\delta} \left\|\sum_{b \in f^{-1}(\{\boldsymbol{a}\})} \lambda_b e_b\right\|,$$

which leads to $\|\boldsymbol{e}^{\boldsymbol{a}}\| \geqslant \alpha^{-\delta} \|e_1\|^{a_1} \cdots \|e_d\|^{a_d}$.

For any

$$s = \sum_{b \in \{1,\ldots,n\}^{\delta}} \mu_b e_b \in E^{\otimes\delta},$$

one has

$$\pi(s) = \sum_{\boldsymbol{a} \in \mathbb{N}^n,\, |\boldsymbol{a}| = \delta} \Bigg(\sum_{b \in f^{-1}(\{\boldsymbol{a}\})} \mu_b\Bigg) \boldsymbol{e}^{\boldsymbol{a}}.$$

Moreover,

$$\begin{aligned}\|s\| &\geqslant \alpha^{\delta} \max_{\substack{\boldsymbol{a}=(a_1,\ldots,a_d)\in\mathbb{N}^d\\ |\boldsymbol{a}|=\delta}} \|e_1\|^{a_1}\cdots\|e_d\|^{a_d} \max_{b\in f^{-1}(\{\boldsymbol{a}\})}|\mu_b| \\ &\geqslant \alpha^{\delta} \max_{\substack{\boldsymbol{a}=(a_1,\ldots,a_d)\in\mathbb{N}^d\\ |\boldsymbol{a}|=\delta}} \|e_1\|^{a_1}\cdots\|e_d\|^{a_d} \bigg|\sum_{b\in f^{-1}(\{\boldsymbol{a}\})}\mu_b\bigg| \\ &\geqslant \alpha^{\delta} \max_{\substack{\boldsymbol{a}=(a_1,\ldots,a_d)\in\mathbb{N}^d\\ |\boldsymbol{a}|=\delta}} \|\boldsymbol{e}^{\boldsymbol{a}}\|\cdot\bigg|\sum_{b\in f^{-1}(\{\boldsymbol{a}\})}\mu_b\bigg|.\end{aligned}$$

Therefore, we obtain that $(e^{\boldsymbol{a}})_{\boldsymbol{a}\in\mathbb{N}^d,\,|\boldsymbol{a}|=\delta}$ forms an α^{δ}-orthogonal basis of $S^{\delta}(V)$, as required. □

REMARK A.2.2. Consider the case where $|\cdot|$ is the trivial absolute value. In this case, the ultrametric norm $\|\cdot\|$ corresponds to a decreasing $\mathbb{R}$-filtration $\mathcal{F}$ on V such that

$$\forall\, t\in\mathbb{R},\quad \mathcal{F}^t(V)=\{x\in V\ :\ \|x\|\leqslant \mathrm{e}^{-t}\}.$$

We can also express this $\mathbb{R}$-filtration as an increasing sequence

$$0=V_0\subsetneq V_1\subsetneq\ldots\subsetneq V_r=V$$

together with a decreasing sequence

$$\mu_1>\ldots>\mu_r,$$

with $\mathcal{F}^t(V)=V_i$ when $t\in\,]\mu_{i+1},\mu_i]\cap\mathbb{R}$, where by convention $\mu_0=+\infty$ and $\mu_{r+1}=-\infty$. For any $t\in\mathbb{R}$, the subquotient

$$\mathrm{sq}^t(V):=\mathcal{F}^t(V)\Big/\bigcup_{\varepsilon>0}\mathcal{F}^{t+\varepsilon}(V)$$

is either the zero vector space when $t\notin\{\mu_1,\ldots,\mu_r\}$, or is equal to V_i/V_{i-1} when $t=\mu_i$.

By [**36**, Proposition 1.2.30], there exists an orthogonal basis $\boldsymbol{e}$ such that $\boldsymbol{e}\cap\mathcal{F}^t(V)$ forms a basis of $\mathcal{F}^t(V)$ for any $t\in\mathbb{R}$. By Proposition A.2.1, we obtain that the elements

$$\boldsymbol{e}^{\boldsymbol{a}},\quad \boldsymbol{a}\in\mathbb{N}^d,\ |\boldsymbol{a}|=\delta$$

form an orthogonal basis of $S^{\delta}(V)$. Moreover, for any $\boldsymbol{a}=(a_1,\ldots,a_n)\in\mathbb{N}^n$ such that $|\boldsymbol{a}|=\delta$, one has

$$\|\boldsymbol{e}^{\boldsymbol{a}}\|=\prod_{i=1}^{d}\|e_i\|^{a_i}.$$

Therefore, if we equip $S^{\delta}(V)$ with the $\mathbb{R}$-filtration induced by the symmetric product norm of $\|\cdot\|$, for any $t\in\mathbb{R}$ one has a natural isomorphism

$$\mathrm{sq}^t(S^{\delta}(V))\cong\bigoplus_{\substack{\boldsymbol{b}=(b_1,\ldots,b_r)\in\mathbb{N}^r\\ |\boldsymbol{b}|=b_1+\cdots+b_r=\delta\\ b_1\mu_1+\cdots+b_r\mu_r=t}} S^{b_1}(V_1/V_0)\otimes\cdots\otimes S^{b_r}(V_r/V_{r-1}).$$

A.2.4. Subquotient metric on symmetric power. In this subsection, we assume that $|\cdot|$ is non-Archimedean and $\|\cdot\|$ is ultrametric. We let $|\cdot|_0$ be the trivial absolute value on k and $\|\cdot\|_0$ be an ultrametric norm on V (with respect to the trivial absolute value), which corresponds to an $\mathbb{R}$-filtration $\mathcal{F}$ on V, or an increasing sequence

$$0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_r = V$$

of vector subspaces of V together with a decreasing sequence

$$\mu_1 > \ldots > \mu_r$$

of real numbers, as explained in Remark A.2.2.

Let δ be a natural integer. We equip $S^\delta(V)$ with the $\mathbb{R}$-filtration corresponding to the symmetric product norm of $\|\cdot\|_0$. As we have seen in Remark A.2.2, for any $t \in \mathbb{R}$, the subquotient $\mathrm{sq}^t(S^\delta(V))$ is isomorphic to

$$\bigoplus_{\substack{\boldsymbol{b}=(b_1,\ldots,b_r)\in\mathbb{N}^r\\ |\boldsymbol{b}|=b_1+\cdots+b_r=\delta\\ b_1\mu_1+\cdots+b_r\mu_r=t}} S^{b_1}(V_1/V_0)\otimes\cdots\otimes S^{b_r}(V_r/V_{r-1}). \tag{A.6}$$

Note that the norm $\|\cdot\|$ induces by passing to subquotient a norm on each V_i/V_{i-1}, which leads to a symmetric power norm on $S^b(V_i/V_{i-1})$ for any $b \in \mathbb{N}$. For $(b_1,\ldots,b_r) \in \mathbb{N}^r$, we equip $S^{b_1}(V_1/V_0)\otimes\cdots\otimes S^{b_r}(V_r/V_{r-1})$ with the tensor product of symmetric power norms (ε-tensor product when $|\cdot|$ is non-Archimedean and orthogonal tensor product when $|\cdot|$ is Archimedean), and the vector space (A.6) with the direct sum norm (ultrametric direct sum if $|\cdot|$ is non-Archimedean and orthogonal direct sum if $|\cdot|$ is Archimedean).

Here we are interested in the comparison between the subquotient norm on $\mathrm{sq}^t(S^\delta(V))$ induced by the symmetric tensor power norm and the direct sum of tensor product norm on (A.6) described above.

Proposition A.2.3. *Assume that the absolute value $|\cdot|$ is non-Archimedean. Then, for any $t \in \mathbb{R}$ the isomorphism*

$$\mathrm{sq}^t(S^\delta(V)) \cong \bigoplus_{\substack{\boldsymbol{b}=(b_1,\ldots,b_r)\in\mathbb{N}^r\\ |\boldsymbol{b}|=b_1+\cdots+b_r=\delta\\ b_1\mu_1+\cdots+b_r\mu_r=t}} S^{b_1}(V_1/V_0)\otimes\cdots\otimes S^{b_r}(V_r/V_{r-1}) \tag{A.7}$$

is an isometry.

Proof. Let α be an element of $]0,1[$. By [**36**, Proposition 1.2.30], for any $i \in \{1,\ldots,r\}$ there exists

$$\boldsymbol{e}^{(i)} = (e_1^{(i)},\ldots,e_{d_i}^{(i)}) \in (V_i \setminus V_{i-1})^{d_i}$$

such that

(a) the images of $e_1^{(i)},\ldots,e_{d_i}^{(i)}$ in V_i/V_{i-1} form a basis of the latter, where $d_i = \dim_k(V_i/V_{i-1})$,
(b) $(\boldsymbol{e}^{(1)},\ldots,\boldsymbol{e}^{(r)})$ forms an α-orthogonal basis of V.

By Proposition A.2.1, the elements

$$(\boldsymbol{e}^{(1)})^{\boldsymbol{a}^{(1)}}\cdots(\boldsymbol{e}^{(r)})^{\boldsymbol{a}^{(r)}}, \quad (\boldsymbol{a}^{(1)},\ldots,\boldsymbol{a}^{(r)}) \in \mathbb{N}^{d_1}\times\cdots\times\mathbb{N}^{d_r},\ \left|\boldsymbol{a}^{(1)}\right|+\cdots\left|\boldsymbol{a}^{(r)}\right| = \delta$$

form an α^δ-orthogonal basis of $S^\delta(V)$. We let $\boldsymbol{x}^{(i)} = (x_1^{(i)},\ldots,x_{d_i}^{(i)}) \in (V_i/V_{i-1})^{d_i}$, where $x_j^{(i)}$ denotes the class of $e_j^{(i)}$ in V_i/V_{i-1}. Since $(\boldsymbol{e}^{(1)},\ldots,\boldsymbol{e}^{(r)})$ forms an α-orthogonal basis of V, we obtain that

$$\alpha\|e_j^{(i)}\| \leqslant \|x_j^{(i)}\| \leqslant \|e_j^{(i)}\|. \tag{A.8}$$

For any $\boldsymbol{b} = (b_1, \ldots, b_r) \in \mathbb{N}^r$ such that $b_1 + \cdots + b_r = \delta$, the vectors

$$(\boldsymbol{x}^{(1)})^{\boldsymbol{a}^{(1)}} \otimes \cdots \otimes (\boldsymbol{x}^{(r)})^{\boldsymbol{a}^{(r)}}, \quad (\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(r)}) \in \mathbb{N}^{d_1} \times \cdots \times \mathbb{N}^{d_r}, \ \forall\, i \in \{1, \ldots, r\},\ |\boldsymbol{a}^i| = b_i,$$

form an α^δ-orthogonal basis of

$$S^{b_1}(V_1/V_0) \otimes \cdots \otimes S^{b_r}(V_r/V_{r-1}).$$

Moreover, by (A.5) and (A.8) we obtain that

$$\left| \ln \frac{\left\|(\boldsymbol{e}^{(1)})^{\boldsymbol{a}^{(1)}} \cdots (\boldsymbol{e}^{(r)})^{\boldsymbol{a}^{(r)}}\right\|}{\|(\boldsymbol{x}^{(1)})^{\boldsymbol{a}^{(1)}} \otimes \cdots \otimes (\boldsymbol{x}^{(r)})^{\boldsymbol{a}^{(r)}}\|} \right| \leqslant 2\delta |\ln(\alpha)|.$$

Therefore, under the isomorphism (A.7), the distance between the norms on the left hand side and the right hand side is bounded from above by $3\delta|\ln(\alpha)|$. Since $\alpha \in \,]0,1[$ is arbitrary, we obtain that (A.7) is actually an isometry. □

A.2.5. Symmetric tensor. Let δ be a positive integer. We denote by

$$\mathrm{sym} : V^{\otimes\delta} \longrightarrow V^{\otimes\delta}$$

sending $x_1 \otimes \cdots \otimes x_\delta$ to

$$\sum_{\sigma\in\mathfrak{S}_\delta} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(\delta)}.$$

Proposition A.2.4. *Assume that the absolute value $|\cdot|$ is non-Archimedean. Let $\delta \in \mathbb{N}_{\geqslant 1}$. We equip $V^{\otimes\delta}$ with the ε-tensor power norm of $\|\cdot\|$. Then the K-linear map $\mathrm{sym} : V^{\otimes\delta} \longrightarrow V^{\otimes\delta}$ has operator norm $\leqslant 1$.*

Proof. Let T be an element of $V^{\otimes\delta}$. If we consider T as a δ-multilinear form on $V^\vee$, then the ε-tensor power norm of T is given by

$$\|T\|_\varepsilon = \sup_{(\alpha_1,\ldots,\alpha_\delta)\in(V^\vee\setminus\{0\})^\delta} \frac{|T(\alpha_1,\ldots,\alpha_\delta)|}{\|\alpha_1\|_* \cdots \|\alpha_\delta\|_*}.$$

Note that the element $\mathrm{sym}(T)$, viewed as a δ-multilinear form on $V^\vee$, is given by

$$\mathrm{sym}(T)(\alpha_1,\ldots,\alpha_\delta) = \sum_{\sigma\in\mathfrak{S}_\delta} T(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(\delta)}).$$

Since the absolute value $|\cdot|$ is non-Archimedean, we obtain

$$|\mathrm{sym}(T)(\alpha_1,\ldots,\alpha_\delta)| \leqslant \max_{\sigma\in\mathfrak{S}_\delta} |T(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(\delta)})| \leqslant \|T\|_\varepsilon \cdot \|\alpha_1\|_* \cdots \|\alpha_\delta\|_*,$$

which shows $\|\mathrm{sym}(T)\|_\varepsilon \leqslant \|T\|_\varepsilon$. □

Proposition A.2.5. *Assume that the absolute value $|\cdot|$ is Archimedean. Let $\delta \in \mathbb{N}_{\geqslant 1}$. We equip $V^{\otimes\delta}$ with the orthogonal tensor power norm of $\|\cdot\|$. Then the K-linear map $\mathrm{sym} : V^{\otimes\delta} \to V^{\otimes\delta}$ has operator norm $\leqslant \delta!$.*

Proof. Let $(e_j)_{j=1}^d$ be an orthonormal basis of V. Recall that an orthonormal basis of $(V^{\otimes\delta}, \|\cdot\|)$ is given by

$$e_{j_1} \otimes \cdots \otimes e_{j_\delta}, \quad (j_1,\ldots,j_\delta) \in \{1,\ldots,d\}^\delta.$$

Let

$$T = \sum_{\lambda=(\lambda_1,\ldots,\lambda_\delta)\in\{1,\ldots,d\}^\delta} a_\lambda e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_\delta} \in V^{\otimes\delta}.$$

One has

$$\|T\|^2 = \sum_{\lambda\in\{1,\ldots,d\}^\delta} |a_\lambda|^2.$$

Let $\mathscr{P}_{\delta,d}$ be the set of vectors $(a_1,\ldots,a_d)\in\mathbb{N}^d$ such that $a_1+\cdots+a_d=\delta$. For each $\boldsymbol{a}=(a_1,\ldots,a_d)\in S_\delta$, let $I_{\boldsymbol{a}}$ be the set of $(\lambda_1,\ldots,\lambda_\delta)\in\{1,\ldots,d\}^\delta$ such that

$$\forall\, j\in\{1,\ldots,d\},\quad a_j=\sum_{\substack{i\in\{1,\ldots,\delta\}\\ \lambda_i=j}} 1.$$

Then the following equality holds

$$\mathrm{sym}(T)=\sum_{\boldsymbol{a}=(a_1,\ldots,a_d)\in\mathscr{P}_{\delta,d}}\bigg(\sum_{\lambda\in I_{\boldsymbol{a}}} a_\lambda\bigg) a_1!\cdots a_d! \sum_{\lambda=(\lambda_1,\ldots,\lambda_\delta)\in I_{\boldsymbol{a}}} e_{\lambda_1}\otimes\cdots\otimes e_{\lambda_\delta},$$

which leads to

$$\begin{aligned}\|\mathrm{sym}(T)\|^2 &= \sum_{\boldsymbol{a}=(a_1,\ldots,a_d)\in\mathscr{P}_{\delta,d}} (a_1!\cdots a_d!)^2\bigg|\sum_{\lambda\in I_{\boldsymbol{a}}} a_\lambda\bigg|^2\frac{\delta!}{a_1!\cdots a_d!}\\ &\leqslant \sum_{\boldsymbol{a}=(a_1,\ldots,a_d)\in\mathscr{P}_{\delta,d}} (a_1!\cdots a_d!)^2\bigg(\sum_{\lambda\in I_{\boldsymbol{a}}}|a_\lambda|^2\bigg)\bigg(\frac{\delta!}{a_1!\cdots a_d!}\bigg)^2\\ &= (\delta!\|T\|)^2.\end{aligned}$$

□

REMARK A.2.6. Note that the k-linear map $\mathrm{sym}: V^{\otimes\delta}\longrightarrow V^{\otimes\delta}$ factors though the symmetric power $S^\delta(V)$. Moreover, in the case where k is of characteristic 0, the unique k-linear map $\mathrm{sym}': S^\delta(V)\to V^{\otimes\delta}$ such that the composition

$$V^{\otimes\delta}\twoheadrightarrow S^\delta(V)\xrightarrow{\mathrm{sym}'} V^{\otimes\delta}$$

identifies with $\mathrm{sym}: V^{\otimes\delta}\to V^{\otimes\delta}$ is injective. The above propositions show that, if we equip $V^{\otimes\delta}$ with the ε-tensor power norm (resp. orthogonal tensor power norm) of $\|\cdot\|$ in the case where $|\cdot|$ is non-Archimedean (resp. Archimedean) and equip $S^\delta(V)$ with the quotient norm, then the operator norm of sym' is bounded from above by 1 (resp. $\delta!$).

A.2.6. Determinant norm. Recall that we have fixed a finite-dimension normed vector space $(V,\|\cdot\|)$ over k. Let r be the dimension of V over k. We denote by $\det(V)$ the exterior power $\Lambda^r(V)$. This is a one-dimensional vector space over k. We equip it with the *determinant norm* $\|\cdot\|_{\det}$, which is defined as

$$\forall\,\eta\in\det(V),\quad \|\eta\|_{\det}=\inf_{\substack{(x_i)_{i=1}^r\in V^r\\ \eta=x_1\wedge\cdots\wedge x_r}}\|x_1\|\cdots\|x_r\|.$$

PROPOSITION A.2.7. *Assume that the absolue value $|\cdot|$ is non-Archimedean and the norm $\|\cdot\|$ is ultrametric. Then the anti-symmetrization map $\det(V)\to V^{\otimes r}$ is an isometry from $(\det(V),\|\cdot\|_{\det})$ to its image, where we consider the ε-tensor power norm on $V^{\otimes r}$.*

PROOF. By the ultrametric Gram-Schmidt procedure, one can approximate the norm $\|\cdot\|$ by a sequence of norms for which V admits an orthogonal basis. Therefore, we may assume without loss of generality that $(V,\|\cdot\|)$ has an orthogonal basis $(e_i)_{i=1}^r$. By [**36**, Proposition 1.2.25], one has

$$\|e_1\wedge\cdots\wedge e_r\|=\|e_1\|\cdots\|e_r\|.$$

Moreover, the anti-symmetrization of $e_1 \wedge \cdots \wedge e_r$ is given by

$$\sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(r)}. \tag{A.9}$$

Since

$$\{e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(r)} \;:\; \sigma \in \mathfrak{S}_r\}$$

is a subset of the orthogonal basis

$$e_{j_1} \otimes \cdots \otimes e_{j_r}, \quad (j_1, \ldots, j_r) \in \{1, \ldots, r\}^r,$$

we obtain that the norm of (A.9) is equal to $\|e_1\| \cdots \|e_r\|$. The proposition is thus proved. □

Remark A.2.8. In the case where $|\cdot|$ is Archimedean and $\|\cdot\|$ is induced by an inner product, the above result is no longer true. The orthogonal tensor power norm of the anti-symmetrization of an element η of $\det(V)$ is equal to $\sqrt{r!}\,\|\eta\|_{\det}$.

A.3. Maximal slopes of symmetric power

In this section, we prove that, on an adelic curve of perfect underlying field and without Archimedean places, the tensor product of semi-stable Hermitian adelic vector bundles remains semi-stable. This allows to justify that the argument used in the proof of [**36**, Proposition 5.3.1] is still valid in the positive characteristic case.

We fix a proper adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ with a perfect underlying field K. We assume in addition that, either the σ-algebra $\mathcal{A}$ is discrete, or the field K is countable.

A.3.1. Tensorial semi-stability. In this subsection, we assume that Ω_∞ is empty, namely the absolute value $|\cdot|_\omega$ is non-Archimedean for any $\omega \in \Omega$. We let $(\overline{E}_i)_{i=1}^d$ be a family of Hermitian adelic vector bundles on S and $\overline{E}$ be the orthogonal tensor product

$$\overline{E}_1 \otimes \cdots \otimes \overline{E}_d.$$

For any $i \in \{1, \ldots, d\}$, we let r_i be the dimension E_i over K. The purpose is to prove the following estimate.

Theorem A.3.1. *Let Q be a one-dimensional quotient vector space of E, equipped with the quotient norm family. Then the following inequality holds:*

$$\widehat{\deg}(\overline{Q}) \geqslant \sum_{i=1}^d \widehat{\mu}_{\min}(\overline{E}_i). \tag{A.10}$$

Proof. Let G be the product of special linear group schemes

$$\mathbb{SL}(E_1) \times \cdots \times \mathbb{SL}(E_d).$$

Note that the algebraic group G acts on the scheme $\mathbb{P}(E)$ and the tautological line bundle $O_E(1)$ is naturally equipped with a G-linear structure. In particular, the group

$$G(K) = \mathrm{SL}(E_1) \times \cdots \times \mathrm{SL}(E_d)$$

acts naturally on the sectional K-algebra

$$\bigoplus_{n \in \mathbb{N}} H^0(\mathbb{P}(E), O_E(n)) = \bigoplus_{n \in \mathbb{N}} S^n(E) = \mathrm{Sym}(E).$$

Let x be the rational point of $\mathbb{P}(E)$ which is represented by the one-dimensional quotient space Q.

Step 1: We suppose firstly that x is semi-stable in the sense of geometric invariant theory with respect to the G-linear line bundle $\mathcal{O}_E(1)$. In other words, we assume that there exists a positive integer δ and a section in $S^\delta(E) = H^0(\mathbb{P}(E), \mathcal{O}_E(\delta))$ invariant by the action of $G(K)$, which does not vanish at x. By Theorem A.1.4, we obtain that δ is divisible by $\operatorname{lcm}(r_1, \ldots, r_d)$ and there exists $(\sigma_1, \ldots, \sigma_d) \in \mathfrak{S}_\delta^d$ such that the following composed map is non-zero

$$\begin{array}{ccc}
\det(E_1)^{\otimes(\delta/r_1)} \otimes \cdots \otimes \det(E_d)^{\otimes(\delta/r_d)} & \longrightarrow & E_1^{\otimes\delta} \otimes \cdots \otimes E_d^{\otimes\delta} \\
& & \Big\downarrow {\scriptstyle \sigma_1\otimes\cdots\otimes\sigma_d} \\
& & E_1^{\otimes\delta} \otimes \cdots \otimes E_d^{\otimes\delta} \cong E^{\otimes\delta} \\
& & \Big\downarrow \\
& & S^\delta(E) \longrightarrow F^{\otimes\delta}
\end{array}$$

Therefore, we obtain

$$\widehat{\deg}(\overline{Q}) \geqslant \sum_{i=1}^d \frac{\delta}{r_i} \widehat{\deg}(\overline{E}_i) = \delta \sum_{i=1}^d \widehat{\mu}(\overline{E}_i) \geqslant \delta \sum_{i=1}^d \widehat{\mu}_{\min}(\overline{E}_i).$$

Step 2: In this step, we assume that x is not semi-stable under the action of G with respect to the G-linear line bundle $\mathcal{O}_E(1)$. Note that this condition is equivalent to the following: x is not semi-stable under the action of

$$\mathbb{GL}(E_1) \times \cdots \times \mathbb{GL}(E_d)$$

with respect to

$$\mathcal{O}_E(r_1 \cdots r_d) \otimes \pi^*(\det(E_1^\vee)^{\otimes b_1} \otimes \cdots \otimes \det(E_d^\vee)^{\otimes b_d}),$$

where $\pi : \mathbb{P}(E) \to \operatorname{Spec} K$ is the structural morphism and, for any $i \in \{1, \ldots, d\}$,

$$b_i := \frac{r_1 \cdots r_d}{r_i}.$$

Then the inequality (A.10) can be obtained following the same argument as in the proof of [**36**, Theorem 5.6.1]. □

Corollary A.3.2. *Let $(\overline{E}_i)_{i=1}^d$ be a family of Hermitian adelic line bundles on S. For any vector subspace F of $E_1 \otimes \cdots \otimes E_d$, one has*

$$\widehat{\mu}(\overline{F}) \leqslant \sum_{i=1}^d \widehat{\mu}_{\max}(\overline{E}_i).$$

In particular, if $\overline{E}_1, \ldots, \overline{E}_d$ are all semi-stable, then $\overline{E}_1 \otimes \cdots \otimes \overline{E}_d$ is also semi-stable.

Proof. We first treat the case where F is of dimension 1. We identify $F^\vee$ with a quotient vector space of $E_1^\vee \otimes \cdots \otimes E_d^\vee$. By Theorem A.10 we obtain

$$\widehat{\mu}(\overline{F}) = \widehat{\deg}(\overline{F}) = -\widehat{\deg}(\overline{F}^\vee) \leqslant -\sum_{i=1}^d \widehat{\mu}_{\min}(\overline{E}_i^\vee) = \sum_{i=1}^d \widehat{\mu}_{\max}(\overline{E}_i),$$

where the last equality comes from [**36**, Corollary 4.3.27].

In the following, we consider the general case. Without loss of generality, we may assume that F is the destabilizing vector subspace of $\overline{E}_1 \otimes \cdots \otimes \overline{E}_d$. In particular, $\overline{F}$ is semi-stable. Let s be the element of $F^\vee \otimes E_1 \otimes \cdots \otimes E_d$ be the element which corresponds to the inclusion map $f : F \to E_1 \otimes \cdots \otimes E_d$. Let L be the one-dimension vector subspace of $F \to E_1 \otimes \cdots \otimes E_d$ spanned by s. By the one-dimensional case of the statement proved above, one has

$$\begin{aligned}\widehat{\deg}(\overline{L}) &\leqslant \widehat{\mu}_{\max}(\overline{F}^\vee) + \sum_{i=1}^d \widehat{\mu}_{\max}(\overline{E}_d) = \widehat{\mu}(\overline{F}^\vee) + \sum_{i=1}^d \widehat{\mu}_{\max}(\overline{E}_d) \\ &= -\widehat{\mu}(\overline{F}) + \sum_{i=1}^d \widehat{\mu}_{\max}(\overline{E}_d)\end{aligned}$$

since $\overline{F}$ is assumed to be semi-stable. Moreover, for any $\omega \in \Omega$, if we denote by $\|\cdot\|_\omega$ the ε-tensor product norm on $F_\omega^\vee \otimes E_{1,\omega} \otimes \cdots \otimes E_{d,\omega}$, then $\|s\|_\omega$ identifies with the operator norm of f_ω, which is bounded from above by 1. Therefore one has $\widehat{\deg}(\overline{L}) \geqslant 0$, which shows that

$$\widehat{\mu}(\overline{F}) \leqslant \sum_{i=1}^d \widehat{\mu}(\overline{E}_d),$$

as required. □

Remark A.3.3. By passing to dual, we obtain from Corollary A.3.2 that the inequality (A.10) actually holds for quotient vector subspace of $\overline{E}_1 \otimes \cdots \otimes \overline{E}_d$ of arbitrary rank. In other words, the following inequality holds:

$$\widehat{\mu}_{\min}(\overline{E}_1 \otimes \cdots \otimes \overline{E}_d) \geqslant \sum_{i=1}^d \widehat{\mu}_{\min}(\overline{E}_i).$$

Therefore, the results of [**36**, Chapter 5] still hold in the case where K is a perfect field of positive characteristic. In particular, if $\overline{F}$ is a vector subspace of $E_1 \otimes \cdots \otimes E_d$, equipped with the restriction of the orthogonal tensor product norm family, then the dual statement of [**36**, Theorem 5.6.1] leads to the following inequality

$$\widehat{\mu}(\overline{F}) \leqslant \widehat{\mu}(F, \|\cdot\|_{0,F})$$

where $\|\cdot\|_{0,F}$ denotes the restriction of the ultrametric norm (where we consider the trivial absolute value on K) on $E_1 \otimes \cdots \otimes E_d$ by taking the ε-tensor product of norms associated with Harder-Narasimhan $\mathbb{R}$-filtrations of $\overline{E}_1, \ldots, \overline{E}_d$.

A.3.2. Slope of a symmetric power.

Proposition A.3.4. *Assume that Ω_∞ is empty. Let $\overline{E}$ be a Hermitian adelic vector bundle on S and δ be a positive number. The following equality holds*

$$\widehat{\mu}(S^\delta(\overline{E})) = \delta\, \widehat{\mu}(\overline{E}).$$

Moreover, if $\overline{E}$ is semi-stable, then $S^\delta(\overline{E})$ is also semi-stable.

Proof. Let r be the dimension of E over K. Without loss of generality, we may assume that $r \geqslant 2$. Let

$$0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_r = E$$

be a complete flag of vector subspaces of E. By Proposition A.2.3, one has

$$\begin{aligned}\widehat{\deg}(S^\delta(\overline{E})) &= \sum_{\substack{(a_1,\ldots,a_r)\in\mathbb{N}^r\\ a_1+\cdots+a_r=\delta}} \sum_{i=1}^r a_i\,\widehat{\deg}(\overline{E_i/E_{i-1}})\\ &= \sum_{i=1}^r \widehat{\deg}(\overline{E_i/E_{i-1}})\sum_{a=0}^{\delta} a\binom{r+\delta-a-2}{r-2}.\end{aligned}$$

Note that

$$\begin{aligned}\sum_{a=0}^{\delta} a\binom{r+\delta-a-2}{r-2} &= \delta\sum_{a=0}^{\delta}\binom{r+\delta-a-2}{r-2} - \sum_{a=0}^{\delta}(\delta-a)\binom{r+\delta-a-2}{\delta-a}\\ &= \delta\binom{\delta+r-1}{r-1} - \sum_{a=0}^{\delta-1}(r-1)\binom{r+\delta-a-2}{\delta-a-1}\\ &= \delta\binom{\delta+r-1}{r-1} - (r-1)\binom{\delta+r-1}{r} = \Big(\delta - \frac{\delta(r-1)}{r}\Big)\binom{\delta+r-1}{r-1}.\end{aligned}$$

Since

$$\dim_K(S^\delta(E)) = \binom{r+\delta-1}{\delta},$$

we obtain

$$\widehat{\mu}(S^\delta(\overline{E})) = \frac{\delta}{r}\sum_{i=1}^r \widehat{\deg}(\overline{E}_i/E_{i-1}) = \delta\,\widehat{\mu}(\overline{E}).$$

In the case where $\overline{E}$ is semi-stable, by Corollary A.3.2 we obtain that $\overline{E}^{\otimes\delta}$ is also semi-stable. Moreover, its slope is also equal to $\delta\,\widehat{\mu}(\overline{E})$. Since any quotient vector space of $S^\delta(E)$ is also a quotient vector space of $E^{\otimes\delta}$, we obtain that, for any quotient vector space Q of $E^{\otimes\delta}$, one has

$$\widehat{\mu}(\overline{Q}) \geqslant \widehat{\mu}(\overline{E}^{\otimes\delta}) = \delta\widehat{\mu}(\overline{E}) = \widehat{\mu}(S^\delta(\overline{E})).$$

Therefore $S^\delta(\overline{E})$ is also semi-stable. □

A.3.3. Symmetric power. In this subsection, we fix a Hermitian adelic vector bundle $\overline{E}$ on S.

Theorem A.3.5. *For any positive integer δ, the following inequality holds:*

$$\widehat{\mu}_{\max}(S^\delta(\overline{E})) \leqslant \delta\,\widehat{\mu}_{\max}(\overline{E}) + \nu(\Omega_\infty)\ln(\delta!) + \frac{1}{2}\nu(\Omega_\infty)\delta\ln(\dim_K(E)). \tag{A.11}$$

Moreover, in the case where Ω_∞ is empty, the norm (where we consider the trivial absolute value on K) on $S^\delta(E)$ associated with the Harder-Narasimhan $\mathbb{R}$-filtration of $S^\delta(\overline{E})$ coincides with the ε-symmetric power of that associated with the Harder-Narasimhan $\mathbb{R}$-filtration of $\overline{E}$.

Proof. We first treat the case where $\Omega_\infty = \varnothing$. Let $\mathcal{F}$ be the Harder-Narasimhan $\mathbb{R}$-filtration of $\overline{E}$, which correspond to a sequence

$$0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_r = E$$

of vector subspaces of E, together with a decreasing sequence

$$\mu_1 > \ldots > \mu_r$$

of successive slopes. We equip $S^\delta(E)$ with the symmetric power of the $\mathbb{R}$-filtration $\mathcal{F}$. Note that the subquotient $\mathrm{sq}^t(S^\delta(E))$ of index t is given by (see §A.2.4)

$$\bigoplus_{\substack{\boldsymbol{b}=(b_1,\ldots,b_r)\in\mathbb{N}^r\\ |\boldsymbol{b}|=b_1+\cdots+b_r=\delta\\ b_1\mu_1+\cdots+b_r\mu_r=t}} S^{b_1}(V_1/V_0)\otimes\cdots\otimes S^{b_r}(V_r/V_{r-1}).$$

By Corollary A.3.2 and Proposition A.3.4, each Hermitian adelic vector bundle

$$S^{b_1}(\overline{E_1/E_0})\otimes\cdots\otimes S^{b_r}(\overline{E_r/E_{r-1}})$$

is semi-stable of slope

$$b_1\mu_1+\cdots+b_r\mu_r=t.$$

Therefore, the symmetric power of the $\mathbb{R}$-filtration $\mathcal{F}$ identifies with the Harder-Narasimhan $\mathbb{R}$-filtration of $S^\delta(\overline{E})$ and the maximal slope of $S^\delta(\overline{E})$ is equal to $\delta\,\widehat{\mu}_{\max}(\overline{E})$.

In the case where Ω_∞ is not empty, the field K is necessarily of characteristic 0. Let $\mathrm{sym}' : S^\delta(E)\to E^{\otimes\delta}$ be the K-linear map induced by the symmetrization map (see Remark A.2.6). Since K is of characteristic 0, this map is injective and hence

$$\widehat{\mu}_{\max}(S^\delta(\overline{E}))\leqslant\widehat{\mu}_{\max}(\overline{E}^{\otimes\delta})+h(\mathrm{sym}')\leqslant\widehat{\mu}_{\max}(\overline{E}^{\otimes\delta})+\nu(\Omega_\infty)\ln(\delta!).$$

By the dual statement of [**36**, Corollary 5.6.2], one has

$$\widehat{\mu}_{\max}(\overline{E}^{\otimes\delta})\leqslant\delta\widehat{\mu}_{\max}(\overline{E})+\frac{1}{2}\nu(\Omega_\infty)\delta\ln(\dim_K(E)).$$

Hence the desired inequality follows. □

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