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ARITHMETIC INTERSECTION THEORY OVER ADELIC CURVES

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CONTENTS

Abstract. — We establish an arithmetic intersection theory in the framework of Arakelov geometry over adelic curves. To each projective scheme over an adelic curve, we associate a multi-homogenous form on the group of adelic Cartier divisors, which can be written as an integral of local intersection numbers along the adelic curve. The integrability of the local intersection number is justified by using the theory of resultants.

INTRODUCTION

Since the seminal work of Dedekind and Weber [**17**], the similarity between number fields and fields of algebraic functions of one variable has been known and has deeply influenced researches in algebraic geometry and number theory. Inspired by the discovery of Hensel and Hasse on embeddings of a number field into diverse local fields, Weil [**65**] considered in the same time all places of a number field, finite or infinite, in his theory of adèles, which made a decisive step toward the unification of number theory and algebraic geometry. Many works have then been done along this direction. On the one hand, the analogue of Diophantine problems (notably Mordell's conjecture) in the function field setting has been studied by Manin [**47**], Grauert [**30**] and Samuel [**60**]; on the other hand, through Weil's height machine [**64**] and the theory of Néron-Tate's height [**51**], methods of algebraic geometry have been systematically applied to the research of Diophantine problems, and it has been realized that the understanding of the arithmetic of algebraic varieties over a number field, which should be analogous to algebraic geometry over a smooth projective curve, is indispensable in the geometrical approach of Diophantine problems. Under such a circonstance Arakelov [**1, 2**] has developed the arithmetic intersection theory for arithmetic surfaces (namely relative curves over $\operatorname{Spec}\mathbb{Z}$). Note that the transcription of the intersection theory into the arithmetic setting is by no means automatic. The key idea of Arakelov is to introduce transcendental objects, notably Hermitian metrics or Green functions, over the infinite places, in order to "compactify" arithmetic surfaces. To each pair of compactified arithmetic divisors, he attached a family of local intersection numbers parametrized by the set of places of the base number field. The global intersection number is obtained by taking the sum of local intersection numbers. Arakelov's idea has soon led to spectacular advancements in Diophantine geometry, especially Faltings' proof [**19**] of Mordell's conjecture.

The fundament of Arakelov geometry for higher dimensional arithmetic varieties has been established by Gillet and Soulé, where an arithmetic intersection theory [**25,**

27] for general arithmetic varieties has been established and an "arithmetic Riemann-Roch theorem" [**26**] has been proved. They have introduced the notion of arithmetic Chow groups, which is a hybride construction of the classic Chow group in algebraic geometry and currents in complex analytic geometry. Applications of arithmetic intersection theory in Diophantine geometry have then been developed, notably to build up an intrinsic height theory for arithmetic projective varieties (see for example [**20, 5**]). Arakelov's height theory becomes now an important tool in arithmetic geometry. Upon the need of including several constructions of local heights (such as canonical local height for subvarieties in an Abelian variety) in the setting of Arakelov geometry, Zhang [**68**] has introduced the notion of adelic metrics for ample line bundles on a projective variety over a number field, which could be considered as uniform limit of Hermitian line bundles (with possibly different integral models).

Inspired by the similarity between Diophantine analysis and Nevanlinna theory, Gubler [**34**] has proposed a vast generalization of height theory in the framework of M-fields. Recall that a M-field is a field K equipped with a measure space M and a map from $K \times M$ to $\mathbb{R}_{\geqslant 0}$ which behaves almost everywhere like absolute values on K. Combining the intersection product of Green currents in the Archimedean case and the local height of Chow forms, he has introduced local heights (parametrized by the measure space M) for a projective variety over an M-field. Assuming the integrability of the function of local heights on the measure space M, he has defined the global height of the variety as the integral of local heights. Interesting examples have been discussed, which show that in many cases the function of local heights is indeed integrable.

In [**12**], we have developed an Arakelov geometry over adelic curves. Our framework is similar to M-field of Gubler, with a slightly different point of view: an adelic curve is a field equipped with a family of absolute values parametrized by a measure space (in particular, we require the absolute values to be defined everywhere). These absolute values play the role of places in algebraic number theory. Hence we can view an adelic curve as a measure space of "places" of a given field, except that we allow possibly equivalent absolute values in the family, or even copies of the same absolute value. Natural examples of adelic curves contain global fields, countably generated fields over global fields (as we will show in the second chapter of the current article), field equipped with copies of the trivial absolute value, and also the amalgamation of different adelic structures of the same field. Our motivation was to establish a theory of adelic vector bundles (generalizing previous works of Stuhler [**63**], Grayson [**31**], Bost [**6**] and Gaudron [**23**]), which is analogous to geometry of numbers and hence provides tools to consider Diophantine analysis in a general and flexible setting. By using the theory of adelic vector bundles, the arithmetic birational invariants are discussed in a systematic way.

The first contribution of the current article is to discuss transcendental coverings of adelic curves. Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve, where K is a countable field, $(\Omega, \mathcal{A}, \nu)$ is a measure space, and $\phi : \omega \mapsto |\cdot|_\omega$ is a map from Ω to the set of all absolute values of K, such that, for any $a \in K^\times$, the function $(\omega \in \Omega) \mapsto \ln |a|_\omega$ is measurable. In [**12**, Chapter 3], for any algebraic extension L/K, we have constructed a measure space $(\Omega_L, \mathcal{A}_L, \nu_L)$, which is fibered over $(\Omega, \mathcal{A}, \nu)$ and admits a family of disintegration probability measures. For each $\omega \in \Omega$, we correspond the fiber $\Omega_{L,\omega}$ to the family of all absolute values of L extending $|\cdot|_\omega$. Thus we obtain a structure of adelic curve on L which is called an *algebraic covering* of S.

In [**12**, §3.2.5], we have illustrated the construction of an adelic curve structure on $\mathbb{Q}(T)$, which takes into account the arithmetic of $\mathbb{Q}$ and the geometry of $\mathbb{P}^1$. In the current article, we generalizes and systemize such a construction on a purely transcendental and countably generated extension of the underlying field K of the adelic curve S. For simplicity, we explain here the case of rational function of finitely many variables. Let n be an integer such that $n \geqslant 1$ and $\boldsymbol{T} = (T_1, \ldots, T_n)$ be variables. Let L be the rational function field $K(\boldsymbol{T}) = K(T_1, \ldots, T_n)$, which is by definition the field of fractions of the polynomial ring $K[\boldsymbol{T}] = K[T_1, \ldots, T_n]$. For each $\omega \in \Omega$ such that the absolute value $|\cdot|_\omega$ is non-Archimedean, by Gauss's lemma, we extends $|\cdot|_\omega$ to be an absolute value on L such that

$$\forall\, f = \sum_{\boldsymbol{d} \in \mathbb{N}^n} a_{\boldsymbol{d}}(f) \boldsymbol{T}^{\boldsymbol{d}} \in K[\boldsymbol{T}], \quad |f|_\omega = \max_{\boldsymbol{d} \in \mathbb{N}^n} |a_{\boldsymbol{d}}|_\omega.$$

We then take $\Omega_{L,\omega}$ to be the one point set $\{\omega\}$, which is equipped with the natural probability measure. In the case where the absolute value $|\cdot|_\omega$ is Archimedean, we fix an embedding $\iota_\omega : K \to \mathbb{C}$ such that $|\cdot|_\omega$ is the composition of the usual absolute value $|\cdot|$ on $\mathbb{C}$ with ι_ω (by a measurable selection argument, we can arrange that the family of ι_ω parametrized by Archimedean places is $\mathcal{A}$-measurable). We let

$$\Omega_{L,\omega} := \left\{ (t_1, \ldots, t_n) \in [0,1]^n \;\middle|\; \begin{array}{l} (e(t_1), \ldots, e(t_n)) \text{ is algebraically} \\ \text{independent over } \iota_\omega(K) \end{array} \right\},$$

where for each $t \in [0,1]$, $e(t)$ denotes $\mathrm{e}^{2\pi i t}$. Note that, if we equip $[0,1]^n$ with the Borel σ-algebra and the uniform probability measure, then $\Omega_{L,\omega}$ is a Borel set of measure 1. Moreover, each element $\boldsymbol{t} = (t_1, \ldots, t_n) \in \Omega_{L,\omega}$ gives rise to an absolute value $|\cdot|_{\boldsymbol{t}}$ on L such that

$$\forall\, f = \sum_{\boldsymbol{d} \in \mathbb{N}^n} a_{\boldsymbol{d}}(f) \boldsymbol{T}^{\boldsymbol{d}} \in K[\boldsymbol{T}], \quad |f|_{\boldsymbol{t}} = \left| \sum_{\boldsymbol{d} \in \mathbb{N}^n} \iota_\omega(a_{\boldsymbol{d}}(f)) e(t_1)^{d_1} \cdots e(t_n)^{d_n} \right|.$$

It turns out that the disjoint union Ω_L of $(\Omega_{L,\omega})_{\omega \in \Omega}$ forms a structure of adelic curve on the field L, which is fibered over that of S, and admits a family of disintegration probability measures. We denote by $S_L = (L, (\Omega_L, \mathcal{A}_{\Omega_L}, \nu_L), \phi_L)$ the corresponding adelic curve.

In the case where the adelic curve S is proper, namely the following equality holds for any $a \in K^\times$

$$\int_\Omega \ln |a|_\omega \, \nu(\mathrm{d}\omega) = 0,$$

it is not true in general that the adelic curve S_L is also proper. In the article, we propose several natural "compactifications" of the adelic curve. Here we explain one of them which has an "arithmetic nature". We say that two irreducible polynomials P and Q in $K[T_1, \ldots, T_n]$ are equivalent if they differ by a factor of non-zero element of K. This is an equivalence relation on the set of all irreducible polynomials. In each equivalence class we pick a representative to form a family $\mathscr{P}$ of irreducible polynomials. Then every non-zero element f of K can be written in a unique way as

$$f = a(f) \prod_{F \in \mathscr{P}} F^{\mathrm{ord}_F(f)},$$

where $a(f) \in K^\times$, and $\mathrm{ord}_F(\cdot) : L \to \mathbb{Z} \cup \{+\infty\}$ is the discrete valuation associated with F, we denote by $|\cdot|_F = \mathrm{e}^{-\mathrm{ord}_F(\cdot)}$ the corresponding absolute value on L. Moreover, the degree function on $K[\boldsymbol{T}]$ extends naturally to L so that $-\deg(\cdot)$ is a discrete valuation on L. Moreover, the following equality holds (see Proposition 2.7.6)

$$\forall\, f \in K(\boldsymbol{T}), \quad \sum_{F \in \mathscr{P}} \deg(F) \, \mathrm{ord}_F(f) = \deg(f).$$

We let $|\cdot|_\infty$ be the absolute value on L such that $|\cdot|_\infty = \mathrm{e}^{\deg(\cdot)}$. Note that, for any $F \in \mathscr{P}$, one has

$$h_{S_L}(F) := \int_\Omega \nu(\mathrm{d}\omega) \int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x) \geqslant 0.$$

We fix a positive real number λ. Let $(\Omega_L^\lambda, \mathcal{A}_L^\lambda, \nu_L^\lambda)$ be the disjoint union of $(\Omega_L, \mathcal{A}_L, \nu_L)$ and $\mathscr{P} \cup \{\infty\}$, which is equipped with the measure ν_L^λ extending ν_L and such that $\nu_L^\lambda(\{\infty\}) = \lambda$ and

$$\forall\, F \in \mathscr{P}, \quad \nu_L^\lambda(\{F\}) = h_{S_L}(F) + \lambda \deg(F).$$

Let ϕ_L^λ be the map from Ω_L^λ to the set of absolute values on L, sending $x \in \Omega_L^\lambda$ to $|\cdot|_x$. Then we establish the following result (see §2.7, notably Propositions 2.7.10 and 2.7.14, see also Proposition 2.5.1 for the general construction).

Theorem A. — *Assume that the adelic curve S is proper.*

(1) *For any $\lambda > 0$, the adelic curve $S_L^\lambda = (L, (\Omega_L^\lambda, \mathcal{A}_L^\lambda, \nu_L^\lambda), \phi_L^\lambda)$ is proper.*
(2) *If the adelic curve S satisfies the Northcott property, namely, for any $C \geqslant 0$, the set*

$$\left\{ a \in K \;\middle|\; \int_\Omega \max\{\ln |a|_\omega, 0\} \, \nu(\mathrm{d}\omega) \leqslant C \right\}$$

is finite, then, for any $\lambda > 0$, the adelic curve S_L^λ satisfies the Northcott property.

Together with the algebraic covering of adelic curves mentioned above. This construction provides a large family of adelic structures for finitely generated extensions of $\mathbb{Q}$, which behave well from the view of geometry of numbers. Note however that the compactification S_L^λ is not fibered over S, but rather fibered over the amalgamation of S with copies of the trivial absolute value on K. This phenomenon suggest that it is a need of dealing with the trivial absolute value in the consideration of the relative geometry of adelic curves.

To build up a more complete picture of Arakelov geometry over an adelic curve, it is important to develop an arithmetic intersection theory and relate it to the heights of projective varieties over an adelic curve. Although the local intersection theory is now well understood, thanks to works such as **[34, 35, 10, 50]**, it remains a challenging problem to show that the local intersection numbers form an integrable function over the parametrizing measure space. In this article, we resolve this integrability problem and thus establish a global intersection theory in the framework of Arakelov geometry over adelic curves. Recall that the function of local heights for an adelic line bundle is only well defined up to the function of absolute values of a non-zero scalar. One way to make explicit the local height function is to fix a family of global sections of the line bundle which intersect properly. Note that each global section determines a Cartier divisor on the projective variety, and the adelic metrics of the adelic line bundle determine a family of Green functions of the Cartier divisor parametrized by the measure space of "places". For this reason, we choose to work in the framework of adelic Cartier divisors.

Let S be an adelic curve, which consists of a field K, a measure space $(\Omega, \mathcal{A}, \nu)$ and a family $(|\cdot|_\omega)_{\omega\in\Omega}$ of absolute values on K parametrized by Ω. Let X be a projective scheme over $\operatorname{Spec} K$ and d be the Krull dimension of X. By adelic Cartier divisor on X, we mean the datum $\overline{D}$ consisting of a Cartier divisor D on X together with a family $g = (g_\omega)_{\omega\in\Omega}$ parametrized by Ω, where g_ω is a Green function of D_ω, the pull back of D on $X_\omega = X \otimes_K K_\omega$, with K_ω being the completion of K with respect to $|\cdot|_\omega$. Conditions of measurability and dominancy (with respect to $\omega \in \Omega$) for the family g are also required (see §§4.1–4.2 for more details). We first introduce the local intersection product for adelic Cartier divisors. More precisely, if $\overline{D}_i = (D_i, g_i)$, $i \in \{0, \ldots, d\}$, form a family of integrable metrized Cartier divisors on X (namely a Cartier divisor equipped with a Green function, which is the difference of two plurisubharmonic Green functions) such that $D_0, \ldots, D_d$ intersect properly, we define, for any $\omega \in \Omega$, a local intersection number

$$(\overline{D}_0, \ldots, \overline{D}_d)_\omega \in \mathbb{R}$$

in a recursive way by using Bedford-Taylor theory **[3]** and its non-Archimedean analogue **[10]**. In the case where $|\cdot|_\omega$ is a trivial absolute value, we need a careful definition of the local intersection number (see Definition 3.10.1, for details). Note the local intersection number is a multi-linear function on the set of $(d+1)$-uplets $(\overline{D}_0, \ldots, \overline{D}_d)$ such that $D_0, \ldots, D_d$ intersect properly.

To establish a global intersection theory, we need to show that the function of local intersection numbers

$$(\omega \in \Omega) \longmapsto (\overline{D}_0, \ldots, \overline{D}_d)_\omega$$

is measurable and integrable with respect to ν, where the measurability part is more subtle. Although the Green function families of $\overline{D}_0, \ldots, \overline{D}_d$ are supposed to be measurable, the corresponding products of Chern currents (or their non-Archimedean analogue) depend on the local analytic geometry relatively to the absolute values $|\cdot|_\omega$. It seems to be a difficult (but interesting) problem to precisely describe the measurability of the local geometry of the analytic spaces X_ω^{an}. For places ω which are Archimedean, as we can embed all local completions K_ω in the same field $\mathbb{C}$, by a measurable selection theorem one can show that the family of Monge-Ampère measures is measurable with respect to ω (see Theorem 4.2.9). However, for non-Archimedean places, such embeddings in a common valued field do not exist in general, and the classic approach of taking a common integral model for all non-Archimedean places is not adequate in the setting of adelic curves, either.

To overcome this difficulty, our approach consists in relating the local intersection number to the local length of the mixed resultant and hence reduce the problem to the measurability of the function of local lengths of the mixed resultant, which is known by the theory of adelic vector bundles developed in [**12**]. This approach is inspired by previous results of Philippon [**55**] on height of algebraic cycles via the theory of Chow forms and the comparison [**56, 57, 62, 5**] between Philippon's height and Faltings height (defined by the arithmetic intersection theory). Note that the similar idea has also been used in [**34**] to construct the local height in the setting of M-fields.

Let us briefly recall the theory of mixed resultant. It is a multi-homogeneous generalization of Chow forms, which allows to describe the interactions of several embeddings of a variety in projective spaces by a multi-homogeneous polynomial. One of its original forms is the discriminant of a quadratic polynomial, or more generally the resultant of $n+1$ polynomials $P_0, \ldots, P_n$ in n variables over an algebraically closed field, which is an irreducible polynomial in the coefficients of $P_0, \ldots, P_n$, that vanishes precisely when these polynomials have a common root. The modern algebraic approach of resultants goes back to the elimination theory of Cayley [**9**], where he related resultant to the determinant of Koszul complex. We use here a geometric reformulation as in the book [**24**] of Gel'fand, Kapranov and Zelevinsky. In Diophantine geometry, mixed resultant has been used by Rémond [**58**] to study multi-projective heights.

We assume that the Cartier divisors D_i are very ample and thus determine closed immersions f_i from X to the projective space of the linear system E_i of the divisor

D_i. By *incidence variety* of $(f_0,\ldots,f_d)$, we mean the closed subscheme I_X of $X\times_K \mathbb{P}(E_0^\vee)\times_K\cdots\times_K\mathbb{P}(E_d^\vee)$ parametrizing points $(x,\alpha_0,\ldots,\alpha_d)$ such that

$$\alpha_0(x)=\cdots=\alpha_d(x)=0.$$

One can also consider I_X as a multi-projective bundle over X (of $E_i^\vee$ quotient by the tautological line subbundle). Therefore, the projection of I_X in $\mathbb{P}(E_0^\vee)\times_K\cdots\times_K\mathbb{P}(E_d^\vee)$ consists of a family of hyperplanes in $\mathbb{P}(E_0),\ldots,\mathbb{P}(E_d)$ respectively, which contain at least one common point of X. It turns out that this projection is actually a multi-homogeneous hypersurface of $\mathbb{P}(E_0^\vee)\times_K\cdots\times_K\mathbb{P}(E_d^\vee)$, which is defined by a multi-homogeneous polynomial $R^X_{f_0,\ldots,f_d}$, called a *resultant* of X with respect to the embeddings of $f_0,\ldots,f_d$. We refer the readers to [**24**, §3.3] for more details, see also [**16**] for applications in arithmetic Nullstellensatz. When K is a number field, the height of the polynomial $R^X_{f_0,\ldots,f_d}$ can be viewed as a height of the arithmetic variety X, and, in the particular case where the image of D_i in the Picard group are colinear, an explicit comparison between the height of resultant and the Faltings height of X has been discussed in [**5**, Theorem 4.3.2] (see also §4.3.4 of *loc. cit.*).

Usually the resultant is well defined up to a factor in $K^\times$. In the classic setting of number field, this is anodyne for the study of the global height, thanks to the product formula. However, in our setting, this dependence on the choice of a non-zero scalar could be annoying, especially when the adelic curve does not satisfy a product formula. In order to obtain a local height equality, we introduce, for each vector

$$(s_0,\ldots,s_d)\in E_0\times\cdots\times E_d$$

such that $\mathrm{div}(s_0),\ldots,\mathrm{div}(s_d)$ intersect properly on X, a specific resultant $R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d}$ of X with respect to the embeddings, which is the only resultant such that

$$R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d}(s_0,\ldots,s_d)=1.$$

We then show that the local height for this resultant coincides with the local height of X defined by the local intersection theory. By using this comparison of local height and properties of adelic vector bundles over an adelic curve (see [**13**, §4.1.4]), we prove the integrability of the local height function on non-Archimedean places. Moreover, the integral of the local height equalities leads to an equality between the global height of the resultant and the arithmetic intersection number (see Remark 4.2.14), which generalizes the height comparison results in [**56, 5**]. In resume, we obtain the following result (see Theorems 3.9.7 and 4.2.12).

Theorem B. — *Let $S=(K,(\Omega,\mathcal{A},\nu),\phi)$ be an adelic curve, X be a projective scheme over S, d be the dimension of X, $D_0,\ldots,D_d$ be Cartier divisors on X, which are equipped with Green function families $g_0,\ldots,g_d$, respectively, such that $(D_{i,\omega},g_{i,\omega})$ is integrable for any $\omega\in\Omega$ and $i\in\{0,\ldots,d\}$.*

(1) *Assume that the Cartier divisors $D_0,\ldots,D_d$ are very ample. For any $i\in\{0,\ldots,d\}$, let $E_i=H^0(X,\mathcal{O}_X(D_i))$, $f_i:X\to\mathbb{P}(E_i)$ be the closed embedding and $s_i\in E_i$ be the regular meromorphic section of $\mathcal{O}_X(D_i)$ corresponding to D_i. Assume that the continuous metric family φ_{g_i} corresponding to the Green function family g_i consists of the orthogonal quotient metrics induces by a Hermitian norm family $\xi_i=(\|\cdot\|_{i,\omega})_{\omega\in\Omega}$ on E_i. Then, for any $\omega\in\Omega$, then following equalities hold.*

(1.a) *In the case where $|\cdot|_\omega$ is non-Archimedean, one has*

$$(\overline{D}_0\cdots\overline{D}_d)_\omega=\ln\left\|R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d}\right\|_{\omega,\varepsilon},$$

where the norm $\|\cdot\|_{\omega,\varepsilon}$ on the space of multi-homogeneous polynomials is the ε-tensor product of ε-symmetric power of $\|\cdot\|_{i,\omega,*}$.

(1.b) *In the case where $|\cdot|_\omega$ is Archimedean, one has*

$$(\overline{D}_0\cdots\overline{D}_d)_\omega=\int_{\mathbb{S}(E_{0,\omega})\times\cdots\times\mathbb{S}(E_{d,\omega})}\ln\left|R^{X,s_0,\ldots s_d}_{f_0,\ldots,f_d}(z_0,\ldots,z_d)\right|_\omega \mathrm{d}z_0\cdots\mathrm{d}z_d + \frac{1}{2}\sum_{i=0}^{d}\delta_i\sum_{\ell=1}^{r_i}\frac{1}{\ell},$$

where $\mathbb{S}(E_{i,\omega})$ denotes the unit sphere of $(E_{i,\omega},\|\cdot\|_{i,\omega})$, $\mathrm{d}z_i$ is the Borel probability measure on $\mathbb{S}(E_{i,\omega})$ invariant by the unitary group, r_i is the dimension of E_i, and δ_i is the intersection number

$$(D_0\cdots D_{i-1}D_{i+1}\cdots D_d).$$

(2) *Assume that, either the σ-algebra $\mathcal{A}$ is discrete, or the field K admits a countable subfield which is dense in each K_ω. If all couples $\overline{D}_i=(D_i,g_i)$ are integrable adelic Cartier divisors on X, the the function*

$$(\omega\in\Omega)\longrightarrow(\overline{D}_0\cdots\overline{D}_d)_\omega$$

is ν-integrable.

As an application, we can define the multi-height of the projective scheme X with respect to $\overline{D}_0,\ldots,\overline{D}_d$ as

$$h_{\overline{D}_0\cdots\overline{D}_d}(X)=\int_\Omega(\overline{D}_0\cdots\overline{D}_d)_\omega\,\nu(\mathrm{d}\omega),$$

and, under the assumptions of the point (1) in the above theorem, we can relate the multi-height with the height of the resultant, by taking the integral of the local height equalities.

From the methodological point of view, the approach of **[56]** works within $\mathbb{P}^N(\mathbb{C})$ and uses elimination theory and complex analysis of the Fubini-Study metric; that of **[5]** relies on a choice of integral model and computations in the arithmetic Chow groups. In our setting, we need to deal with general non-Archimedean metrics. Hence these approaches do not fit well with the framework of adelic curves. Our method consists in computing the local height of

$$X \times_K \mathbb{P}(E_0^\vee) \times_K \cdots \times_K \mathbb{P}(E_d^\vee)$$

in two ways (see Lemma 3.9.6 for details). We first consider this scheme as a fibration of multi-projective space over X and relate this local height to that of X by taking the local intersection along the fibers. We then relate the height of this product scheme to that of the incidence subscheme I_X and then use the identification of I_X with a multi-projective bundle over X to compute recursively the height of I_X. Our method allows to obtain a local height equality in considering the Archimedean case and the non-Archimedean case in a uniform way.

It is worth mentioning that an intersection theory of arithmetic cycles and a Riemann-Roch theory could be expected for the setting of adelic curves. However, new ideas are needed to establish a good formulation of the measurability for various arithmetic objects arising in such a theory.

The rest of the article is organized as follows. In the first chapter, we remind several basic constructions used in the article, including multi-linear subsets and multi-linear functions, Cartier divisors on general scheme, proper intersection of Cartier divisors on a projective scheme, multi-homogeneous polynomials, incidence subscheme and resultants, and linear projections of closed subschemes in a projective space. The second chapter is devoted to the construction of adelic structures. After a brief reminder on the definition of adelic curves and their algebraic covers, we introduce transcendental fibrations of adelic curves and their compactifications. These constructions provide a large family of examples of adelic curves. In the third chapter, we consider the local intersection theory in the setting of projective schemes over a complete valued field. We first remind the notions of continuous metrics on an invertible sheaf and its semi-positivity. Then we explain the notion of Green functions of Cartier divisors and their relation with continuous metrics. The construction of Monge-Ampère mesures and local intersection numbers is then discussed. The last sections are devoted to establish the link between the local intersection number and the length (in the non-Archimedean case) or Mahler measure (in the Archimedean case) of the corresponding resultant, respectively. In the fourth and last chapter, we prove the integrability of the local height function and construct the global multi-height.

CHAPTER 1

MULTILINEAR ALGEBRA AND RESULTANTS

The purpose of this chapter is preliminaries of this book, especially, we review basics of a multilinear algebra and resultants.

1.1. Symmetric and multi-linear subsets

In this section, we fix a commutative and unitary ring k, and a non-negative integer d.

1.1.1. Definition. — Let V be a k-module. We say that a subset S of V^{d+1} is *multi-linear* if, for any $j \in \{0, \ldots, d\}$ and for any $(x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in V^d$, the subset

$$\{x_j \in V \mid (x_0, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) \in S\}$$

of V is either empty or a sub-k-module. If in addition

$$(x_0, \ldots, x_d) \in S \implies (x_{\sigma(0)}, \ldots, x_{\sigma(d)}) \in S$$

for any bijection $\sigma : \{0, \ldots, d\} \to \{0, \ldots, d\}$, we say that the multi-linear subset S is *symmetric*.

1.1.2. Proposition. — *Let V be a k-module and S be a multi-linear subset of V^{d+1}. For any $j \in \{0, \ldots, d\}$, let I_j be a non-empty finite set, $(x_{j,i})_{i \in I_j}$ be a family of elements of V, $(\lambda_{j,i})_{i \in I_j}$ be a family of elements of k, and $y_j = \sum_{i \in I_j} \lambda_{j,i} x_{j,i}$. Assume that, for any $(i_0, \ldots, i_d) \in I_0 \times \cdots \times I_d$, one has $(x_{0,i_0}, \ldots, x_{d,i_d}) \in S$. Then $(y_0, \ldots, y_d) \in S$.*

Proof. — We reason by induction on d. In the case where $d = 0$, S is a sub-k-module of V when it is not empty. Since y_0 is a k-linear combination of elements of S, we obtain that $y_0 \in S$.

We now assume that $d \geqslant 1$ and that the statement holds for multi-linear subsets of V^d. Let

$$S' = \{(z_0, \dots, z_{d-1}) \in V^d \mid (z_0, \dots, z_{d-1}, y_d) \in S\}.$$

Since S is a multi-linear subset of V^{d+1}, for any $(i_0, \dots, i_{d-1}) \in I_0 \times \cdots \times I_{d-1}$, one has $(x_{i_0}, \dots, x_{i_{d-1}}, y_d) \in S$ and hence $(x_{i_0}, \dots, x_{i_{d-1}}) \in S'$. Moreover, S' is a multi-linear subset of V^d. Hence the induction hypothesis leads to $(y_0, \dots, y_{d-1}) \in S'$ and thus $(y_0, \dots, y_d) \in S$. □

1.1.3. Definition. — Let V and W be two k-modules, and S be a multi-linear subset of V^{d+1}. We say that a map $f : S \to W$ is *multi-linear* if, for any $j \in \{0, \dots, d\}$ and for any $(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in V^d$, the map

$$\{x_j \in V \mid (x_0, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d) \in S\} \longrightarrow W, \quad x_j \mapsto f(x_0, \dots, x_d),$$

is k-linear once

$$\{x_j \in V \mid (x_0, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d) \in S\}$$

is not empty. If in addition S is symmetric and $f(x_0, \dots, x_d) = f(x_{\sigma(0)}, \dots, x_{\sigma(d)})$ for any $(x_0, \dots, x_d) \in S$ and any bijection $\sigma : \{0, \dots, d\} \to \{0, \dots, d\}$, we say that f is a *symmetric multi-linear map.*

1.1.4. Proposition. — *Let V and W be two k-modules, S be a multi-linear subset of V^{d+1}, and $f : S \to W$ be a multi-linear map. Let $(x_{j,i})_{(j,i) \in \{0,\dots,d\}^2}$ be a matrix consisting of elements of V such that $(x_{0,i_0}, \dots, x_{d,i_d}) \in S$ for any $(i_0, \dots, i_d) \in \{0, \dots, d\}^{d+1}$. Then*

$$\begin{aligned}\sum_{\sigma \in \mathfrak{S}(\{0,\dots,d\})} & f(x_{0,\sigma(0)}, \dots, x_{d,\sigma(d)}) \\ &= \sum_{\varnothing \neq I \subseteq \{0,\dots,d\}} (-1)^{d+1-\#I} f\Big(\sum\nolimits_{i_0 \in I} x_{0,i_0}, \dots, \sum\nolimits_{i_d \in I} x_{d,i_d}\Big),\end{aligned} \tag{1.1}$$

where $\mathfrak{S}(\{0, \dots, d\})$ is the permutation group of $\{0, \dots, d\}$.

Proof. — By the multi-linearity of f, we can rewrite the right-hand side of the equality (1.1) as

$$\begin{aligned}&\sum_{\varnothing \neq I \subseteq \{0,\dots,d\}} (-1)^{d+1-\#I} \sum_{(i_0,\dots,i_d) \in I^{d+1}} f(x_{0,i_0}, \dots, x_{d,i_d}) \\ &= \sum_{(i_0,\dots,i_d) \in \{0,\dots,d\}^{d+1}} \Bigg(\sum_{\{i_0,\dots,i_d\} \subseteq I \subseteq \{0,\dots,d\}} (-1)^{d+1-\#I} \Bigg) f(x_{0,i_0}, \dots, x_{d,i_d}).\end{aligned}$$

Note that, for $(i_0, \dots, i_d) \in \{0, \dots, d\}^{d+1}$ such that $\{i_0, \dots, i_d\} \subsetneq \{0, \dots, d\}$, one has

$$\sum_{\{i_0,\dots,i_d\} \subseteq I \subseteq \{0,\dots,d\}} (-1)^{d+1-\#I} = (-1)^{d+1-\#\{i_0,\dots,i_d\}} \sum_{J \subseteq \{0,\dots,d\} \setminus \{i_0,\dots,i_d\}} (-1)^{-\#J} = 0$$

since

$$\sum_{J\subseteq\{0,\ldots,d\}\setminus\{i_0,\ldots,i_d\}}(-1)^{-\#J}=(1+(-1))^{d+1-\#\{i_0,\ldots,i_d\}}=0.$$

Therefore the equality (1.1) holds. $\square$

1.1.5. Lemma. — *Let P be a subset of an abelian group G with the following properties:*

(1) *For $x, y \in P$, $x + y \in P$.*
(2) *For $x \in G$, there exist $x', x'' \in P$ such that $x = x' - x''$.*

Let d be a positive integer, A be an abelian group and $f : P^d \to A$ be a map such that

$$f(x_1,\ldots,x_i+y_i,\ldots,x_d)=f(x_1,\ldots,x_i,\ldots,x_d)+f(x_1,\ldots,y_i,\ldots,x_d)$$

for all $i \in \{1,\ldots,d\}$ and $x_1,\ldots,x_i,y_i,\ldots,x_d \in P$. Then there exists a unique multi-linear map $\tilde{f} : G^d \to A$ such that $\tilde{f}\big|_{P^d} = f$.

Proof. — For $x_1,\ldots,x_d \in G$, we can find $x_1', x_1'',\ldots,x_d',x_d'' \in P$ such that $x_i = x_i' - x_i''$ for all $i \in \{1,\ldots,d\}$. We would like to define $\tilde{f}(x_1,\ldots,x_d)$ to be

$$\tilde{f}(x_1,\ldots,x_d):=\sum_{I\subseteq\{1,\ldots,d\}}(-1)^{\mathrm{card}(I)}f(x_{1,I},\ldots,x_{d,I}),$$

where

$$x_{i,I}=\begin{cases}x_i'' & \text{if } i\in I,\\ x_i' & \text{if } i\in\{1,\ldots,d\}\setminus I.\end{cases}$$

It is sufficient to show that if $x_1', x_1'', y_1', y_1'',\ldots,x_d',x_d'',y_d',y_d'' \in P$ and $x_i' - x_i'' = y_i' - y_i''$ for all $i \in \{1,\ldots,d\}$, then

$$\sum_{I\subseteq\{1,\ldots,d\}}(-1)^{\mathrm{card}(I)}f(x_{1,I},\ldots,x_{d,I})=\sum_{I\subseteq\{1,\ldots,d\}}(-1)^{\mathrm{card}(I)}f(y_{1,I},\ldots,y_{d,I}).$$

We prove it by induction on d. We assume that $d = 1$. As $x_1' + y_1'' = x_1'' + y_1'$, one has $f(x_1') + f(y_1'') = f(x_1'') + f(y_1')$. Thus the assertion follows. We assume that $d > 1$. Then, by using the hypothesis of induction,

$$\begin{aligned}
&\sum_{I\subseteq\{1,\ldots,d\}}(-1)^{\mathrm{card}(I)}f(x_{1,I},\ldots,x_{d,I})\\
&=\sum_{\substack{I\subseteq\{1,\ldots,d\}\\ d\notin I}}(-1)^{\mathrm{card}(I)}f(x_{1,I},\ldots,x_{d,I})+\sum_{\substack{I\subseteq\{1,\ldots,d\}\\ d\in I}}(-1)^{\mathrm{card}(I)}f(x_{1,I},\ldots,x_{d,I})\\
&=\sum_{I'\subseteq\{1,\ldots,d-1\}}(-1)^{\mathrm{card}(I')}(f(x_{1,I'},\ldots,x_{d-1,I'},x_d')-f(x_{1,I'},\ldots,x_{d-1,I'},x_d''))\\
&=\sum_{I'\subseteq\{1,\ldots,d-1\}}(-1)^{\mathrm{card}(I')}(f(y_{1,I'},\ldots,y_{d-1,I'},x_d')-f(y_{1,I'},\ldots,y_{d-1,I'},x_d''))\\
&=\sum_{I'\subseteq\{1,\ldots,d-1\}}(-1)^{\mathrm{card}(I')}(f(y_{1,I'},\ldots,y_{d-1,I'},y_d')-f(y_{1,I'},\ldots,y_{d-1,I'},y_d''))
\end{aligned}$$

$$= \sum_{I \subseteq \{1,\ldots,d\}} (-1)^{\operatorname{card}(I)} f(y_{1,I}, \ldots, y_{d,I}),$$

as required. □

1.2. Cartier divisors

In this section, let us recall the notion of Cartier divisor on a general scheme. The main references are [**33**, IV_4, §§20-21] and [**43**].

1.2.1. Definition. — Let X be a locally ringed space. We denote by $\mathcal{O}_X$ the structural sheaf of X. Let $\mathscr{M}_X$ be the sheaf of *meromorphic functions on* X. Recall that $\mathscr{M}_X$ is the sheaf of commutative and unitary rings associated with the presheaf

$$U \longmapsto \mathcal{O}_X(U)[S_X(U)^{-1}],$$

where $S_X(U)$ denotes the multiplicative sub-monoid of $\mathcal{O}_X(U)$ consisting of local non-zero-divisors of $\mathcal{O}_X(U)$, that is, $s \in \mathcal{O}_X(U)$ such that the homothety

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}, \quad a \longmapsto as_x$$

is injective for any $x \in U$ (here s_x denotes the canonical image of s in the local ring $\mathcal{O}_{X,x}$). We refer the readers to [**43**] for a clarification on the construction of the sheaf of meromorphic functions comparing to [**33**, IV_4.(20.1.3)].

1.2.2. Remark. — Note that, for any $x \in X$, $\mathscr{M}_{X,x}$ identifies with $\mathcal{O}_{X,x}(S_{X,x}^{-1})$, where $S_{X,x}$ denotes the direct limit of $S_X(U)$ with U running over the set of open neighbourhoods of x, viewed as a multiplicative submonoid of $\mathcal{O}_{X,x}$, which is contained in the sub-monoid of non-zero-divisors. Therefore, $\mathscr{M}_{X,x}$ could be considered as a subring of the total fraction ring of $\mathcal{O}_{X,x}$, namely the localization of $\mathcal{O}_{X,x}$ with respect to the set of non-zero-divisors. In general the local ring $\mathscr{M}_{X,x}$ is different from the ring of total fractions of $\mathcal{O}_{X,x}$ even if X is an affine scheme. The equality holds notably when X is a locally Noetherian scheme or a reduced scheme whose set of irreducible component is locally finite. We refer the readers to [**43**] for counter-examples and more details.

1.2.3. Definition. — Let X be a locally ringed space. We denote by $\mathscr{M}_X^\times$ the subsheaf of multiplicative monoids of $\mathscr{M}_X$ consisting of invertible elements. In other words, for any open subset U of X, $\mathscr{M}_X^\times(U)$ is consisting of sections $s \in \mathscr{M}_X^\times(U)$ such that, for any $x \in U$, the homothety

$$\mathscr{M}_{X,x} \longrightarrow \mathscr{M}_{X,x}, \quad a \longmapsto as_x$$

is an isomorphism of $\mathscr{M}_{X,x}$-modules. An element of $\mathscr{M}_X^\times(U)$ is called a *regular meromorphic function* on X. Similarly, let $\mathcal{O}_X^\times$ be the subsheaf of multiplicative monoids of $\mathcal{O}_X$ consisting of invertible elements : for any open subset U of X, $\mathcal{O}_X^\times(U)$ consists of sections $s \in \mathcal{O}_X(U)$ such that, for any $x \in U$, the homothety

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}, \quad a \longmapsto as_x$$

is an isomorphism of $\mathcal{O}_{X,x}$-modules. Note that, for each $s \in \mathcal{O}_X(U)$, the homothety $s_x : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ induces by passing to localisation an homothety $\mathscr{M}_{X,x} \to \mathscr{M}_{X,x}$, which is an isomorphism of $\mathscr{M}_{X,x}$-modules if $s_x : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is an isomorphism. Therefore, the canonical morphism $\mathcal{O}_X \to \mathscr{M}_X$ induces a morphism of sheaves of abelian groups $\mathcal{O}_X^\times \to \mathscr{M}_X^\times$.

1.2.4. Definition. — We call *Cartier divisor* on X any global section of the sheaf $\mathscr{M}_X^\times/\mathcal{O}_X^\times$. By definition, a Cartier divisor D is represented by the following data: (i) an open cover $X = \bigcup_i U_i$ of X and (ii) $f_i \in \mathscr{M}_X^\times(U_i)$ for each i such that $f_i/f_j \in \mathcal{O}_X^\times$ on $U_i \cap U_j$ for all i, j. The regular meromorphic function f_i is called a *local equation* of D over U_i. The group of Cartier divisors is denoted by $\mathrm{Div}(X)$ and the group law of $\mathrm{Div}(X)$ is written additively. Note that the exact sequence

$$1 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathscr{M}_X^\times \longrightarrow \mathscr{M}_X^\times/\mathcal{O}_X^\times \longrightarrow 0$$

induces an exact sequence of cohomological groups

$$1 \longrightarrow \Gamma(X, \mathcal{O}_X^\times) \longrightarrow \Gamma(X, \mathscr{M}_X^\times) \longrightarrow \mathrm{Div}(X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^1(X, \mathscr{M}_X^\times)\ . \quad (1.2)$$

We denote by $\mathrm{div}(\cdot)$ the group homomorphism $\Gamma(X, \mathscr{M}_X^\times) \to \mathrm{Div}(X)$ in this exact sequence. Since the group law of $\mathrm{Div}(X)$ is written additively, one has

$$\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$$

for any couple of regular meromorphic functions f and g on X. A Cartier divisor belonging to the image of $\mathrm{div}(\cdot)$ is said to be *principal*. If D_1 and D_2 are two Cartier divisors such that $D_1 - D_2$ is principal, we say that D_1 and D_2 are *linearly equivalent*, denoted by $D_1 \sim D_2$.

1.2.5. Remark. — Recall that $H^1(X, \mathcal{O}_X^\times)$ identifies with the Picard group $\mathrm{Pic}(X)$ of X, namely the group of isomorphism classes of invertible $\mathcal{O}_X$-modules (see [**32**, **0**.(5.6.3)]). Similarly, $H^1(X, \mathscr{M}_X^\times)$ identifies with the group of isomorphism classes of invertible $\mathscr{M}_X$-modules. If L is an invertible $\mathcal{O}_X$-module, then $\mathscr{M}_X \otimes_{\mathcal{O}_X} L$ is an invertible $\mathscr{M}_X$-module. The homomorphism $H^1(X, \mathcal{O}_X^\times) \to H^1(X, \mathscr{M}_X^\times)$ sends the isomorphism class of an invertible $\mathcal{O}_X$-module L to that of the invertible $\mathscr{M}_X$-module $\mathscr{M}_X \otimes_{\mathcal{O}_X} L$.

1.2.6. Definition. — Let L be an invertible $\mathcal{O}_X$-module and U be a non-empty open subset of X. We call *regular meromorphic section* of L on U any element of $\Gamma(U, \mathscr{M}_X \otimes_{\mathcal{O}_X} L)$ which defines an isomorphism from $\mathscr{M}_U$ to $\mathscr{M}_U \otimes_{\mathcal{O}_U} L|_U$. Therefore, $\mathscr{M}_X \otimes_{\mathcal{O}_X} L$ is isomorphic as $\mathscr{M}_X$-module to $\mathscr{M}_X$ if and only if L admits a regular meromorphic section on X.

1.2.7. Remark. — Let X be a locally Noetherian scheme or a reduced scheme whose set of irreducible component is locally finite. For any $x \in X$, the local ring $\mathscr{M}_{X,x}$ identifies with the ring of total fractions of $\mathcal{O}_{X,x}$. Therefore, if L is an invertible $\mathcal{O}_X$-module and if U is an open subset of X, an element $s \in \Gamma(U, \mathscr{M}_X \otimes_{\mathcal{O}_X} L)$ is a regular meromorphic section of L on U if and only if it defines an injective homomorphism from $\mathcal{O}_U$ to $\mathscr{M}_U \otimes_{\mathcal{O}_U} L$. In particular, an element $s \in \Gamma(U, L)$ defines a regular meromorphic section of L on U if and only if, for any $x \in U$, $s_x \in \mathcal{O}_{X,x} \otimes_{\mathcal{O}_X} L$ is of the form $f_x s_{0,x}$, where f_x is a non-zero-divisor of $\mathcal{O}_{X,x}$, and $s_{0,x}$ is a local trivialization of L at x. This condition is also equivalent to $s(y) \neq 0$ for any associate point $y \in U$. Recall that a point $y \in X$ is called an *associated point* if there exists $a \in \mathcal{O}_{X,y}$ such that the maximal ideal of $\mathcal{O}_{X,y}$ identifies with

$$\mathrm{ann}(a) := \{f \in \mathcal{O}_{X,y} \,|\, af = 0\}.$$

Let x be a point of X. Assume that $s_x = f_x s_{0,x}$ where f_x is a zero-divisor in $\mathcal{O}_{X,x}$, then f_x belongs to an associated prime ideal of $\mathcal{O}_{X,x}$, which corresponds to an associated point $y \in X$ such that $x \in \overline{\{y\}}$ and $s(y) = 0$.

By [**33**, IV$_4$.(21.3.5)], if X is a Noetherian scheme, which admits an ample invertible $\mathcal{O}_X$-module, then the set of all associated points of X is contained in an affine open subset of X, and any invertible $\mathcal{O}_X$-module admits a regular meromorphic section.

1.2.8. Definition. — Let D be a Cartier divisor on X. The homomorphism $\mathrm{Div}(X) \to H^1(X, \mathcal{O}_X^\times)$ in the exact sequence (1.2) sends D to an isomorphism class of invertible $\mathcal{O}_X$-modules. One can actually construct explicitly an invertible $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ in this class as follows. Let $(U_i)_{i \in I}$ be an open cover of the topological space such that D is represented on each U_i by a regular meromorphic function $f_i \in \Gamma(U_i, \mathscr{M}_{U_i}^\times)$. For any couple $(i,j) \in I^2$, $f_i|_{U_i \cap U_j} f_j|_{U_i \cap U_j}^{-1}$ defines an isomorphism

$$(f_i^{-1}\mathcal{O}_{U_i})|_{U_i \cap U_j} \longrightarrow (f_j^{-1}\mathcal{O}_{U_j})|_{U_i \cap U_j}.$$

Moreover, these isomorphisms clearly satisfy the cocycle condition. Thus the gluing of the sheaves $f_i^{-1}\mathcal{O}_{U_i}$ leads to an invertible sub-$\mathcal{O}_X$-module of $\mathscr{M}_X$ which we denote by $\mathcal{O}_X(D)$. Note that the gluing of meromorphic sections

$$f_i \otimes f_i^{-1} \in \Gamma(U_i, \mathscr{M}_{U_i} \otimes \mathcal{O}_X(D))$$

leads to a global regular meromorphic section of $\mathcal{O}_X(D)$, which we denote by s_D and call *canonical regular meromorphic section* of $\mathcal{O}_X(D)$. Hence $\mathscr{M}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is canonically isomorphic to $\mathscr{M}_X$. Note that two Cartier divisors D_1 and D_2 are linearly equivalent if and only if the invertible $\mathcal{O}_X$-modules $\mathcal{O}_X(D_1)$ and $\mathcal{O}_X(D_2)$ are isomorphic.

Conversely, the exactness of the diagram (1.2) shows that, an invertible $\mathcal{O}_X$-module L is isomorphic to an invertible $\mathcal{O}_X$-module of the form $\mathcal{O}_X(D)$ if and only if it admits a regular meromorphic section on X. One can also construct explicitly a Cartier divisor from a regular meromorphic section s of L. In fact, let $(U_i)_{i \in I}$ be an open cover of X such that each $L|_{U_i}$ is trivialized by a section $s_i \in L(U_i)$. For any $i \in I$, let f_i be the unique regular meromorphic function on U_i such that $s = f_i s_i$. Then the family $(f_i)_{i \in I}$ of regular meromorphic functions defines a Cartier divisor on X which we denote by $\mathrm{div}(L; s)$, or by $\mathrm{div}(s)$ for simplicity.

1.2.9. Remark. — In the case where X is a quasi-projective scheme over a field, any invertible $\mathcal{O}_X$-module admits a global regular meromorphic section and therefore is isomorphic to an invertible $\mathcal{O}_X$-module of the form $\mathcal{O}_X(D)$, where D is a Cartier divisor. Hence one has an exact sequence

$$1 \longrightarrow \Gamma(X, \mathcal{O}_X^\times) \longrightarrow \Gamma(X, \mathscr{M}_X^\times) \longrightarrow \mathrm{Div}(X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow 1\ .$$

1.2.10. Remark. — Let X be a 0-dimensional projective scheme over a field k. Then there is a k-algebra A which is finite-dimensional as a vector space over k, and such that $X = \mathrm{Spec}(A)$. Note that the canonical homomorphism $A \to \bigoplus_{x \in X} A_x$ is an isomorphism. Let f_x be a regular element of A_x. As the homotethy map $A_x \to A_x$, $a \mapsto f_x a$, is injective and A_x is a finite-dimensional vector space over k, this homothety map is actually an isomorphism, that is, $f_x \in A_x^\times$. Thus $\mathscr{M}_X^\times = \mathcal{O}_X^\times$. Therefore, every Cartier divisor on X can be represented by $1 \in A$.

1.2.11. Remark. — Let X be a Noetherian scheme. We denote by $X^{(1)}$ the set of all height 1 points of X, that is, $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 1$. For $x \in X^{(1)}$ and a regular element f of $\mathcal{O}_{X,x}$, we set

$$\mathrm{ord}_x(f) := \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/f\mathcal{O}_{X,x}).$$

Then $\mathrm{ord}_x(fg) = \mathrm{ord}_x(f) + \mathrm{ord}_x(g)$ for all regular elements f, g of $\mathcal{O}_{X,x}$ (cf [**49**, the last paragraph of Section 1.3]), so that $\mathrm{ord}_x(\cdot)$ extends to a homomorphism $\mathscr{M}_{X,x}^\times \to \mathbb{Z}$. Let D be a Cartier divisor on X and f be a local equation of D at x. Then it is easy to see that $\mathrm{ord}_x(f)$ does not depend on the choice of f, so that $\mathrm{ord}_x(f)$ is denoted by $\mathrm{ord}_x(D)$. We call the cycle

$$\sum_{x \in X^{(1)}} \mathrm{ord}_x(D)\overline{\{x\}}$$

the cycle associated with D, which is denoted by $z(D)$. Let $X_1, \ldots, X_\ell$ be the irreducible components of X and $\eta_1, \ldots, \eta_\ell$ be the generic points of $X_1, \ldots, X_\ell$, respectively. Then

$$z(D) = \sum_{j=1}^{\ell} \mathrm{length}_{\mathcal{O}_{X,\eta_j}}(\mathcal{O}_{X,\eta_j}) z(D|_{X_j}). \tag{1.3}$$

Indeed, by [**49**, (6) of Lemma 1.7], $\mathrm{ord}_x(D) = \sum_{j \in J_x} b_j \,\mathrm{ord}_x(D|_{X_j})$, where $b_j = \mathrm{length}_{\mathcal{O}_{X,\eta_j}}(\mathcal{O}_{X,\eta_j})$ and $J_x = \{j \mid x \in X_j\}$. Thus if we set

$$a_{x,j} = \begin{cases} \mathrm{ord}_x(D|_{X_j}) & \text{if } x \in X_j, \\ 0 & \text{if } x \notin X_j, \end{cases}$$

then $\operatorname{ord}_x(D) = \sum_{j=1}^{\ell} a_{x,j} b_j$. Thus

$$\begin{aligned} z(D) &= \sum_{x \in X^{(1)}} \operatorname{ord}_x(D)\overline{\{x\}} = \sum_{x \in X^{(1)}} \Big(\sum_{j=1}^{\ell} a_{x,j} b_j\Big)\overline{\{x\}} \\ &= \sum_{j=1}^{\ell} b_j \sum_{x \in X^{(1)}} a_{x,j}\overline{\{x\}} = \sum_{j=1}^{\ell} b_j \sum_{x \in X_j^{(1)}} \operatorname{ord}_x(D|_{X_j})\overline{\{x\}} = \sum_{j=1}^{\ell} b_j z(D|_{X_j}), \end{aligned}$$

as required.

Let L be an invertible $\mathcal{O}_X$-module and s be a regular meromorphic section of L over X. For $x \in X^{(1)}$, $\operatorname{ord}_x(s)$ is defined by $\operatorname{ord}_x(f)$, where f is given by $s = f\omega$ for some local basis ω of L around x. Note that $\operatorname{ord}_x(s)$ does not depend on the choice of the local basis ω around x. Then the cycle $z(L; s)$ associated with $\operatorname{div}(L; s)$ is defined by

$$z(L; s) := \sum_{x \in X^{(1)}} \operatorname{ord}_x(s)\overline{\{x\}}.$$

1.2.12. Definition. — Let $\varphi : X \to Y$ be a morphism of locally ringed space. If U is an open subset of Y, we denote by $S_\varphi(U)$ the preimage of $S_X(\varphi^{-1}(U))$ by the structural ring homomorphism

$$\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\varphi^{-1}(U)).$$

We denote by $\mathscr{M}_\varphi$ the sheaf of commutative and unitary rings associated with the presheaf

$$U \longmapsto \mathcal{O}_Y(U)[S_\varphi(U)^{-1}].$$

It is a subsheaf of $\mathscr{M}_Y$. Moreover, the structural morphism of sheaves $\mathcal{O}_Y \to \varphi_*(\mathcal{O}_X)$ induces by localization a morphism $\mathscr{M}_\varphi \to \varphi_*(\mathscr{M}_X)$, which defines a morphism of locally ringed spaces $(X, \mathscr{M}_X) \to (Y, \mathscr{M}_\varphi)$.

1.2.13. Remark. — There are several situations in which $\mathscr{M}_\varphi$ identifies with $\mathscr{M}_Y$, notably when one of the following conditions is satisfied (see [**33**, IV_4.(21.4.5)]):

(1) φ is flat, namely for any $x \in X$, the morphism of rings $\varphi_x : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ defines a structure of flat $\mathcal{O}_{Y,\varphi(x)}$-algebra on $\mathcal{O}_{X,x}$,
(2) X and Y are locally Noetherian schemes, and f sends any associated point of X to an associated point of Y,
(3) X and Y are schemes, the set of irreducible components of Y is locally finite, X is reduced, and any irreducible component of X dominates an irreducible component of Y.

1.2.14. Definition. — Let $\varphi : X \to Y$ be a morphism of locally ringed spaces, and D be a Cartier divisor on Y. Assume that both D and $-D$ are global sections of $(\mathscr{M}_Y^\times \cap \mathscr{M}_\varphi)/\mathcal{O}_X^\times$, or equivalently, for any local equation f of D over an open subset

U of Y, one has $\{f, f^{-1}\} \subset \mathscr{M}_\varphi(U)$. Then the canonical regular meromorphic section s_D of $\mathcal{O}_Y(D)$ actually defines an isomorphism

$$\mathscr{M}_\varphi \longrightarrow \mathscr{M}_\varphi \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D).$$

which induces an isomorphisme

$$\varphi^*(s_D) : \mathscr{M}_X \longrightarrow \mathscr{M}_X \otimes_{\mathcal{O}_X} \varphi^*(\mathcal{O}_Y(D)).$$

We denote by $\varphi^*(D)$ the Cartier divisor $\mathrm{div}(\varphi^*(\mathcal{O}_Y(D)); \varphi^*(s_D))$ corresponding to this regular meromorphic section, and call it the *pull-back* of D by φ. In the case where φ is an immersion, the Cartier divisor $\varphi^*(D)$ is also denoted by $D|_X$.

Finally let us consider the following lemmas.

1.2.15. Lemma. — *Let $\mathfrak{o}$ be an integral domain, A be an $\mathfrak{o}$-algebra and $S := \mathfrak{o} \setminus \{0\}$. If A is flat over $\mathfrak{o}$, then we have the following:*

(1) *For $s \in S$, the homomorphism $s\cdot : A \to A$ given by $a \mapsto s \cdot a$ is injective. In particular, the structure homomorphism $\mathfrak{o} \to A$ is injective, so that in the following, $\mathfrak{o}$ is considered as a subring of A.*
(2) *The natural homomorphism $A \to A_S$ is injective.*
(3) *For $a \in A$, a is a non-zero-divisor in A if and only if $a/1$ is a non-zero-divisor in A_S. In particular, a non-zero-divisor of A_S can be written in the form of a/s, where a is a non-zero-divisor of A and $s \in S$.*
(4) *Let $Q(A)$ and $Q(A_S)$ be the total quotient rings of A and A_S, respectively. The homomorphism $Q(A) \to Q(A_S)$ induced by $A \to A_S$ is well-defined and bijective. In particular, $Q(A)^\times = Q(A_S)^\times$.*

Proof. — (1) is obvious because $\mathfrak{o}$ is an integral domain and A is flat over $\mathfrak{o}$. (2) follows from (1).

(3) The assertion follows from (1) and the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & A_S \\ {\scriptstyle a\cdot}\big\downarrow & & \big\downarrow{\scriptstyle a\cdot} \\ A & \longrightarrow & A_S \end{array}$$

(4) By (3), if $a \in A$ is a non-zero-divisor, then $a/1$ is a non-zero-divisor in A_S, so that $Q(A) \to Q(A_S)$ is well-defined. The injectivity of $Q(A) \to Q(A_S)$ follows from (2). For its surjectivity, observe the following:

$$\frac{b/t}{a/s} = \frac{(st/1)(b/t)}{(st/1)(a/s)} = \frac{sb/1}{ta/1}.$$

□

1.2.16. Lemma. — *Let X be an integral projective scheme over a field k, L be an invertible $\mathcal{O}_X$-module and F be a coherent $\mathcal{O}_X$-module. We assume that there exist a surjective morphism $f : X \to Y$ of integral projective schemes over k and an* *ample invertible $\mathcal{O}_Y$-module A such that $f^*(A) = L$. Then $R = \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})$ is a finitely generated algebra over k and $M = \bigoplus_{n=0}^{\infty} H^0(X, F \otimes L^{\otimes n})$ is a finitely generated R-module.*

Proof. — By [**49**, §1.8], there exist positive integers d and n_0 such that

$$H^0(Y, A^{\otimes d}) \otimes H^0(Y, A^{\otimes n} \otimes f_*(F)) \longrightarrow H^0(Y, A^{\otimes(d+n)} \otimes f_*(F))$$

is surjective for all $n \geqslant n_0$, and hence

$$H^0(X, L^{\otimes d}) \otimes H^0(X, L^{\otimes n} \otimes F) \longrightarrow H^0(X, L^{\otimes(d+n)} \otimes F)$$

is surjective for all $n \geqslant n_0$ because $f_*(L^{\otimes n}) = A^{\otimes n} \otimes f_*(\mathcal{O}_X)$, $f_*(L^{\otimes n} \otimes F) = A^{\otimes n} \otimes f_*(F)$, $\mathcal{O}_Y \subseteq f_*(\mathcal{O}_X)$. Thus, by the arguments in [**49**, §1.8], one can see the assertion. □

1.3. Proper intersection

Let d be a non-negative integer and X be a d-dimensional scheme of finite type over a field k. Let D be a Cartier divisor on X. We define the support of D to be

$$\mathrm{Supp}(D) := \{x \in X \mid f_x \notin \mathcal{O}_{X,x}^{\times}\},$$

where f_x is a local equation of D at x. Note that the above definition does not depend on the choice of f_x since two local equations of D at x differ by a factor in $\mathcal{O}_{X,x}^{\times}$.

1.3.1. Proposition. — (1) $\mathrm{Supp}(D)$ *is a Zariski closed subset of* X.
(2) $\mathrm{Supp}(D + D') \subseteq \mathrm{Supp}(D) \cup \mathrm{Supp}(D')$.

Proof. — (1) Clearly we may assume that X is affine and D is principal, that is, $X = \mathrm{Spec}(A)$ and D is defined by a regular meromorphic function f on X, which could be considered as an element of the total fraction ring of A (that is, the localization of A with respect to the subset of non-zero-divisors). By [**43**], for any prime ideal $\mathfrak{p}$ of A, there is a canonical ring homomorphism from the total fraction ring of A to that of $A_{\mathfrak{p}}$. We set $\mathfrak{a} = \{a \in A \mid af \in A\}$ and $\mathfrak{b} = \mathfrak{a}f$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A. Note that, for $\mathfrak{p} \in \mathrm{Spec}(A)$,

$$\mathfrak{a}_{\mathfrak{p}} = \{u \in A_{\mathfrak{p}} \mid uf \in A_{\mathfrak{p}}\}.$$

In fact, clearly one has $\mathfrak{a}_{\mathfrak{p}} \subseteq \{u \in A_{\mathfrak{p}} \mid uf \in A_{\mathfrak{p}}\}$. Conversely, if $u = a/s$ (with $a \in A$ and $s \in A \setminus \mathfrak{p}$) is an element of $A_{\mathfrak{p}}$ such that $uf \in A_{\mathfrak{p}}$, then there exists $t \in A \setminus \mathfrak{p}$ such that $at \in \mathfrak{a}$ and hence $u = at/st \in \mathfrak{a}_{\mathfrak{p}}$. Thus

$$\mathfrak{p} \notin \mathrm{Supp}(D) \iff f \in A_{\mathfrak{p}}^{\times} \iff \mathfrak{a}_{\mathfrak{p}} = A_{\mathfrak{p}} \text{ and } \mathfrak{b}_{\mathfrak{p}} = A_{\mathfrak{p}} \iff \mathfrak{p} \notin V(\mathfrak{a}) \cup V(\mathfrak{b}),$$

that is, $\mathrm{Supp}(D) = V(\mathfrak{a}) \cup V(\mathfrak{b})$, as desired.

(2) Let f_x and f'_x be local equations of D and D' at x, respectively. Then

$$\begin{aligned}x \notin \operatorname{Supp}(D) \cup \operatorname{Supp}(D') \Longrightarrow f_x, f'_x \in \mathcal{O}_{X,x}^{\times} \Longrightarrow f_x f'_x \in \mathcal{O}_{X,x}^{\times} \\ \Longrightarrow x \notin \operatorname{Supp}(D + D'),\end{aligned}$$

as required. □

1.3.2. Definition. — Let n be an integer such that $0 \leqslant n \leqslant d$. Let $D_0, \ldots, D_n$ be Cartier divisors on X. We say that $D_0, \ldots, D_n$ *intersect properly* if, for any non-empty subset J of $\{0, \ldots, n\}$,

$$\dim\Big(\bigcap\nolimits_{j\in J} \operatorname{Supp}(D_j)\Big) \leqslant d - \operatorname{card}(J).$$

By convention, $\dim(\varnothing)$ is defined to be -1. We set

$$\mathcal{IP}_X^{(n)} := \{(D_0, \ldots, D_n) \in \operatorname{Div}(X)^{n+1} \mid D_0, \ldots, D_n \text{ intersect properly}\}.$$

In the case where $n = d$, we often denote $\mathcal{IP}_X^{(n)}$ by $\mathcal{IP}_X$.

1.3.3. Lemma. — *Let k'/k be an extension of fields. Let A be a k-algebra and $A' := A \otimes_k k'$. Let $\pi : \operatorname{Spec}(A') \to \operatorname{Spec}(A)$ be the morphism induced by the natural homomorphism $A \to A'$. Let $Q(A)$ (resp. $Q(A')$) be the total fraction ring of A (resp. A'). Let $\alpha \in Q(A)^{\times}$ and $\alpha' := \alpha \otimes_k 1 \in Q(A) \otimes_k k'$. If we set*

$$\begin{cases}\operatorname{Supp}(\alpha) := \{P \in \operatorname{Spec}(A) \mid \alpha \notin A_P^{\times}\}, \\ \operatorname{Supp}(\alpha') := \{P' \in \operatorname{Spec}(A') \mid \alpha' \notin A'^{\times}_{P'}\},\end{cases}$$

then $\operatorname{Supp}(\alpha') = \pi^{-1}(\operatorname{Supp}(\alpha))$.

Proof. — First of all, note that $Q(A) \otimes_k k' \subseteq Q(A')$ and $\alpha' \in (Q(A) \otimes_k k')^{\times} \subseteq Q(A')^{\times}$ because π is flat. Let $I := \{a \in A \mid a\alpha \in A\}$, $J := I\alpha$, $I' := \{a' \in A' \mid a'\alpha' \in A'\}$ and $J' := I'\alpha'$. Then one has the following.

1.3.4. Claim. — (1) $\operatorname{Supp}(\alpha) = \operatorname{Spec}(A/I) \cup \operatorname{Spec}(A/J)$ *and* $\operatorname{Supp}(\alpha') = \operatorname{Spec}(A'/I') \cup \operatorname{Spec}(A'/J')$.

(2) $I' = I \otimes_k k'$ *and* $J' = J \otimes_k k'$.

(3) $\operatorname{Spec}(A'/I') = \pi^{-1}(\operatorname{Spec}(A/I))$ *and* $\operatorname{Spec}(A'/J') = \pi^{-1}(\operatorname{Spec}(A/J))$.

Proof. — Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a basis of k' over k. Note that $V \otimes_k k' = \bigoplus_{\lambda \in \Lambda} V \otimes_k kx_\lambda$ for any k-module V.

(1) It is sufficient to prove the first equality. The second is similar to the first. Note that $I_P = \{a \in A_P \mid a\alpha \in A_P^{\times}\}$. Thus, if $\alpha \in A_P^{\times}$, then $I_P = J_P = A_P$, so that $P \notin \operatorname{Spec}(A/I) \cup \operatorname{Spec}(A/J)$. Conversely, we assume that $P \notin \operatorname{Spec}(A/I) \cup \operatorname{Spec}(A/J)$, that is, $I \not\subseteq P$ and $J \not\subseteq P$. Thus $I_P = J_P = A_P$, and hence $\alpha \in A_P^{\times}$.

(2) Obviously $I \otimes_k k' \subseteq I'$. We assume $a' \in I'$. Then there exists $(a_\lambda)_{\lambda \in \Lambda} \in A^{\Lambda}$ such that $a' = \sum_\lambda a_\lambda \otimes x_\lambda$. By our assumption, we can find $(b_\lambda)_{\lambda \in \Lambda} \in A^{\Lambda}$ such that

$$\sum\nolimits_\lambda a_\lambda \alpha \otimes x_\lambda = a'\alpha' = \sum\nolimits_\lambda b_\lambda \otimes x_\lambda,$$

so that $a_\lambda \alpha = b_\lambda \in A$ for all λ. Thus $a_\lambda \in I$. Therefore the first assertion follows. The second is a consequence of the first.

(3) follows from (2). □

By using (1) and (3) of the above claim,

$$\begin{aligned}\pi^{-1}(\mathrm{Supp}(\alpha)) &= \pi^{-1}(\mathrm{Spec}(A/I) \cup \mathrm{Spec}(A/J)) \\ &= \pi^{-1}(\mathrm{Spec}(A/I)) \cup \pi^{-1}(\mathrm{Spec}(A/J)) \\ &= \mathrm{Supp}(A'/I') \cup \mathrm{Supp}(A'/J') = \mathrm{Supp}(\alpha'),\end{aligned}$$

as required. □

1.3.5. Remark. — Let k'/k be an extension of fields, $X_{k'} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k'$ and $\pi : X_{k'} \to X$ be the morphism of projection. Since the canonical morphism $\mathrm{Spec}\, k' \to \mathrm{Spec}\, k$ is flat, so is the morphism of projection π (see [**33**, IV_1.(2.1.4)]). Therefore, for any Cartier divisor D on X, the pull-back $\pi^*(D)$ is well defined as a Cartier divisor on $X_{k'}$, which we denote by $D_{k'}$.

By Lemma 1.3.3, one has

$$\mathrm{Supp}(D_{k'}) = \pi^{-1}(\mathrm{Supp}(D)).$$

In particular, if $D_0, \ldots, D_n$ are Cartier divisors on X, which intersect properly, then, for any subset J of $\{0, \ldots, n\}$, one has (see for example [**28**, Proposition 5.38] for the equality in the middle)

$$\begin{aligned}\dim\Big(\bigcap\nolimits_{j\in J} \mathrm{Supp}(D_{j,k'})\Big) &= \dim\Big(\pi^{-1}\Big(\bigcap\nolimits_{j\in J} \mathrm{Supp}(D_j)\Big)\Big) \\ &= \dim\Big(\bigcap\nolimits_{j\in J} \mathrm{Supp}(D_j)\Big) \leqslant d - \mathrm{card}(J).\end{aligned}$$

Therefore, the Cartier divisors $D_{0,k'}, \ldots, D_{n,k'}$ on $X_{k'}$ intersect properly.

1.3.6. Lemma. — *The set $\mathcal{IP}_X^{(n)}$ forms a symmetric and multi-linear subset of* $\mathrm{Div}(X)^{n+1}$ *in the sense of Definition 1.1.1.*

Proof. — It is sufficient to show that if $(D_0, D_1, \ldots, D_n), (D_0', D_1, \ldots, D_n) \in \mathcal{IP}_X^{(n)}$, then $(D_0 + D_0', D_1, \ldots, D_n) \in \mathcal{IP}_X^{(n)}$. We set

$$D_i'' = \begin{cases} D_0 + D_0', & \text{if } i = 0, \\ D_i, & \text{if } i \geqslant 1. \end{cases}$$

If $(D_0'', D_1'', \ldots, D_n'') \notin \mathcal{IP}_X^{(n)}$, then there is a non-empty subset J of $\{0, \ldots, n\}$ such that

$$\dim\Big(\bigcap\nolimits_{j\in J} \mathrm{Supp}\left(D_j''\right)\Big) > d - \#(J).$$

Clearly $0 \in J$. We can find a schematic point $P \in X$ such that $\dim \overline{\{P\}} > d - \#(J)$ and $P \in \mathrm{Supp}\left(D_j''\right)$ for all $j \in J$, so that $P \in \mathrm{Supp}(D_0 + D_0')$ and $P \in \mathrm{Supp}(D_j)$ for

$j \in J \setminus \{0\}$. Thus, by Proposition 1.3.1, $P \in \mathrm{Supp}(D_0)$ or $P \in \mathrm{Supp}(D'_0)$, which is a contradiction. $\square$

1.3.7. Lemma. — *We assume that X is projective. Let n be an integer such that $0 \leqslant n \leqslant d$.*

(1) *Let $L_0, \ldots, L_n$ be invertible $\mathcal{O}_X$-modules. Then there are regular meromorphic sections $s_0, \ldots, s_n$ of $L_0, \ldots, L_n$, respectively, such that, if we set $D_i = \mathrm{div}(L_i; s_i)$ for $i \in \{0, \ldots, n\}$, then $D_0, \ldots, D_n$ intersect properly.*

(2) *If $(D_0, D_1, \ldots, D_n), (D'_0, D'_1, \ldots, D'_n) \in \mathcal{IP}_X^{(n)}$ and $D_0 \sim D'_0$, then there is D''_0 such that $D''_0 \sim D_0\,(\sim D'_0)$ and $(D''_0, D_1, \ldots, D_n), (D''_0, D'_1, \ldots, D'_n) \in \mathcal{IP}_X^{(n)}$.*

Proof. — (1) We prove it by induction on n in incorporating the proof of the initial case in the induction procedure. By the hypothesis of induction (when $n \geqslant 1$), there are regular meromorphic sections $s_0, \ldots, s_{n-1}$ of $L_0, \ldots, L_{n-1}$, respectively, such that if we set $D_i = \mathrm{div}(L_i; s_i)$ for $i \in \{0, \ldots, n-1\}$, then $D_0, \ldots, D_{n-1}$ intersect properly.

We now introduce the following claim, which (in the case where $n = 0$) also proves the initial case of induction.

1.3.8. Claim. — *There exist very ample invertible $\mathcal{O}_X$-modules L'_n and L''_n, and global sections s'_n and s''_n of L'_n and L''_n, which satisfy the following conditions :*

(i) $L_n = L'_n \otimes L_n''^{-1}$,

(ii) *s'_n and s''_n define regular meromorphic sections of L'_n and L''_n, respectively,*

(iii) *if we set $D'_n = \mathrm{div}(L'_n; s'_n)$ and $D''_n = \mathrm{div}(L''_n; s''_n)$, then both families of Cartier divisors $D_0, \ldots, D_{n-1}, D'_n$ and $D_0, \ldots, D_{n-1}, D''_n$ intersect properly.*

Proof of Claim 1.3.8. — Since X is projective, there exists a very ample $\mathcal{O}_X$-module L. By [**33**, II.(4.5.5)], there exists an integer $\ell_0 \in \mathbb{N}_{\geqslant 1}$ such that both invertible $\mathcal{O}_X$-modules $L^{\otimes \ell_0}$ and $L^{\otimes \ell_0} \otimes L_n^{-1}$ are generated by global sections. Let Σ be the set of generic points of

$$\bigcap_{i=0}^{n-1} \mathrm{Supp}(D_i).$$

We equip the set $\Sigma \cup \mathrm{Ass}(X)$ with the order $\succ$ of generalization, namely $x \succ y$ if and only if y belongs to the Zariski closure of $\{x\}$. We denote by $\{y_1, \ldots, y_b\}$ the set of all minimal elements of the set $\Sigma \cup \mathrm{Ass}(X)$.

For any $i \in \{1, \ldots, b\}$, one has

$$y_i \in X \setminus \bigcup_{\substack{j \in \{1,\ldots,b\} \\ j \neq i}} \overline{\{y_j\}}.$$

By [**33**, II.(4.5.4)], for any $i \in \{1, \ldots, b\}$, there exists $\ell_i \in \mathbb{N}_{\geqslant 1}$ and a section $t_i \in H^0(X, L^{\otimes \ell_i})$ such that $t_i(y_i) \neq 0$ and that $t_i(y_j) = 0$ for any $j \in \{1, \ldots, b\} \setminus \{i\}$. Moreover, by replacing the global sections $t_1, \ldots, t_b$ by suitable powers, we may assume, without loss of generality, that all $\ell_1, \ldots, \ell_b$ are equal to a positive integer ℓ.

For any $i \in \{1,\dots,b\}$, let $u_i \in H^0(X, L^{\otimes \ell_0})$ and $v_i \in H^0(X, L^{\otimes \ell_0} \otimes L_n^{-1})$ be such that $u_i(y_i) \neq 0$ and $v_i(y_i) \neq 0$. These sections exist since the invertible $\mathcal{O}_X$-modules $L^{\otimes \ell_0}$ and $L^{\otimes \ell_0} \otimes L_n^{-1}$ are generated by global sections. Now we take

$$L'_n = L^{\otimes(\ell_0+\ell)}, \qquad L''_n = L'_n \otimes L_n^{-1} = (L^{\ell_0} \otimes L_n^{-1}) \otimes L^{\otimes \ell},$$

and

$$s'_n = \sum_{i=1}^{b} u_i t_i, \qquad s''_n = \sum_{i=1}^{b} v_i t_i.$$

Then, for any $i \in \{1,\dots,b\}$, one has $s'_n(y_i) \neq 0$ and $s''_n(y_i) \neq 0$. In particular, s'_n and s''_n do not vanish on any of the associated points of X and hence are regular meromorphic sections (see Remark 1.2.7). Moreover, since these sections do not vanish at any point of Σ, we obtain the condition (iii) above. □

Thus, by Lemma 1.3.6, we can see that $D_1,\dots,D_{n-1},D_n$ intersect properly, where $D_n = D'_n - D''_n = \operatorname{div}(L_n; s_n \otimes s_n{}^{-1})$, as required.

(2) We can find very ample Cartier divisors A and B on X such that $D_0 = A - B$. Then, by the same argument as the induction procedure in the proof of (1), we obtain that there are A' and B' such that $A' \sim A$, $B' \sim B$ and

$$(A', D_1,\dots,D_n), (A', D'_1,\dots,D'_n), (B', D_1,\dots,D_n), (B', D'_1,\dots,D'_n) \in \mathcal{IP}_X^{(n)}.$$

Thus if we set $D''_0 = A' - B'$, then, by Lemma 1.3.6, one has the conclusion. □

1.3.9. Remark. — Claim 1.3.8 has its own interest and will be used in further chapters in the following way. Let X be a d-dimensional projective scheme over $\operatorname{Spec} k$ and $D_0,\dots,D_d$ be Cartier divisors on X. We suppose that $D_0,\dots,D_d$ intersect properly. Let $D_0 = A_0 - A'_0$ be a decomposition of D_0 into the difference of two very ample Cartier divisors. A priori $A_0, D_1,\dots,D_d$ do not intersect properly. However, by Claim 1.3.8, one can find a very ample invertible $\mathcal{O}_X$-module L and a global section s of $L \otimes \mathcal{O}_X(A_0)$ defining a regular meromorphic section, such that $\operatorname{div}(L;s), D_1,\dots,D_d$ intersect properly. Let $B = \operatorname{div}(L;s) - A_0$. This is a very ample Cartier divisor since $\mathcal{O}_X(B)$ is isomorphic to L. Moreover, both $(A_0 + B, D_1,\dots,D_d)$ and $(A'_0 + B, D_1,\dots,D_d)$ belong to $\mathcal{IP}_X^{(d)}$ since the former one and their difference do.

1.3.10. Lemma. — *We assume that $D_0,\dots,D_n$ are effective and ample. Then $D_0,\dots,D_n$ intersect properly if and only if* $\dim\bigl(\bigcap_{i=0}^n \operatorname{Supp}(D_i)\bigr) \leqslant d-n-1$.

Proof. — Obviously if $D_0,\dots,D_n$ intersect properly, then $\dim\bigl(\bigcap_{i=0}^n \operatorname{Supp}(D_i)\bigr) \leqslant d-n-1$. Conversely, let J be a subset of $\{0,\dots,n\}$. If we set $Z = \bigcap_{j\in J} \operatorname{Supp}(D_j)$ and $I = \{0,\dots,n\} \setminus J$, then

$$\dim\Bigl(Z \cap \bigcap\nolimits_{i\in I} \operatorname{Supp}(D_i)\Bigr) \geqslant \dim Z - \operatorname{card}(I)$$

because D_i is effective and ample for all $i \in I$. Thus, by our assumption, one has

$$d-n-1 \geqslant \dim Z - \operatorname{card}(I),$$

which implies $\dim Z \leqslant d - \mathrm{card}(J)$. □

1.4. Multi-homogeneous polynomials

Let k be a field and $(E_i)_{i=0}^d$ be a family of finite-dimensional vector spaces over k. Let $(\delta_0, \ldots, \delta_d)$ be a multi-index in $\mathbb{N}^{d+1}$.

1.4.1. Definition. — We call *multi-homogeneous polynomial of multi-degree* $(\delta_0, \ldots, \delta_d)$ on $E_0 \times \cdots \times E_d$ any element of

$$S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee),$$

where $S^{\delta_i}(E_i^\vee)$ denotes the δ_i-th symmetric power of the vector space $E_i^\vee$.

Recall that the dual vector space of $S^{\delta_i}(E_i^\vee)$ is given by

$$\Gamma^{\delta_i}(E_i) := (E_i^{\otimes \delta_i})^{\mathfrak{S}_{\delta_i}},$$

where $\mathfrak{S}_{\delta_i}$ is the symmetric group on $\{1, \ldots, \delta_i\}$, which acts on $E_i^{\otimes \delta_i}$ by permuting tensor factors (see [**7**, Chapitre IV, §5, no. 11, proposition 20]). Therefore, the dual vector space of $S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)$ is given by

$$\Gamma^{\delta_0}(E_0) \otimes_k \cdots \otimes_k \Gamma^{\delta_d}(E_d).$$

If $R \in S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)$ is a multi-homogeneous polynomial of multi-degree $(\delta_0, \ldots, \delta_d)$, for any $(s_0, \ldots, s_d) \in E_0 \times \cdots \times E_d$, we denote by $R(s_0, \ldots, s_d)$ the value

$$R(s_0^{\otimes \delta_0} \otimes \cdots \otimes s_d^{\otimes \delta_d})$$

in k. Thus R determines a function on $E_0 \times \cdots \times E_d$ valued in K (which we still denote by R by abuse of notation). In the case where the field k is infinite, as an element of $S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)$, R is uniquely determined by the corresponding function on $E_0 \times \cdots \times E_d$ since each vector space $\Gamma^{\delta_i}(E_i)$ is spanned over k by elements of the form $s_i^{\otimes \delta_i}$, $s_i \in E_i$ (see [**7**, Chapitre IV, §5, no. 5, proposition 5]). Moreover, for any $i \in \{0, \ldots, d\}$ and $s_i \in E_i$, we denote by

$$R(\cdots, \underset{\substack{\uparrow \\ i\text{-th coordinate}}}{s_i}, \cdots)$$

of R at s_i as an element of

$$S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_{i-1}}(E_{i-1}^\vee) \otimes_k S^{\delta_{i+1}}(E_{i+1}^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)$$

or as a multi-homogeneous polynomial function on

$$E_0 \times \cdots \times E_{i-1} \times E_{i+1} \times \cdots \times E_d,$$

according to the context.

1.4.2. Remark. — Note that an element of $S^{\delta_0}(E_0^\vee)\otimes_k\cdots\otimes_k S^{\delta_d}(E_d^\vee)$ yields a multi-homogeneous polynomial function on $\mathbb{A}(E_0^\vee)\times_k\cdots\times_k\mathbb{A}(E_d^\vee)$ and the set of k-rational points of $\mathbb{A}(E_0^\vee)\times_k\cdots\times_k\mathbb{A}(E_d^\vee)$ is naturally isomorphic to $E_0\times\cdots\times E_d$, where $\mathbb{A}(E_i^\vee)=\operatorname{Spec}(\bigoplus_{\delta=0}^\infty S^\delta(E_i^\vee))$ for each i.

1.5. Incidence subscheme

Let k be a field and E be a finite-dimensional vector space over k. We denote $\operatorname{Proj}(\bigoplus_{\delta=0}^\infty S^\delta(E))$ by $\mathbb{P}(E)$. Recall that the projective space $\mathbb{P}(E)$ represents the contravariant functor from the category of k-schemes to that of sets, which sends a k-scheme $\varphi:S\to\operatorname{Spec}k$ to the set of isomorphism classes of invertible quotient $\mathcal{O}_S$-modules of $\varphi^*(E)$ (cf. [**33**, EGA2, Théorèm 4.2.4]). In particular,

$$\mathbb{P}(E)(k)=(E^\vee\setminus\{0\})/\sim,$$

where, for $\theta_1,\theta_2\in E^\vee\setminus\{0\}$, $\theta_1\sim\theta_2$ if and only if $\theta_1=a\theta_2$ for some $a\in k^\times$. Thus an element of $S^\delta(E)$ yields a homogeneous polynomial of degree δ on $\mathbb{P}(E)(k)$. Moreover, if we denote by $\pi_E:\mathbb{P}(E)\to\operatorname{Spec}k$ the structural scheme morphism, then the universal object of the representation of the above functor by $\mathbb{P}(E)$ is the isomorphism class of a quotient $\mathcal{O}_{\mathbb{P}(E)}$-module of $\pi_E^*(E)$, which we denote by $\mathcal{O}_E(1)$ and which we call *universal invertible sheaf* on $\mathbb{P}(E)$. For any positive integer n, we let $\mathcal{O}_E(n):=\mathcal{O}_E(1)^{\otimes n}$ and $\mathcal{O}_E(-n):=(\mathcal{O}_E(1)^\vee)^{\otimes n}$. Note that the quotient homomorphism $\pi_E^*(E)\to\mathcal{O}_E(1)$ induces by passing to dual modules an injective homomorphism

$$\mathcal{O}_E(-1)\longrightarrow\pi_E^*(E^\vee).$$

We now consider the fibre product of projective spaces $\mathbb{P}(E)\times_k\mathbb{P}(E^\vee)$. Let

$$p_1:\mathbb{P}(E)\times_k\mathbb{P}(E^\vee)\longrightarrow\mathbb{P}(E)\quad\text{and}\quad p_2:\mathbb{P}(E)\times_k\mathbb{P}(E^\vee)\longrightarrow\mathbb{P}(E^\vee)$$

be morphisms of projection. Note that the following diagram of scheme morphisms is cartesian

$$\begin{tikzcd}
\mathbb{P}(E)\times_k\mathbb{P}(E^\vee) \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathbb{P}(E^\vee) \arrow[d, "\pi_{E^\vee}"] \\
\mathbb{P}(E) \arrow[r, "\pi_E"'] & \operatorname{Spec}k
\end{tikzcd}$$

The composition of the homomorphisms

$$p_1^*(\mathcal{O}_E(-1))\longrightarrow p_1^*(\pi_E^*(E^\vee))\cong p_2^*(\pi_{E^\vee}^*(E^\vee))\longrightarrow p_2^*(\mathcal{O}_{E^\vee}(1)) \tag{1.4}$$

determines a global section of the invertible sheaf

$$\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1):=p_1^*(\mathcal{O}_E(1))\otimes p_2^*(\mathcal{O}_{E^\vee}(1)).$$

1.5.1. Definition. — We call *incidence subscheme* of $\mathbb{P}(E)\times_k\mathbb{P}(E^\vee)$ and we denote by I_E the closed subscheme of $\mathbb{P}(E)\times_k\mathbb{P}(E^\vee)$ defined by the vanishing of the global section of $\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1)$ determined by (1.4). In particular, the cycle class of I_E modulo the linear equivalence is

$$c_1(\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1))\cap[\mathbb{P}(E)\times_k\mathbb{P}(E^\vee)].$$

The following proposition shows that the incidence subscheme can be realized as a projective bundle over $\mathbb{P}(E)$.

1.5.2. Proposition. — *Let $Q_{E^\vee}$ be the quotient sheaf of $\pi_E^*(E^\vee)$ by the canonical image of $\mathcal{O}_E(-1)$. Then the incidence subscheme I_E is isomorphic as a $\mathbb{P}(E)$-scheme to the projective bundle $\mathbb{P}(Q_{E^\vee}\otimes\mathcal{O}_E(1))$. Moreover, under this isomorphism, the restriction of $\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1)$ to I_E is isomorphic to the universal invertible sheaf of the projective bundle $\mathbb{P}(Q_{E^\vee}\otimes\mathcal{O}_E(1))$.*

Proof. — It suffices to identify $p_1:\mathbb{P}(E)\times_k\mathbb{P}(E^\vee)\to\mathbb{P}(E)$ with the projective bundle

$$\mathbb{P}(\pi_E^*(E^\vee)\otimes\mathcal{O}_E(1))\longrightarrow\mathbb{P}(E).$$

Note that the universal invertible sheaf of this projective bundle is isomorphic to $\mathcal{O}_E(1)\boxtimes\mathcal{O}_{E^\vee}(1)$. Under this identification, the vanishing locus of (1.4) coincides with the projective bundle $\mathbb{P}(Q_{E^\vee}\otimes\mathcal{O}_E(1))$. □

1.5.3. Remark. — As a scheme over $\mathbb{P}(E)$, the incident subscheme I_E also identifies with the projective bundle $\mathbb{P}(Q_{E^\vee})$. However, the universal invertible sheaf of this projective bundle is the restriction of $p_2^*(\mathcal{O}_{E^\vee}(1))$. Moreover, we can also consider the morphism of projection from the incidence subscheme to $\mathbb{P}(E^\vee)$. By the duality between E and $E^\vee$, the incidence subscheme I_E also identifies with the projective bundle of $Q_E:=\pi_{E^\vee}^*(E)/\mathcal{O}_{E^\vee}(-1)$ over $\mathbb{P}(E^\vee)$. In particular, if x is a point of $\mathbb{P}(E^\vee)$, then the fibre of the incidence subscheme I_E over x identifies with

$$\mathbb{P}((E\otimes_k\kappa(x))/x^*\mathcal{O}_E(-1)),$$

which is a hyperplane in $\mathbb{P}(E\otimes_k\kappa(x))$ defined by the vanishing locus of any non-zero element of the one-dimensional $\kappa(x)$-vector subspace of $E\otimes_k\kappa(x)$ defining the point x.

1.6. Resultants

Let k be a field and X be an integral projective k-scheme, and d be the Krull dimension of X. For any $i\in\{0,\ldots,d\}$, we fix a finite-dimensional vector space E_i over k and a closed embedding $f_i:X\to\mathbb{P}(E_i)$, and we denote by L_i the pull-back of $\mathcal{O}_{E_i}(1)$ by f_i. For each $i\in\{0,\ldots,d\}$, we let r_i be the Krull dimension of $\mathbb{P}(E_i)$, which identifies with $\dim_k(E_i)-1$. For each $i\in\{0,\ldots,d\}$, we let δ_i be the intersection number

$$\deg\big(c_1(L_0)\cdots c_1(L_{i-1})c_1(L_{i+1})\cdots c_1(L_d)\cap[X]\big)$$

Let $\mathbb{P}=\mathbb{P}(E_0)\times_k\cdots\times_k\mathbb{P}(E_d)$ be the product of k-schemes $(\mathbb{P}(E_i))_{i=0}^d$. The family $(f_i)_{i=0}^d$ induces a closed embedding $f:X\to\mathbb{P}$. Let

$$\check{\mathbb{P}}:=\mathbb{P}(E_0^\vee)\times_k\cdots\times_k\mathbb{P}(E_d^\vee)$$

be the product of dual projective spaces. We identify $\mathbb{P}\times_k\check{\mathbb{P}}$ with

$$(\mathbb{P}(E_0)\times_k\mathbb{P}(E_0^\vee))\times_k\cdots\times_k(\mathbb{P}(E_d)\times_k\mathbb{P}(E_d^\vee))$$

and we denote by

$$I_{\mathbb{P}}:=I_{E_0}\times_k\cdots\times_k I_{E_d}$$

the fibre product of incidence subschemes, so that the class of $I_{\mathbb{P}}$ modulo the linear equivalence coincides with the intersection product

$$c_1(r_0^*(\mathcal{O}_{E_0}(1)\boxtimes\mathcal{O}_{E_0^\vee}(1)))\cdots c_1(r_n^*(\mathcal{O}_{E_n}(1)\boxtimes\mathcal{O}_{E_n^\vee}(1)))\cap[\mathbb{P}\times_k\check{\mathbb{P}}],$$

where $r_i:\mathbb{P}\times_k\check{\mathbb{P}}\to\mathbb{P}(E_i)\times_k\mathbb{P}(E_i^\vee)$ is the i-th projection. By Proposition 1.5.2 (see also Remark 1.5.3), $I_{\mathbb{P}}$ is isomorphic to a fiber product of projective bundles

$$\mathbb{P}(Q_{E_0})\times_k\cdots\times_k\mathbb{P}(Q_{E_d}).$$

1.6.1. Definition. — We denote by I_X the fibre product $X\times_{\mathbb{P}}I_{\mathbb{P}}$, called the *incidence subscheme* of $X\times_k\check{\mathbb{P}}$. As an X-scheme, it identifies with

$$\mathbb{P}(Q_{E_0}|_X)\times_X\cdots\times_X\mathbb{P}(Q_{E_d}|_X).$$

and hence is an integral closed subscheme of dimension

$$d+(r_0-1)+\cdots+(r_d-1)=r_0+\cdots+r_d-1$$

of $\mathbb{P}\times_k\check{\mathbb{P}}$. In particular, for any extension K of k and any element

$$(x,\alpha_0,\ldots,\alpha_d)\in X(K)\times\mathbb{P}(E_0^\vee)(K)\times\cdots\times\mathbb{P}(E_d^\vee)(K),$$

if we denote by H_i the hyperplane in $\mathbb{P}(E_{i,K})$ defined by the vanishing of α_i, then $(x,\alpha_0,\ldots,\alpha_d)$ belongs to $I_X(K)$ if and only if $f_{i,K}(x)\in H_i$ for any $i\in\{1,\ldots,d\}$. In addition, the cycle class of I_X modulo the linear equivalence is the intersection product

$$c_1\left(p^*(L_0)\otimes q^*q_0^*(\mathcal{O}_{E_0^\vee}(1))\right)\cdots c_1\left(p^*(L_d)\otimes q^*q_d^*(\mathcal{O}_{E_d^\vee}(1))\right)\cap[X\times_k\check{\mathbb{P}}],\qquad(1.5)$$

where $p:X\times_k\check{\mathbb{P}}\to X$, $q:X\times_k\check{\mathbb{P}}\to\check{\mathbb{P}}$ and $q_i:\check{\mathbb{P}}\to\mathbb{P}(E_i^\vee)$ are the projections.

1.6.2. Proposition. — *The direct image by the projection $q:X\times_k\check{\mathbb{P}}\to\check{\mathbb{P}}$ of I_X is a multi-homogeneous hypersurface of multi-degree $(\delta_0,\ldots,\delta_d)$.*

Proof. — It is sufficient to see that $q_*(I_X)$ belongs to the cycle class

$$c_1(\mathcal{O}_{E_0^\vee}(\delta_0)\boxtimes\cdots\boxtimes\mathcal{O}_{E_d^\vee}(\delta_d))\cap[\check{\mathbb{P}}].$$

Note that, for any $(i_1,\ldots,i_n)\in\{0,\ldots,d\}^n$ such that $i_1,\ldots,i_n$ are distinct,

$$q_*(c_1(p^*(L_{i_1}))\cdots c_1(p^*(L_{i_n}))\cap[X\times_k\check{\mathbb{P}}])$$

is equal to

$$c_1(\mathcal{O}_{E_{i_1}^\vee}(1)\boxtimes\cdots\boxtimes\mathcal{O}_{E_{i_n}^\vee}(1))\cap[\check{\mathbb{P}}]$$

if $n=d$, and is equal to the zero cycle class otherwise. Therefore, the assertion follows from (1.5). □

1.6.3. Proposition. — *For $(\alpha_0,\ldots,\alpha_d)\in(E_0\setminus\{0\})\times\cdots\times(E_d\setminus\{0\})$, the following are equivalent:*

(1) *For all i, $f_i(X)\not\subseteq\mathrm{Supp}(\mathrm{div}(\alpha_i))$, and $\mathrm{div}(f_0^*(\alpha_0))),\ldots,\mathrm{div}(f_d^*(\alpha_d)))$ intersect properly on X.*

(2) *One has $([\alpha_0],\ldots,[\alpha_d])\notin q(I_X)$, where $[\alpha_i]$ denotes the class of α_i in $\mathbb{P}(E_i^\vee)(k)$.*

Proof. — (1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (1) : We set $J=\{j\in\{0,\ldots,d\}\mid f_j(X)\subseteq\mathrm{Supp}(\mathrm{div}(\alpha_j))\}$. We assume that $J\neq\varnothing$. Then, as $f_j^*(\alpha_j)=0$ for all $j\in J$ and $\mathrm{div}(f_i^*(\alpha_i))$ is ample for all $i\notin J$, one has $\bigcap_{i=0}^d\mathrm{Supp}(\mathrm{div}(f_i^*(\alpha_i)))\neq\varnothing$. If we choose $x\in\bigcap_{i=0}^d\mathrm{Supp}(\mathrm{div}(f_i^*(\alpha_i)))$, then $(x,[\alpha_0],\ldots,[\alpha_d])\in I_X$, which is a contradiction. Therefore $J=\varnothing$. Note that $\mathrm{div}(f_i^*(\alpha_i))$ is ample for every i and $\bigcap_{i=0}^d\mathrm{Supp}(\mathrm{div}(f_i^*(\alpha_i)))=\varnothing$. Thus, by Lemma 1.3.10, $\mathrm{div}(f_0^*(\alpha_0))),\ldots,\mathrm{div}(f_d^*(\alpha_d)))$ intersect properly on X. □

1.6.4. Definition. — Let X be an integral projective k-scheme of dimension d. We call *resultant of X with respect to* $(f_i)_{i=0}^d$ any multi-homogeneous polynomial of multi-degree $(\delta_0,\ldots,\delta_d)$ on $E_0\times\cdots\times E_d$, whose vanishing cycle in

$$\mathbb{P}(E_0^\vee)\times_k\cdots\times_k\mathbb{P}(E_d^\vee)$$

identifies with the projection of the cycle associated with the incidence subscheme I_X. Note that the resultant of X with respect to $(f_i)_{i=0}^d$ is unique up to a factor of scalar in $k\setminus\{0\}$ as an element of $S^{\delta_0}(E_0^\vee)\otimes_k\cdots\otimes_k S^{\delta_d}(E_d^\vee)$.

In general, if X is a projective k-scheme of dimension d and if

$$\sum_{i=1}^n m_iX_i$$

is the d-dimensional part of the fundamental cycle of X, where $X_1,\ldots,X_n$ are d-dimensional irreducible components of X, and m_i is the local multiplicity of X at the generic point of X_i, we define the *resultant* of X with respect to $(f_i)_{i=0}^d$ as any multi-homogeneous polynomial of the form

$$(R^{X_1}_{f_0|_{X_1},\ldots,f_d|_{X_1}})^{m_1}\cdots(R^{X_n}_{f_0|_{X_n},\ldots,f_d|_{X_n}})^{m_n},$$

where each $R^{X_i}_{f_0|_{X_i},\ldots,f_d|_{X_i}}$ is a resultant of X_i with respect to $(f_i|_{X_i})_{i=0}^d$. Note that, in the case where X is equidimensional, the projection to $\check{\mathbb{P}}$ of the incidence subscheme $X \times_{\mathbb{P}} I_{\mathbb{P}}$, is the hypersurface defined by the resultant of X.

1.6.5. Example. — We consider the particular case where $d = 0$. Let $f_0 : X \to \mathbb{P}(E_0)$ be a close embedding. We first assume that X is integral. In this case f_0 sends X to a closed point x of $\mathbb{P}(E_0)$. Let $\kappa(x)$ be the residue field of x and $\delta_0 = [\kappa(x) : k]$ be the degree of x. Let s_0 be an element of E_0. We assume that, if we view s_0 as a global section of $\mathcal{O}_{E_0}(1)$, one has $s_0(x) \neq 0$. We construct an element $R^{X,s_0}_{f_0} \in S^{\delta_0}(E_0^\vee)$ as follows. Let

$$\varphi_0 : E_0 \otimes_K \kappa(x) \longrightarrow \mathcal{O}_{E_0}(1)(x)$$

be the surjective $\kappa(x)$-linear map corresponding to the closed point x, and

$$\varphi_0^\vee : \mathcal{O}_{E_0}(-1)(x) \longrightarrow E_0^\vee \otimes_K \kappa(x)$$

be the dual $\kappa(x)$-linear map of φ_0, which is an injective linear map. Let $s_0(x)^\vee$ be the unique $\kappa(x)$-linear form on $\mathcal{O}_{E_0}(1)(x)$ taking the value 1 at $s_0(x)$. We let

$$R^{X,s_0}_{f_0} := N_{\kappa(x)/K}(\varphi_0^\vee(s_0(x)^\vee)) \in S^{\delta_0}(E_0^\vee),$$

which is defined as the determinant of the following homothety endomorphism of the free module $\mathrm{Sym}(E_0^\vee) \otimes_K \kappa(x)$ of rank δ_0 over the symmetric algebra $\mathrm{Sym}(E_0^\vee)$

$$\mathrm{Sym}(E_0^\vee) \otimes_K \kappa(x) \xrightarrow{\varphi_0^\vee(s_0(x)^\vee)} \mathrm{Sym}(E_0^\vee) \otimes_K \kappa(x).$$

Note that

$$\varphi_0^\vee(s_0(x)^\vee)(s_0 \otimes 1) = s_0(x)^\vee(s_0(x)) = 1.$$

Therefore the following equality holds

$$R^{X,s_0}_{f_0}(s_0) = 1.$$

Assume that X is not irreducible. We let $X_1, \ldots, X_n$ be irreducible components of X (namely points of X). For each $i \in \{1, \ldots, n\}$, let $x_i = f_0(X_i)$ and a_i be the local multiplicity of X at X_i. Then

$$a_1 x_1 + \cdots + a_n x_n$$

is the decomposition of $f(X)$ as a zero-dimensional cycle in $\mathbb{P}(E_0)$, where $x_1, \ldots, x_n$ are closed points of $\mathbb{P}(E_0)$ and $a_1, \ldots, a_n$ are positive integers. If s_0 is a global section of $\mathcal{O}_{E_0}(1)$, which does not vanish on any of the points $x_1, \ldots, x_n$, we define

$$R^{X,s_0}_{f_0} := \prod_{i=1}^n (R^{X_i,s_0}_{f_0|_{X_i}})^{a_i}.$$

Then $R^{X,s_0}_{f_0}$ is a resultant of X with respect to the closed embedding f_0, which satisfies $R^{X,s_0}_{f_0}(s_0) = 1$.

1.6.6. Example. — Let n and m be positive integers, and let

$$f : \mathbb{P}^1_k \longrightarrow \mathbb{P}^n_k, \qquad (x_0 : x_1) \longmapsto (x_0^n : x_0^{n-1}x_1 : \cdots : x_1^n)$$

and

$$g : \mathbb{P}^1_k \longrightarrow \mathbb{P}^m_k, \qquad (x_0 : x_1) \longmapsto (x_0^m : x_0^{m-1}x_1 : \cdots : x_1^m)$$

be the Veronese embeddings of degree n and m, respectively. Note that the standard resultant of $\mathbb{P}^1_k$ with respect to f and g is the Sylvester resultant $R^{\mathrm{Syl}}_{n,m}$, that is,

$$R^{\mathrm{Syl}}_{n,m}(a_0x_0^n + a_1x_0^{n-1}x_1 + \cdots + a_nx_1^n, b_0x_0^m + b_1x_0^{m-1}x_1 + \cdots + b_mx_1^m) =$$

$$\det \begin{pmatrix} a_0 & a_1 & \cdots & & a_n & & & \\ & a_0 & a_1 & \cdots & & a_n & 0 & \\ & 0 & \ddots & & & & \ddots & \\ & & & a_0 & a_1 & \cdots & & a_n \\ b_0 & b_1 & \cdots & & b_m & & & \\ & b_0 & b_1 & \cdots & & b_m & 0 & \\ & 0 & \ddots & & & & \ddots & \\ & & & b_0 & b_1 & \cdots & & b_m \end{pmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} m \text{ rows} \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \text{ rows} \end{array}$$

Note that $R^{\mathrm{Syl}}_{n,m}(x_0^n, x_1^m) = 1$.

1.6.7. Example. — Let $n, d_0, \ldots .d_n$ be positive integers and

$$\mathbb{P}^n_k = \mathrm{Proj}(k[T_0, \ldots, T_n]).$$

For each $i \in \{0, \ldots, n\}$, let $\psi_i : \mathbb{P}^n_k \hookrightarrow \mathbb{P}_k^{\binom{n+d_i}{n}-1}$ be the Veronese emdedding of degree d_i. Let R be a resultant with respect to $\psi_0, \ldots, \psi_n$. If we give the normalization condition $R(T_0^{d_n}, \ldots, T_n^{d_n}) = 1$, then R is uniquely dertermінded. It is denoted by $R_{d_0,\ldots,d_n}$ and is called the *multipolynomial resultant* or the *Macaulay resultant* (cf. [**24**, Example 4.17] and [**15**, §3.2]). Note that

$$R_{d_0,\ldots,d_n} \in H^0((\check{\mathbb{P}}^n_k)^{n+1}, \mathcal{O}_{\check{\mathbb{P}}^n_k}(\delta_0) \boxtimes \cdots \boxtimes \mathcal{O}_{\check{\mathbb{P}}^n_k}(\delta_d)),$$

where $\delta_i = (d_0 \cdots d_n)/d_i$ for $i \in \{0, \ldots, 0\}$. In the case where $n = 1$, $R_{d_0,d_1} = R^{\mathrm{Syl}}_{d_0,d_1}$. The following facts are well known (cf. [**15**, §3.2]):

(1) If $L_0 = \sum_{i=0}^d a_{0i}x_i$, …, $L_d = \sum_{i=0}^d a_{di}x_i$ are linear forms, then

$$R_{1,\ldots,1}(L_0, \ldots, L_d) = \det(a_{ij}).$$

(2) Let F_i' and F_i'' be homogeneous polynomials of degree d_i' and d_i'', respectively such that $d_i' + d_i'' = d_i$, then

$$\begin{aligned} &R_{d_0,\ldots,d_i,\ldots,d_n}(F_0, \ldots, F_i'F_i'', \ldots, F_d) \\ &\qquad = R_{d_0,\ldots,d_i',\ldots,d_n}(F_0, \ldots, F_i', \ldots, F_d)R_{d_0,\ldots,d_i'',\ldots,d_n}(F_0, \ldots, F_i'', \ldots, F_d). \end{aligned}$$

1.6.8. Remark. — Let $R^X_{f_0,\ldots,f_d}$ be a resultant of X with respect to $(f_i)_{i=0}^d$. If K/k is an extension and if s is an element of $E_d \otimes_k K$, defining a global section of $\mathcal{O}_{\mathbb{P}(E_d\otimes_k K)}(1)$, which intersects properly with all irreducible components $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$, then, viewed as a multi-homogeneous polynomial on

$$(E_0 \otimes_k K) \times \cdots \times (E_d \otimes_k K)$$

by extension of scalars, the resultant $R^X_{f_0,\ldots,f_d}$ specified on the last coordinate at s, is a resultant of $\operatorname{div}(s) \cap X_K$ with respect to $(f_{i,K})_{i=0}^{d-1}$. This observation motivates the following explicit construction of the resultant polynomial by induction.

1.6.9. Definition. — Let $(s_0, \ldots, s_d) \in E_0 \times \cdots \times E_d$. We assume that, for any irreducible component Z of X, the divisors $\operatorname{div}(s_0), \ldots, \operatorname{div}(s_d)$ intersect properly on Z. We denote by $R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d}$ the unique resultant of X with respect to $f_0, \ldots, f_d$ such that

$$R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d}(s_0, \ldots, s_d) = 1.$$

1.6.10. Remark. — Let k'/k be an extension of fields. For any $i \in \{0, \ldots, d\}$, the morphism $f_i : X \to \mathbb{P}(E_i)$ induces by base change a closed embedding f_i' from $X' := X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ to $\mathbb{P}(E_i')$, where $E_i' := E_i \otimes_k k'$. Note that the incidence subscheme of

$$X' \times_{k'} \mathbb{P}(E_0'^\vee) \times_{k'} \cdots \times_{k'} \mathbb{P}(E_d'^\vee)$$

identifies with $I_X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$. Therefore, if $R^X_{f_0,\ldots,f_d}$ is a resultant of X with respect to $(f_i)_{i=0}^d$, then

$$R^X_{f_0,\ldots,f_d} \otimes 1 \in (S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)) \otimes_k k' \cong S^{\delta_0}(E_0'^\vee) \otimes_{k'} \cdots \otimes_{k'} S^{\delta_d}(E_d'^\vee)$$

is a resultant of X' with respect to $(f_i')_{i=0}^d$. Similarly, if $(s_0, \ldots, s_d)$ is an element of $E_0 \times \cdots \times E_d$ such that the divisors $\operatorname{div}(s_0), \ldots, \operatorname{div}(s_d)$ intersect properly on each irreducible component of X, then the following equality holds

$$R^{X',s_0',\ldots,s_d'}_{f_0',\ldots,f_d'} = R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d} \otimes 1,$$

where for each $i \in \{0, \ldots, d\}$, s_i' denotes the element $s_i \otimes 1$ in $E_i' = E_i \otimes_k k'$.

1.7. Projection to a projective space

Let k be an infinite field, n be an integer such that $n \geqslant 1$, and V be a vector space of dimension $n+1$ over k. Let $\mathbb{P}(V)$ be the projective space associated with the k-vector space V and $\mathcal{O}_V(1)$ be the universal invertible sheaf on $\mathbb{P}(V)$. Recall that for any k-algebra A, any k-point of $\mathbb{P}(V)$ valued in A corresponds to a quotient invertible A-module of $V \otimes_k A$. In particular, if x is a scheme point of $\mathbb{P}(V)$ and $\kappa(x)$ is the residue field of x, then the scheme point x corresponds to a non-zero $\kappa(x)$-linear map $p_x : V \otimes_k \kappa(x) \to \kappa(x)$, which is unique up to a unique homothety $\kappa(x) \to \kappa(x)$ by an element of $\kappa(x)^\times$.

1.7.1. Definition. — We call *rational linear subspace* of $\mathbb{P}(V)$ any Zariski closed subset of $\mathbb{P}(V)$ defined by the vanishing of sections in a k-linear subspace of $V = H^0(\mathbb{P}(V), \mathcal{O}_V(1))$. If Y is a rational linear subspace of $\mathbb{P}(V)$ which is of codimension 1, we say that Y is a *rational hyperplane* in $\mathbb{P}(V)$.

1.7.2. Example. — (1) The scheme $\mathbb{P}(V)$ is a rational linear subspace of $\mathbb{P}(V)$. It is defined by the vanishing of the zero vector in V.
(2) Let x be a rational point of $\mathbb{P}(V)$, which corresponds to a non-zero k-linear map $\pi_x : V \to k$. Then $\{x\}$ is the vanishing locus of sections in $\mathrm{Ker}(\pi_x)$ and hence is a rational linear subspace of $\mathbb{P}(V)$.
(3) The empty subset of $\mathbb{P}(V)$ is a rational linear subspace, which identifies with the vanishing locus of all sections in V. By convention, the dimension of the empty subset of $\mathbb{P}(V)$ is defined as -1.

1.7.3. Remark. — If Y is a rational linear subspace of $\mathbb{P}(V)$ which is the vanishing locus of a k-vector subspace W of V, then the k-scheme Y is isomorphic to $\mathbb{P}(V/W)$. We call *linear projection with center* Y the k-morphisme $\pi_Y : \mathbb{P}(V) \setminus Y \to \mathbb{P}(W)$ which sends, for any commutative k-algebra A, any quotient invertible A-module $p_L : V \otimes_k A \to L$ in $(\mathbb{P}(V) \setminus Y)(A)$ to the composition

$$W \otimes_k A \hookrightarrow V \otimes_k A \xrightarrow{p_L} L,$$

which is an element of $\mathbb{P}(W)(A)$.

We assume that $Y = \{y\}$ is the set of one rational point of $\mathbb{P}(V)$, which corresponds to a non-zero k-linear map $p_y : V \to k$ whose kernel is W. Let z be a scheme point of $\mathbb{P}(V)$, $\kappa(z)$ be the residue field of z, and $p_z : V \otimes_k \kappa(z) \to \kappa(z)$ be the non-zero $\kappa(z)$-linear map corresponding to the scheme point z. Note that $\kappa(z)$ is generated by elements of the form $p_z(f \otimes 1)/p_z(g \otimes 1)$, where f and g are elements of V such that $p_z(g \otimes 1) \neq 0$. Assume that y does not belong the Zariski closure of $\{z\}$. Then there exists at least an element $s \in V \setminus W$ such that $p_z(s \otimes 1) = 0$. Let z' be the image of z by the linear projection π_Y. The residue field of z' identifies with the sub-extension of $\kappa(x)/k$ generated by elements of the form $p_z(f' \otimes 1)/p_z(g' \otimes 1)$, where f' and g' are elements of W such that $p_z(g' \otimes 1) \neq 0$. As W is of codimension 1 in V and s is an element of $V \setminus W$ such that $p_z(s \otimes 1) = 0$, we obtain that, for any $f \in V$, there exists $f' \in W$ such that $p_z(f \otimes 1) = p_z(f' \otimes 1)$. Therefore we obtain that $\kappa(z) = \kappa(z')$. In particular, if X is a closed subset of $\mathbb{P}(V)$ which does not contain y, then $\pi_Y(X)$ has the same dimension as X.

1.7.4. Proposition. — *Let $d \in \{0, \ldots, n\}$. Let X be a Zariski closed set of $\mathbb{P}(V)$ such that $\dim(X) \leqslant d$. Then we have the following:*

(1) *There is a rational linear subspace M of $\mathbb{P}(V)$ such that $\dim(M) = n - 1 - d$ and $X \cap M = \varnothing$.*

(2) *Let T be a rational linear subspace of $\mathbb{P}(V)$ such that $\dim(T) > n-d-1$, and that X and T meet properly. Then there is a rational linear subspace M of $\mathbb{P}(V)$ such that $M \subseteq T$, $\dim(M) = n-1-d$ and $X \cap M = \varnothing$.*

(3) *We assume that X is irreducible and $\dim(X) = d$. Let M be a rational linear subspace of $\mathbb{P}(V)$ such that $\dim(M) = n-1-d$ and $M \cap X = \varnothing$, which is the vanishing locus of a vector space W of V. Let $\pi_M : \mathbb{P}(V) \setminus M \to \mathbb{P}(W)$ be the projection with the center M. Then $\pi := \pi_M\big|_X : X \to \mathbb{P}^d_k$ is finite and surjective and $\pi^*(\mathcal{O}_{\mathbb{P}^d_k}(1)) = \mathcal{O}_{\mathbb{P}^n_k}(1)\big|_X$.*

Proof. — (1) We prove the assertion by induction on $n-d$. If $n = d$, then the assertion is obvious by choosing M as the empty set, so that we assume that $n > d$. Since $X \neq \mathbb{P}(V)$ and k is an infinite field, there is a rational point $x \in \mathbb{P}(V)$ which does not belong to X. Let W be the set of sections $s \in V = H^0(\mathbb{P}(V), \mathcal{O}_V(1))$ which vanish at x. This is a vector subspace of V. Let $\pi : \mathbb{P}(V) \setminus \{x\} \to \mathbb{P}(W)$ be the projection with center $\{x\}$. Since $x \notin X$, by Remark 1.7.3 we obtain that X and X' have the same dimension. In particular, $\dim(X') \leqslant d$. As $(n-1)-d < n-d$, by the hypothesis of induction, there is a linear subspace M' in $\mathbb{P}(W)$ such that $\dim(M') = n-2-d$ and $X' \cap M' = \varnothing$. Thus if we set $M = \pi^{-1}(M') \cup \{x\}$, then one has the desired subspace.

(2) Assume that T is defined by the vanishing of sections in a k-vector subspace W of V. If we set $X' = X \cap T$ and $t = \dim T$, then $\dim X' \leqslant d - (n-t)$ and $T \simeq \mathbb{P}(V/W)$. As

$$t - (d-(n-t)) = n - d \geqslant 0,$$

by (1), there is linear subspace M in T such that $\dim M = t - 1 - (d-(n-t))$ and $M \cap X' = \varnothing$. Thus one has (2).

(3) Let T be a linear subspace of $\mathbb{P}(V)$ such that $M \subseteq T$ and $\dim(T) = n-d$. It is sufficient to show that $\dim(T \cap X) = 0$. Note that M is a rational hyperplane in T, so that if $\dim(T \cap X) \geqslant 1$, then $M \cap X \neq \varnothing$. Therefore $\dim(T \cap X) = 0$. $\square$

CHAPTER 2

ADELIC CURVES AND THEIR CONSTRUCTIONS

In this chapter, we recall an adelic structure of a field, and give a "standard" construction of an adelic structure for a countable field of characteristic zero.

2.1. Adelic structures

Let K be a field. An *adelic structure* of K consists of data $((\Omega, \mathcal{A}, \nu), \phi)$ satisfying the following properties:

(1) $(\Omega, \mathcal{A}, \nu)$ is a measure space, that is, $\mathcal{A}$ is a σ-algebra of Ω and ν is a measure on $(\Omega, \mathcal{A})$.
(2) The last ϕ is a map from Ω to M_K, where M_K is the set of all absolute values of K. For any $\omega \in \Omega$, we denote the absolute value $\phi(\omega)$ by $|\cdot|_\omega$.
(3) For any $\omega \in \Omega$ and any $a \in K^\times$, the function $(\omega \in \Omega) \mapsto \ln |a|_\omega$ is ν-integrable.

The field K equipped with an adelic structure is called an *adelic curve*. Moreover, the adelic structure $((\Omega, \mathcal{A}, \nu), \phi)$ is said to be *proper* if

$$\int_\Omega \ln |a|_\omega \, \nu(\mathrm{d}\omega) = 0 \tag{2.1}$$

holds for all $a \in K^\times$. If the adelic structure $((\Omega, \mathcal{A}, \nu), \phi)$ is proper, we also say that the adelic curve $(K, (\Omega, \mathcal{A}, \nu), \phi)$ is *proper*. The equation (2.1) is called *product formula*. For details, see [**13**, Chapter 3]. We denote the set of all $\omega \in \Omega$ with $|\cdot|_\omega$ Archimedean (resp. non-Archimedean) by Ω_∞ (resp. Ω_{fin}). The restriction of $\mathcal{A}$ to Ω_∞ (resp. Ω_{fin}) is denoted by $\mathcal{A}_\infty$ (resp. $\mathcal{A}_{\mathrm{fin}}$). Note that Ω_∞ and Ω_{fin} belong to $\mathcal{A}$ (see [**13**, Proposition 3.1.1]). For each $\omega \in \Omega_\infty$, there exist an embedding $\iota_\omega : K \to \mathbb{C}$ and $\kappa_\omega \in (0, 1]$ such that $|a|_\omega = |\iota_\omega(a)|^{\kappa_\omega}$ for all $a \in K$, where $|\cdot|$ is the usual absolute value of $\mathbb{C}$. Note that the exponent κ_ω does not depend on the choice of the embedding $\iota_\omega : K \to \mathbb{C}$. From now on, we always assume that $\kappa_\omega = 1$ for all $\omega \in \Omega_\infty$.

For $(a_1,\ldots,a_n)\in K^n\setminus\{(0,\ldots,0)\}$, the height $h_S(a_1,\ldots,a_n)$ of $(a_1,\ldots,a_n)$ with respect to the adelic curve $S=(K,(\Omega,\mathcal{A},\nu),\phi)$ is defined to be

$$h_S(a_1,\ldots,a_n):=\int_\Omega \ln(\max\{|a_1|_\omega,\ldots,|a_n|_\omega\})\nu(\mathrm{d}\omega). \tag{2.2}$$

Note that if S is proper, then $h_S(a)=0$ for all $a\in K^\times$.

2.1.1. Remark. — Many classic constructions in algebraic geometry and arithmetic geometry, such as algebraic curves, rings of algebraic integers, polarized projective varieties and arithmetic varieties, can be interpreted as adelic curves. For example, on the filed $\mathbb{Q}$ of rational numbers there is an adelic structure consisting of all places of $\mathbb{Q}$ (namely the set $\Omega_{\mathbb{Q}}$ of all prime numbers and ∞) equipped with the discrete σ-algebra and the measure ν such that $\nu(\{\omega\})=1$ for any $\omega\in\Omega_{\mathbb{Q}}$, where $|\cdot|_\infty$ is the usual absolute value on $\mathbb{Q}$ and $|\cdot|_p$ is the p-adic absolute value for any prime number p. The product formula for this adelic curve is just the logarithmic version of the usual product formula for rational numbers

$$\forall\, a\in\mathbb{Q}^\times,\quad |a|_\infty\cdot\prod_p |a|_p=1.$$

We call this adelic structure *the standard adelic structure on* $\mathbb{Q}$. We refer the readers to [**13**, §3.2] for more examples.

2.1.2. Definition. — Let $S=(K,(\Omega,\mathcal{A},\nu),\phi)$ and $S'=(K',(\Omega',\mathcal{A}',\nu'),\phi')$ be two adelic curves. We call *morphism* from S' to S any triplet $\alpha=(\alpha^\#,\alpha_\#,I_\alpha)$, where

(1) $\alpha^\#:K\to K'$ is a field homomorphism,

(2) $\alpha_\#:(\Omega',\mathcal{A}')\to(\Omega,\mathcal{A})$ is a measurable map, such that, for any $\omega'\in\Omega'$,

$$|\cdot|_{\omega'}\circ\alpha^\#=|\cdot|_{\alpha_\#(\omega')},$$

and that the direct image of ν' by $\alpha_\#$ coincides with ν, namely, for any $\mathcal{A}$-measurable function $g:\Omega\to\mathbb{R}$ which is non-negative (resp. integrable), the function $g\circ\alpha_\#$ is also non-negative (resp. integrable), and one has

$$\int_{\Omega'} g(\alpha_\#(\omega'))\,\nu'(\mathrm{d}\omega')=\int_\Omega g(\omega)\,\nu(\mathrm{d}\omega).$$

If in addition ν' admits a disintegration with respect to the fibration $\alpha_\#$, namely there exists an $\mathbb{R}$-linear map

$$I_\alpha:\mathscr{L}^1(\Omega',\mathcal{A}',\nu')\longrightarrow\mathscr{L}^1(\Omega,\mathcal{A},\nu)$$

sending positive integrable functions on $(\Omega',\mathcal{A}',\nu')$ to positive integrable functions on $(\Omega,\mathcal{A},\nu)$ such that

$$\forall\, f\in\mathscr{L}^1(\Omega',\mathcal{A}',\nu'),\quad \int_\Omega I_\alpha(f)(\omega)\,\nu(\mathrm{d}\omega)=\int_{\Omega'} f(\omega')\,\nu'(\mathrm{d}\omega'),$$
$$\forall\, g\in\mathscr{L}^1(\Omega,\mathcal{A},\nu),\quad I_\alpha(g\circ\alpha_\#)=g,$$

we say that $\alpha:S'\to S$ is a *covering* of adelic curves.

2.2. Algebraic coverings of adelic curves

Adelic curves are very flexible constructions. On a field there exist many adelic structures. It is also possible to construct new adelic structures from given ones. Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve. In [**13**, §3.2] it has been explained how to construct, for any algebraic extension L/K, a natural adelic curve

$$S \otimes_K L = (L, (\Omega_L, \mathcal{A}_L, \nu_L), \phi_L)$$

on L such that $\Omega_L = \Omega \times_{M_K, \phi} M_L$. The projection map $\pi_{L/K} : \Omega_L \to \Omega$ satisfies the relation

$$\nu = (\pi_{L/K})_*(\nu_L).$$

Moreover, for any $\omega \in \Omega$, the fibre $\pi_{L/K}^{-1}(\{\omega\})$ is equipped with a natural σ-algebra and a probability measure $\nu_{L,\omega}$, such that, for any positive $\mathcal{A}_L$-measurable function f on Ω_L, one has

$$\int_{\Omega_L} g(x)\, \nu_L(\mathrm{d}x) = \int_{\Omega} \nu(\mathrm{d}\omega) \int_{\pi_{L/K}^{-1}(\omega)} g(x)\, \nu_{L,\omega}(\mathrm{d}x).$$

In other words, the family of measures $(\nu_{L,\omega})_{\omega \in \Omega}$ form a disintegration of ν_L over ν. If the adelic curve S is proper, then also is $S \otimes_K L$, see [**13**, Proposition 3.4.10]. If we denote by $i_{K,L} : K \to L$ the inclusion map, and

$$I_{L/K} : \mathscr{L}^1(\Omega_L, \mathcal{A}_L, \nu_L) \longrightarrow \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$$

the linear map of fiber integrals, which sends $g \in L^1(\Omega_L, \mathcal{A}_L, \nu_L)$ to the function

$$(\omega \in \Omega) \longmapsto \int_{\pi_{L/K}^{-1}(\omega)} g(x)\, \nu_{L,\omega}(\mathrm{d}x),$$

then the triplet $(i_{K,L}, \pi_{L/K}, I_{L/K})$ forms a covering of adelic curves in the sense of Definition 2.1.2.

2.2.1. Lemma. — *Let K' be an algebraic extension of K and $S \otimes K' := (K', (\Omega', \mathcal{A}', \nu'), \phi')$. Suppose that K and Ω_{fin} are countable sets. If $(\Omega_{\mathrm{fin}}, \mathcal{A}_{\mathrm{fin}})$ is discrete, so is $(\Omega'_{\mathrm{fin}}, \mathcal{A}'_{\mathrm{fin}})$.*

Proof. — Since K'/K is an algebraic extension, and K and Ω_{fin} are countable sets, we obtain that the sets K' and Ω'_{fin} are countable, so that it is sufficient to see that $\{\omega'\} \in \mathcal{A}'_{\mathrm{fin}}$ for all $\omega' \in \Omega'_{\mathrm{fin}}$.

First we consider the case where K' is finite over K. Let $\omega' \in \Omega'$ and $\omega = \pi(\omega')$, where $\pi : \Omega' \to \Omega$ is the canonical map. Then as $\{\omega\}$ is $\mathcal{A}_{\mathrm{fin}}$-measurable and π is measurable, $\pi^{-1}(\{\omega\}) \in \mathcal{A}'_{\mathrm{fin}}$. If $|\cdot|_\omega$ is trivial, then $\pi^{-1}(\{\omega\}) = \{\omega'\}$, so that the assertion is obvious. Next we assume that $|\cdot|_\omega$ is non-trivial. Let us see that, for any $(x, x') \in \pi^{-1}(\{\omega\})^2$ with $x \neq x'$, $|\cdot|_x$ is not equivalent to $|\cdot|_{x'}$. Otherwise, there is $\kappa \in \mathbb{R}_{>0}$ such that $|\cdot|_{x'} = |\cdot|_x^\kappa$. As $|\cdot|_\omega$ is non-trivial, there is $a \in K$ such that $|a|_\omega < 1$. Then

$$|a|_\omega = |a|_{x'} = |a|_x^\kappa = |a|_\omega^\kappa,$$

and hence $\kappa = 1$, which is a contradiction. Therefore, there is $a' \in K'$ such that $|a'|_{\omega'} < 1$ and $|a'|_x > 1$ for all $x \in \pi^{-1}(\{\omega\})\setminus\{\omega'\}$ (cf. [**52**, the proof of Theorem 3.4]). Note that

$$\Delta := \{\chi \in \Omega' \,:\, |a'|_\chi < 1\}$$

is $\mathcal{A}'_{\mathrm{fin}}$-measurable, so that $\{\omega'\} = \pi^{-1}(\{\omega\}) \cap \Delta$ is $\mathcal{A}'_{\mathrm{fin}}$-measurable.

In general, for $a \in K'$, let

$$(K(a), (\Omega_{K(a)}, \mathcal{A}_{K(a)}, \nu_{K(a)}), \phi_{K(a)}) = S \otimes K(a)$$

and let $\pi_{K'/K(a)} : \Omega' \to \Omega_{K(a)}$ be the canonical map. By the previous case, $\{\pi_{K'/K(a)}(\omega')\} \in \mathcal{A}_{K(a)}$, so that $\pi^{-1}_{K'/K(a)}(\{\pi_{K'/K(a)}(\omega')\}) \in \mathcal{A}'$. Therefore, as K' is countable,

$$\bigcap_{a \in K'} \pi^{-1}_{K'/K(a)}(\pi_{K'/K(a)}(\omega')).$$

belongs to $\mathcal{A}'$. Thus it suffices to prove

$$\{\omega'\} = \bigcap_{a \in K'} \pi^{-1}_{K'/K(a)}(\pi_{K'/K(a)}(\omega')). \tag{2.3}$$

Indeed, if $x \in \bigcap_{a\in K'} \pi^{-1}_{K'/K(a)}(\{\pi_{K'/K(a)}(\omega')\})$, then, for any $a \in K'$, $\pi_{K'/K(a)}(x) = \pi_{K'/K(a)}(\omega')$, so that $|a|_x = |a|_{\omega'}$, which means that $x = \omega'$. □

2.3. Transcendental fibrations of adelic curves

The purpose of this section is to discuss the extension of an adelic structure to a transcendental extension of the field. We fix an adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$. For any $\omega \in \Omega$, let K_ω be the completion of K with respect to the absolute value $|\cdot|_\omega$. We begin with an example which illustrates a construction of pure transcendental fibration of transcendence degree 1 over the adelic curve.

2.3.1. An illustrative example. — Assume that the field K is countable when Ω_∞ is not empty. For $\omega \in \Omega_{\mathrm{fin}}$, we extend the absolute value $|\cdot|_\omega$ to $K(T)$ by taking the Gauss norm. Recall that for any polynomial

$$F(T) = a_0 + a_1T + \cdots + a_nT^n \in K[T],$$

one has

$$|F|_\omega = \max\{|a_0|_\omega, \ldots, |a_n|_\omega\}.$$

If Ω_∞ is not empty, for any $\omega \in \Omega_\infty$, K_ω identifies with $\mathbb{R}$ or $\mathbb{C}$. We let $\iota_\omega : K \to \mathbb{C}$ be corresponding embedding (in the case where $K_\omega \cong \mathbb{C}$ we need to choose an embedding between two conjugated ones), and let $\Omega_{K,\omega}$ be the set of $t \in [0,1]$ such that $\mathrm{e}^{2\pi i t}$ is transcendental over K with respect to the embedding ι_ω. Note that the complementary of $\Omega_{K(T),\omega}$ in $[0,1]$ is countable since K is assumed to be countable in the case where $\Omega_\infty \neq \varnothing$. Note that For each $t \in \Omega_{K(T),\omega}$, the evaluation of polynomials

$$\Big(F = \sum_{\ell=0}^{n} a_\ell T^\ell \in K[T]\Big) \longmapsto F_\omega(\mathrm{e}^{2\pi i t}) := \sum_{\ell=0}^{n} \iota_\omega(a_\ell)\mathrm{e}^{2\pi i \ell t} \in \mathbb{C}$$

defines a ring homomorphism from $K[T]$ to $\mathbb{C}$ which is injective since $\mathrm{e}^{2\pi i t}$ is transcendental over K. Thus it induces a field embedding from $K(T)$ to $\mathbb{C}$, which we denote by

$$(f \in K(T)) \longmapsto f_\omega(\mathrm{e}^{2\pi i t}).$$

Therefore, the usual absolute value on $\mathbb{C}$ induces by restriction an absolute value on $K(T)$ which we denote by $|\cdot|_t$.

We denote by $\Omega_{K(T)}$ the disjoint union

$$\Omega_{\mathrm{fin}} \amalg \coprod_{\omega \in \Omega_\infty} \Omega_{K(T),\omega}.$$

Clearly the set $\Omega_{K(T)}$ is fibered over Ω, where the projection map sends the elements of $\Omega_{K(T),\omega}$ to ω. We equip $\Omega_{K(T),\omega}$ (which is a subset of measure 1 of $[0,1]$) with the Borel σ-algebra and the Lebesgue measure. Then the fiber integral defines a σ-algebra $\mathcal{A}_{K(T)}$ as follows. A real-valued function f on $\Omega_{K(T)}$ is $\mathcal{A}_{K(T)}$-measurable if it satisfies the following conditions:

(1) for any $\omega \in \Omega_\infty$, the restriction of f to $\Omega_{K(T),\omega}$ is Borel measurable,
(2) the fibre integral of f, which is defined as

$$(\omega \in \Omega) \longrightarrow \begin{cases} f(\omega), & \text{if } \omega \in \Omega_{\mathrm{fin}}, \\ \displaystyle\int_{\Omega_{K(T),\omega}} f(t)\,\mathrm{d}t, & \text{if } \omega \in \Omega_\infty, \end{cases}$$

is $\mathcal{A}$-measurable.

Moreover, the fibre integral also defines a measure $\nu_{K(T)}$ on $(\Omega_{K(T)}, \mathcal{A}_{K(T)})$ such that, for any non-negative function f on $\Omega_{K(T)}$, one has

$$\int_{\Omega_{K(T)}} f(x)\,\nu_{K(T)}(\mathrm{d}x) = \int_{\Omega_{\mathrm{fin}}} f(\omega)\,\nu(\mathrm{d}\omega) + \int_{\Omega_\infty}\int_{\Omega_{K(T),\omega}} f(t)\,\mathrm{d}t\,\nu(\mathrm{d}\omega).$$

Thus we obtain an adelic curve with $K(T)$ as its underlying field. Moreover, the canonical embedding $K \to K(T)$ and the projection map $\Omega_{K(T)} \to \Omega$ defines a covering of adelic curves as described in Definition 2.1.2.

Note that, in the case where the adelic curve S is proper, it is not true in general that the adelic curve constructed above is also proper. However, it admits a natural compactification that we will explain below. We denote by $\mathscr{P}$ the set of irreducible monic polynomials in $K[T]$. For any $P \in \mathscr{P}$, let $|\cdot|_P$ be the absolute value on the field $K(T)$ of rational functions defined as

$$|\cdot|_P = \mathrm{e}^{-\operatorname{ord}_P(\cdot)}.$$

We denote by $(\Omega^*_{K(T)},\mathcal{A}^*_{K(T)},\nu^*_{K(T)})$ the disjoint union of $(\Omega_{K(T)},\mathcal{A}_{K(T)},\nu_{K(T)})$ with $\mathscr{P}$, where $\mathscr{P}$ is equipped with the discrete σ-algebra and the measure such that

$$\forall\, P\in\mathscr{P},\quad \nu^*_{K(T)}(\{P\})=\int_{\Omega_{\mathrm{fin}}}\ln|P|_\omega\,\nu(\mathrm{d}\omega)+\int_{\Omega_\infty}\int_0^1\ln|P_\omega(\mathrm{e}^{2\pi it})|\,\mathrm{d}t\,\nu(\mathrm{d}\omega).$$

Then the measure space $(\Omega^*_{K(T)},\mathcal{A}^*_{K(T)},\nu^*_{K(T)})$ together with the family $(|\cdot|_x)_{x\in\Omega_{K(T)}}$ form a proper adelic curve.

The above compactification is not unique. Let us consider its λ-twisted variant as follows. Let $|\cdot|_\infty$ be the absolute value on $K(T)$ such that, for any $F\in K[T]$, one has

$$|F|_\infty=\mathrm{e}^{\deg(F)}.$$

Let λ be a positive real number. We denote by $(\Omega^\lambda_{K(T)},\mathcal{A}^\lambda_{K(T)},\nu^\lambda_{K(T)})$ the disjoint union of $(\Omega_{K(T)},\mathcal{A}_{K(T)},\nu_{K(T)})$ with $\mathscr{P}\amalg\{\infty\}$, where $\mathscr{P}\amalg\{\infty\}$ is equipped with the discrete σ-algebra and the measurable such that

$$\begin{aligned}\forall\, P\in\mathscr{P},\quad \nu^\lambda_{K(T)}\{P\}&=\nu^*_{K(T)}(\{P\})+\lambda\deg(P)\\ \nu^\lambda_{K(T)}(\{\infty\})&=\lambda.\end{aligned}$$

Then the measure space $(\Omega^\lambda_{K(T)},\mathcal{A}^\lambda_{K(T)},\nu^\lambda_{K(T)})$ together with the family of absolute values $(|\cdot|_\omega)_{\omega\in\Omega^\lambda_{K(T)}}$ form a proper adelic curve, which is called the λ-twisted compactification of $(K(T),(\Omega_{K(T)},\mathcal{A}_{K(T)},\nu_{K(T)}),(|\cdot|_\omega)_{\omega\in\Omega_{K(T)}})$.

2.3.2. A general construction of transcendental fibration. — Let B be a K-algebra. Note that B is not necessarily of finite type over K. We assume that B is a unique factorization domain and the set $B^\times$ of units in B coincides with $K^\times$. We say that two irreducible elements of B are *equivalent* if they differ by a unit as a factor. This defines an equivalence relation on the set of all irreducible elements of B. We pick a representative in each of the equivalence classes to form a subset $\mathscr{P}_B$ of B consisting of non-equivalent irreducible elements. Let L be the field of fractions of B. Recall that any non-zero element $g\in L$ can be written in a unique way as

$$c(g)\prod_{F\in\mathscr{P}_B}F^{\mathrm{ord}_F(g)},$$

where $c(g)$ is an element of $K^\times=B^\times$, and for each $F\in\mathscr{P}_B$, $\mathrm{ord}_F(g)$ is an integer. Note that $\mathrm{ord}_F(\cdot)$ is a discrete valuation on the field L, and $\mathrm{ord}_F(a)=0$ for any $a\in K^\times=B^\times$.

2.3.1. Definition. — For any $\omega\in\Omega$, let $S_{L,\omega}=(L,(\Omega_{L,\omega},\mathcal{A}_{L,\omega},\nu_{L,\omega}),\phi_{L,\omega})$ be an adelic curve such that $\nu_{L,\omega}$ is a probability measure. We say that the family $(S_{L,\omega})_{\omega\in\Omega}$ is an *admissible fibration with respect to* $(B,\mathscr{P}_B)$ over the adelic curve S if the following conditions are satisfied:

(a) for any $\omega\in\Omega$ and any $x\in\Omega_{L,\omega}$, the absolute value $\phi_{L,\omega}(x)$ on L is an extension of $\phi(\omega)$ on K,

(b) for any element $g \in B \setminus \{0\}$, any finite family $(F_j)_{j=1}^n$ of elements of $\mathscr{P}_B$ containing $\{F \in \mathscr{P}_B \mid \mathrm{ord}_F(g) \neq 0\}$ and any $(C_j)_{j=1}^n \in \mathbb{R}^n_{\geqslant 0}$, the function

$$(\omega \in \Omega) \longmapsto \int_{\Omega_{L,\omega}} |g|_x \mathbb{1}_{|F_1|_x \leqslant C_1, \ldots, |F_n|_x \leqslant C_n} \, \nu_{L,\omega}(\mathrm{d}x)$$

is $\mathcal{A}$-measurable,

(c) for any $\omega \in \Omega$ and any element F of $\mathscr{P}_B$, the function

$$(\omega \in \Omega) \longmapsto \int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x)$$

is integrable with respect to ν.

Let $(S_{L,\omega})_{\omega \in \Omega}$ be an admissible fibration over the adelic curve S. We define Ω_L as the disjoint union of $(\Omega_{L,\omega})_{\omega \in \Omega}$ and let ϕ_L be the map from Ω_L to the set of all absolute values on L, whose restriction on each $\Omega_{L,\omega}$ is equal to $\phi_{L,\omega}$. Let $\pi_{L/K} : \Omega_L \to \Omega$ be the projection map, sending the elements of $\Omega_{L,\omega}$ to ω. We equip Ω_L with the σ-algebra $\mathcal{A}_L$ generated by the projection map $\pi_{L/K}$ and all functions of the form $(x \in \Omega_L) \mapsto |g|_x$, where g runs over the set L.

2.3.2. Proposition. — *Let f be a non-negative $\mathcal{A}_L$-measurable function on Ω_L. For any $\omega \in \Omega$, the function f is $\mathcal{A}_{L,\omega}$-measurable on $\Omega_{L,\omega}$. Moreover, the function*

$$(\omega \in \Omega) \longmapsto \int_{\Omega_{L,\omega}} f(x) \, \nu_{L,\omega}(\mathrm{d}x) \in [0, +\infty]$$

is $\mathcal{A}$-measurable.

Proof. — Let $\mathcal{H}$ be the set of all bounded non-negative $\mathcal{A}_L$-measurable functions g on Ω_L which is $\mathcal{A}_{L,\omega}$-measurable on $\Omega_{L,\omega}$ for any $\omega \in \Omega$ and such that the function

$$(\omega \in \Omega) \longmapsto \int_{\Omega_{L,\omega}} f(x) \, \nu_{L,\omega}(\mathrm{d}x)$$

is $\mathcal{A}$-measurable. Note that, for any non-negative bounded $\mathcal{A}$-measurable function φ on Ω, one has $\varphi \circ \pi \in \mathcal{H}$ since it is constant on each fiber $\Omega_{L,\omega}$ and

$$\int_{\Omega_{L,\omega}} \varphi(\pi(x)) \, \nu_{L,\omega}(\mathrm{d}x) = \int_{\Omega_{L,\omega}} \varphi(\omega) \, \nu_{L,\omega}(\mathrm{d}x) = \varphi(\omega).$$

In particular, all non-negative constant functions belong to $\mathcal{H}$. Clearly, for any $(g_1, g_2) \in \mathcal{H} \times \mathcal{H}$ and any $(a_1, a_2) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$, one has $a_1 g_1 + a_2 g_2 \in \mathcal{H}$. For any increasing sequence of functions $(g_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$, the pointwise limit of $(g_n)_{n \in \mathbb{N}}$ belongs to $\mathcal{H}$. Moreover, for functions g_1 and g_2 in $\mathcal{H}$ such that $g_2 \geqslant g_1$, one has $g_2 - g_1 \in \mathcal{H}$.

Let $\mathcal{S}$ be the set of functions of the form

$$(x \in \Omega_L) \longmapsto |g|_x \mathbb{1}_{|F_1|_x \leqslant C_1, \ldots, |F_n|_x \leqslant C_n} \varphi(\pi(x)),$$

where g is an element of $B\setminus\{0\}$, $(F_j)_{j=1}^n$ is a finite family of elements of $\mathscr{P}_B$ containing $\{F\in\mathscr{P}_B \mid \mathrm{ord}_F(g)\neq 0\}$, $(C_j)_{j=1}^n$ is a family of positive constants and φ is a non-negative and bounded $\mathcal{A}$-measurable function on Ω. Clearly the set $\mathcal{S}$ is stable by multiplication. Note that the function sending $\omega\in\Omega$ to

$$\begin{aligned}&\int_{\Omega_{L,\omega}} |g|_x \mathbb{1}_{|F_1|_x\leqslant C_1,\ldots,|F_n|_x\leqslant C_n}\varphi(\pi(x))\,\nu_{L,\omega}(\mathrm{d}x)\\ &= \varphi(\omega)\int_{\Omega_{L,\omega}} |g|_x \mathbb{1}_{|F_1|_x\leqslant C_1,\ldots,|F_n|_x\leqslant C_n}\,\nu_{L,\omega}(\mathrm{d}x)\end{aligned}$$

takes real values and is $\mathcal{A}$-measurable by the condition (b) above. Therefore, $\mathcal{S}$ is a subset of $\mathcal{H}$. Since the σ-algebra $\mathcal{A}_L$ is generated by $\mathcal{S}$, by monotone class theorem (see [**66**, §2.2], see also [**13**, §A.1]), $\mathcal{H}$ contains all bounded non-negative $\mathcal{A}_L$-measurable functions. Finally, since any non-negative $\mathcal{A}_L$-measurable function f can be written as the limit of an increasing sequence of bounded non-negative $\mathcal{A}_L$-measurable functions, the assertion of the proposition is true. □

2.3.3. Definition. — Let $(S_{L,\omega})_{\omega\in\Omega}$ be an admissible fibration over S (see Definition 2.3.1), where $S_{L,\omega}=(L,(\Omega_{L,\omega},\mathcal{A}_{L,\omega},\nu_{L,\omega}),\phi_{L,\omega})$. By Proposition 2.3.2, there is a measure ν_L on the measurable space $(\Omega_L,\mathcal{A}_L)$ such that, for any non-negative $\mathcal{A}_L$-measurable function f on Ω_L, one has

$$\int_{\Omega_L} f(x)\,\nu_L(\mathrm{d}x)=\int_{\Omega}\nu(\mathrm{d}\omega)\int_{\Omega_{L,\omega}} f(x)\,\nu_{L,\omega}(\mathrm{d}x).$$

Therefore $S_L:=(L,(\Omega_L,\mathcal{A}_L,\nu_L),\phi_L)$ is an adelic curve, called the *adelic curve associated with the admissible fibration* $(S_{L,\omega})_{\omega\in\Omega}$. Since $\nu_{L,\omega}$ are probability measures, if we denote by $i_{K,L}:K\to L$ the inclusion map, by $\pi_{L/K}:\Omega_L\to\Omega$ the map sending the elements of $\Omega_{L,\omega}$ to ω, and by

$$I_{L/K}:\mathscr{L}^1(\Omega_L,\mathcal{A}_L,\nu_L)\longrightarrow\mathscr{L}^1(\Omega,\mathcal{A},\nu)$$

the linear map of fiber integrals, then the triplet $(i_{K,L},\pi_{L/K},I_{L/K})$ forms a covering of adelic curves in the sense of Definition 2.1.2.

2.4. Intrinsic compactification of admissible fibrations

Let $S=(K,(\Omega,\mathcal{A},\nu),\phi)$ be a proper adelic curve, B be a K-algebra which is a unique factorization domain, and $\mathscr{P}_B$ be a representative family of irreducible elements as in the previous section. Let L be the field of fractions of B and

$$\big(S_{L,\omega}=(L,(\Omega_{L,\omega},\mathcal{A}_{L,\omega},\nu_{L,\omega}),\phi_{L,\omega})\big)_{\omega\in\Omega}$$

be an admissible fibration with respect to $(B,\mathscr{P}_B)$. In the previous section, we have constructed an adelic curve $S_L:=(L,(\Omega_L,\mathcal{A}_L,\nu_L),\phi_L)$ which fibers over S and such that the measure ν_L disintegrates over ν by the family of measures $(\nu_{L,\omega})_{\omega\in\Omega}$ on the fibers. This construction looks similar to algebraic coverings of adelic curves.

However, even in the case where the adelic structure $((\Omega, \mathcal{A}, \nu), \phi)$ is proper, the adelic structure $((\Omega_L, \mathcal{A}, \nu_L), \phi_L)$ is not necessarily proper. In this section, we show that, under a mild condition on the admissible fibration $(S_\omega)_{\omega\in\Omega}$ over S, we can naturally "compactify" the adelic structure $((\Omega_L, \mathcal{A}_L, \nu_L), \phi_L)$. For any element $F \in \mathscr{P}_B$, we denote by $|\cdot|_F$ the absolute value on $K(T)$ such that

$$\forall\, g \in L^\times, \quad |g|_F := \mathrm{e}^{-\operatorname{ord}_F(g)}.$$

Thus we obtain a map ϕ'_L from $\mathscr{P}_B$ to M_L sending F to $|\cdot|_F$. Let $(\Omega_L^*, \mathcal{A}_L^*)$ be the disjoint union of the measurable spaces $(\Omega_L, \mathcal{A}_L)$ and $\mathscr{P}_B$ equipped with the discrete σ-algebra. Let $\phi_L^* : \Omega_L^* \to M_L$ be the map extending ϕ_L on Ω_L and ϕ'_L on $\mathscr{P}_B$.

2.4.1. Proposition. — *Let $(S_\omega)_{\omega\in\Omega}$ be an admissible fibration over S. We assume that, for any element $F \in \mathscr{P}_B$,*

$$h_{S_L}(F) := \int_\Omega \nu(\mathrm{d}\omega) \int_{\Omega_{L,\omega}} \ln|F|_x\, \nu_{L,\omega}(\mathrm{d}x) \geqslant 0. \tag{2.4}$$

Let ν_L^* be the measure on $(\Omega_L^*, \mathcal{A}_L^*)$ which coincides with ν_L on $(\Omega_L, \mathcal{A}_L)$ and such that

$$\forall\, F \in \mathscr{P}_B, \quad \nu_L^*(\{F\}) = h_{S_L}(F).$$

Then $S_L^* := (L, (\Omega_L^*, \mathcal{A}_L^*, \nu_L^*), \phi_L^*)$ is a proper adelic curve.

Proof. — For any $g \in L^\times$, one has

$$\begin{aligned}\int_{\Omega_L} \ln|g|_x\, \nu_L(\mathrm{d}x) &= \int_\Omega \nu(\mathrm{d}\omega) \int_{\Omega_{L,\omega}} \ln|g|_x\, \nu_{L,\omega}(\mathrm{d}x) \\ = \sum_{F\in\mathscr{P}_B} \operatorname{ord}_F(g) &\int_\Omega \nu(\mathrm{d}\omega) \int_{\Omega_{L,\omega}} \ln|F|_x\, \nu_{L,\omega}(\mathrm{d}x) = \sum_{F\in\mathscr{P}_B} \operatorname{ord}_F(g) h_{S_L}(F).\end{aligned} \tag{2.5}$$

Thus

$$\int_{\Omega_L^*} \ln|g|_x\, \nu_L^*(\mathrm{d}x) = \int_{\Omega_L} \ln|g|_x\, \nu_L(\mathrm{d}x) + \sum_{F\in\mathscr{P}_B} h_{S_L}(F) \ln|g|_F = 0.$$

□

2.4.2. Definition. — Under the assumption (2.4), the adelic curve S_L^* is called the *canonical compactification* of S_L.

2.4.3. Remark. — Let $\mathcal{A}_B$ be the discrete σ-algebra on $\mathscr{P}_B$, ν_B be the measure on $(\mathscr{P}_B, \mathcal{A}_B)$ such that

$$\nu_B(\{F\}) = h_{S_L}(F)$$

for any $F \in \mathscr{P}_B$, and $\phi_B : \mathscr{P}_B \to M_K$ be the map sending any element of $\mathscr{P}_B$ to the trivial absolute value on K. Then $S_B := (K, (\mathscr{P}_B, \mathcal{A}_B, \nu_B), \phi_B)$ forms an adelic curve having K as the underlying field. Let S^* be the amalgamation of S and S_B. Then, the inclusion map $K \to L$, the projection

$$\pi_{L/K} \amalg \mathrm{Id}_{\mathscr{P}_B} : \Omega_L^* = \Omega_L \amalg \mathscr{P}_B \longrightarrow \Omega^* = \Omega \amalg \mathscr{P}_B$$

and the integral along fibers form a covering of adelic curves.

2.5. Non-intrinsic compactification of admissible fibrations

We keep the notation of the previous section. In this section, we assume that the family of absolute values $(|\cdot|_F)_{F\in\mathscr{P}_B}$ can be included in a proper adelic structure. We will show that a weaker positivity condition than (2.4) would be enough to ensure the existence of (non-intrinsic) compactifications of the adelic structure $((\Omega_L, \mathcal{A}_L, \nu_L), \phi_L)$. In the rest of the subsection, we assume that there exists a *proper* adelic structure $((\Omega_L', \mathcal{A}_L', \nu_L'), \phi_L')$ on L which satisfies the following conditions:

(1) Ω_L' contains $\mathscr{P}_B$ as a discrete measurable sub-space and $\nu_L'(\{F\}) > 0$ for any $F \in \mathscr{P}_B$,
(2) for any $F \in \mathscr{P}_B$, one has $\phi_L'(F) = |\cdot|_F$.

Note that the existence of such an adelic structure is is true when K is of characteristic 0 and $\operatorname{Spec} B$ is a smooth K-scheme of finite type. In this case there exists a projective K-scheme X and an open immersion from B into X. Then one can construct an adelic structure consisting of prime divisors of X, by choosing a polarization on X. We refer the readers to [**13**, §3.2.4] for more details.

2.5.1. Proposition. — *Let $(S_\omega)_{\omega\in\Omega}$ be an admissible fibration over S. For any element $F \in \mathscr{P}_B$, let*

$$h_{S_L}(F) := \int_\Omega \nu(\mathrm{d}\omega) \int_{\Omega_{L,\omega}} \ln|F|_x \, \nu_{L,\omega}(\mathrm{d}x). \tag{2.6}$$

Let δ be a positive constant. We assume that

$$\forall\, F \in \mathscr{P}_B, \quad h_{S_L}(F) + \delta\nu_L'(\{F\}) \geqslant 0.$$

Let $(\Omega_L'', \mathcal{A}_L'')$ be the disjoint union of $(\Omega_L, \mathcal{A}_L)$ and $(\Omega_L', \mathcal{A}_L')$, $\phi_L'' : \Omega_L'' \to M_L$ be the map extending ϕ_L and ϕ_L', and ν_L^δ be the measure on $(\Omega_L'', \mathcal{A}_L'')$ which coincides with ν_L on $(\Omega_L, \mathcal{A}_L)$ and coincides with

$$\delta\nu_L' + \sum_{F\in\mathscr{P}_B} h_{S_L}(F)\,\mathrm{Dirac}_F$$

on (Ω_L', ν_L'), where Dirac_F denotes the Dirac measure at F. Then $((\Omega_L'', \mathcal{A}_L'', \nu_L^\delta), \phi_L'')$ is a proper adelic structure on L.

Proof. — For any $g \in L^\times$, one has

$$\int_{\Omega_L^*} \ln|g|_x \, \nu_L^\delta(\mathrm{d}x) = \int_{\Omega_L} \ln|g|_x \, \nu_L(\mathrm{d}x) + \delta \int_{\Omega_L'} \ln|g|_x \, \nu_L'(\mathrm{d}x) + \sum_{F\in\mathscr{P}_B} h_{S_L}(F) \ln|g|_F.$$

By (2.5), one has

$$\int_{\Omega_L} \ln|g|_x \, \nu_L(\mathrm{d}x) + \sum_{F\in\mathscr{P}_B} h_{S_L}(F)\ln|g|_F = 0.$$

Moreover, since $((\Omega'_L, \mathcal{A}'_L, \nu'_L), \phi'_L)$ is a proper adelic structure, one has

$$\int_{\Omega'_L} \ln|g|_x \, \nu'_L(\mathrm{d}x) = 0.$$

Therefore we obtain

$$\int_{\Omega''_L} \ln|g|_x \, \nu^{\delta}_L(\mathrm{d}x) = 0.$$

□

2.6. Purely transcendental fibration of adelic curves

In this section, we apply the results obtained in previous sections to the study of adelic structures on a purely transcendental extension of the underlying field of an adelic curve. Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve and I be a non-empty set. We consider the polynomial ring $K[\boldsymbol{T}_I]$ spanned by I, where $\boldsymbol{T}_I = (T_i)_{i\in I}$ denotes the variables. Let $\mathbb{N}^{\oplus I}$ be the set of vectors $\boldsymbol{d} = (d_i)_{i\in I} \in \mathbb{N}^I$ such that $d_i = 0$ for all but a finite number of $i \in I$. For any vector $\boldsymbol{d} = (d_i)_{i\in I} \in \mathbb{N}^{\oplus I}$, we denote by $\boldsymbol{T}^{\boldsymbol{d}}$ the monomial

$$\prod_{i\in I,\, d_i>0} T_i^{d_i}.$$

If g is an element of $K[\boldsymbol{T}_I]$, for any $\boldsymbol{d} \in \mathbb{N}^{\oplus I}$ we denote by $a_{\boldsymbol{d}}(g)$ the coefficient of $\boldsymbol{T}^{\boldsymbol{d}}$ in the writing of g as a K-linear combination of monomials. For convenience, $K[\boldsymbol{T}_I]$ means K in the case where $I = \varnothing$.

2.6.1. Lemma. — (1) *Let J be a subset of I. If f and g are two elements of $K[\boldsymbol{T}_I]$ such that fg belongs to $K[\boldsymbol{T}_J]$, then both polynomials f and g belong to $K[\boldsymbol{T}_J]$.*

(2) *The ring $K[\boldsymbol{T}_I]$ is a unique factorization domain and $K[\boldsymbol{T}_I]^\times = K^\times$.*

Proof. — (1) For $i \in I$ and $\varphi \in K[\boldsymbol{T}_I]$, the degree of φ with respect to T_i is denoted by $\deg_i(\varphi)$. Note that the function $\deg_i(\cdot)$ satisfies the equality $\deg_i(fg) = \deg_i(f) + \deg_i(g)$, so that $\deg_i(f) = \deg_i(g) = 0$ once $i \in I \setminus J$, which means that g and h belong to $K[\boldsymbol{T}_J]$.

(2) For any finite subset J of I, it is well known that $K[\boldsymbol{T}_J]$ is a unique factorization domain. Moreover, for $f \in K[\boldsymbol{T}_I] \setminus \{0\}$, there is a finite subset J of I such that $f \in K[\boldsymbol{T}_J]$. Thus the first assertion follows from (1). The second assertion is a direct consequence of (1) in the particular case where $J = \varnothing$. □

Let $L = K(\boldsymbol{T}_I)$ be the field of fractions of $K[\boldsymbol{T}_I]$. As in §2.3, we pick in each equivalence class of irreducible polynomials in $K[\boldsymbol{T}_I]$, a representative to form a subset $\mathscr{P}_{K[\boldsymbol{T}_I]}$. For each element $F \in \mathscr{P}_{K[\boldsymbol{T}_I]}$, we let $\mathrm{ord}_F(\cdot)$ be the discrete valuation on L defined by F and let $|\cdot|_F := \mathrm{e}^{-\mathrm{ord}_F(\cdot)}$ be the corresponding absolute value. Let $\deg(\cdot)$ be the degree function on $K[\boldsymbol{T}_I]$. Note that for any $(f,g) \in K[\boldsymbol{T}_I]^2$ one has

$$\deg(f+g) \leqslant \max\{\deg(f), \deg(g)\}, \quad \deg(fg) = \deg(f) + \deg(g).$$

Therefore the function $-\deg(\cdot)$ extends to a discrete valuation on L. Denote by $|\cdot|_\infty$ the corresponding absolute value, defined as

$$|\cdot|_\infty = \mathrm{e}^{\deg(\cdot)}.$$

Note that the following product formula holds

$$\forall\, g \in L \setminus \{0\}, \quad \ln|g|_\infty + \sum_{F \in \mathscr{P}_{K[\boldsymbol{T}_I]}} \deg(F) \ln|g|_F = 0.$$

In other words, if we equip $\Omega'_L := \mathscr{P}_{K[\boldsymbol{T}_I]} \amalg \{\infty\}$ with the discrete σ-algebra $\mathcal{A}'_L$ and the measure ν'_L such that

$$\nu'_L(\{\infty\}) = 1 \text{ and } \nu'_L(\{F\}) = \deg(F)$$

for any $F \in \mathscr{P}_{K[\boldsymbol{T}_I]}$, then $(L, (\Omega'_L, \mathcal{A}'_L, \nu'_L), \phi'_L)$ forms a proper adelic curve, where

$$\phi'_L : \mathscr{P}_{K[\boldsymbol{T}_I]} \amalg \{\infty\} \to M_L$$

sends x to $|\cdot|_x$.

2.6.2. Remark. — Let $\boldsymbol{X}_{I\cup\{\infty\}} = \{X_i\}_{i\in I} \cup \{X_\infty\}$ be the variables indexed by $I \cup \{\infty\}$. Let $\varphi : K[\boldsymbol{X}_{I\cup\{\infty\}}] \to K[\boldsymbol{T}_I]$ be the homomorphism given by $\varphi(f) = f((T_i)_{i\in I}, 1)$. If f is an irreducible homogeneous polynomial in $K[\boldsymbol{X}_{I\cup\{\infty\}}]$ and $f \neq X_\infty$, then $\varphi(f)$ is an irreducible polynomial in $K[\boldsymbol{T}_I]$. Moreover, for any irreducible polynomial g in $K[\boldsymbol{T}_I]$, there is an irreducible homogeneous polynomial f in $K[\boldsymbol{X}_{I\cup\{\infty\}}]$ such that $\varphi(f) = g$. Note that the above $|\cdot|_\infty$ comes from the irreducible polynomial X_∞, so that the corresponding element is $1 = \varphi(X_\infty)$.

***2.6.3. Lemma* (Gauss's Lemma).** — *Let $|\cdot|$ be a non-Archimedean absolute value on K. We fix $\boldsymbol{e} = (e_i)_{i\in I} \in \mathbb{R}^I_{>0}$. For $\boldsymbol{d} = (d_i)_{i\in I} \in \mathbb{N}^{\oplus I}$, we set $\boldsymbol{e}^{\boldsymbol{d}} := \prod_{i\in I} e_i^{d_i}$. We denote by $|\cdot|_{\boldsymbol{e},L}$ the function on $K[\boldsymbol{T}_I]$ sending $f \in K[\boldsymbol{T}_I]$ to*

$$\max_{\boldsymbol{d}\in\mathbb{N}^{\oplus I}} |a_{\boldsymbol{d}}(f)|\boldsymbol{e}^{\boldsymbol{d}}.$$

Then, for any $(f,g) \in K[\boldsymbol{T}_I]^2$ one has

$$|fg|_{\boldsymbol{e},L} = |f|_{\boldsymbol{e},L} \cdot |g|_{\boldsymbol{e},L} \quad \text{and} \quad |f+g|_{\boldsymbol{e},L} \leqslant \max\{|f|_{\boldsymbol{e},L}, |g|_{\boldsymbol{e},L}\}.$$

In particular, $|\cdot|_{\boldsymbol{e},L}$ extends to an absolute value on $L = K(\boldsymbol{T}_I)$.

Proof. — If we set $f = \sum_{\boldsymbol{d}' \in \mathbb{N}^{\oplus I}} a_{\boldsymbol{d}'} \boldsymbol{T}^{\boldsymbol{d}'}$ and $g = \sum_{\boldsymbol{d}'' \in \mathbb{N}^{\oplus I}} b_{\boldsymbol{d}''} \boldsymbol{T}^{\boldsymbol{d}''}$, then

$$fg = \sum_{\boldsymbol{d} \in \mathbb{N}^{\oplus I}} \left(\sum_{\substack{\boldsymbol{d}', \boldsymbol{d}'' \in \mathbb{N}^{\oplus I}, \\ \boldsymbol{d}' + \boldsymbol{d}'' = \boldsymbol{d}}} a_{\boldsymbol{d}'} b_{\boldsymbol{d}''} \right) \boldsymbol{T}^{\boldsymbol{d}} \quad \text{and} \quad f + g = \sum_{\boldsymbol{d} \in \mathbb{N}^{\oplus I}} (a_{\boldsymbol{d}} + b_{\boldsymbol{d}}) \boldsymbol{T}^{\boldsymbol{d}}$$

Thus it is easy to see

$$\begin{cases} |fg|_{\boldsymbol{e},L} \leqslant |f|_{\boldsymbol{e},L} \cdot |g|_{\boldsymbol{e},L}, \\ |f+g|_{\boldsymbol{e},L} \leqslant \max\{|f|_{\boldsymbol{e},L}, |g|_{\boldsymbol{e},L}\}. \end{cases} \tag{2.7}$$

Let $\Sigma_f = \{\boldsymbol{d}' \in \mathbb{N}^{\oplus I} \mid |a_{\boldsymbol{d}'}| \boldsymbol{e}^{\boldsymbol{d}'} = |f|_{\boldsymbol{e},L}\}$ and $\Sigma_g = \{\boldsymbol{d}'' \in \mathbb{N}^{\oplus I} \mid |b_{\boldsymbol{d}''}| \boldsymbol{e}^{\boldsymbol{d}''} = |g|_{\boldsymbol{e},L}\}$. Let $\leqslant_{\mathrm{lex}}$ be the lexicographic order on $\mathbb{N}^{\oplus I}$. We choose $\boldsymbol{\delta}(f) \in \Sigma_f$ and $\boldsymbol{\delta}(g) \in \Sigma_g$ such that $\boldsymbol{d}' \leqslant_{\mathrm{lex}} \boldsymbol{\delta}(f)$ and $\boldsymbol{d}'' \leqslant_{\mathrm{lex}} \boldsymbol{\delta}(g)$ for all $\boldsymbol{d}' \in \Sigma_f$ and $\boldsymbol{d}'' \in \Sigma_g$.

2.6.4. Claim. — *One has $|a_{\boldsymbol{d}'}| \cdot |b_{\boldsymbol{d}''}| \leqslant |a_{\boldsymbol{\delta}(f)}| \cdot |b_{\boldsymbol{\delta}(g)}|$ for all $\boldsymbol{d}', \boldsymbol{d}'' \in \mathbb{N}^{\oplus I}$ with $\boldsymbol{d}' + \boldsymbol{d}'' = \boldsymbol{\delta}(f) + \boldsymbol{\delta}(g)$. Moreover, the equality holds if and only if $\boldsymbol{d}' = \boldsymbol{\delta}(f)$ and $\boldsymbol{d}'' = \boldsymbol{\delta}(g)$.*

Proof. — As $|a_{\boldsymbol{d}'}| \boldsymbol{e}^{\boldsymbol{d}'} \leqslant |f|_{\boldsymbol{e},L}$ and $|b_{\boldsymbol{d}''}| \boldsymbol{e}^{\boldsymbol{d}''} \leqslant |g|_{\boldsymbol{e},L}$, one has

$$|a_{\boldsymbol{d}'}| \cdot |b_{\boldsymbol{d}''}| \leqslant \frac{|f|_{\boldsymbol{e},L} |g|_{\boldsymbol{e},L}}{\boldsymbol{e}^{\boldsymbol{d}' + \boldsymbol{d}''}} = \frac{|a_{\boldsymbol{\delta}(f)}| \boldsymbol{e}^{\boldsymbol{\delta}(f)} |b_{\boldsymbol{\delta}(g)}| \boldsymbol{e}^{\boldsymbol{\delta}(g)}}{\boldsymbol{e}^{\boldsymbol{d}' + \boldsymbol{d}''}} = |a_{\boldsymbol{\delta}(f)}| \cdot |b_{\boldsymbol{\delta}(g)}|.$$

We assume that the equality holds. Then $\boldsymbol{d}' \in \Sigma_f$ and $\boldsymbol{d}'' \in \Sigma_g$, so that $\boldsymbol{d}' \leqslant_{\mathrm{lex}} \boldsymbol{\delta}(f)$ and $\boldsymbol{d}'' \leqslant_{\mathrm{lex}} \boldsymbol{\delta}(g)$. Therefore, one has the assertion because $\boldsymbol{d}' + \boldsymbol{d}'' = \boldsymbol{\delta}(f) + \boldsymbol{\delta}(g)$. $\square$

The above claim implies that

$$\left| \sum_{\substack{\boldsymbol{d}', \boldsymbol{d}'' \in \mathbb{N}^{\oplus I}, \\ \boldsymbol{d}' + \boldsymbol{d}'' = \boldsymbol{\delta}(f) + \boldsymbol{\delta}(g)}} a_{\boldsymbol{d}'} b_{\boldsymbol{d}''} \right| \boldsymbol{e}^{\boldsymbol{\delta}(f) + \boldsymbol{\delta}(g)} = |a_{\boldsymbol{\delta}(f)}| \boldsymbol{e}^{\boldsymbol{\delta}(f)} |b_{\boldsymbol{\delta}(g)}| \boldsymbol{e}^{\boldsymbol{\delta}(g)} = |f|_{\boldsymbol{e},L} |g|_{\boldsymbol{e},L},$$

which means that $|fg|_{\boldsymbol{e},L} \geqslant |f|_{\boldsymbol{e},L} |g|_{\boldsymbol{e},L}$, as required. $\square$

For any $\omega \in \Omega \setminus \Omega_\infty$, let $|\cdot|_{\omega,L}$ be the absolute value on L such that

$$\forall\, g = \sum_{\boldsymbol{d} \in \mathbb{N}^{\oplus I}} a_{\boldsymbol{d}}(g) \boldsymbol{T}_I^{\boldsymbol{d}} \in K[\boldsymbol{T}_I], \quad |g|_{\omega,L} := \sup_{\boldsymbol{d} \in \mathbb{N}^{\oplus I}} |a_{\boldsymbol{d}}(g)|_\omega.$$

By Lemma 2.6.3, this absolute value is an extension of $|\cdot|_\omega$ on K. Let

$$((\Omega_{L,\omega}, \mathcal{A}_{L,\omega}, \nu_{L,\omega}), \phi_{L,\omega})$$

be the adelic structure on L which consists of a single copy of the absolute value $|\cdot|_{\omega,L}$, equipped with the unique probability measure. We denote by $S_{L,\omega}$ the adelic curve $(L, (\Omega_{L,\omega}, \mathcal{A}_{L,\omega}, \nu_{L,\omega}), \phi_{L,\omega})$.

2.6.5. Proposition. — *If $\Omega_\infty = \varnothing$, then family $(S_{L,\omega})_{\omega \in \Omega}$ is an admissible fibration over S.*

Proof. — Let g be a non-zero element of $K[\boldsymbol{T}_I]$, $(F_j)_{j=1}^n$ be a finite family of elements of $\mathscr{P}_{K[\boldsymbol{T}_I]}$ containing $\{F \in \mathscr{P}_{K[\boldsymbol{T}_I]} \mid \mathrm{ord}_F(g) \neq 0\}$ and $(C_j)_{j=1}^n$ be a family of non-negative constants. One has

$$\int_{\Omega_{L,\omega}} |g|_x \mathbb{1}_{|F_1|_x \leqslant C_1, \ldots, |F_n|_x \leqslant C_n} \, \nu_{L,\omega}(\mathrm{d}x) = \max_{\boldsymbol{d} \in \mathbb{N}^{\oplus I}} |a_{\boldsymbol{d}}(g)|_\omega \cdot \prod_{j=1}^n \prod_{\boldsymbol{d} \in \mathbb{N}^{\oplus I}} \mathbb{1}_{|a_{\boldsymbol{d}}(F_j)|_\omega \leqslant C_j}.$$

Therefore the function

$$(\omega \in \Omega) \longmapsto \int_{\Omega_{L,\omega}} |g|_x \mathbb{1}_{|F_1|_x \leqslant C_1, \ldots, |F_n|_x \leqslant C_n} \, \nu_{L,\omega}(\mathrm{d}x).$$

is $\mathcal{A}$-measurable. Moreover, for any element F of $\mathscr{P}_{K[\boldsymbol{T}_I]}$, one has

$$\int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x) = \max_{\boldsymbol{d} \in \mathbb{N}^{\oplus I},\, a_{\boldsymbol{d}}(F) \neq 0} \ln |a_{\boldsymbol{d}}(F)|_\omega. \tag{2.8}$$

Therefore the function

$$(\omega \in \Omega) \longmapsto \int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x)$$

is ν-integrable. □

2.6.6. Remark. — In the case where $\Omega_\infty = \varnothing$ and the adelic curve S is proper, for any $\boldsymbol{d}$ such that $a_{\boldsymbol{d}}(F) \neq 0$, one has

$$\int_{\omega \in \Omega} \ln |a_{\boldsymbol{d}}(F)|_\omega \, \nu(\mathrm{d}\omega) = 0,$$

and hence

$$h_{S_L}(F) = \int_{\Omega} \nu(\mathrm{d}\omega) \int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x) \geqslant 0.$$

2.7. Arithmetic adelic structure

In this section, we provides a "standard" construction of an adelic structure for a countable field of characteristic zero. More precisely, for any countable field E of characteristic zero, we will construct an adelic curve $S_E = (E, (\Omega_E, \mathcal{A}_E, \nu_E), \phi_E)$, which satisfies the following properties:

(1) S_E is proper.
(2) For any $\omega \in \Omega_E$, the absolute value $\phi_E(\omega)$ is not trivial.
(3) The set $\Omega_{E,\mathrm{fin}}$ of $\omega \in \Omega$ such that $\phi_E(\omega)$ is non-Archimedean is infinite but countable.
(4) Let E^{ac} be an algebraic closure of E. If E_0 is a subfield of E^{ac} such that E_0 is finitely generated over $\mathbb{Q}$, then

$$\{a \in E^{\mathrm{ac}} \mid h_{S_E \otimes_E E^{\mathrm{ac}}}(1, a) \leqslant C \text{ and } [E_0(a) : E_0] \leqslant \delta\}$$

is finite for all $C \in \mathbb{R}_{\geqslant 0}$ and $\delta \in \mathbb{Z}_{\geqslant 1}$.

2.7.1. Definition. — Let K be a countable field of characteristic 0. An adelic structure of K which satisfies the above conditions (1)–(4) is said to be *arithmetic*.

2.7.2. Remark. — Note that the condition (4) is analogous to Northcott's property in Diophantine geometry. In Arakelov geometry of adelic curve, we say that an adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ *has Northcott property* if the set

$$\{a \in K \mid h_S(1, a) \leqslant C\}$$

is finite for any $C \geqslant 0$ (see [**13**, Definition 3.5.2]). In the case where the adelic curve S is proper and has Northcott property, an analogue of Northcott's theorem holds (see [**13**, Definition 3.5.3])

In the remaining of the section, we fix a countable field K of characteristic 0 and a countable non-empty set I. We equip K with an adelic structure $((\Omega, \mathcal{A}, \nu), \phi)$ to form an adelic curve, which we denote by S. We also fix a family $(\iota_\omega)_{\omega\in\Omega_\infty}$ of embeddings from K to $\mathbb{C}$ such that $|\cdot|_\omega = |\iota_\omega(\cdot)|$ for any $\omega \in \Omega_\infty$ and that the map $(\omega \in \Omega_\infty) \mapsto \iota_\omega(a)$ is measurable for each $a \in K$ (see [**13**, Step 1 in Theorem 4.1.26]). For any element $f \in K[\boldsymbol{T}_I]$, we denote by $\iota_\omega(f)$ the polynomial in $\mathbb{C}[\boldsymbol{T}_I]$ defined as

$$\iota_\omega(f) := \sum_{\boldsymbol{d}\in\mathbb{N}^{\oplus I}} \iota_\omega(a_{\boldsymbol{d}}(f))\boldsymbol{T}_I^{\boldsymbol{d}}.$$

This defines a ring homomorphism from $K[\boldsymbol{T}_I]$ to $\mathbb{C}[\boldsymbol{T}_I]$, which extends to a homomorphism of fields form $K(\boldsymbol{T}_I)$ to $\mathbb{C}(\boldsymbol{T}_I)$, which we still denote by $\iota_\omega(\cdot)$.

2.7.3. Notation. — For convenience, for any $f \in K[\boldsymbol{T}_I]$, the complex polynomial $\iota_\omega(f) \in \mathbb{C}[\boldsymbol{T}_I]$ is often denoted by f_ω.

For any $t \in [0, 1]$, we denote by $e(t)$ the complex number $\mathrm{e}^{2\pi t\sqrt{-1}}$. For any $\omega \in \Omega_\infty$, we denote by $\Omega_{L,\omega}$ the set

$$\Omega_{L,\omega} := \left\{ (t_i)_{i\in I} \in [0,1]^I \;\middle|\; \begin{array}{l} (e(t_i))_{i\in I} \text{ is algebraically} \\ \text{independent over } \iota_\omega(K) \end{array} \right\}.$$

Note that by definition one has

$$[0,1]^I \setminus \Omega_{L,\omega} = \bigcup_{f\in K[\boldsymbol{T}_I]\setminus\{0\}} \{(t_i)_{i\in I} \in [0,1]^I \;:\; f_\omega\big((e(t_i))_{i\in I}\big) = 0\}. \tag{2.9}$$

We equip $[0, 1]^I$ with the product σ-algebra (namely the smallest σ-algebra making measurable the projection maps to the coordinates) and the product of the uniform probability measure on $[0, 1]$, denoted by η_I (see [**45**, §4.2] for the product of an arbitrary family of probability spaces).

2.7.4. Lemma. — *For any $\omega \in \Omega_\infty$, the subset $\Omega_{L,\omega}$ of $[0, 1]^I$ is measurable, and $[0, 1]^I \setminus \Omega_{L,\omega}$ is η_I-negligible.*

Proof. — The measurability of $\Omega_{L,\omega}$ follows from (2.9).

For any non-zero element of $K[\boldsymbol{T}_I]$, let

$$V_I(f) = \{(t_i)_{i\in I} \in [0,1]^I \mid f_\omega\big((e(t_i))_{i\in I}\big) = 0\}.$$

Since K and I are countable, $K[\boldsymbol{T}_I]$ is a countable set. Therefore, by (2.9), to prove the second statement it suffices to show that $\eta_I(V_I(f)) = 0$. We first treat the case where I is a finite set. Without loss of generality, we assume that $I = \{1,\ldots,n\}$, where $n \in \mathbb{N}$. The case where $n = 0$ (namely $I = \varnothing$) is trivial since in this case $V_I(f)$ is empty. Assume that $n \geqslant 1$. For $t \in [0,1]$, let f_t be the polynomial

$$\iota_\omega(f)(T_1,\ldots,T_{n-1},e(t)) \in \iota_\omega(K)(e(t))[T_1,\ldots,T_{n-1}].$$

Then by Fubini's theorem, one has

$$\eta_{\{1,\ldots,n\}}(V_{\{1,\ldots,n\}}(f)) = \int_{[0,1]} \eta_{\{1,\ldots,n-1\}}(V_{\{1,\ldots,n-1\}}(f_t))\,\mathrm{d}t = 0,$$

where the second equality comes from the induction hypothesis.

We now consider the general case. Let J be a finite subset of I such that $f \in K[(T_i)_{i\in J}]$. By the definition of the product measure, one has

$$\eta_I(V_I(f)) = \eta_J(V_J(f)) = 0.$$

□

For any $\omega \in \Omega_\infty$, we equip $\Omega_{L,\omega}$ with the restriction of the product σ-algebra on $[0,1]^I$ and the restriction of the product probability measure η_I to obtain a probability space denoted by $(\Omega_{L,\omega},\mathcal{A}_{L,\omega},\nu_{L,\omega})$. Let $\phi_{L,\omega} : \Omega_{L,\omega} \to M_L$ be the map sending $x = (t_i)_{i\in I} \in \Omega_{L,\omega}$ to the absolute value

$$(f \in L) \longmapsto |f|_x := \Big|f_\omega\big((e(t_i))_{i\in I}\big)\Big|.$$

Thus we obtain an adelic curve $S_{L,\omega} := (L,(\Omega_{L,\omega},\mathcal{A}_{L,\omega},\nu_{L,\omega}),\phi_{L,\omega})$.

We recall Jensen's formula for Mahler measure of polynomials (see **[41]** for a proof).

***2.7.5. Lemma* (Jensen's formula).** — *Let*

$$P(T) = a_d(T-\alpha_1)\cdots(T-\alpha_d) \in \mathbb{C}[T]$$

be a complex polynomial of one variable T, with $a_d \in \mathbb{C}\setminus\{0\}$ and $(\alpha_1,\ldots,\alpha_d) \in \mathbb{C}^d$. One has

$$\int_0^1 \ln|P(e(t))|\,\mathrm{d}t = \ln|a_d| + \sum_{j=1}^d \ln(\max\{1,|\alpha_j|\}) \geqslant \ln|a_d|.$$

2.7.6. Proposition. — *The family of adelic curves $(S_{L,\omega})_{\omega\in\Omega}$ is an admissible fibration over the adelic curve S. Moreover, in the case where the adelic curve S is proper, for any $F \in \mathscr{P}_{K[T]}$, one has*

$$h_{S_L}(F) := \int_\Omega \nu(\mathrm{d}\omega)\int_{\Omega_{L,\omega}} \ln|F|_x\,\nu_{L,\omega}(\mathrm{d}x) \geqslant 0.$$

Proof. — **Step 1.** By construction, for any $\omega \in \Omega$ and any $x \in \Omega_{L,\omega}$, the absolute value $\phi_{L,\omega}(x)$ on L extends the absolute value $\phi(\omega)$ on K.

Step 2. Let g be a non-zero element of $K[\boldsymbol{T}_I]$, $(F_j)_{j=1}^n$ be elements of $\mathscr{P}_{K[\boldsymbol{T}_I]}$ containing

$$\{F \in \mathscr{P}_{K[\boldsymbol{T}_I]} \,:\, \mathrm{ord}_F(g) \neq 0\},$$

and $(C_j)_{j=1}^n \in \mathbb{R}_{\geqslant 0}^n$. We show that the function

$$(\omega \in \Omega) \longmapsto \int_{\Omega_{L,\omega}} |g|_x \mathbb{1}_{|F_1|_x \leqslant C_1, \ldots, |F_n|_x \leqslant C_n} \, \nu_{L,\omega}(\mathrm{d}x) \tag{2.10}$$

is $\mathcal{A}$-measurable. We choose a finite subset J of I such that g, $F_1, \ldots, F_n$ belong to $K[(T_i)_{i\in J}]$. By Lemma 2.7.4, one has

$$\begin{aligned}
&\int_{\Omega_{L,\omega}} |g|_x \mathbb{1}_{|F_1|_x \leqslant C_1, \ldots, |F_n|_x \leqslant C_n} \, \nu_{L,\omega}(\mathrm{d}x) \\
&= \int_{[0,1]^I} \Big| g_\omega\big((e(t_i))_{i\in I}\big)\Big| \prod_{j=1}^n \mathbb{1}_{|F_{j,\omega}((e(t_i))_{i\in I})| \leqslant C_j} \, \eta_I(\mathrm{d}(t_i)_{i\in I}) \\
&= \int_{[0,1]^J} \Big| g_\omega\big((e(t_i))_{i\in J}\big)\Big| \prod_{j=1}^n \mathbb{1}_{|F_{j,\omega}((e(t_i))_{i\in I})| \leqslant C_j} \, \eta_J(\mathrm{d}(t_i)_{i\in J})
\end{aligned}$$

Note that $[0,1]^J$ is a separable compact metric space. By the criterion of measurability for functions on product measurable space proved in [**46**, Lemma 9.2] and the measurability of integrals with parameter (see [**42**, Lemma 1.26]), we obtain the measurability of the function (2.10) on Ω_∞. The measurability of this function on $\Omega \setminus \Omega_\infty$ follows from Proposition 2.6.5.

Step 3. It remains to show that the function

$$(\omega \in \Omega) \longmapsto \int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x) \tag{2.11}$$

is well defined and is integrable for any $F \in \mathscr{P}_{K[T]}$. By Proposition 2.6.5 again, it suffices to show its integrability on Ω_∞. Let

$$\Theta := \{\boldsymbol{d} \in \mathbb{N}^{\oplus I} \,:\, a_{\boldsymbol{d}}(F) \neq 0\}.$$

One has

$$\ln |F|_x \leqslant \max_{\boldsymbol{d}\in\Theta} \ln |a_{\boldsymbol{d}}(F)|_\omega + \ln(\mathrm{card}(\Theta)).$$

Therefore, for $\omega \in \Omega_\infty$, the integral

$$\int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x)$$

is well defined and the following inequality holds:

$$\int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x) \leqslant \max_{\boldsymbol{d}\in\Theta} \ln |a_{\boldsymbol{d}}(F)|_\omega + \ln(\mathrm{card}(\Theta)). \tag{2.12}$$

Moreover, by an argument similar to that in Step 2, it can be shown that the function

$$(\omega \in \Omega_\infty) \longrightarrow \int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x)$$

is measurable. Finally, by writing

$$\int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x)$$

as successive integrals, and then by applying Jensen's formula in a recursive way, we obtain that there exists $\boldsymbol{d}_0 \in \Theta$ such that

$$\forall\, \omega \in \Omega_\infty, \quad \int_{\Omega_{L,\omega}} \ln |F|_x \, \nu_{L,\omega}(\mathrm{d}x) \geqslant \ln |a_{\boldsymbol{d}_0}(F)|_\omega. \tag{2.13}$$

Combining this inequality with (2.12) and the fact that $\nu(\Omega_\infty) < +\infty$ (see [**13**, Proposition 3.1.2]), we obtain the integrability of the function (2.11) on Ω_∞. Finally, applying (2.8) to $\omega \in \Omega \setminus \Omega_\infty$, the inequality (2.13) leads to

$$h_{S_L}(F) \geqslant \int_{\omega \in \Omega} \ln |a_{\boldsymbol{d_0}}(F)|_\omega \, \nu(\mathrm{d}\omega) = 0$$

provided that the adelic curve S is proper. The proposition is thus proved. □

2.7.7. Remark. — Note that, for $f \in L$,

$$h_{S_L}(f) = \int_{\Omega_\infty} \nu(\mathrm{d}\omega) \int_{\Omega_{L,\omega}} \ln \Big| f_\omega\big((e(t_i))_{i\in I}\big)\Big| \, \eta_I(\mathrm{d}(t_i)_{i\in I}) + \int_{\Omega_{\mathrm{fin}}} \ln |f|_\omega \, \nu(\mathrm{d}\omega).$$

Thus $h_{S_L}(1) = 0$ and $h_{S_L}(T_i) = 0$ for all $i \in I$.

2.7.8. Definition. — As a corollary, to the admissible fibration $(S_{L,\omega})_{\omega\in\Omega}$ one can associate an adelic structure $((\Omega_L, \mathcal{A}_L, \nu_L), \phi_L)$ on L as in Definition 2.3.3. We fix $\lambda \in \mathbb{R}_{\geqslant 0}$. Let $S_L^\lambda := (L, (\Omega_L^\lambda, \mathcal{A}_L^\lambda, \nu_L^\lambda), \phi_L^\lambda)$ be an adelic curve with underlying field L such that

(1) $(\Omega_L^\lambda, \mathcal{A}_L^\lambda, \nu_L^\lambda)$ is the disjoint union of $(\Omega_L, \mathcal{A}_L, \nu_L)$ and $\mathscr{P}_{K[\boldsymbol{T}_I]} \cup \{\infty\}$ equipped with the discrete σ-algebra and the measure satisfying

$$\nu_L^\lambda(\{F\}) = h_{S_L}(F) + \lambda \deg(F) \quad \text{and} \quad \nu_L^\lambda(\{\infty\}) = \lambda$$

for any $F \in \mathscr{P}_{K[\boldsymbol{T}_I]}$.

(2) the map $\phi_L^\lambda : \Omega_L^\lambda \to M_L$ extends ϕ_L and the map

$$(x \in \mathscr{P}_{K[\boldsymbol{T}_I]} \cup \{\infty\}) \longmapsto |\cdot|_x.$$

The adelic curve S_L^λ is called the λ-*twisted compactification* of S_L.

2.7.9. Remark. — Note that if $\lambda = 0$, then $S_L^\lambda = S_L^*$. Moreover, if K and Ω_{fin} are countable and $\mathcal{A}_{\Omega_{\mathrm{fin}}}$ is discrete, then L and $\Omega_{L,\mathrm{fin}}^*$ are countable and $\mathcal{A}_{\Omega_{L,\mathrm{fin}}^*}$ is discrete.

2.7.10. Proposition. — *The adelic curve* $S_L^\lambda = (L, (\Omega_L^\lambda, \mathcal{A}_L^\lambda, \nu_L^\lambda), \phi_L^\lambda)$ *is proper.*

Proof. — If $\lambda = 0$, then the assertion follows from Proposition 2.4.1 and Proposition 2.7.6. Note that

$$\deg(g) = \sum_{F \in \mathscr{P}_{K[\boldsymbol{T}_I]}} \deg(F) \operatorname{ord}_F(g) \tag{2.14}$$

for $g \in L^{\times}$, so that

$$\sum_{F \in \mathscr{P}_{K[\boldsymbol{T}_I]}} (h_{S_L}(F) + \lambda \deg(F))(-\operatorname{ord}_F(g)) + \lambda \deg(g) = \sum_{F \in \mathscr{P}_{K[\boldsymbol{T}_I]}} h_{S_L}(F)(-\operatorname{ord}_F(g)),$$

as required. $\square$

2.7.11. Remark. — The above result can be considered as a particular case of Proposition 2.5.1. In fact, if we equip $\mathscr{P}_{K[\boldsymbol{T}_I]} \cup \{\infty\}$ with the discrete σ-algebra $\mathcal{A}'$ and the measure ν' such that $\nu'(\{\infty\}) = 1$ and $\nu'(\{F\}) = \deg(F)$, then

$$(L, (\mathscr{P}_{K[\boldsymbol{T}_I]} \cup \{\infty\}, \mathcal{A}', \nu'), \phi')$$

forms an adelic curve, where ϕ' sends $x \in \mathscr{P}_{K[\boldsymbol{T}_I]} \cup \{\infty\}$ to the absolute value $|\cdot|_x$. Then the equality (2.14) shows that this adelic curve is proper. Note that the restriction of ν_L^{λ} on $\mathscr{P}_{K[\boldsymbol{T}_I]} \cup \{\infty\}$ coincides with

$$\lambda \nu'_L + \sum_{F \in \mathscr{P}_{K[\boldsymbol{T}_I]}} h_{S_L}(F) \operatorname{Dirac}_F .$$

Therefore the statement of Proposition 2.7.10 follows from Proposition 2.5.1.

2.7.12. Lemma. — (1) *If $F_0, \ldots, F_r \in K[\boldsymbol{T}_I]$ with $(F_0, \ldots, F_r) \neq (0, \ldots, 0)$, then*

$$\begin{aligned} h_{S_L^{\lambda}}(F_0, \ldots, F_r) \leqslant \int_{\Omega_{L,\infty}} & \ln \max\{|F_0|_x, \ldots, |F_r|_x\} \nu_{L,\infty}(\mathrm{d}x) \\ & + \int_{\Omega_{\mathrm{fin}}} \ln \max\{|F_0|_\omega, \ldots, |F_r|_\omega\} \nu_{\mathrm{fin}}(\mathrm{d}\omega) \\ & \qquad + \lambda \max\{\deg(F_0), \ldots, \deg(F_r)\}. \end{aligned}$$

Moreover, if G.C.D$(F_0, \ldots, F_r) = 1$*, then the equality holds.*

(2) *Fix $n \in I$ and let $I' = I \setminus \{n\}$ and $L' = K(\boldsymbol{T}_{I'})$. For $F \in K[\boldsymbol{T}_I] \setminus \{0\}$, if we set $F = a_0 T_n^d + a_1 T_n^{d-1} + \cdots + a_d$ such that $a_0, a_1, \ldots, a_d \in K[\boldsymbol{T}_{I'}]$ and $a_0 \neq 0$, then*

$$h_{S_{L'}^{\lambda}}(a_0, \ldots, a_d) \leqslant h_{S_L}(F) + \deg(F)(\lambda + \ln(2)\nu(\Omega_\infty)).$$

Proof. — (1) Note that

$$\max\{|F_0|_\xi, \ldots, |F_r|_\xi\} \begin{cases} \leqslant 1 & \text{in general,} \\ = 1 & \text{if G.C.D}(F_0, \ldots, F_r) = 1, \end{cases}$$

for $\xi \in \mathscr{P}_{K[\boldsymbol{T}_I]}$, so that the assertion follows.

(2) Note that $d \leqslant \deg(F)$. We set $f = F/a_0$. For $y \in \Omega_{L',\infty}$, let

$$f_y = T_n^d + \iota_y(a_1/a_0)T_n^{d-1} + \cdots + \iota_y(a_d/a_0) = (T_n - \alpha_1)\cdots(T_n - \alpha_d)$$

be the irreducible decomposition in $\mathbb{C}[T_n]$. Then,

$$\iota_y(a_k/a_0) = (-1)^k \sum_{1\leqslant i_1<\cdots<i_k\leqslant d} \alpha_{i_1}\cdots\alpha_{i_k},$$

so that

$$\begin{aligned} |a_k/a_0|_y &\leqslant \sum_{1\leqslant i_1<\cdots<i_k\leqslant d} |\alpha_{i_1}|\cdots|\alpha_{i_k}| \leqslant \sum_{1\leqslant i_1<\cdots<i_k\leqslant d} \max\{1,|\alpha_{i_1}|\}\cdots\max\{1,|\alpha_{i_k}|\} \\ &\leqslant \sum_{1\leqslant i_1<\cdots<i_k\leqslant d} \max\{1,|\alpha_1|\}\cdots\max\{1,|\alpha_d|\} \\ &\leqslant 2^{\deg(F)}\max\{1,|\alpha_1|\}\cdots\max\{1,|\alpha_d|\} \end{aligned}$$

because $\binom{d}{k} \leqslant 2^d \leqslant 2^{\deg(F)}$, and hence one has

$$\max\{1,|a_k/a_0|_y\} \leqslant 2^{\deg(F)}\max\{1,|\alpha_1|\}\cdots\max\{1,|\alpha_d|\}.$$

On the other hand, by Jensen's formula,

$$\int_0^1 \ln|f_y(e(t_n))|\,\mathrm{d}t_n = \sum_{i=1}^d \ln\max\{1,|\alpha_i|\}.$$

Therefore, one obtains

$$\ln\max\{1,|a_k/a_0|_y\} \leqslant \int_0^1 \ln|f_y(e(t_n))|\,\mathrm{d}t_n + \deg(F)\ln(2)$$

for all $k \in \{1,\ldots,d\}$, so that

$$\begin{aligned} &\ln\max\{|a_0|_y,|a_1|_y,\ldots,|a_d|_y\} \\ &\qquad = \ln|a_0|_y + \ln\max\{1,|a_1/a_0|_y,\ldots,|a_d/a_0|_y\} \\ &\qquad \leqslant \ln|a_0|_y + \int_0^1 \ln|f_y(e(t_n))|\,\mathrm{d}t_n + \deg(F)\ln(2) \\ &\qquad = \int_0^1 \ln|F_y(e(t_n))|\,\mathrm{d}t_n + \deg(F)\ln(2). \end{aligned}$$

Thus, by Fubini's theorem,

$$\begin{aligned} \int_{\Omega_{L,\infty}} \ln|F|_x \nu_{L,\infty}(\mathrm{d}x) &= \int_{\Omega_{L',\infty}\times[0,1]} \ln|F_y(e(t_n))|\,\nu_{L'}(\mathrm{d}y)\,\mathrm{d}t_n \\ &= \int_{\Omega_{L',\infty}} \left(\int_0^1 \ln|F_y(e(t_n))|\,\mathrm{d}t_n\right)\nu_{L'}(\mathrm{d}y) \\ &\geqslant \int_{\Omega_{L',\infty}} \left(\ln\max\{|a_0|_y,\ldots,|a_d|_y\} - \deg(F)\ln(2)\right)\nu_{L'}(\mathrm{d}y) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega_{L',\infty}} \ln \max\{|a_0|_y, \ldots, |a_d|_y\}\, \nu_{L'}(\mathrm{d}y) - \deg(F)\ln(2)\nu_{L'}(\Omega_{L',\infty}) \\
&= \int_{\Omega_{L',\infty}} \ln \max\{|a_0|_y, \ldots, |a_d|_y\}\, \nu_{L'}(\mathrm{d}y) - \deg(F)\ln(2)\nu(\Omega_\infty).
\end{aligned}
$$

On the other hand, note that

$$|F|_\omega = \max\{|a_0|_\omega, \ldots, |a_d|_\omega\}$$

for $\omega \in \Omega_{\mathrm{fin}}$, so that

$$
\begin{aligned}
\int_{\Omega_{L',\infty}} & \ln \max\{|a_0|_y, \ldots, |a_d|_y\}\, \nu_{L'}(\mathrm{d}y) \\
&\qquad + \int_{\Omega_{\mathrm{fin}}} \ln \max\{|a_0|_\omega, \ldots, |a_d|_\omega\}\, \nu(\mathrm{d}\omega) \\
&\qquad\qquad \leqslant h_{S_L}(F) + \deg(F)\ln(2)\,\nu(\Omega_\infty).
\end{aligned}
$$

$$
\begin{aligned}
h_{S^\lambda_{L'}}(a_0, \ldots, a_d) &\leqslant \int_{\Omega_{L',\infty}} \ln \max\{|a_0|_y, \ldots, |a_d|_y\}\, \nu_{L'}(\mathrm{d}y) \\
&\qquad + \int_{\Omega_{\mathrm{fin}}} \ln \max\{|a_0|_\omega, \ldots, |a_d|_\omega\}\, \nu(\mathrm{d}\omega) \\
&\qquad\qquad + \lambda \max\{\deg(a_0), \ldots, \deg(a_d)\} \\
&\leqslant h_{S_L}(F) + \deg(F)(\lambda + \ln(2)\nu(\Omega_\infty)),
\end{aligned}
$$

as required. □

Fix $n \in I$ and let $I' = I \setminus \{n\}$ and $L' = K(\boldsymbol{T}_{I'})$. For $F \in K[\boldsymbol{T}_I] \setminus \{0\}$, we set $F = a_0 T_n^d + \cdots + a_d$ such that $a_0, \ldots, a_d \in K[\boldsymbol{T}_{I'}]$ and $a_0 \neq 0$. We define $\nu(F)$ to be

$$\nu(F) := F/a_0 = T_n^d + (a_1/a_0)T_n^{d-1} + \cdots + (a_d/a_0).$$

Note that $\nu(F)$ is a monic polynomial over L'.

2.7.13. Proposition. — *If $S^\lambda_{L'}$ has Northcott's property, then, for $C \in \mathbb{R}$ and $\delta \in \mathbb{Z}_{\geqslant 1}$, then the set*

$$\{\nu(F) \mid F \in K[\boldsymbol{T}_I] \setminus \{0\},\ h_{S_L}(F) \leqslant C \text{ and } \deg(F) \leqslant \delta\}$$

is finite.

Proof. — Let $\Theta := \{F \in K[\boldsymbol{T}_I] \setminus \{0\} \mid h_{S_L}(F) \leqslant C \text{ and } \deg(F) \leqslant \delta\}$ and $\vartheta : \Theta \to \mathbb{P}^\delta(L')$ be a map given by the following way: for

$$F = a_0 T_n^d + \cdots + a_d \in \Theta \quad (a_0, \ldots, a_d \in K[\boldsymbol{T}_I] \text{ and } a_0 \neq 0),$$

$$\vartheta(F) := \overbrace{(a_0 : \cdots : a_d : 0 : \cdots : 0)}^{\delta+1} \in \mathbb{P}^\delta(L').$$

By Lemma 2.7.12,

$$h_{S^\lambda_{L'}}(\vartheta(F)) \leqslant h_{S_L}(F) + \deg(F)(\lambda + \ln(2)\nu(\Omega_\infty)) \leqslant C + \delta(\lambda + \ln(2)\nu(\Omega_\infty)).$$

Thus the assertion of the proposition is a consequence of Northcott's property of $S^\lambda_{L'}$. □

2.7.14. Proposition. — *If S has Northcott's property,* $\mathrm{card}(I) < \infty$ *and $\lambda > 0$, then S^λ_L has also Northcott's property.*

Proof. — We prove it by induction on $\mathrm{card}(I)$. If $\mathrm{card}(I) = 0$, then the assertion is obvious because $S^\lambda_L = S$. Fix $n \in I$ and let $I' = I \setminus \{n\}$ and $L' = K(\boldsymbol{T}_{I'})$. It is sufficient to see that $\{f \in L^\times \mid h_{S^\lambda_L}(f, 1) \leqslant C\}$ is finite for any C. For $f \in L^\times$, let us choose $F_1, F_2 \in K[\boldsymbol{T}_I] \setminus \{0\}$ such that $f = F_1/F_2$, and F_1 and F_2 are relatively prime. We set

$$\begin{cases} F_1 = a_{1,0}T_n^{d_1} + \cdots + a_{1,d_1} & (a_{1,0}, \ldots, a_{1,d_1} \in K[\boldsymbol{T}_{I'}] \text{ and } a_{1,0} \neq 0), \\ F_2 = a_{2,0}T_n^{d_2} + \cdots + a_{2,d_2} & (a_{2,0}, \ldots, a_{2,d_2} \in K[\boldsymbol{T}_{I'}] \text{ and } a_{2,0} \neq 0). \end{cases}$$

2.7.15. Claim. — *If $h_{S^\lambda_L}(f, 1) \leqslant C$, then one has the following:*

(1) $\max\{\deg(F_1), \deg(F_2)\} \leqslant C/\lambda$ *and* $\max\{h_{S_L}(F_1), h_{S_L}(F_2)\} \leqslant C$.

(2) $h_{S^\lambda_{L'}}(a_{10}, a_{20}) \leqslant C$.

Proof. — (1) As $C \geqslant h_{S^\lambda_L}(f, 1) = h_{S^\lambda_L}(F_1, F_2)$ and F_1 and F_2 are relatively prime, by (1) in Lemma 2.7.12, one has

$$C \geqslant \lambda \max\{\deg(F_1), \deg(F_2)\} + \int_{\Omega_{L,\infty}} \ln\max\{|F_1|_x, |F_2|_x\}\,\nu_L(\mathrm{d}x) + \int_{\Omega_{\mathrm{fin}}} \ln\max\{|F_1|_\omega, |F_2|_\omega\}\,\nu(\mathrm{d}\omega). \quad (2.15)$$

Thus,

$$C \geqslant \lambda \max\{\deg(F_1), \deg(F_2)\} + \max\{h_{S_L}(F_1), h_{S_L}(F_2)\}$$

Therefore, (1) follows because $h_{S_L}(F_1), h_{S_L}(F_2) \geqslant 0$.

(2) By (1) in Lemma 2.7.12,

$$h_{S^\lambda_{L'}}(a_{1,0}, a_{2,0}) \leqslant \lambda \max\{\deg(a_{1,0}), \deg(a_{2,0})\} + \int_{\Omega_{L',\infty}} \ln\max\{|a_{1,0}|_y, |a_{2,0}|_y\}\,\nu_{L'}(\mathrm{d}y) + \int_{\Omega_{\mathrm{fin}}} \ln\max\{|a_{1,0}|_\omega, |a_{2,0}|_\omega\}\,\nu(\mathrm{d}\omega).$$

Therefore, by (2.15), it is sufficient to see the following:

$$\int_{\Omega_{L,\infty}} \ln\max\{|F_1|_x, |F_2|_x\}\nu_{L,\infty}(\mathrm{d}x) \geqslant \int_{\Omega_{L',\infty}} \ln\max\{|a_{1,0}|_y, |a_{2,0}|_y\}\,\nu_{L'}(\mathrm{d}y) \quad (2.16)$$

and

$$\int_{\Omega_{\mathrm{fin}}} \ln\max\{|F_1|_\omega, |F_2|_\omega\}\,\nu(\mathrm{d}\omega) \geqslant \int_{\Omega_{\mathrm{fin}}} \ln\max\{|a_{1,0}|_\omega, |a_{2,0}|_\omega\}\,\nu(\mathrm{d}\omega). \tag{2.17}$$

Indeed, by Jensen's formula together with Fubini's theorem,

$$\begin{aligned}
&\int_{\Omega_{L,\infty}} \ln\max\{|F_1|_x, |F_2|_x\}\,\nu_L(\mathrm{d}x)\\
&\qquad = \int_{\Omega_{L',\infty}\times[0,1]} \ln\max_{i=1,2}\{|F_{i,y}(e(t_n))|\}\,\nu_{L'}(\mathrm{d}y)\mathrm{d}t_n\\
&\qquad = \int_{\Omega_{L',\infty}} \left(\int_0^1 \ln\max_{i=1,2}\{|F_{i,y}(e(t_n))|\}\mathrm{d}t_n\right)\nu_{L'}(\mathrm{d}y)\\
&\qquad \geqslant \int_{\Omega_{L',\infty}} \max_{i=1,2}\left\{\int_0^1 \ln|F_{i,y}(e(t_n))|\mathrm{d}t_n\right\}\nu_{L'}(\mathrm{d}y)\\
&\qquad \geqslant \int_{\Omega_{L',\infty}} \max_{i=1,2}\{\ln|\iota_y(a_{i,0})|\}\,\nu_{L'}(\mathrm{d}y)\\
&\qquad = \int_{\Omega_{L',\infty}} \ln\max\{|a_{1,0}|_y, |a_{2,0}|_y\}\,\nu_{L'}(\mathrm{d}y),
\end{aligned}$$

as required for (2.16). Further, since $|F_1|_\omega \geqslant |a_{1,0}|_\omega$ and $|F_2|_\omega \geqslant |a_{2,0}|_\omega$, one has (2.17). $\square$

If we set

$$\begin{cases}\Delta = \{\nu(F) \mid F \in K[\boldsymbol{T}_I]\setminus\{0\},\ h_{S_L}(F) \leqslant C \text{ and } \deg(F) \leqslant C/\lambda\ \},\\ \Delta' = \{a \in K(\boldsymbol{T}_{I'}) \mid h_{S^\lambda_{L'}}(a,1) \leqslant C\},\end{cases}$$

then, by Proposition 2.7.13 together with the hypothesis of induction, Δ and Δ' are finite. Moreover, by Claim 2.7.15, if $h_{S^\lambda_L}(f,1) \leqslant C$, then

$$\nu(F_1), \nu(F_2) \in \Delta \quad \text{and} \quad a_{1,0}/a_{2,0} \in \Delta'.$$

Thus the assertion follows because $f = (a_{1,0}/a_{2,0})(\nu(F_1)/\nu(F_2))$. $\square$

2.7.16. Remark. — (1) Note that $h_{S^\lambda_L}(1, T_n) = \lambda$ for all $n \in I$, so that Northcott's property does not hold for S^λ_L if I is infinite.

(2) Let $S_{\mathbb{Q}}$ be the standard adelic structure of $\mathbb{Q}$. Then, it is easy to see that

$$h_{(S_{\mathbb{Q}})^*_{\mathbb{Q}(T)}}(1, T^n - 1) = \int_0^1 \ln\max\{1, |e(nt) - 1|\}\,\mathrm{d}t \leqslant \ln 2$$

for all $n \geqslant 0$, so that the Northcott's property does not hold for $S^*_{\mathbb{Q}(T)}$.

2.7.17. Theorem. — *We use the same notation as in Section 2.6. We assume that S has Northcott's propery and $\lambda > 0$. Let E be an algebraic closure of $L = K(\boldsymbol{T}_I)$.*

If E_0 is a subfield of E such that E_0 is finitely generated over K, then $S_L^\lambda \otimes_L E$ has Northcott's property over E_0, that is,

$$\Big\{a \in E \mid h_{S_L^\lambda \otimes E}(1,a) \leqslant C \text{ and } [E_0(a):E_0] \leqslant \delta\Big\}$$

is finite for any $C \in \mathbb{R}_{\geqslant 0}$ and $\delta \in \mathbb{Z}_{\geqslant 1}$.

Proof. — Since E_0 is finitely generated over K and E is algebraic over L, we can choose a finite subset I' of I such that $E_0(\boldsymbol{T}_{I'})$ is finite over $K(\boldsymbol{T}_{I'})$. It is sufficient to see that the set

$$\Big\{\alpha \in E \mid h_{S_L^\lambda \otimes E}(1,\alpha) \leqslant C \text{ and } [K(\boldsymbol{T}_{I'})(\alpha):K(\boldsymbol{T}_{I'})] \leqslant \delta\Big\} \tag{2.18}$$

is finite for any $C \in \mathbb{R}_{\geqslant 0}$ and $\delta \in \mathbb{Z}_{\geqslant 1}$. Indeed, note that

$$\begin{aligned}[K(\boldsymbol{T}_{I'})(\alpha):K(\boldsymbol{T}_{I'})] &\leqslant [E_0(\boldsymbol{T}_{I'})(\alpha):K(\boldsymbol{T}_{I'})]\\ &= [E_0(\boldsymbol{T}_{I'})(\alpha):E_0(\boldsymbol{T}_{I'})][E_0(\boldsymbol{T}_{I'}):K(\boldsymbol{T}_{I'})]\\ &\leqslant [E_0(\alpha):E_0][E_0(\boldsymbol{T}_{I'}):K(\boldsymbol{T}_{I'})],\end{aligned}$$

so that

$$\begin{aligned}&\Big\{a \in E \mid h_{S_L^\lambda \otimes E}(1,a) \leqslant C \text{ and } [E_0(a):E_0] \leqslant \delta\Big\}\\ &\qquad\subseteq \Big\{\alpha \in E \mid h_{S_L^\lambda \otimes E}(1,\alpha) \leqslant C \text{ and}\\ &\qquad\qquad [K(\boldsymbol{T}_{I'})(\alpha):K(\boldsymbol{T}_{I'})] \leqslant \delta[E_0(\boldsymbol{T}_{I'}):K(\boldsymbol{T}_{I'})]\Big\}.\end{aligned}$$

Let α be an element of the set (2.18). Let $f(t)$ be the minimal polynomial of α over $K(\boldsymbol{T}_{I'})$. As $K(\boldsymbol{T}_I)$ is a regular extension over $K(\boldsymbol{T}_{I'})$,

$$K(\boldsymbol{T}_I)[t]/f(t)K(\boldsymbol{T}_I)[t] \simeq (K(\boldsymbol{T}_{I'})[t]/f(t)K(\boldsymbol{T}_{I'})[t]) \otimes_{K(\boldsymbol{T}_{I'})} K(\boldsymbol{T}_I)$$

is an integral domain, so that $f(t)$ is irreducible over $K(\boldsymbol{T}_I)$, and hence $f(t)$ is also the minimal polynomial of α over $K(\boldsymbol{T}_I)$. We set

$$f = t^d + a_1 t^{d-1} + \cdots + a_d \quad (a_1,\ldots,a_d \in K(\boldsymbol{T}_{I'})).$$

Then, in the same arguments as [**13**, Theorem 3.5.3], one has

$$h_{S_L^\lambda}(1,a_1,\ldots,a_d) \leqslant \delta C + (\delta-1)\ln(2)\nu(\Omega_\infty),$$

so that $h_{S^\lambda_{K(\boldsymbol{T}_{I'})}}(1,a_1,\ldots,a_d) \leqslant \delta C + (\delta-1)\ln(2)\nu(\Omega_\infty)$. Therefore, the assertion is a consequence of Proposition 2.7.14. □

2.7.18. Theorem. — *If E is a countable field of characteristic zero, then E has an arithmetic adelic structure (see Definition 2.7.1).*

Proof. — We denote by S the standard adelic curve with $\mathbb{Q}$ as underlying field. Recall that the measure space of S is given by the set of all places of $\mathbb{Q}$ equipped with the discrete σ-algebra and the counting measure. Let $\{x_n\}_{n=1}^N$ be a transcendental basis of E over $\mathbb{Q}$. Note that N might be $+\infty$. Moreover, E is algebraic over $L :=$

$\mathbb{Q}((x_n)_{n=1}^N)$. Let λ be a positive number. Starting from the adelic curve S, by the way in Subsetion 2.6, let S_L^λ be the λ-twisted compactification of S_L. We claim that the adelic curve $S_L^\lambda \otimes_L E$ satisfies the properties (1) – (4) characterizing an arithmetic adelic curve. The property (1) follows from Proposition 2.7.10 and [**13**, Proposition 3.4.10]. The property (2) is obvious. For (3), see Lemma 2.2.1 and Remark 2.7.9. Finally the property (4) follows from Theorem 2.7.17. □

2.7.1. Density of Fermat property over arithmetic function fields. — In this subsection, let us consider a simple application of Theorem 2.7.18 together with Faltings' theorem [**21**]. Let K be a field. We denote by $\mu(K)$ the subgroup of $K^\times$ consisting of roots of unity in K, that is,

$$\mu(K) := \{a \in K \mid a^n = 1 \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

Let N be a positive integer and let $F_N := \operatorname{Spec}(\mathbb{Z}[X,Y]/(X^N + Y^N - 1))$. We say that F_N *has Fermat's property over* K if $x, y \in \mu(K) \cup \{0\}$ for all $(x, y) \in F_N(K)$. Then one has the following theorem.

2.7.19. Theorem. — *If K is an arithmetic function field, then*

$$\lim_{m\to\infty} \frac{\#\{N \in \mathbb{Z} \mid 1 \leqslant N \leqslant m \text{ and } F_N \text{ has Fermat's property over } K\}}{m} = 1.$$

Proof. — Let S be a proper adelic structure of K with Northcott's property (cf. Theorem 2.7.18). Let us begin with the following claim:

2.7.20. Claim. — (1) *For $x, y \in K$, $h_S(x, y, 1) = 0$ if and only if $x, y \in \mu(K) \cup \{0\}$.*

(2) *If $N \geq 4$, then there is a positive integer m_0 such that F_{Nm} has Fermat's property of every integer $m \geq m_0$.*

Proof. — (1) We assume that $h_S(x, y, 1) = 0$ for $x, y \in K$. Then $h_S(x^n, y^n, 1) = nh_S(x, y, 1) = 0$ for all $n \in \mathbb{Z}_{>0}$, so that, by Northcott's property,

$$\{(x^n, y^n) \mid n \in \mathbb{Z}_{>0}\}$$

is finite. Therefore, there are $n, n' \in \mathbb{Z}_{>0}$ such that $n > n'$ and $(x^n, y^n) = (x^{n'}, y^{n'})$, and hence $x, y \in \mu(K) \cup \{0\}$. The converse is obvious.

(2) First of all, note that $F_N(K)$ is finite by Faltings' theorem [**21**]. We set

$$\begin{cases} H := \max\{h_S(x, y, 1) \mid (x, y) \in F_N(K)\}, \\ a := \inf\{h_S(x, y, 1) \mid x, y \in K \text{ and } h_S(x, y, 1) > 0\}. \end{cases}$$

Note that $a > 0$ by Northcott's property. For a positive integer m with $m \geqslant \exp(H/a)$, we assume that $h_S(x, y, 1) > 0$ for some $(x, y) \in F_{Nm}(K)$. Then, as $(x^m, y^m) \in F_N(K)$,

$$H \geqslant h_S(x^m, y^m, 1) = mh_S(x, y, 1) \geqslant ma,$$

so that $\exp(H/a) \geqslant \exp(m)$, and hence $m \geqslant \exp(m)$. This is a contradiction. Therefore, $h_S(x, y, 1) = 0$ for all $(x, y) \in F_{Nm}(K)$. Thus, by (1), F_{Nm} has Fermat's property. □

By (2) together with [**40**, Lemma 5.16], one can conclude the assertion of the theorem. □

In the case where $K = \mathbb{Q}$, it was proved by [**22, 29, 39**] (cf. [**59**]). A general number field case is treated in [**40**]. The above theorem gives an evidence of the following conjecture:

2.7.21. Conjecture **(Fermat's conjecture over an arithmetic function field)** *Let K be an arithmetic function field. Then is there a positive integer N_0 depending on K such that F_N has Fermat's property over K for all $N \geq N_0$?*

2.8. Polarized adelic structure

In this subsection, we recall an adelic structure induced by a polarization of a field. Let K be a finitely generated field over $\mathbb{Q}$ and n be the transcendental degree of K over $\mathbb{Q}$. Let $\mathscr{B} \to \operatorname{Spec}\mathbb{Z}$ be a normal projective arithmetic variety such that the function field of $\mathscr{B}$ is K. Note that $\dim \mathscr{B} = n + 1$. Let

$$\left(\mathscr{B}; \overline{\mathscr{H}}_1 = (\mathscr{H}_1, h_1), \ldots, \overline{\mathscr{H}}_n = (\mathscr{H}_n, h_n)\right)$$

be data with the following properties:

(1) $\mathscr{H}_1, \ldots, \mathscr{H}_n$ are invertible $\mathcal{O}_{\mathscr{B}}$-modules that are nef along all fibers of $\mathscr{B} \to \operatorname{Spec}(\mathbb{Z})$.
(2) The second entries $h_1, \ldots, h_n$ are semipositive metrics of $\mathscr{H}_1, \ldots, \mathscr{H}_n$ on $\mathscr{B}(\mathbb{C})$, respectively.
(3) For each $i = 1, \ldots, n$, the associated height function with $\overline{\mathscr{H}}_i$ is non-negative

According to [**48**], the data $(\mathscr{B}; \overline{\mathscr{H}}_1, \ldots, \overline{\mathscr{H}}_n)$ is called a *polarization of* K.

Let x be a $\mathbb{C}$-valued point of $\mathscr{B}$, that is, there are a unique scheme point $p_x \in \mathscr{B}$ and a unique homomorphism $\phi_x : \mathcal{O}_{\mathscr{B},p_x} \to \mathbb{C}$ such that x is given by ϕ_x. We say x is *generic* if p_x is the generic point of $\mathscr{B}$. We denote the set of all generic $\mathbb{C}$-valued points by $\mathscr{B}(\mathbb{C})_{\mathrm{gen}}$. Note that the measure of $\mathscr{B}(\mathbb{C}) \setminus \mathscr{B}(\mathbb{C})_{\mathrm{gen}}$ is zero.

The polarization $(\mathscr{B}; \overline{\mathscr{H}}_1, \ldots, \overline{\mathscr{H}}_n)$ yields a proper adelic structure of K in the following way. First of all, we set

$$\begin{cases} \Omega_\infty := \mathscr{B}(\mathbb{C})_{\mathrm{gen}}, \\ \Omega \setminus \Omega_\infty := \text{the set of all prime divisors on } \mathscr{B}. \end{cases}$$

For each element of $\omega \in \Omega$, $|\cdot|_\omega$ is give by

$$\begin{cases} |f|_x := |\phi_x(f)| & \text{if } x \in \Omega_\infty, \\ |f|_\Gamma := \exp\left(-\operatorname{ord}_\Gamma(f)\right) & \text{if } \Gamma \in \Omega \setminus \Omega_\infty \end{cases}$$

for $f \in K$. Note that Ω_∞ is a measurable subset of a projective space, so that one can give the standard measurable space structure and a measure on Ω_∞ is given by $c_1(\overline{\mathscr{H}}_1) \wedge \cdots \wedge c_1(\overline{\mathscr{H}}_n)$. The measurable space structure on $\Omega \setminus \Omega_\infty$ is discrete and a measure ν on $\Omega \setminus \Omega_\infty$ is given by $\nu(\{\Gamma\}) = (\overline{\mathscr{H}}_1 \cdots \overline{\mathscr{H}}_n \cdot (\Gamma, 0))$. This adelic structure is called the *polarized adelic structure by the polarization* $(\mathscr{B}; \overline{\mathscr{H}}_1, \ldots, \overline{\mathscr{H}}_n)$.

2.8.1. Example. — Let h be the metric of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ on $\mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj}(\mathbb{C}[T_0, T_1])$ given by

$$|aT_0 + bT_1|_h(\zeta_0, \zeta_1) := \frac{|a\zeta_0 + b\zeta_1|}{\max\{|\zeta_0|, |\zeta_1|\}}.$$

Then $(\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1), h)$ gives rise to a semipositive metrized invertible $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}$-module, so that

$$\Big((\mathbb{P}^1_{\mathbb{Z}})^n; p_1^*(\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1), h), \ldots, p_n^*(\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1), h)\Big)$$

yields to an adelic structure of the purely transcendental extension $\mathbb{Q}(x_1, \ldots, x_n)$ over $\mathbb{Q}$, where $p_i : (\mathbb{P}^1_{\mathbb{Z}})^n \to \mathbb{P}^1_{\mathbb{Z}}$ is the projection to the i-th factor. Note that it is nothing more than the adelic structure described in Section 2.6 and Section 2.7.

CHAPTER 3

LOCAL INTERSECTION NUMBER AND LOCAL HEIGHT

In this chapter, we fix a field k equipped with an absolute value $|\cdot|$, such that k is complete under the topology induced by the absolute value $|\cdot|$. In the case where $|\cdot|$ is Archimedean, k is equal to $\mathbb{R}$ or $\mathbb{C}$. In this case we always assume that $|\cdot|$ is the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$. Note that the absolute value $|\cdot|$ extends in a unique way to any algebraic extension of k (see **[52]** Chapter II, Theorem 6.2). In particular, we fix an algebraic closure k^{ac}, on which the absolute value $|\cdot|$ extends in a unique way. Throughout this chapter, we denote the pair $(k, |\cdot|)$ by v. In the case where $|\cdot|$ is non-Archimedean, we denote by $\mathfrak{o}_v$ the valuation ring of $v = (k, |\cdot|)$, and by $\mathfrak{m}_v$ the maximal ideal of $\mathfrak{o}_v$.

3.1. Reminder on completion of an algebraic closure

We denote by $\mathbb{C}_k$ the completion of an algebraic closure k^{ac} of k, on which the absolute value $|\cdot|$ extends by continuity. Recall that $\mathbb{C}_k$ is algebraically closed. A proof for the case where $k = \mathbb{Q}_p$ can for example be found in [**53**, (10.3.2)], by using Krasner's lemma. The positive characteristic case is quite similar, but a supplementary argument is needed to show that there is no inseparable algebraic extension of $\mathbb{C}_k$. For the convenience of the readers, we include the proof here (see also [**61**, Theorem 17.1] for another proof).

3.1.1. Lemma. — *Let K be a field equipped with an absolute value $|\cdot|$ and $\widehat{K}$ be the completion of K. If the field K is perfect, then also is $\widehat{K}$.*

Proof. — Clearly it suffices to treat the case where the characteristic of K is $p > 0$. To prove that the completed field $\widehat{K}$ is perfect, we need to show that any element a of $\widehat{K}$ has a p-th root in $\widehat{K}$. We choose a sequence $(a_n)_{n\in\mathbb{N}}$ of elements of K which converges to a. Since K is supposed to be perfect, for each $n \in \mathbb{N}$ we can choose

$b_n \in K$ such that $b_n^p = a_n$. For any $(n,m) \in \mathbb{N}^2$ one has

$$|b_n - b_m|^p = |(b_n - b_m)^p| = |b_n^p - b_m^p| = |a_n - a_m|.$$

Hence $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in K, which converges to an element $b \in \widehat{K}$. Therefore

$$b^p = \lim_{n\to+\infty} b_n^p = \lim_{n\to+\infty} a_n = a,$$

as required. □

3.1.2. Proposition. — *The field $\mathbb{C}_k$ is algebraically closed.*

Proof. — It suffices to treat the case where the absolute value $|\cdot|$ is non-Archimedean. We begin with proving that the field $\mathbb{C}_k$ is separably closed. Let $\mathbb{C}_k^s$ be a separable closure of $\mathbb{C}_k$, on which $|\cdot|$ extends in a unique way. Let α be a non-zero element of $\mathbb{C}_k^s$ and

$$f(T) = T^r + a_1T^{r-1} + \cdots + a_r \in \mathbb{C}_k[T]$$

be the minimal polynomial of α. Assume that $r \geqslant 2$. Let $\alpha_2, \ldots, \alpha_r$ be conjugates of α in $\mathbb{C}_k^s$ which are different from α, and let

$$\varepsilon = \min_{j\in\{2,\ldots,r\}} |\alpha - \alpha_j|.$$

Since k^{ac} is dense in $\mathbb{C}_k$, there exists a polynomial

$$g(T) = T^r + b_1T^{r-1} + \cdots + b_r \in k^{\mathrm{ac}}[T]$$

such that

$$\max_{i\in\{1,\ldots,r\}} |\alpha|^{r-i}|b_i - a_i| < \varepsilon^r.$$

Since k^{ac} is algebraically closed, there exist elements $\beta_1, \ldots, \beta_r$ such that

$$g(T) = (T - \beta_1)\cdots(T - \beta_r).$$

One has

$$\prod_{i=1}^r |\alpha - \beta_i| = |g(\alpha)| = |g(\alpha) - f(\alpha)| \leqslant \max_{i\in\{1,\ldots,r\}} |\alpha|^{r-i}|b_i - a_i| < \varepsilon^r.$$

Hence there exists $\beta \in \{\beta_1, \ldots, \beta_r\}$ such that $|\alpha - \beta| < \varepsilon$. However, for any $\sigma \in \mathrm{Gal}(\mathbb{C}_k^s/\mathbb{C}_k)$, one has

$$|\alpha - \beta| = |\sigma(\alpha - \beta)| = |\sigma(\alpha) - \beta|.$$

This implies $|\alpha - \sigma(\alpha)| < \varepsilon$, which leads to a contradiction. Therefore one has $r = 1$, or equivalently, $\alpha \in \mathbb{C}_k$.

To show that $\mathbb{C}_k$ is algebraic closed, it suffices to check that $\mathbb{C}_k$ does not admit any algebraic inseparable extension, or equivalently, $\mathbb{C}_k$ is a perfect field. Note that any algebraic closed field is perfect (see [**8**, Chapitre V, §1, no.5, Proposition 5]). Hence the result follows from Lemma 3.1.1. □

3.2. Reminder on norms

Let E be a finite-dimensional vector space over k. If $\|\cdot\|$ is a norm on E, we denote by $\|\cdot\|_*$ the *dual norm* of $\|\cdot\|$ on the dual vector space $E^\vee$, which is defined as follows:

$$\forall\, f \in E^\vee, \quad \|f\|_* = \sup_{s\in E\setminus\{0\}} \frac{|f(s)|}{\|s\|}.$$

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms of E. Let $\|\cdot\|_{1,*}$ and $\|\cdot\|_{2,*}$ be the dual norm of $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then we define $d(\|\cdot\|_1, \|\cdot\|_2)$ and $d_*(\|\cdot\|_1, \|\cdot\|_2)$ to be

$$\begin{cases} d(\|\cdot\|_1, \|\cdot\|_2) := \sup\limits_{s\in E\setminus\{0\}} \Big| \ln \|s\|_1 - \ln \|s\|_2 \Big|, \\ d_*(\|\cdot\|_1, \|\cdot\|_2) := d(\|\cdot\|_{1,*}, \|\cdot\|_{2,*}). \end{cases}$$

Note that if $\dim_k E = 1$, then $d(\|\cdot\|_1, \|\cdot\|_2) = d_*(\|\cdot\|_1, \|\cdot\|_2)$. It is easy to see that d and d_* satisfy the triangle inequality.

3.2.1. Lemma. — *Let $0 \to F \to E \to Q \to 0$ be an exact sequence of finite-dimensional vector spaces over k. Let $\|\cdot\|_{1,F}$ and $\|\cdot\|_{2,F}$ be restricted norms of $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, and $\|\cdot\|_{1,Q}$ and $\|\cdot\|_{2,Q}$ be quotient norms of $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then one has the following:*

$$\begin{cases} d(\|\cdot\|_{1,F}, \|\cdot\|_{2,F}) \leqslant d(\|\cdot\|_1, \|\cdot\|_2), \ d(\|\cdot\|_{1,Q}, \|\cdot\|_{2,Q}) \leqslant d(\|\cdot\|_1, \|\cdot\|_2), \\ d_*(\|\cdot\|_{1,F}, \|\cdot\|_{2,F}) \leqslant d_*(\|\cdot\|_1, \|\cdot\|_2), \ d_*(\|\cdot\|_{1,Q}, \|\cdot\|_{2,Q}) \leqslant d_*(\|\cdot\|_1, \|\cdot\|_2). \end{cases} \tag{3.1}$$

Proof. — See [**13**, Proposition 1.1.42]. □

3.2.2. Lemma **(Abstract form of Fubini-Study metric)**
Let $\pi : E \to Q$ be a surjective homomorphism of finite-dimensional vector spaces over k such that $\dim_k Q = 1$. Let $\|\cdot\|_E$ be a norm on E and $\|\cdot\|_Q$ be the quotient norm of Q induced by the homomorphism $\pi : E \to Q$ and $\|\cdot\|_E$. Let $\phi \in E^\vee \setminus \{0\}$ such that $\phi|_{\operatorname{Ker}\pi} = 0$. Then, for any $s \in E$,

$$\|\pi(s)\|_Q = \frac{|\phi(s)|}{\|\phi\|_{E,*}}.$$

Proof. — Note that the dual norm $\|\cdot\|_{Q,*}$ of $Q^\vee$ is equal to the sub-norm $\|\cdot\|_{E,*,Q^\vee\hookrightarrow E^\vee}$ of $Q^\vee$ induced by the injective homomorphism $Q^\vee \to E^\vee$, $\alpha \mapsto \alpha\circ\pi$ and the dual norm $\|\cdot\|_{E,*}$ (cf. [**13**, Proposition 1.1.20]). As $\phi|_{\operatorname{Ker}\pi} = 0$, there is $\varphi \in Q \setminus \{0\}$ such that $\varphi \circ \pi = \phi$, and one has $\|\phi\|_{E,*} = \|\varphi\|_{Q,*}$. Since Q is of dimension 1 over k, for any $q \in Q$, one has

$$\|q\|_Q = \frac{|\varphi(q)|}{\|\varphi\|_{Q,*}} = \frac{|\varphi(q)|}{\|\phi\|_{E,*}}.$$

In particular, for any $s \in E$, the following equality holds:

$$\|\pi(s)\|_Q = \frac{|\varphi(\pi(s))|}{\|\phi\|_{E,*}} = \frac{|\phi(s)|}{\|\phi\|_{E,*}},$$

which concludes the lemma. □

3.3. Continuous metrics

If X is a projective k-scheme, we denote by X^{an} the analytification of X. If $k = \mathbb{C}$ and $|\cdot|$ is the usual absolute value, then X^{an} is a complex analytic space; if $|\cdot|$ is non-Archimedean, then the analytification X^{an} is defined in the sense of Berkovich (see [**4**, §4.3]). Recall that any element x of X^{an} consists of a scheme point of X and an absolute value $|\cdot|_x$ on the residue field of the scheme point, which extends the absolute value $|\cdot|$ on k. We denote by $\widehat{\kappa}(x)$ the completion of the residue field of the scheme point with respect to the absolute value $|\cdot|_x$, on which the absolute value extends by continuity. In the remaining of the section, we fix a projective k-scheme X.

3.3.1. Definition. — Let E be a locally free $\mathcal{O}_X$-module. We call *continuous metric* on E any family $\varphi = (|\cdot|_\varphi(x))_{x\in X^{\mathrm{an}}}$, where for each $x \in X^{\mathrm{an}}$, $|\cdot|_\varphi(x)$ is a norm on $E(x) := E \otimes_{\mathcal{O}_X} \widehat{\kappa}(x)$, such that, for any section s of E on a Zariski open subset U of X, the map $|s|_\varphi$ from U^{an} to $\mathbb{R}_{\geqslant 0}$ sending $(x \in U^{\mathrm{an}})$ to $|s(x)|_\varphi(x)$ is a continuous function on U^{an}. Let L be an invertible $\mathcal{O}_X$-module. If φ and ψ are continuous metrics on L, we define

$$d(\varphi,\psi) := \sup_{x\in X^{\mathrm{an}}} \left| \ln \frac{|\cdot|_\varphi(x)}{|\cdot|_\psi(x)} \right|,$$

where

$$\frac{|\cdot|_\varphi(x)}{|\cdot|_\psi(x)} := \frac{|\ell|_\varphi(x)}{|\ell|_\psi(x)} \quad \text{for any } \ell \in L(x) \setminus \{0\}.$$

3.3.2. Example. — (1) Let L be an invertible $\mathcal{O}_X$-module and n be a positive integer. Let $(E, \|\cdot\|)$ be a finite-dimensional normed vector space over k. We assume that $p : E \otimes_k \mathcal{O}_X \to L^{\otimes n}$ is a surjective homomorphism of $\mathcal{O}_X$-modules, which induces a k-morphism $f : X \to \mathbb{P}(E)$ such that $L^{\otimes n}$ is isomorphic to $f^*(\mathcal{O}_E(1))$, where $\mathcal{O}_E(1)$ denotes the universal invertible sheaf on the projective space $\mathbb{P}(E)$ (see [**33**, II.(4.2.3)]). For each point $x \in X^{\mathrm{an}}$ the norm $\|\cdot\|$ induces a quotient norm $|\cdot|(x)$ on $L(x)$ such that, for any $\ell \in L(x) \setminus \{0\}$,

$$|\ell|(x) = \inf_{\substack{s\in E,\, \lambda\in\widehat{\kappa}(x)^\times \\ p(s)(x)=\lambda\ell^{\otimes n}}} \left(|\lambda|_x^{-1}\|s\|\right)^{1/n}.$$

The quotient norms $(|\cdot|(x))_{x\in X^{\mathrm{an}}}$ define a continuous metric on L, called the *quotient metric induced by* $\|\cdot\|$. By definition, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on

E, and if φ_1 and φ_2 are quotient metrics induced by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, then one has

$$d(\varphi_1,\varphi_2)\leqslant d(\|\cdot\|_1,\|\cdot\|_2). \tag{3.2}$$

(2) Let L be an invertible $\mathcal{O}_X$-module and $\varphi=(|\cdot|_\varphi(x))_{x\in X^{\mathrm{an}}}$ be a continuous metric on L. The dual norms of $|\cdot|_\varphi(x)$ on $L(x)^\vee$ form a continuous metric on $L^\vee$, which we denote by $-\varphi$. Recall that for any $\ell\in L(x)\setminus\{0\}$, one has

$$|\ell^\vee|_{-\varphi}=|\ell|_\varphi^{-1},$$

where $\ell^\vee$ denotes the linear form on $L(x)$ such that $\ell^\vee(\lambda\ell)=\lambda$ for any $\lambda\in\widehat{\kappa}(x)$.

(3) Let L_1 and L_2 be invertible $\mathcal{O}_X$-modules, and φ_1 and φ_2 be continuous metrics on L_1 and L_2 respectively. Then the tensor product norms of $|\cdot|_{\varphi_1}(x)$ and $|\cdot|_{\varphi_2}(x)$ form a continuous metric on $L_1\otimes L_2$, which we denote by $\varphi_1+\varphi_2$. Note that, for any $\ell_1\in L_1(x)$ and $\ell_2\in L_2(x)$, one has

$$|\ell_1\otimes\ell_2|_{\varphi_1+\varphi_2}(x)=|\ell_1|_{\varphi_1}(x)\cdot|\ell_2|_{\varphi_2}(x).$$

(4) Let $f:Y\to X$ be a k-morphism of projective k-schemes. We denote by $f^{\mathrm{an}}:Y^{\mathrm{an}}\to X^{\mathrm{an}}$ the continuous map of analytifications induced by f. Let L be an invertible $\mathcal{O}_X$-module, equipped with a continuous metric φ. Then the metric φ induces by pull-back a continuous metric $f^*(\varphi)$ on $f^*(L)$ such that, for any $y\in Y^{\mathrm{an}}$ and any $\ell\in L(f^{\mathrm{an}}(y))$, one has

$$|f^*(\ell)|_{f^*(\varphi)}(y)=|\ell|_\varphi(f^{\mathrm{an}}(y)).$$

The metric $f^*(\varphi)$ is called the *pull-back* of φ by f.

(5) Let k'/k be an extension of fields. We assume that the absolute value $|\cdot|$ extends to k' and that the field k' is complete with respect to the topology induced by the extended absolute value. Let $X_{k'}$ be the fiber product $X\times_{\operatorname{Spec}k}\operatorname{Spec}k'$. We denote by $\pi:X_{k'}\to X$ the morphism of projection. Then the map

$$\pi^\natural:X_{k'}^{\mathrm{an}}\longrightarrow X^{\mathrm{an}}, \tag{3.3}$$

sending any point $x'=(j(x'),|\cdot|_{x'})\in X_{k'}^{\mathrm{an}}$ to the pair consisting of the scheme point $\pi(j(x'))$ of X and the restriction of $|\cdot|_{x'}$ to the residue field of $\pi(j(x'))$, is continuous (see [**13**, Proposition 2.1.17]), where $j:X_{k'}^{\mathrm{an}}\to X_{k'}$ denotes the map sending a point in the analytic space to its underlying scheme point.

Let L be an invertible $\mathcal{O}_X$-module, equipped with a continuous metric φ. Let $L_{k'}$ be the pull-back of L by the morphism of projection π. The continuous metric φ induces a continuous metric $\varphi_{k'}$ on $L_{k'}$ such that, for any $x'\in X_{k'}^{\mathrm{an}}$ and any $\ell\in L(\pi^\natural(x'))$, one has

$$\forall\,a\in\widehat{\kappa}(x'),\quad |a\otimes\ell|_{\varphi_{k'}}(x')=|a|_{x'}\cdot|\ell|_\varphi(\pi^\natural(x')).$$

In particular, if ψ is another continuous metric on L, then one has

$$d(\varphi_{k'},\psi_{k'})\leqslant d(\varphi,\psi). \tag{3.4}$$

3.3.3. Definition. — Let $(E,\|\cdot\|)$ be a finite-dimensional normed vector space over k. We assume that the norm $\|\cdot\|$ is either ultrametric or induced by an inner product. Let k'/k be an extension of fields, on which the absolute value $|\cdot|$ extends. We assume that the field k' is complete with respect to the extended absolute value. We denote by $\|\cdot\|_{k'}$ the following norm on $E_{k'}:=E\otimes_k k'$.

(1) In the case where the absolute value $|\cdot|$ is non-Archimedean and the norm $\|\cdot\|$ is ultrametric, $\|\cdot\|_{k'}$ is the ε-extension of scalars of the norm $\|\cdot\|$. Namely, for any $t=s_1\otimes\lambda_1+\cdots+s_m\otimes\lambda_m\in E\otimes_k k'$
$$\|t\|_{k'}:=\sup_{f\in E^\vee\setminus\{0\}}\frac{|\lambda_1 f(s_1)+\cdots+\lambda_m f(s_m)|}{\|f\|_*},$$
where $\|\cdot\|_*$ denotes the dual norm of $\|\cdot\|$, which is defined as
$$\|f\|_*:=\sup_{x\in E\setminus\{0\}}\frac{|f(s)|}{\|s\|}.$$
This is an ultrametric norm on $E_{k'}$ such that $\|s\otimes a\|_{k'}=\|s\|\cdot|a|$ (see [**13**, Proposition 1.3.1]). Moreover, if $(e_i)_{i=1}^r$ is an orthonormal basis of $(E,\|\cdot\|)$, then $(e_i\otimes 1)_{i=1}^r$ is an orthonormal basis of $(E_{k'},\|\cdot\|_{k'})$ (see [**13**, Proposition 1.3.13]).

(2) In the case where the absolute value $|\cdot|$ is Archimedean, $k=\mathbb{R}$, $k'=\mathbb{C}$, and $\|\cdot\|$ is induced by an inner product $\langle\,,\rangle$, $\|\cdot\|_{\mathbb{C}}$ is the orthogonal extension of scalars of $\|\cdot\|$. Namely, for any $(s,t)\in E\times E$,
$$\|s\otimes 1+t\otimes\sqrt{-1}\|_{\mathbb{C}}:=(\|s\|^2+\|t\|^2)^{1/2}.$$
Clearly, for any $s\in E$ one has $\|s\otimes 1\|_{\mathbb{C}}=\|s\|$. Note that the norm $\|\cdot\|_{\mathbb{C}}$ is induced by an inner product $\langle\,,\rangle_{\mathbb{C}}$ on $E_{\mathbb{C}}$ such that, for any $u=s\otimes 1+t\otimes\sqrt{-1}$ and $u'=s'\otimes 1+t'\otimes\sqrt{-1}$ in $E_{\mathbb{C}}$,
$$\langle u,u'\rangle=\langle s,s'\rangle+\langle t,t'\rangle+\sqrt{-1}(\langle s,t'\rangle-\langle t,s'\rangle).$$
Moreover, if $(e_i)_{i=1}^r$ is an orthonormal basis of $(E,\|\cdot\|)$, then $(e_i\otimes 1)_{i=1}^r$ is an orthonormal basis of $(E_{\mathbb{C}},\|\cdot\|_{\mathbb{C}})$.

3.3.4. Remark. — Let n be a positive integer. Assume that $p:E\otimes_k\mathcal{O}_X\to L^{\otimes n}$ is a surjective homomorphism of $\mathcal{O}_X$-modules, which induces a k-morphism $f:X\to\mathbb{P}(E)$ such that $L^{\otimes n}=f^*(\mathcal{O}_E(1))$. We equip L with the quotient metric φ induced by $\|\cdot\|$. In the case where the absolute value $|\cdot|$ is non-Archimedean, for any point $x\in X^{\mathrm{an}}$, the norm $|\cdot|_{n\varphi}(x)$ on $L^{\otimes n}(x)$ coincides with the quotient norm on $L^{\otimes n}(x)$ induced by the norm $\|\cdot\|_{\widehat{\kappa}(x)}$ on $E\otimes_k\widehat{\kappa}(x)$ and the quotient map $p_x:E\otimes_k\widehat{\kappa}(x)\to L^{\otimes n}$. We refer the readers to [**13**, Proposition 1.3.26 (i)] for a proof. As for the Archimedean case with $k=\mathbb{R}$ and $\widehat{\kappa}(x)=\mathbb{C}$, note that, if s and t are elements of E and a and b are complex numbers such that
$$p_x(s)=a\ell^{\otimes n},\quad p_x(t)=b\ell^{\otimes n},$$

where ℓ is a fixed non-zero element of $L(x)$. Then one has

$$p_x(s \otimes 1 + t \otimes \sqrt{-1}) = (a + b\sqrt{-1})\ell^{\otimes n}$$

and hence

$$\frac{(\|s\|^2 + \|t\|^2)^{\frac{1}{2}}}{|a + b\sqrt{-1}|} \geqslant \frac{(\|s\|^2 + \|t\|^2)^{\frac{1}{2}}}{|a| + |b|} \geqslant \frac{1}{\sqrt{2}} \frac{\|s\| + \|t\|}{|a| + |b|} \geqslant \frac{1}{\sqrt{2}} |\ell|_{n\varphi}(x).$$

Therefore, the quotient norm on $L^{\otimes n}$ induced by $\|\cdot\|_{\widehat{\kappa}(x)}$ and the quotient map

$$p_x : E \otimes_k \widehat{\kappa}(x) \longrightarrow L^{\otimes n}(x),$$

which is bounded from above by $|\cdot|_{n\varphi}(x)$ by definition, is actually bounded from below by $(1/\sqrt{2})|\cdot|_{n\varphi}(x)$.

Let k'/k be a valued extension of $(k, |\cdot|)$ which is complete. By extension of scalars, we obtain a surjective homomorphism of $\mathcal{O}_{X_{k'}}$-modules

$$p_{k'} : E_{k'} \otimes_{k'} \mathcal{O}_{X_{k'}} \longrightarrow L_{k'}^{\otimes n},$$

which corresponds to the k'-morphism $f_{k'} : X_{k'} \to \mathbb{P}(E_{k'})$. Let φ be the quotient metric on L induced by $\|\cdot\|$. In the case where $|\cdot|$ is non-Archimedean, it turns out that the quotient metric on $L_{k'}$ induced by $\|\cdot\|_{k'}$ coincides with $\varphi_{k'}$. This fact follows from [**13**, Proposition 1.3.15 (i)] and the above identification of the quotient metric to a family of quotient norms. In the Archimedean case with $k = \mathbb{R}$ and $k' = \mathbb{C}$, by the above estimate, in general the quotient metric φ' on $L_{\mathbb{C}}$ induced by $\|\cdot\|_{\mathbb{C}}$ is different from $\varphi_{\mathbb{C}}$. The above estimate actually shows that, for any $x \in X_{\mathbb{C}}^{\mathrm{an}}$ one has

$$2^{-\frac{1}{2n}} |\cdot|_{\varphi_{\mathbb{C}}}(x) \leqslant |\cdot|_{\varphi'}(x) \leqslant |\cdot|_{\varphi_{\mathbb{C}}}(x).$$

Note that the metric $\varphi_{\mathbb{C}}$ is still a quotient metric. In fact, if we consider the π-extension of scalars $\|\cdot\|_{\mathbb{C},\pi}$ on $E_{\mathbb{C}}$ defined as

$$\forall\, t \in E_{\mathbb{C}}, \quad \|t\|_{\mathbb{C},\pi} := \inf_{t = s_1 \otimes \lambda_1 + \cdots + s_m \otimes \lambda_m} \sum_{i=1}^{m} |\lambda_i| \cdot \|s_i\|.$$

Then the metric $\varphi_{\mathbb{C}}$ identifies with the quotient metric induced by $\|\cdot\|_{\mathbb{C},\pi}$.

3.3.5. Definition. — Let L be an invertible $\mathcal{O}_X$-module and n be a positive integer. Let $(E, \|\cdot\|)$ be a finite-dimensional normed vector space over k. We assume that the norm $\|\cdot\|$ is either ultrametric or induced by an inner product. Let $p : E \otimes_k \mathcal{O}_X \to L^{\otimes n}$ be a surjective homomorphism of $\mathcal{O}_X$-modules, which induces a k-morphism $f : X \to \mathbb{P}(E)$ such that $L^{\otimes n}$ is isomorphic to $f^*(\mathcal{O}_E(1))$. For each point $x \in X^{\mathrm{an}}$, the norm $\|\cdot\|_{\widehat{\kappa}(x)}$ on $E \otimes_k \widehat{\kappa}(x)$ induces by quotient a norm $|\cdot|(x)$ on $L^{\otimes n}(x)$. There then exists a unique continuous metric φ on L such that $|\cdot|_{n\varphi}(x) = |\cdot|(x)$ for any $x \in X^{\mathrm{an}}$. The metric φ is called the *orthogonal quotient metric induced by* $\|\cdot\|$. Note that, in the case where $|\cdot|$ is non-Archimedean or $(k, |\cdot|)$ is $\mathbb{C}$ equipped with the usual absolute value, the orthogonal quotient metric identifies with the quotient metric induced by $\|\cdot\|$ introduced in Example 3.3.2 (1). Moreover, for any complete valued extension

k'/k, the metric $\varphi_{k'}$ identifies with the orthogonal quotient metric induced by $\|\cdot\|_{k'}$ (see Remark 3.3.4 above).

3.3.6. Definition. — Let L be a semi-ample invertible $\mathcal{O}_X$-module and φ be a continuous metric on L. If there exists a sequence of quotient metrics φ_n on L such that

$$\lim_{n\to+\infty} d(\varphi_n, \varphi) = 0,$$

we say that the metric φ is *semi-positive* (see [**12**, §2.2]). In the case where $|\cdot|$ is Archimedean and $k = \mathbb{C}$, this definition is equivalent to the plurisubharmonicity of the metric φ (see for example [**67**, Theorem 3.5]).

3.3.7. Remark. — Let L be an invertible $\mathcal{O}_X$-module. Let k'/k be a complete valued extension of k, $X_{k'}$ be the fiber product $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ and $\pi^\natural : X_{k'}^{\mathrm{an}} \to X^{\mathrm{an}}$ be the map defined in (3.3). If φ and ψ are two continuous metrics on L, then the metrics $\varphi_{k'}$ and $\psi_{k'}$ satisfy the relation (see (3.4))

$$d(\varphi_{k'}, \psi_{k'}) \leqslant d(\varphi, \psi).$$

Therefore, if φ is a semi-positive metric on L, then $\varphi_{k'}$ is also a semi-positive metric.

3.3.8. Definition. — Let L be a very ample invertible $\mathcal{O}_X$-module and φ be a continuous metric on L. For any positive integer m, the continuous metric φ induces a seminorm $\|\cdot\|_{m\varphi}$ on $H^0(X, L^{\otimes m})$ as follows:

$$\forall\, s \in H^0(X, L^{\otimes m}), \quad \|s\|_{m\varphi} = \sup_{x\in X^{\mathrm{an}}} |s|_{m\varphi}(x).$$

This seminorm is a norm notably when the scheme X is reduced. For each point $x \in X^{\mathrm{an}}$, the seminorm $\|\cdot\|_{m\varphi}$ induces a quotient seminorm $|\cdot|_{\varphi^{(m)}}(x)$ on $L(x)$ such that, for any $\ell \in L(x) \setminus \{0\}$

$$|\ell|_{\varphi^{(m)}}(x) = \inf_{\substack{s\in H^0(X,L^{\otimes m}),\, \lambda\in\widehat{\kappa}(x)^\times \\ s(x)=\lambda\ell^{\otimes m}}} (|\lambda|^{-1}\|s\|_{m\varphi})^{1/m}.$$

This seminorm is actually a norm and is bounded from below by $|\cdot|_\varphi(x)$. The norms $(|\cdot|_{\varphi^{(m)}}(x))_{x\in X^{\mathrm{an}}}$ form a continuous metric on L, which we denote by $\varphi^{(m)}$.

3.3.9. Proposition. — *Let L be a very ample invertible $\mathcal{O}_X$-module. If φ_1 and φ_2 are two continuous metrics on L, then the following inequalities hold:*

$$\forall\, m \in \mathbb{N}_{\geqslant 1}, \quad d(\varphi_1^{(m)}, \varphi_2^{(m)}) \leqslant d(\varphi_1, \varphi_2).$$

Proof. — By definition, one has

$$\sup_{\substack{s\in H^0(X,L^{\otimes m}) \\ \|s\|_{m\varphi_1}\neq 0}} \left| \ln \frac{\|s\|_{m\varphi_1}}{\|s\|_{m\varphi_2}} \right| \leqslant d(m\varphi_1, m\varphi_2) = m\, d(\varphi_1, \varphi_2).$$

Therefore,

$$d(\varphi_1^{(m)}, \varphi_2^{(m)}) \leqslant \frac{1}{m} d(m\varphi_1, m\varphi_2) = d(\varphi_1, \varphi_2).$$

□

3.3.10. Remark. — Let $(E, \|\cdot\|)$ be a finite-dimensional vector space over k, m be a positive integer and $p : \pi^*(E) \to L^{\otimes m}$ be a surjective homomorphism of $\mathcal{O}_X$-modules, where $\pi : X \to \operatorname{Spec} k$ denotes the structural morphism of schemes. Let φ be the quotient metric induced by $\|\cdot\|$. Note that p induces by adjunction between π^* and π_* a k-linear map $\alpha : E \to H^0(X, L^{\otimes m})$. Let s be an element of $H^0(X, L^{\otimes m})$. For any $x \in X^{\mathrm{an}}$, one has

$$|s|_{m\varphi}(x) = \inf_{\substack{t \in E,\, \lambda \in \widehat{\kappa}(x)^\times \\ \alpha(t)(x) = \lambda s(x)}} \frac{\|t\|}{|\lambda|_x}.$$

In particular, for any s in the image of the linear map α, one has

$$\|s\|_{m\varphi} \leqslant \inf_{t \in E,\, \alpha(t) = s} \|t\|.$$

Therefore, for $x \in X^{\mathrm{an}}$ and $\ell \in L(x) \setminus \{0\}$, one has

$$|\ell|_{\varphi^{(m)}}(x) = \inf_{\substack{s \in H^0(X, L^{\otimes m}),\, \lambda \in \widehat{\kappa}(x)^\times \\ s(x) = \lambda \ell^{\otimes m}}} \Big(\frac{\|s\|_{m\varphi}}{|\lambda|_x}\Big)^{1/m} \leqslant \inf_{\substack{t \in E,\, \lambda \in \widehat{\kappa}(x)^\times \\ \alpha(t)(x) = \lambda \ell^{\otimes m}}} \Big(\frac{\|t\|}{|\lambda|_x}\Big)^{1/m} = |\ell|_\varphi(x).$$

Combining with the inequality $|\ell|_{\varphi^{(m)}}(x) \geqslant |\ell|_\varphi(x)$, we obtain the equality $\varphi^{(m)} = \varphi$.

3.3.11. Proposition. — *Let L be a very ample invertible $\mathcal{O}_X$-module, equipped with a continuous metric φ. Let $\|\cdot\|$ be a norm on the vector space $H^0(X, L^{\otimes n})$. For any $a > 0$, let $\|\cdot\|_a$ be the norm on $H^0(X, L^{\otimes n})$ defined by*

$$\forall\, s \in H^0(X, L^{\otimes n}), \quad \|s\|_a = \max\{\|s\|_\varphi, a\|s\|\} = \max\Big\{\sup_{x \in X^{\mathrm{an}}} |s|_\varphi(x), a\|s\|\Big\},$$

and let φ_a be the quotient metric on L induced by $\|\cdot\|_a$. Then, for any $x \in X^{\mathrm{an}}$

$$|\cdot|_{\varphi^{(1)}}(x) \leqslant |\cdot|_{\varphi_a}(x), \tag{3.5}$$

and there exists $a_0 > 0$ such that $\varphi_a = \varphi^{(1)}$ when $0 < a \leqslant a_0$.

Proof. — By definition, one has $\|\cdot\|_a \geqslant \|\cdot\|_\varphi$. Hence the inequality (3.5) holds.

Let $N_{\|\cdot\|_\varphi}$ be the null space of the seminorm $\|\cdot\|_\varphi$, which is defined as

$$N_{\|\cdot\|_\varphi} = \{s \in H^0(X, L) \mid \|s\|_\varphi = 0\}.$$

Let E be the quotient vector space $H^0(X, L)/N_{\|\cdot\|_\varphi}$ and $\pi : H^0(X, L) \to E$ be the projection map. We denote by $\|\cdot\|_E$ the quotient norm of $\|\cdot\|$ on E and $\|\cdot\|_{\varphi,E}$ be the quotient seminorm of $\|\cdot\|_\varphi$ on E, which is actually a norm satisfying the relation

$$\forall\, s \in H^0(X, L), \quad \|\pi(s)\|_{\varphi,E} = \|s\|_\varphi. \tag{3.6}$$

Since all norms on E are equivalent, there exists $C > 0$ such that $\|\cdot\|_E \leqslant C\|\cdot\|_{\varphi,E}$. Therefore, for any $x \in X^{\mathrm{an}}$, and any $\ell \in L(x) \setminus \{0\}$ one has

$$\begin{aligned}|\ell|_{\varphi_a}(x) &= \inf_{\substack{s\in H^0(X,L^{\otimes n}),\,\lambda\in\widehat{\kappa}(x)^\times\\ s(x)=\lambda\ell^{\otimes n}}} \Big(\frac{\max\{\|s\|_\varphi, a\|s\|\}}{|\lambda|}\Big)^{\frac{1}{n}}\\ &= \inf_{\substack{s\in H^0(X,L^{\otimes n}),\,\lambda\in\widehat{\kappa}(x)^\times\\ s(x)=\lambda\ell^{\otimes n}}} \Big(\frac{\max\{\|\pi(s)\|_{\varphi,E}, a\|\pi(s)\|_E\}}{|\lambda|}\Big) = |\ell|_{\varphi^{(1)}}(x)\end{aligned}$$

once $a < C^{-1}$, where the second equality comes from the fact that $s(x) = 0$ when $s \in N_{\|\cdot\|_\varphi}$. □

3.3.12. Proposition. — *Let L be a very ample invertible $\mathcal{O}_X$-module, equipped with a semi-positive continuous metric φ. Then one has*

$$\lim_{m\to+\infty} d(\varphi^{(m)}, \varphi) = 0.$$

Proof. — First of all, for positive integers m and m', one has

$$\forall\, x \in X^{\mathrm{an}},\ \forall\, \ell \in L(x) \setminus \{0\},\quad |\ell|^{m+m'}_{\varphi^{(m+m')}}(x) \leqslant |\ell|^m_{\varphi^{(m)}} \cdot |\ell|^{m'}_{\varphi^{(m')}}.$$

Therefore

$$(m+m')d(\varphi^{(m+m')}, \varphi) \leqslant md(\varphi^{(m)}, \varphi) + m'd(\varphi^{(m')}, \varphi).$$

By Fekete's lemma we obtain that the sequence

$$d(\varphi^{(m)}, \varphi),\quad m \in \mathbb{N},\ m \geqslant 1$$

converges to a non-negative real number, which is also equal to

$$\inf_{m\in\mathbb{N},\ m\geqslant 1} d(\varphi^{(m)}, \varphi).$$

Moreover, since the metric φ is semi-positive, there exist a sequence of positive integers $(m_n)_{n\in\mathbb{N}}$, a sequence of finite-dimensional normed vector spaces $((E_n, \|\cdot\|_n))_{n\in\mathbb{N}}$ and surjective homomorphisms of $\mathcal{O}_X$-modules $p_n : E_n \otimes_k \mathcal{O}_X \to L^{\otimes m_n}$ such that, if we denote by φ_n the quotient metric on L induced by $\|\cdot\|_n$, then one has

$$\lim_{n\to+\infty} d(\varphi_n, \varphi) = 0.$$

By Remark 3.3.10, one has $\varphi_n^{(m_n)} = \varphi_n$ and hence

$$d(\varphi^{(m_n)}, \varphi) \leqslant d(\varphi^{(m_n)}, \varphi_n) + d(\varphi_n, \varphi) = d(\varphi^{(m_n)}, \varphi_n^{(m_n)}) + d(\varphi_n, \varphi) \leqslant 2d(\varphi_n, \varphi),$$

where the last inequality comes from Proposition 3.3.9. By taking the limite when $n \to +\infty$, we obtain that

$$\inf_{m\in\mathbb{N},\, m\geqslant 1} d(\varphi^{(m)}, \varphi) = 0.$$

□

3.3.13. Definition. — Let (L, φ) be a metrized invertible $\mathcal{O}_X$-module. We say that (L, φ) is *integrable* if there exist ample invertible $\mathcal{O}_X$-modules L_1 and L_2 equipped with semi-positive metrics φ_1 and φ_2 respectively, such that $L = L_1 \otimes L_2^\vee$ and $\varphi = \varphi_1 - \varphi_2$.

3.3.14. Definition. — We assume that v is non-Archimedean. Let (L, φ) be a metrized invertible $\mathcal{O}_X$-module. We say φ is a *model metric* if there are a positive integer n and a model $(\mathscr{X}, \mathscr{L})$ of $(X, L^{\otimes n})$ such that $n\varphi$ coincides with the metric arising from the model $(\mathscr{X}, \mathscr{L})$ (cf. [**13**, Subsection 2.3.2]). In the above definition, we may assume that $\mathscr{X}$ is flat over $\mathfrak{o}_v$ (for details, see [**13**, Subsection 2.3.2]). In the case where L is nef, if $\mathscr{L}$ is nef along the special fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_v)$, then the model $(\mathscr{X}, \mathscr{L})$ is said to be *nef* and φ is called a *nef model metric*.

3.3.15. Remark. — Let $(\mathscr{X}, \mathscr{L})$ be a model of (X, L), $\mathscr{X}_{\mathrm{red}}$ be the reduced scheme associated with $\mathscr{X}$ and $\mathscr{L}_{\mathrm{red}} := \mathscr{L}|_{X_{\mathrm{red}}}$. For $x \in X^{\mathrm{an}}$, the morphism $\mathrm{Spec}(\mathfrak{o}_x) \to \mathscr{X}$ factors through $\mathrm{Spec}(\mathfrak{o}_x) \to \mathscr{X}_{\mathrm{red}} \to \mathscr{X}$, and hence $\varphi_{\mathscr{L}}$ coincides with $\varphi_{\mathscr{L}_{\mathrm{red}}}$. Moreover, $\mathscr{L}$ is nef with respect to $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_v)$ if and only if $\mathscr{L}_{\mathrm{red}}$ is nef with respect to $\mathscr{X}_{\mathrm{red}} \to \mathrm{Spec}(\mathfrak{o}_v)$.

3.3.16. Definition. — Let (L, φ) be a metrized invertible $\mathcal{O}_X$-module. We say that φ is *smooth* if one of the following conditions is satisfied:

(i) if v is Archimedean, φ is a C^∞-metric;
(ii) if v is non-Archimedean, φ is a model metric.

If L is nef and v is non-Archimedean, then φ is said to be *M-semi-positive* if there is a sequence $(\varphi_m)_{m=1}^\infty$ of nef model metrics of L such that $\lim_{m\to\infty} d(\varphi, \varphi_m) = 0$.

3.3.17. Lemma. — *We assume that v is non-Archimedean. Let L be an invertible $\mathcal{O}_X$-module and $(\mathscr{X}, \mathscr{L})$ be a model of (X, L). Then there is a model $(\mathscr{X}', \mathscr{L}')$ of (X, L) with the following properties:*

(1) $\mathscr{X}' \to \mathrm{Spec}(\mathfrak{o}_v)$ *is finitely presented, that is, $(\mathscr{X}', \mathscr{L}')$ is a coherent model of (X, L) (cf.* [**13**, Subsection 2.3.2]).
(2) $\mathscr{X}$ *is a closed subscheme of* $\mathscr{X}'$.
(3) *The special fiber of* $\mathscr{X}' \to \mathrm{Spec}(\mathfrak{o}_v)$ *coincides with the special fiber of* $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_v)$.
(4) $\mathscr{L}'|_{\mathscr{X}} = \mathscr{L}$.

Proof. — By [**38**, Corollary 5.16 in Chapter II], there are a polynomial ring $A := \mathfrak{o}_v[T_0, \ldots, T_N]$ over $\mathfrak{o}_v$ and a homogeneous ideal I of A such that $\mathscr{X} = \mathrm{Proj}(A/I)$. We set $R := A/I$. Let $p : A \to R$ and $\pi : A \to A \otimes_{\mathfrak{o}_v} (\mathfrak{o}_v/\mathfrak{m}_v) = (\mathfrak{o}_v/\mathfrak{m}_v)[T_0, \ldots, T_N]$ be the natural homomorphisms. There are homogeneous elements $h_1, \ldots, h_e$ of R and $g_{ij} \in R_{(h_i h_j)}$ ($(i,j) \in \{1, \ldots, e\}^2$) such that $\mathscr{X} = \bigcup_{i=1}^e D_+(h_i)$ and $(g_{ij})_{(i,j)\in\{1,\ldots,e\}^2}$ gives transition functions of $\mathscr{L}$, where $R_{(h)}$ (for a homogenous element h) is the homogeneous localization with respect to h. We choose a homogeneous element H_i of A such that $p(H_i) = h_i$. Since

$$\varnothing = \bigcap_{i=1}^{e} V_+(h_i) = V_+(h_1 R + \cdots + h_e R),$$

we have $R_+ \subseteq \mathrm{rad}(h_1 R + \cdots + h_e R)$ by [**44**, Lemma 3.35 in Section 2.3], that is, there is a positive integer a such that $p(T_0)^a, \ldots, p(T_N)^a \in h_1 R + \cdots + h_e R$, so that

$$T_0^a, \ldots, T_N^a \in H_1 A + \cdots + H_e A + I. \tag{3.7}$$

We also choose $G_{ij} \in A_{(H_i H_j)}$ such that $p(G_{ij}) = g_{ij}$ and $G_{ii} = 1$. As $g_{ij} g_{jl} = g_{il}$ on $R_{(h_i h_j h_l)}$, one can see

$$G_{ij} G_{jl} - G_{il} \in I_{(H_i H_j H_l)} \tag{3.8}$$

for all $(i, j, l) \in \{1, \ldots, e\}^3$. Let $S = \mathfrak{o}_v \setminus \{0\}$. Since I_S and $\pi(I)$ are homogeneous ideals of $k[T_0, \ldots, T_N]$ and $(\mathfrak{o}_v/\mathfrak{m}_v)[T_0, \ldots, T_N]$, respectively, I_S and $\pi(I)$ are finitely generated ideals. Therefore, by using (3.7) and (3.8), one can find a finitely generated homogeneous ideal I' of A such that

$$\begin{cases} I' \subseteq I, \quad I'_S = I_S, \quad \pi(I') = \pi(I), \\ T_0^a, \ldots, T_N^a \in H_1 A + \cdots + H_e A + I', \\ G_{ij} G_{jl} - G_{il} \in I'_{(H_i H_j H_l)} \quad (\forall\, i, j, l \in \{1, \ldots, e\}). \end{cases}$$

Let $R' := A/I'$, $\mathscr{X}' := \mathrm{Proj}(R')$ and $p' : A \to R'$ be the natural homomorphism. Obviously $\mathscr{X}$ is a closed subscheme of $\mathscr{X}'$. We set $h'_i = p'(H_i)$ and $g'_{ij} = p'(G_{ij})$. Then $p'(T_0)^a, \ldots, p'(T_N)^a \in h'_1 R' + \cdots + h'_e R'$, which means that $\mathscr{X}' = \bigcup_{i=1}^{e} D_+(h'_i)$ by [**44**, Lemma 3.35 in Section 2.3]. Moreover, $g'_{ij} g'_{il} = g'_{il}$. In particular, $g'_{ij} g'_{ji} = g'_{ii} = 1$, so that $g'_{ij} \in R'^{\times}_{(h'_i h'_j)}$. This means that $\{g'_{ij}\}_{i,j \in \{1,\ldots,e\}}$ gives rise to an invertible $\mathcal{O}_{\mathscr{X}'}$-module $\mathscr{L}'$ such that $\mathscr{L}'|_{\mathscr{X}} = \mathscr{L}$. Moreover, $(\mathscr{X}', \mathscr{L}')$ is a model of (X, L) and the special fiber of $\mathscr{X}' \to \mathrm{Spec}(\mathfrak{o}_v)$ is same as the special fiber of $\mathscr{X}' \to \mathrm{Spec}(\mathfrak{o}_v)$, as required. □

3.3.18. Proposition. — *Let $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_v)$ be a model of X and $\mathscr{L}$ be an invertible $\mathcal{O}_{\mathscr{X}}$-module. If $\mathscr{L}$ is ample on every fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_v)$, then $\mathscr{L}$ is ample.*

Proof. — By Lemma 3.3.17, there are a coherent model of $\mathscr{X}'$ of X and an invertible $\mathcal{O}_{\mathscr{X}'}$-module $\mathscr{L}'$ such that $\mathscr{X}$ is a closed subscheme of $\mathscr{X}'$, $\mathscr{L}'|_{\mathscr{X}} = \mathscr{L}$ and the special fiber of $\mathscr{X}' \to \mathrm{Spec}(\mathfrak{o}_v)$ coincides with the special fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_v)$. Note that $\mathscr{L}'$ is ample on every fiber of $\mathscr{X}' \to \mathrm{Spec}(\mathfrak{o}_v)$, and hence $\mathscr{L}'$ is ample by [**33**, IV-3, Corollaire (9.6.4)] because $\mathscr{X}' \to \mathrm{Spec}(\mathfrak{o}_v)$ is finitely presented. Therefore $\mathscr{L}$ is ample. □

3.3.19. Theorem. — *We assume that v is non-Archimedean and $|\cdot|$ is not trivial. Let L be a semi-ample invertible $\mathcal{O}_X$-module and φ be a continuous metric of L. Then φ is semi-positive if and only if φ is M-semi-positive.*

Proof. — First we assume that φ is semi-positive. By Remark 3.3.15, we may assume that X is reduced. As L is semi-positive, there is a positive integer n_0 such that $L^{\otimes n_0}$ is generated by global sections, so we may assume that L is generated by global sections, and hence $L^{\otimes n}$ is generated by global sections for all $n \geqslant 1$. Fix $\lambda \in]0,1[$ such that $\lambda < \sup\{|a| | a \in k^\times, |a| < 1\}$. By [**13**, Proposition 1.2.22], there is a finitely generated lattice $\mathcal{E}_n$ of $H^0(X, L^{\otimes n})$ such that $d(\|\cdot\|_{\mathcal{E}_n}, \|\cdot\|_{n\varphi}) \leqslant \log(\lambda^{-1})$. Note that there is a morphism $f_n : X \to \mathbb{P}(H^0(X, L^{\otimes n}))$ with $f_n^*(\mathcal{O}_{\mathbb{P}(H^0(X,L^{\otimes n}))}(1)) = L^{\otimes n}$, so we can find a morphism $\mathcal{F}_n : \mathscr{X}_n \to \mathbb{P}(\mathcal{E}_n)$ over $\mathfrak{o}_v$ such that $\mathscr{X}_n$ is flat and projective over $\mathfrak{o}_v$ and $\mathcal{F}_n$ is an extension of f_n over $\mathfrak{o}_v$. If we set $\mathscr{L}_n = \mathcal{F}_n^*(\mathcal{O}_{\mathbb{P}(\mathcal{E}_n)}(1))$, then $(\mathscr{X}_n, \mathscr{L}_n)$ is a flat model of $(X, L^{\otimes n})$. As $\mathcal{E}_n \otimes_{\mathfrak{o}_v} \mathcal{O}_{\mathbb{P}(\mathcal{E}_n)} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_n)}(1)$ is surjective, one also has the sujectivity of $\mathcal{E}_n \otimes_{\mathfrak{o}_v} \mathcal{O}_{\mathscr{X}_n} \to \mathscr{L}_n$. Therefore, by [**13**, Proposition 2.3.12], the model metric $\varphi_{\mathscr{L}_n}$ coincides with the quotient metric induced by $\|\cdot\|_{\mathcal{E}_n}$. Therefore, if we denote by φ_n the quotient metic induced by $\|\cdot\|_{n\varphi}$, then, by [**13**, Proposition 2.2.20],

$$d(\varphi_{\mathscr{L}_n}, \varphi_n) \leqslant d(\|\cdot\|_{\mathcal{E}_n}, \|\cdot\|_{n\varphi}) \leqslant \log(\lambda^{-1}),$$

which implies

$$d(\tfrac{1}{n}\varphi_{\mathscr{L}_n}, \varphi) \leqslant d(\tfrac{1}{n}\varphi_{\mathscr{L}_n}, \tfrac{1}{n}\varphi_n) + d(\tfrac{1}{n}\varphi_n, \varphi) \leqslant \tfrac{1}{n}\log(\lambda^{-1}) + d(\tfrac{1}{n}\varphi_n, \varphi),$$

and hence $\lim_{n\to\infty} d(\frac{1}{n}\varphi_{\mathscr{L}_n}, \varphi) = 0$. Thus φ is M-semi-positive because $\mathscr{L}_n$ is nef.

Let us see the converse. Let $(\mathscr{X}, \mathscr{L})$ be a model of (X, L) such that $\mathscr{L}$ is nef along the special fiber of $\mathscr{X} \to \operatorname{Spec}(\mathfrak{o}_v)$. Let $\varphi_{\mathscr{L}}$ be the metric arising from the model $(\mathscr{X}, \mathscr{L})$. It is sufficient to see that $\varphi_{\mathscr{L}}$ is semi-positive. Let $\mathscr{A}$ be an ample invertible $\mathcal{O}_{\mathscr{X}}$-module. Then, for $n \geqslant 1$, $\mathscr{A} \otimes \mathscr{L}^{\otimes n}$ is ample on every fiber of $\mathscr{X} \to \operatorname{Spec}(\mathfrak{o}_v)$, and hence, by Proposition 3.3.18, $\mathscr{A} \otimes \mathscr{L}^{\otimes n}$ is ample on $\mathscr{X}$ for all $n \geqslant 1$. Therefore, by [**13**, Proposition 2.3.17], $\varphi_{\mathscr{L}}$ is semi-positive. □

3.4. Green functions

In this section, we fix a projective k-scheme X.

3.4.1. Definition. — Let D be a Cartier divisor on X. We call *Green function* of D any real-valued continuous function on $(X \setminus \operatorname{Supp}(D))^{\mathrm{an}}$ such that, for any regular meromorphic function $f \in \Gamma(U, \mathscr{M}_X^\times)$ which defines the Cartier divisor locally on a Zariski open subset U, the function $g + \log|f|$ on $(U \setminus \operatorname{Supp}(D))^{\mathrm{an}}$ extends to a continuous function on U^{an}. A pair (D, g) consisting of a Cartier divisor D on X and a Green function g of D is called a *metrized Cartier divisor*. We denote by $\widehat{\mathrm{Div}}(X)$ the set of all metrized Cartier divisors on X. Further g is said to be *smooth* if $(\mathcal{O}_X(D), |\cdot|_g)$ is smooth. A smooth Green function of $D = 0$ is called a *smooth function* on X^{an}.

3.4.2. Example. — In the case where D is the zero Cartier divisor, Green functions of D are continuous functions on X^{an}. In particular, if the Krull dimension of X is zero, then X^{an} consists of isolated points. In this case any Cartier divisor D on X is trivial (see Remark 1.2.10) and hence Green functions identify with elements in the real vector space spanned by X^{an}.

In the case where D is a principal Cartier divisor, namely a Cartier divisor of the form $\mathrm{div}(f)$, where f is a regular meromorphic function, then by definition $-\ln|f|$ is a Green function of $\mathrm{div}(f)$. We denote by $\widehat{\mathrm{div}}(f)$ the pair $(\mathrm{div}(f), -\ln|f|)$. Such a metrized Cartier divisor is said to be *principal*.

3.4.3. Remark. — Metrized Cartier divisors are closely related to metrized invertible sheafs. Let D be a Cartier divisor on X. We denote by $\mathcal{O}_X(D)$ the sub-$\mathcal{O}_X$-module of $\mathscr{M}_X$ generated by $-D$. Let $(U_i)_{i\in I}$ be an open covering of X such that, on each U_i the Cartier divisor is defined by a regular meromorphic function s_i. Then the restriction of $\mathcal{O}_X(D)$ at U_i is given by $\mathcal{O}_{U_i}s_i^{-1}$. If g is a Green function of D, then it induces a continuous metric $\varphi_g = (|\cdot|_g(x))_{x\in X^{\mathrm{an}}}$ on $\mathcal{O}_X(D)$ such that

$$|s_i^{-1}|_g := \exp(-g - \ln|s_i|) \text{ on } U_i^{\mathrm{an}}.$$

Note that the metric of the canonical regular meromorphic section (see Definition 1.2.8) is given by

$$|s_D|_g = |s_i \otimes s_i^{-1}|_g = \exp(-g) \text{ on } U_i.$$

Conversely, given an invertible $\mathcal{O}_X$-module L, any non-zero rational section s of L defines a Cartier divisor $\mathrm{div}(L; s)$. Moreover, if φ is a continuous metric on L, then $-\ln|s|_\varphi$ is a Green function of $\mathrm{div}(L; s)$. We denote by $\widehat{\mathrm{div}}(\overline{L}; s)$ (or by $\widehat{\mathrm{div}}(s)$ for simplicity) the metrized Cartier divisor $(\mathrm{div}(L; s), -\ln|s|_\varphi)$.

The above relation between metrized Cartier divisors and metrized invertible sheaves is important to define the following composition law on the set of metrized Cartier divisors. Let (D_1, g_1) and (D_2, g_2) be metrized Cartier divisors. Note that $\mathcal{O}_X(D_1 + D_2)$ is canonically isomorphic to $\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)$. Moreover, under the canonical isomorphism

$$\mathcal{O}_X(D_1 + D_2) \xrightarrow{\sim} \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2),$$

the regular meromorphic section $s_{D_1+D_2}$ corresponds to $s_{D_1} \otimes s_{D_2}$. We equip the invertible sheaf $\mathcal{O}_X(D_1)$ and $\mathcal{O}_X(D_2)$ with the metrics $\varphi_{g_1} = (|\cdot|_{g_1}(x))_{x\in X^{\mathrm{an}}}$ and $\varphi_{g_2} = (|\cdot|_{g_2}(x))_{x\in X^{\mathrm{an}}}$ respectively, and $\mathcal{O}_X(D_1 + D_2)$ with the tensor product metric $\varphi_{g_1} \otimes \varphi_{g_2}$. We then denote by $g_1 + g_2$ the Green function in the metrized Cartier divisor $\widehat{\mathrm{div}}(s_{D_1+D_2})$. Clearly, for any $x \in \big(X \setminus (\mathrm{Supp}(D_1) \cup \mathrm{Supp}(D_2))\big)^{\mathrm{an}}$, one has

$$(g_1 + g_2)(x) = g_1(x) + g_2(x).$$

Note that the set $\widehat{\mathrm{Div}}(X)$ of metrized Cartier divisors equipped with this composition law forms a commutative group.

3.4.4. Definition. — Let (A, g) be a metrized Cartier divisor such that $\mathcal{O}_X(A)$ is an ample invertible $\mathcal{O}_X$-module (namely the Cartier divisor A is ample). We say that the Green function g is *plurisubharmonic* if the metric $|\cdot|_g$ on $\mathcal{O}_X(A)$ is semi-positive. We refer to [**11**, §6.8] and [**37**, §6] for a local version of positivity conditions.

We say that a metrized Cartier divisor (D,g) is *integrable* if there are ample Cartier divisors A_1 and A_2 together with plurisubharmonic Green functions g_1 and g_2 of A_1 and A_2, respectively, such that $(D,g)=(A_1,g_1)-(A_2,g_2)$. We denote by $\widehat{\mathrm{Int}}(X)$ the set of all integrable metrized Cartier divisors. This is a subgroup of the group $\widehat{\mathrm{Div}}(X)$ of metrized Cartier divisors.

3.4.5. Remark. — Let k'/k be a valued extension which is complete. Let $X_{k'}$ be the fiber product $X\times_{\mathrm{Spec}\,k}\mathrm{Spec}\,k'$, and $\pi:X_{k'}\to X$ be the morphism of projection. Let (D,g) be a metrized Cartier divisor on X. Then the pull-back $D_{k'}$ of D by the morphisme π is well defined (see Definition 1.2.14 and Remark 1.3.5). Note that $\mathcal{O}_{X_{k'}}(D_{k'})$ is isomorphic with the pull-back of $\mathcal{O}_X(D)$ by π, and the canonical meromorphic section $s_{D_{k'}}$ of $D_{k'}$ identifies with the pull-back of s_D by π. Let φ_g be the continuous metric on $\mathcal{O}_X(D)$ induced by the Green function g. We denote by $g_{k'}$ the Green function of $D_{k'}$ defined as

$$g_{k'}=-\ln|s_{D_{k'}}|_{\varphi_{g,k'}},$$

where $\varphi_{g,k'}$ is the continuous metric on $\pi^*(\mathcal{O}_X(D))\cong\mathcal{O}_{X_{k'}}(D_{k'})$ induced by φ_g (see Example 3.3.2 (5)). Note that, for any element $x'\in X_{k'}^{\mathrm{an}}$ such that $\pi^\natural(x')\in(X\setminus\mathrm{Supp}(D))^{\mathrm{an}}$, one has

$$g_{k'}(x')=g(\pi^\natural(x')).$$

Moreover, the composition of g with the restriction of $\pi^\natural$ to $(X_{k'}\setminus\mathrm{Supp}(D_{k'}))^{\mathrm{an}}$ forms a Green function of $D_{k'}$. We denote by $g_{k'}$ this Green function. By Remark 3.3.7, if $\mathcal{O}_X(D)$ is semi-ample and g is plurisubharmonic, then $g_{k'}$ is also plurisubharmonic. If (D,g) is integrable, then $(D_{k'},g_{k'})$ is also integrable. Therefore the correspondance $(D,g)\mapsto(D_{k'},g_{k'})$ defines a group homomorphism from $\widehat{\mathrm{Div}}(X)\to\widehat{\mathrm{Div}}(X_{k'})$, whose restriction to $\widehat{\mathrm{Int}}(X)$ defines a group homomorphism $\widehat{\mathrm{Int}}(X)\to\widehat{\mathrm{Int}}(X_{k'})$.

3.4.6. Theorem. — *Let X be a d-dimensional projective and integral scheme over k. Let D be a nef and effective Cartier divisor and g be a Green function of D such that either*

(a) *if v is Archimedean, the metric of $|\cdot|_g$ of $\mathcal{O}_X(D)$ is C^∞ and semi-positive, or*
(b) *if v is non-Archimedean, the metric of $|\cdot|_g$ of $\mathcal{O}_X(D)$ is a nef model metric.*

Then there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ of smooth functions on X^{an} with the following properties:

(1) *for all $n\in\mathbb{N}$, $\psi_n\leqslant g$, $\psi_n\leqslant\psi_{n+1}$.*
(2) *for each point $x\in X^{\mathrm{an}}$, $\sup\{\psi_n(x)\,|\,n\in\mathbb{N}\}=g(x)$.*
(3) *for all $n\in\mathbb{N}$, $g-\psi_n$ is a Green function of D such that either*

(3.a) *if v is Archimedean, the metric of $|\cdot|_{g-\psi_n}$ of $\mathcal{O}_X(D)$ is C^∞ and semi-positive, or*

(3.b) *if v is non-Archimedean, the metric of $|\cdot|_{g-\psi_n}$ of $\mathcal{O}_X(D)$ is a nef model metric.*

Proof. — This theorem is nothing more than [**10**, Théorème 3.1]. In the case where v is non-Archimedean, it is proved under the additional assumption that v is discrete. However, their proof works well by slight modifications. For reader's convenience, we reprove it here.

We may assume that v is non-Archimedean. If the theorem holds for (mD, mg) for some positive number m, then it also holds for (D, g), so that we may assume that there is a flat model $(\mathscr{X}, \mathscr{L})$ of $(X, \mathcal{O}_X(D))$ such that $|\cdot|_g = |\cdot|_{\varphi_{\mathscr{L}}}$ and $\mathscr{L}$ is nef along the special fiber of $\mathscr{X} \to \operatorname{Spec}(\mathfrak{o}_v)$. By Lemma 1.2.15, there is a Cartier divisor $\mathscr{D}$ on $\mathscr{X}$ such that $\mathcal{O}_{\mathscr{X}}(\mathscr{D}) = \mathscr{L}$, $\mathscr{D}|_X = D$ and g is the Green function arising from $(\mathscr{X}, \mathscr{D})$. Let $\mathscr{X} = \bigcup_{i=1}^N \operatorname{Spec}(\mathscr{A}_i)$ be an affine open covering of $\mathscr{X}$ such that $\mathscr{D}$ is given by a local equation f_i on $\operatorname{Spec}(\mathscr{A}_i)$. Since D is effective, one has $f_i \in (\mathscr{A}_i)_S$, that is, $s_i f_i \in \mathscr{A}_i$ for some $s_i \in S$, where $S := \mathfrak{o}_v \setminus \{0\}$, so that if we set $s = s_1 \cdots s_N$, then $sf_i \in \mathscr{A}_i$ for all $i = 1, \ldots, N$. Let

$$g' := g - \log|s|, \quad \mathscr{L}' := \mathscr{L} \otimes \mathcal{O}_{\mathscr{X}} s^{-1} \quad \text{and} \quad \mathscr{D}' := \mathscr{D} + \operatorname{div}(s).$$

Then $\mathscr{D}'$ is effective, $\mathcal{O}_{\mathscr{X}}(\mathscr{D}') = \mathscr{L}'$ and $|\cdot|_{g'} = |\cdot|_{\mathscr{L}'}$. Thus, if the theorem holds for g', then one has the assertion for g, and hence we may further assume that $\mathscr{D}$ is effective.

Fix $a \in S$ such that $|a| < 1$, and set

$$\psi_n = \min\{g, -n \log|a|\} \quad (\forall\, n \in \mathbb{N}).$$

The properties (1) and (2) are obvious, so we need to see (3). Let $\mathscr{I}_n$ be the ideal sheaf of $\mathcal{O}_{\mathscr{X}}$ generated by a local equation of $\mathscr{D}$ and a^n. Let $p_n : \mathscr{Y}_n \to \mathscr{X}$ be the blowing-up in terms of the ideal sheaf $\mathscr{I}_n$. Note that $\mathscr{I}_n \mathcal{O}_{\mathscr{Y}}$ is a locally principal ideal sheaf of $\mathcal{O}_{\mathscr{Y}_n}$ whose support is contained in the special fiber of $\mathscr{Y}_n \to \operatorname{Spec}(\mathfrak{o}_v)$, that is, there is an effective Cartier divisor $\mathscr{E}_n$ on $\mathscr{Y}_n$ such that $\mathcal{O}_{\mathscr{Y}_n}(-\mathscr{E}_n) = \mathscr{I}_n \mathcal{O}_{\mathscr{Y}_n}$ and $\mathscr{E}_n|_X = 0$. Obviously ψ_n is a smooth function arising from the model $(\mathscr{Y}_n, \mathscr{E}_n)$. Therefore, it is sufficient to show that $p_n^*(\mathscr{D}) - \mathscr{E}_n$ is nef along the special fiber $\mathscr{Y}_n \to \operatorname{Spec}(\mathfrak{o}_v)$. Let $\mathscr{X} = \bigcup_{i=1}^N \operatorname{Spec}(\mathscr{A}_i)$ be an affine open covering of $\mathscr{X}$ as before. Note that $\mathscr{D}$ is given by $f_i \in \mathscr{A}_i$ on $\operatorname{Spec} \mathscr{A}_i$ for each i. Then

$$p_n^{-1}(\operatorname{Spec} \mathscr{A}_i) = \operatorname{Proj}(\mathscr{A}_i[T_0, T_1]/(f_i T_0 - a^n T_1)).$$

If we set $p_n^{-1}(\operatorname{Spec} \mathscr{A}_i)_\alpha = \{T_\alpha \neq 0\}$ for $\alpha \in \{0, 1\}$, then $f_i = a^n (T_1/T_0)$ on $p_n^{-1}(\operatorname{Spec} \mathscr{A}_i)_0$ and $a^n = f_i (T_0/T_1)$ on $p_n^{-1}(\operatorname{Spec} \mathscr{A}_i)_1$, so that

$$\begin{cases} \mathcal{O}_{\mathscr{Y}_n}(-\mathscr{E}_n)|_{p_n^{-1}(\operatorname{Spec} \mathscr{A}_i)_0} = a^n \mathcal{O}_{p_n^{-1}(\operatorname{Spec} \mathscr{A}_i)_0}, \\ \mathcal{O}_{\mathscr{Y}_n}(-\mathscr{E}_n)|_{p_n^{-1}(\operatorname{Spec} \mathscr{A}_i)_1} = f_i \mathcal{O}_{p_n^{-1}(\operatorname{Spec} \mathscr{A}_i)_1}. \end{cases} \tag{3.9}$$

Therefore, one can see that $p_n^*(\mathscr{D}) - \mathscr{E}_n$ and $\operatorname{div}(a^n) - \mathscr{E}_n$ are effective. Let us see $(p_n^*(\mathscr{D}) - \mathscr{E}_n \cdot C) \geqslant 0$ for any irreducible curve C on the special fiber of $\mathscr{Y}_n \to \operatorname{Spec}(\mathfrak{o}_v)$. Let ξ be the generic point of C. We choose i such that $\xi \in p_n^{-1}(\operatorname{Spec}\mathscr{A}_i)$. If $\xi \notin \operatorname{Supp}(p_n^*(\mathscr{D}) - \mathscr{E}_n)$, then the assertion is obvious because $p_n^*(\mathscr{D}) - \mathscr{E}_n$ is effective. Otherwise, by (3.9), $\xi \in p_n^{-1}(\operatorname{Spec}\mathscr{A}_i)_0$. Then, by (3.9) again, $\xi \notin \operatorname{Supp}(\operatorname{div}(a^n) - \mathscr{E}_n)$, so that $((\operatorname{div}(a^n) - \mathscr{E}_n) \cdot C) \geqslant 0$ by the reason of the effectivity of $\operatorname{div}(a^n) - \mathscr{E}_n$. Note that $p_n^*(\mathscr{D}) - \mathscr{E}_n$ is linearly equivalent to $p_n^*(\mathscr{D}) + (\operatorname{div}(a^n) - \mathscr{E}_n)$. Thus it is sufficient to show that $(p_n^*(\mathscr{D}) \cdot C) \geqslant 0$, which is obvious because of the projection formula and the nefness of $\mathscr{D}$. □

3.5. Local measures

In this section, we assume that k is *algebraically closed*. Let X be a projective k-scheme and let d be the dimension of X. Assume given a family $(L_i)_{i=1}^d$ of semi-ample invertible $\mathcal{O}_X$-modules. For any $i \in \{1, \ldots, d\}$, let φ_i be a semi-positive continuous metric on L_i. First we assume that X is integral. In the case where $|\cdot|$ is Archimedean (and hence $k = \mathbb{C}$), by Bedford-Taylor theory **[3]** one can construct a Borel measure

$$c_1(L_1, \varphi_1) \cdots c_1(L_d, \varphi_d)$$

having

$$\deg(c_1(L_1) \cdots c_1(L_d) \cap [X])$$

as its total mass. In the non-Archimedean case, an analoguous measure has been proposed by Chambert-Loir **[10]**, assuming that the field k admits a dense countable subfield (see also **[11**, §5**]** for a general non-Archimedean analogue of Bedford-Taylor theory). In any case, the measure $c_1(L_1, \varphi_1) \cdots c_1(L_d, \varphi_d)$ is also denoted by $\mu_{(L_1,\varphi_1)\cdots(L_d,\varphi_d)}$. Note that the measure $\mu_{(L_1,\varphi_1)\cdots(L_d,\varphi)}$ is additive with respect to each (L_i, φ_i). More precisely, if $i \in \{1, \ldots, d\}$ and if (M_i, ψ_i) is another semi-positively metrized invertible $\mathcal{O}_X$-module, then the measure

$$\mu_{(L_1,\varphi_1)\cdots(L_{i-1},\varphi_{i-1})(L_i\otimes M_i,\varphi_i\otimes\psi_i)(L_{i+1},\varphi_{i+1})\cdots(L_d,\varphi_d)}$$

is equal to

$$\mu_{(L_1,\varphi_1)\cdots(L_{i-1},\varphi_{i-1})(L_i,\varphi_i)(L_{i+1},\varphi_{i+1})\cdots(L_d,\varphi_d)} + \mu_{(L_1,\varphi_1)\cdots(L_{i-1},\varphi_{i-1})(M_i,\psi_i)(L_{i+1},\varphi_{i+1})\cdots(L_d,\varphi_d)}$$

Moreover, for any permutation $\sigma : \{1, \ldots, d\} \to \{1, \ldots, d\}$, one has

$$\mu_{(L_{\sigma(1)},\varphi_{\sigma(1)})\cdots(L_{\sigma(d)},\varphi_{\sigma(d)})} = \mu_{(L_1,\varphi_1)\cdots(L_d,\varphi_d)}.$$

In general, let $X_1, \ldots, X_n$ be irreducible components of X which are of dimension d, and $\eta_1, \ldots, \eta_n$ the generic points of $X_1, \ldots, X_n$, respectively. Let $\xi_i : X_i \hookrightarrow X$ be the canonical closed embedding for each i. Then a measure $\mu_{(L_1,\varphi_1)\cdots(L_d,\varphi_d)}$ on X^{an} is defined to be

$$\mu_{(L_1,\varphi_1)\cdots(L_d,\varphi_d)} := \sum_{j=1}^{n} \mathrm{length}_{\mathcal{O}_{X,\eta_j}}(\mathcal{O}_{X,\eta_j})(\xi_j^{\mathrm{an}})_*\Big(c_1\big(\xi_j^*(L_1,\varphi_1)\big)\cdots c_1\big(\xi_j^*(L_d,\varphi_d)\big)\Big). \quad (3.10)$$

3.5.1. Definition. — Let $(L_1,\varphi_1),\ldots,(L_d,\varphi_d)$ be a family of integrable metrized invertible $\mathcal{O}_X$-modules. For each $i\in\{1,\ldots,d\}$, we let (L_i',φ_i') and (L_i'',φ_i'') be ample invertible $\mathcal{O}_X$-modules equipped with semi-positive metrics, such that $L_i = L_i'\otimes(L_i'')^\vee$ and $\varphi_i = \varphi_i'\otimes(\varphi_i'')^\vee$. We define a signed Radon measure $\mu_{(L_1,\varphi_1)\cdots(L_d,\varphi_d)}$ on X^{an} as follows:

$$\mu_{(L_1,\varphi_1)\cdots(L_d,\varphi_d)} := \sum_{I\subseteq\{1,\ldots,d\}} (-1)^{\mathrm{card}(I)}\mu_{(L_{1,I},\varphi_{1,I})\cdots(L_{d,I},\varphi_{d,I})},$$

where $(L_{j,I},\varphi_{j,I}) = (L_j'',\varphi_j'')$ if $j\in I$, and $(L_{j,I},\varphi_{j,I}) = (L_j',\varphi_j')$ if $j\in\{1,\ldots,d\}\setminus I$ (cf. Lemma 1.1.5).

3.5.2. Example. — We recall the explicit construction of Chambert-Loir's measure in a particular case as explained in [**10**, §2.3]. Assume that the absolute value $|\cdot|$ is non-Archimedean and that the k-scheme X is integral and normal. Let k° be the valuation ring of $(k,|\cdot|)$ and $\mathfrak{m}$ be the maximal ideal of k°. Suppose given an integral model of X, namely, a flat and normal projective k°-scheme $\mathscr{X}$ such that

$$\mathscr{X}\times_{\operatorname{Spec} k^\circ}\operatorname{Spec} k \cong X.$$

Let $\mathscr{X}_{\mathfrak{m}}$ be the fibre of $\mathscr{X}$ over the closed point of $\operatorname{Spec} k^\circ$. It turns out that the reduction map from X^{an} to $\mathscr{X}_{\mathfrak{m}}$ is surjective. Let $Z_1,\ldots,Z_n$ be irreducible components of $\mathscr{X}_{\mathfrak{m}}$. For any $j\in\{1,\ldots,n\}$, there exists a unique point $z_j\in X^{\mathrm{an}}$ whose reduction identifies with the generic point of Z_j.

Assume that each metric φ_j is induced by an integral model $\mathscr{L}_i$, which is an invertible sheaf on $\mathscr{X}$ such that $\mathscr{L}_i|_X\cong L_i$. Then the measure

$$c_1(L_1,\varphi_1)\cdots c_1(L_d,\varphi_d)$$

is given by

$$\sum_{j=1}^{d}\mathrm{mult}_{Z_j}(\mathscr{X}_{\mathfrak{m}})\deg(c_1(\mathscr{L}_1|_{\mathscr{X}_{\mathfrak{m}}})\cdots c_1(\mathscr{L}_d|_{\mathscr{X}_{\mathfrak{m}}})\cap[Z_j])\,\mathrm{Dirac}_{z_j},$$

where $\mathrm{mult}_{Z_j}(\mathscr{X}_{\mathfrak{m}})$ is the multiplicity of Z_j in $\mathscr{X}_{\mathfrak{m}}$, and Dirac_{z_j} denotes the Dirac measure at z_j.

3.5.3. Remark. — We assume that X is integral. Let $(\varphi_{1,n})_{n=1}^\infty,\ldots,(\varphi_{d,n})_{n=1}^\infty$ be sequences of semi-positive metrics of $L_1,\ldots,L_d$, respectively such that

$$\lim_{n\to\infty} d(\varphi_{i,n},\varphi_i) = 0$$

for all $i \in \{1, \ldots, d\}$. Then, by using [**18**, Corollary (3.6)] and [**11**, Corollaire (5.6.5)], one can see

$$\lim_{n\to\infty} \int_{X^{\mathrm{an}}} f\mu_{(L_1,\varphi_{1,n})\cdots(L_d,\varphi_{d,n})} = \int_{X^{\mathrm{an}}} f\mu_{(L_1,\varphi_1)\cdots(L_d,\varphi_d)}$$

for any smooth function f on X^{an}.

3.5.4. Definition. — Let $\overline{D}_1 = (D_1, g_1), \ldots, \overline{D}_d = (D_d, g_d)$ be a family of integrable metrized Cartier divisors on X. Note that $(\mathcal{O}_X(D_i), |\cdot|_{g_i})$ is an integrable metrized invertible $\mathcal{O}_X$-module for each $i \in \{1, \ldots, d\}$, so that we define a signed Radon measure $\mu_{\overline{D}_1\cdots\overline{D}_d}$ on X^{an} to be

$$\mu_{\overline{D}_1\cdots\overline{D}_d} := \mu_{(\mathcal{O}_X(D_1),|\cdot|_{g_1})\cdots(\mathcal{O}_X(D_d),|\cdot|_{g_d})}.$$

For any $i \in \{1, \ldots, d\}$, we write (D_i, g_i) as the difference of two metrized Cartier divisors $(D'_i, g'_i) - (D''_i, g''_i)$, where D'_i and D''_i are ample, and g'_i and g''_i are plurisubharmonic. Then we can see

$$\mu_{\overline{D}_1\cdots\overline{D}_d} := \sum_{I\subseteq\{1,\ldots,d\}} (-1)^{\mathrm{card}(I)} \mu_{\overline{D}_{1,I}\cdots\overline{D}_{d,I}},$$

where $\overline{D}_{j,I} = (D''_j, g''_j)$ if $j \in I$, and $\overline{D}_{j,I} = (D'_j, g'_j)$ if $j \in \{1, \ldots, d\} \setminus I$ (cf. Lemma 1.1.5, Definition 3.4.4 and Definition 3.5.1).

Let $X_1, \ldots, X_n$ be irreducible components of X and $\eta_1, \ldots, \eta_n$ be the generic points of $X_1, \ldots, X_n$, respectively. Let $\xi_j : X_j \hookrightarrow X$ be the canonical closed embedding. Then it is easy to see

$$\mu_{(D_1,g_1)\cdots(D_d,g_d)} = \sum_{j=1}^{n} \mathrm{length}_{\mathcal{O}_{X,\eta_j}}(\mathcal{O}_{X,\eta_j})(\xi_j^{\mathrm{an}})_*\Big(\mu_{\xi_j^*(D_1,g_1)\cdots\xi_j^*(D_d,g_d)}\Big). \qquad (3.11)$$

3.5.5. Proposition. — *Let $\pi : Y \to X$ be a surjective morphism between integral projective schemes over k. We set $e = \dim X$ and $d = \dim Y$. Let $(L_1, \varphi_1), \ldots, (L_d, \varphi_d)$ be integrable metrized invertible $\mathcal{O}_X$-modules. Then one has the following:*

(1) *If $d > e$, then $\pi_*(\mu_{\pi^*(L_1,\varphi_1)\cdots\pi^*(L_d,\varphi_d)}) = 0$.*
(2) *If $d = e$, then $\pi_*(\mu_{\pi^*(L_1,\varphi_1)\cdots\pi^*(L_d,\varphi_d)}) = (\deg \pi)\mu_{(L_0,\varphi_0)\cdots(L_d,\varphi_d)}$.*

Proof. — We may assume that $L_1, \ldots, L_d$ are ample and $\varphi_1, \ldots, \varphi_d$ are semi-positive. If $\varphi_1, \ldots, \varphi_d$ are smooth, then the assertion is well-known (cf. [**36**, Proposition 10.4]). Let $(\varphi_{1,n})_{n=1}^{\infty}, \ldots, (\varphi_{d,n})_{n=1}^{\infty}$ be regularizations of $\varphi_1, \ldots, \varphi_d$, that is, $\varphi_{1,n}, \ldots, \varphi_{d,n}$ are smooth and semi-positive for $i \in \{1, \ldots, d\}$ and $n \geqslant 1$, and $\lim_{n\to\infty} d(\varphi_i, \varphi_{i,n}) = 0$ for $i \in \{1, \ldots, d\}$ (for example, see [**14**] for the Archimedean case and Theorem 3.3.19 for the non-Archimedean case). Let f be a smooth function on X^{an} (namely the metric on $\mathcal{O}_X$ induced by f is smooth). Then, by using [**18**, Corollary (3.6)] and [**11**, Corollaire (5.6.5)], one can see that

$$\lim_{n\to\infty}\int_{X^{\mathrm{an}}}\pi^*(f)\mu_{\pi^*(L_1,\varphi_{1,n})\cdots\pi^*(L_d,\varphi_{d,n})}=\int_{X^{\mathrm{an}}}\pi^*(f)\mu_{\pi^*(L_1,\varphi_1)\cdots\pi^*(L_d,\varphi_d)}$$

and if $d=e$, then

$$\lim_{n\to\infty}\int_{Y^{\mathrm{an}}}f\mu_{(L_1,\varphi_{1,n})\cdots(L_d,\varphi_{d,n})}=\int_{X^{\mathrm{an}}}f\mu_{(L_1,\varphi_1)\cdots(L_d,\varphi_d)}.$$

Thus the assertions follow. □

3.5.6. Remark. — Let X and Y be two projective schemes over $\operatorname{Spec} k$, of Krull dimension d and n, respectively. Let $\overline{L}_1,\ldots,\overline{L}_d$ be integrable metrized invertible $\mathcal{O}_X$-modules, $\overline{M}_1,\ldots,\overline{M}_n$ be integrable metrized invertible $\mathcal{O}_Y$-modules. We consider the fiber product $X\times_k Y$ and let $\pi_1: X\times_k Y\to X$ and $\pi_2: X\times_k Y\to Y$ be the two morphisms of projection. In the case where k is Archimedean, the analytic space $(X\times_k Y)^{\mathrm{an}}$ is homeomorphic to $X^{\mathrm{an}}\times Y^{\mathrm{an}}$ and the measure

$$\mu_{\pi_1^*(\overline{L}_1)\cdots\pi_1^*(\overline{L}_d)\pi_2^*(\overline{M}_1)\cdots\pi_2^*(\overline{M}_n)}$$

on $(X\times_k Y)^{\mathrm{an}}$ identifies with

$$\mu_{\overline{L}_1\cdots\overline{L}_d}\otimes\mu_{\overline{M}_1\cdots\overline{M}_n}.$$

In the case where $|\cdot|$ is non-Archimedean, in general the topological space $(X\times_k Y)^{\mathrm{an}}$ is not homeomorphic to $X^{\mathrm{an}}\times Y^{\mathrm{an}}$. However, there is a natural continuous map

$$\alpha:(X\times_k Y)^{\mathrm{an}}\longrightarrow X^{\mathrm{an}}\times Y^{\mathrm{an}}.$$

Then the following equality holds (see [**10**, §2.8])

$$\alpha_*\Big(\mu_{\pi_1^*(\overline{L}_1)\cdots\pi_1^*(\overline{L}_d)\pi_2^*(\overline{M}_1)\cdots\pi_2^*(\overline{M}_n)}\Big)=\mu_{\overline{L}_1\cdots\overline{L}_d}\otimes\mu_{\overline{M}_1\cdots\overline{M}_n}.$$

In particular, if g is a measurable function on Y^{an} which is integrable with respect to $\mu_{\overline{M}_1\cdots\overline{M}_n}$, one has

$$\int_{(X\times_k Y)^{\mathrm{an}}}(g\circ\pi_2^{\mathrm{an}})\,\mathrm{d}\mu_{\pi_1^*(\overline{L}_1)\cdots\pi_1^*(\overline{L}_d)\pi_2^*(\overline{M}_1)\cdots\pi_2^*(\overline{M}_n)}=\int_{Y^{\mathrm{an}}}g\,\mathrm{d}\mu_{\overline{M}_1\cdots\overline{M}_n}.\tag{3.12}$$

3.5.7. Definition. — Let E be a finite-dimensional vector space over k. We say that a norm $\|\cdot\|$ on E is *orthonormally decomposable* if

(1) in the case where $|\cdot|$ is non-Archimedean, the norm $\|\cdot\|$ is ultrametric, and $(E,\|\cdot\|)$ admits an orthonormal basis $(e_j)_{j=0}^r$, namely,

$$\forall\,(\lambda_j)_{j=0}^r\in k^{r+1},\quad \|\lambda_0e_0+\cdots+\lambda_re_r\|=\max_{j\in\{0,\ldots,r\}}|\lambda_j|;$$

(2) in the case where $|\cdot|$ is Archimedean, the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$.

Note that for each valued extension $(k', |\cdot|')$ of $(k, |\cdot|)$, there is a unique norm $\|\cdot\|_{k'}$ on $E \otimes_k k'$, which is either ultrametric or induced by an inner product, such that any orthonormal basis of $(E, \|\cdot\|)$ is also an orthonormal basis of the extended normed vector space $(E \otimes_k k', \|\cdot\|_{k'})$ (see Definition 3.3.3).

3.5.8. Remark. — Let E be a finite-dimensional vector space over k, and $\|\cdot\|$ be an orthonormally decomposable norm on E. For any $s \in E$, the real number $\|s\|$ belongs to the image of the absolute value $|\cdot|$. In particular, if s is non-zero, then there exists $\lambda \in k$ such that $\|\lambda s\| = 1$.

In the case where the absolute value $|\cdot|$ is non-Archimedean, it is *not* true that any ultrametrically normed vector space admits an orthonormal basis (see [**54**, Example 2.3.26]). However, if $(E, \|\cdot\|)$ is a finite-dimensional ultrametrically normed vector space over k, for any $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$, there exists an α-orthogonal basis of E (cf. [**54**, §2.3], see also [**13**, §1.2.6] for details), namely a basis $(e_i)_{i=1}^r$ such that, for any $(\lambda_i)_{i=1}^r \in k^r$,

$$\alpha \max_{i \in \{1,\ldots,r\}} |\lambda_i| \cdot \|e_i\| \leqslant \|\lambda_1 e_1 + \cdots + \lambda_r e_r\| \leqslant \max_{i \in \{1,\ldots,r\}} |a_i| \cdot \|e_i\|.$$

Moreover, since k is assumed to be algebraically closed, in the case where absolute value $|\cdot|$ is non-trivial, the image of $|\cdot|$ is dense in $\mathbb{R}$. In fact, if a is an element of k such that $|a| \neq 1$, for any non-zero rational number p/q with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$, any element $x \in k$ satisfying the polynomial equation

$$x^q = a^p$$

has $|a|^{p/q}$ as absolute value. Therefore, by possibly delating the vectors $(e_i)_{i=1}^r$ we may assume that

$$\alpha \leqslant \|e_i\| \leqslant 1$$

for any $i \in \{1, \ldots, r\}$. Therefore, if we denote by $\|\cdot\|_\alpha$ the norm on E under which $(e_i)_{i=1}^r$ is an orthonormal basis of E, then for any $x = \lambda_1 e_1 + \cdots + \lambda_r e_r$ in E, one has

$$\|x\|_\alpha = \max_{i \in \{1,\ldots,r\}} |\lambda_i| \leqslant \alpha^{-1} \max_{i \in \{1,\ldots,r\}} |\lambda_i| \cdot \|e_i\| \leqslant \alpha^{-2} \|x\|,$$

and

$$\|x\| \leqslant \max_{i \in \{1,\ldots,r\}} |\lambda| \cdot \|e_i\| \leqslant \max_{i \in \{1,\ldots,r\}} |\lambda_i| = \|x\|_\alpha.$$

Therefore, one has

$$d(\|\cdot\|_\alpha, \|\cdot\|) := \sup_{x \in E \setminus \{0\}} \Big| \ln \|x\|_\alpha - \ln \|x\| \Big| \leqslant -2 \ln(\alpha).$$

Thus we can approximate the ultrametric norm $\|\cdot\|$ by a sequence of ultrametric norms which are orthonormally decomposable.

3.5.9. Proposition. — *Let $(E, \|\cdot\|)$ be a finite-dimensional vector space over k, equipped with an orthonormally decomposable norm. Then any element $s_0 \in E$ such* *that* $\|s_0\| = 1$ *belongs to an orthonormal basis. Moreover, for any quotient vector space* G *of* E*, the quotient norm on* G *is orthonormally decomposable.*

Proof. — The statement is classic when $|\cdot|$ is Archimedean, which follows from the Gram-Schmidt process. In the following, we assume that $|\cdot|$ is non-Archimedean. Let k° be the valuation ring of $(k, |\cdot|)$.

Let $(e_j)_{j=0}^r$ be an orthonormal basis of $(E, \|\cdot\|)$. Without loss of generality, we may assume that $s_0 = \lambda_0 e_0 + \cdots + \lambda_r e_r$ with $(\lambda_0, \ldots, \lambda_r) \in (k^\circ)^{r+1}$ and $|\lambda_0| = 1$. We then construct an upper triangular matrix A of size $(r+1) \times (r+1)$, such that the first row A is $(\lambda_0, \ldots, \lambda_r)$ and the diagonal coordinates of A are elements of absolute value 1 in k. Then the matrix A belongs to $\mathrm{GL}_{r+1}(k^\circ)$. Let $(s_j)_{j=0}^r$ be the basis of E such that

$$(s_0, \ldots, s_r)^T = A(e_0, \ldots, e_r)^T.$$

For any $j \in \{0, \ldots, r\}$, one has $\|s_j\| = 1$. Moreover, for any $(b_0, \ldots, b_r) \in k^r$, one has

$$b_0 s_0 + \cdots + b_r s_r = (b_0, \ldots, b_r) A (e_0, \ldots, e_r)^T.$$

Let $(c_0, \ldots, c_r) = (b_0, \ldots, b_r)A$. Since $(e_0, \ldots, e_r)$ is an orthonormal basis, one has

$$\|b_0 s_0 + \cdots + b_r s_r\| = \max_{j \in \{0, \ldots, r\}} |c_j|.$$

Note that $(b_0, \ldots, b_r) = (c_0, \ldots, c_r)A^{-1}$. Since A^{-1} belongs to $\mathrm{GL}_{r+1}(k^\circ)$, one has

$$\forall\, i \in \{0, \ldots, r\}, \quad |b_i| \leqslant \max_{j \in \{0, \ldots, r\}} |c_j|.$$

Therefore one obtains

$$\|b_0 s_0 + \cdots + b_r s_r\| \geqslant \max_{i \in \{0, \ldots, r\}} |b_i|.$$

Combined with the strong triangle inequality, we obtain

$$\|b_0 s_0 + \cdots + b_r s_r\| = \max_{i \in \{0, \ldots, r\}} |b_i|.$$

Therefore $(s_j)_{j=0}^r$ is an orthonormal basis of $(E, \|\cdot\|)$. In particular, the image of $(s_1, \ldots, s_r)$ in E/ks_0 forms an orthonormal basis of E/ks_0 with respect to $\|\cdot\|$. Therefore the quotient norm on E/ks_0 is orthonormally decomposable. By induction we can show that all quotient norms of $\|\cdot\|$ are orthonormally decomposable. □

In the remaining of this section, we fix a finite-dimensional vector space E equipped with an orthonormally decomposable norm $\|\cdot\|$. We also choose an orthonormal basis $(e_j)_{j=1}^r$ of $(E, \|\cdot\|)$. Let $\mathbb{P}(E)$ be the projective space of E and $\mathcal{O}_E(1)$ be the universal invertible sheaf on $\mathbb{P}(E)$. We equip $\mathcal{O}_E(1)$ with the orthogonal quotient metric $(|\cdot|(x))_{x \in \mathbb{P}(E)^{\mathrm{an}}}$ (see Definition 3.3.5) and denote by $\overline{\mathcal{O}_E(1)}$ the corresponding metrized invertible sheaf. Recall that each point $x \in \mathbb{P}(E)^{\mathrm{an}}$ corresponds to a one-dimensional quotient vector space

$$E \otimes_K \widehat{\kappa}(x) \longrightarrow \mathcal{O}_E(1)(x),$$

where $\widehat{\kappa}(x)$ denotes the completed residue field of x. Then the norm $|\cdot|(x)$ on $\mathcal{O}_E(1)(x)$ is by definition the quotient norm of $\|\cdot\|_{\widehat{\kappa}(x)}$.

3.5.10. Definition. — Assume that $|\cdot|$ is non-Archimedean. We denote by ξ the point in $\mathbb{P}(E)^{\mathrm{an}}$ which is the generic point of $\mathbb{P}(E)^{\mathrm{an}}$ equipped with the absolute value

$$|\cdot|_\xi : k\big(\tfrac{e_0}{e_r}, \dots, \tfrac{e_{r-1}}{e_r}\big) \longrightarrow \mathbb{R}_{\geqslant 0}$$

such that, for any

$$P = \sum_{\boldsymbol{a}=(a_0,\dots,a_{r-1})\in\mathbb{N}^d} \lambda_{\boldsymbol{a}} \Big(\frac{e_0}{e_r}\Big)^{a_0} \cdots \Big(\frac{e_{r-1}}{e_r}\Big)^{a_{r-1}} \in k\big[\tfrac{e_0}{e_r}, \dots, \tfrac{e_{r-1}}{e_r}\big],$$

one has

$$|P|_\xi = \max_{\boldsymbol{a}\in\mathbb{N}^d} |\lambda_{\boldsymbol{a}}|.$$

Note that the point ξ does not depend on the choice of the orthonormal basis $(e_j)_{j=0}^r$. In fact, the norm $\|\cdot\|$ induces a symmetric algebra norm on $k[E]$ (which is often called a *Gauss norm*) and hence defines an absolute value on the fraction field of $k[E]$. The restriction of this absolute value to the field of rational functions on $\mathbb{P}(E)$ identifies with $|\cdot|_\xi$. Hence ξ is called the *Gauss point* of $\mathbb{P}(E)^{\mathrm{an}}$.

3.5.11. Proposition. — *Assume that the absolute value $|\cdot|$ is non-Archimedean. The following equality holds*

$$c_1(\overline{\mathcal{O}_E(1)})^r = \mathrm{Dirac}_\xi,$$

where Dirac_ξ *denotes the Dirac measure at* ξ.

Proof. — Let k° be the valuation ring of $(k, |\cdot|)$, $\mathfrak{m}$ be the maximal ideal of k°, and $\kappa = k^\circ/\mathfrak{m}$ be the residue field of k°. Let $\mathcal{E}$ be the free k°-module generated by $\{e_0, \dots, e_r\}$. Then $\mathbb{P}(\mathcal{E})$ is a projective flat k°-scheme such that

$$\mathbb{P}(\mathcal{E}) \times_{\operatorname{Spec} k^\circ} \operatorname{Spec} k \cong \mathbb{P}(E).$$

Note that the fibre product

$$\mathbb{P}(\mathcal{E}) \times_{\operatorname{Spec} k^\circ} \operatorname{Spec} \kappa$$

is isomorphic to $\mathbb{P}(\mathcal{E} \otimes_{k^\circ} \kappa)$, which is an integral κ-scheme. Therefore, one has (see Example 3.5.2)

$$c_1(\overline{\mathcal{O}_E(1)})^r = \deg(c_1(\mathcal{O}_{\mathcal{E}_\kappa}(1))^r \cap \mathbb{P}(\mathcal{E}_\kappa))\,\mathrm{Dirac}_\xi = \mathrm{Dirac}_\xi\,.$$

□

3.5.12. Remark. — Assume that $k = \mathbb{C}$ and $|\cdot|$ is the usual absolute value. Let $(E, \|\cdot\|)$ be a Hermitian space and

$$\mathbb{S}(E^\vee, \|\cdot\|_*) = \{\alpha \in E^\vee \mid \|\alpha\|_* = 1\}$$

be the unit sphere in E, where $\|\cdot\|_*$ denotes the dual norm of $\|\cdot\|$, which is also a Hermitian norm. Note that $\mathbb{P}(E)^{\mathrm{an}}$ identifies with the quotient of $\mathbb{S}(E^\vee, \|\cdot\|_*)$ by the action of the unit sphere $\mathbb{S}(\mathbb{C}) = \{z \in \mathbb{C} \mid |z| = 1\}$ in $\mathbb{C}$. We equip the universal invertible sheaf $\mathcal{O}_E(1)$ with the orthogonal quotient metric induced by $\|\cdot\|$ and equip $\mathbb{S}(E^\vee, \|\cdot\|_*)$ with the unique $U(E^\vee, \|\cdot\|_*)$-invariant Borel probability measure $\eta_{\mathbb{S}(E^\vee,\|\cdot\|_*)}$ which is locally equivalent to Lebesgue measure. Then the measure

$$c_1(\overline{\mathcal{O}_E(1)})^{\dim_{\mathbb{C}}(E)-1}$$

identifies with the direct image of $\eta_{\mathbb{S}(E^\vee,\|\cdot\|_*)}$ by the projection map from $\mathbb{S}(E^\vee, \|\cdot\|_*)$ to $\mathbb{P}(E)^{\mathrm{an}}$ (see for example [**5**, (1.4.7)] for more details).

3.5.13. Theorem. — *Let $\overline{L} = (L, \varphi), \overline{L}_1 = (L_1, \varphi_1), \ldots, \overline{L}_d = (L_d, \varphi_d)$ be integrable metrized invertible $\mathcal{O}_X$-modules. Let s be a regular meromorphic section of L. Then $g = -\log|s|_\varphi$ is integrable with respect to $\mu_{\overline{L}_1\cdots\overline{L}_d}$.*

Proof. — The proof of this theorem is same as [**10**, Théorème 4.1]. We prove it without using the local intersection numbers.

Clearly we may assume that X is integral, $L, L_1, \ldots, L_d$ are ample and $\overline{L}, \overline{L}_1, \ldots, \overline{L}_d$ are semi-positive. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{O}_X$ given by

$$\mathcal{I}_x = \{a \in \mathcal{O}_{X,x} \mid as_x \in L_x\}.$$

Choose a positive number m and a non-zero section $t_1 \in H^0(X, \mathcal{I}L^{\otimes m}) \setminus \{0\}$. If we set $t_2 = t_1 \otimes s$, then $s = t_2 \otimes t_1^{-1}$ and $t_2 \in H^0(X, L^{\otimes m+1}) \setminus \{0\}$ and $g = -\log|t_2|_{(m+1)\varphi} + \log|t_1|_{m\varphi}$, so that we may assume that $s \in H^0(X, L) \setminus \{0\}$. Let φ' be a metric of L such that either (a) if v is Archimedean, φ' is C^∞ and semi-positive, or (b) if v is non-Archimedean, φ' is a nef model metric. Then $-\log|s|_\varphi + \log|s|_{\varphi'}$ is a continuous function, so that we may assume that $\varphi = \varphi'$. By Theorem 3.4.6, there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ of smooth functions on X^{an} with the following properties:

(1) for all $n \in \mathbb{N}$, $\psi_n \leqslant g$, $\psi_n \leqslant \psi_{n+1}$.
(2) for each point $x \in X^{\mathrm{an}}$, $\sup\{\psi_n(x) \mid n \in \mathbb{N}\} = g(x)$.
(3) for all $n \in \mathbb{N}$, $g - \psi_n$ is a Green function of D such that either
    (3.a) if v is Archimedean, the metric of $|\cdot|_{g-\psi_n}$ of L is C^∞ and semi-positive, or
    (3.b) if v is non-Archimedean, the metric of $|\cdot|_{g-\psi_n}$ of L is a nef model metric.

We prove the assertion by induction on the number

$$e := \mathrm{Card}\{i \in \{1, \ldots, d\} \mid \varphi_i \text{ is not smooth}\}.$$

If $e = 0$, that is, φ_i is smooth for all i, then the assertion is obvious. We assume that $e > 0$. Obviously we may assume that φ_1 is not smooth. Let φ_1' be a semi-positive and smooth metric of L_1. If we choose a continuous function ϑ such that $|\cdot|_{\varphi_1} = \exp(-\vartheta)|\cdot|_{\varphi_1'}$, then $c_1(\overline{L}_1) = c_1(\overline{L}_1') + dd^c(\vartheta)$, where $\overline{L}_1' = (L_1, \varphi_1')$.

Let us consider the following integral:

$$I_n := \int_{X^{\mathrm{an}}} \psi_n\, c_1(\overline{L}_1) \cdots c_1(\overline{L}_d).$$

Note that ψ_n and ϑ are locally written by differences of plurisubharmonic functions, so that, by [**10**, Proposition 2.3],

$$\begin{aligned}I_n &= \int_{X^{\mathrm{an}}} \psi_n\, c_1(\overline{L}_1') \cdots c_1(\overline{L}_d) + \int_{X^{\mathrm{an}}} \psi_n\, \mathrm{dd}^{\mathrm{c}}(\vartheta) c_1(\overline{L}_2) \cdots c_1(\overline{L}_d) \\ &= \int_{X^{\mathrm{an}}} \psi_n\, c_1(\overline{L}_1') \cdots c_1(\overline{L}_d) + \int_{X^{\mathrm{an}}} \vartheta\, \mathrm{dd}^{\mathrm{c}}(\psi_n) c_1(\overline{L}_2) \cdots c_1(\overline{L}_d).\end{aligned}$$

By the hypothesis of induction,

$$\lim_{n\to\infty} \int_{X^{\mathrm{an}}} \psi_n\, c_1(\overline{L}_1') \cdots c_1(\overline{L}_d)$$

exists. Moreover, by the same arguments as the last part of [**10**, Théorèm 4.1], one can see

$$\begin{aligned}\lim_{n\to\infty} &\int_{X^{\mathrm{an}}} \vartheta\, \mathrm{dd}^{\mathrm{c}}(\psi_n) c_1(\overline{L}_2) \cdots c_1(\overline{L}_d) \\ &= \int_{X^{\mathrm{an}}} \vartheta\, c_1(\overline{L}) c_1(\overline{L}_2) \cdots c_1(\overline{L}_d) - \int_{\mathrm{div}(s)^{\mathrm{an}}} \vartheta\, c_1(\overline{L}_2) \cdots c_1(\overline{L}_d).\end{aligned}$$

Therefore $\lim_{n\to\infty} I_n$ exists, as required. □

3.6. Local intersection number over an algebraically closed field

Let k be an algebraically closed field equipped with a non-trivial absolute value $|\cdot|$ such that k is complete with respect to the topology defined by $|\cdot|$. The pair $(k, |\cdot|)$ is denoted by v. Let X be a projective scheme over k and d be its dimension. Recall that any element x of X^{an} consists of a scheme point of X and an absolute value $|\cdot|_x$ of the residue field of the scheme point. We denote by $\widehat{\kappa}(x)$ the completion of the residue field of the scheme point with respect to the absolute value $|\cdot|_x$, on which the absolute value extends by continuity.

3.6.1. Definition. — Let $(D_0, g_0), \ldots, (D_d, g_d)$ be integrable metrized Cartier divisors on X. We assume that $D_0, \ldots, D_d$ intersect properly, that is, $(D_0, \ldots, D_d) \in \mathcal{IP}_X$ (see Definition 1.3.2). According to [**10**], we define *the local intersection number* $\big((D_0, g_0) \cdots (D_d, g_d)\big)_v$ *at* v as follows.

In the case where $d = 0$, one has $X = \mathrm{Spec}(A)$ for some k-algebra with $\dim_k(A) < \infty$. By Remark 1.2.10 and Example 3.4.2,

$$A = \bigoplus_{x \in \mathrm{Spec}(A)} A_x \quad \text{and} \quad (D_0, g_0) = \sum_{x \in \mathrm{Spec}(A)} (0, a_x),$$

where $a_x \in \mathbb{R}$ for all $x \in \mathrm{Spec}(A)$. Then

$$\big((D_0, g_0)\big)_v := \sum_{x \in \mathrm{Spec}(A)} \mathrm{length}_{A_x}(A_x)\, a_x. \tag{3.13}$$

Note that $\mathrm{length}_{A_x}(A_x) = \dim_k(A_x)$ because k is algebraically closed.

If $d > 0$ and $\sum_{i=1}^{n} a_i Z_i$ is the cycle associated with D_d (cf. Remark 1.2.11), then the local intersection number $\big((D_0, g_0)\cdots(D_d, g_d)\big)_v$ is defined in a recursive way with respect to $d = \dim(X)$ as

$$\sum_{i=1}^{n} a_i \Big((D_0, g_0)|_{Z_i} \cdots (D_{d-1}, g_{d-1})|_{Z_i} \Big)_v + \int_{X^{\mathrm{an}}} g_d(x)\, \mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})}(\mathrm{d}x). \quad (3.14)$$

For the integrability of g_d with respect to the measure $\mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})}$, see Theorem 3.5.13.

3.6.2. Proposition. — *Let $X_1, \ldots, X_\ell$ be irreducible components of X and $\eta_1, \ldots, \eta_\ell$ be the generic points of $X_1, \ldots, X_\ell$, respectively. Then*

$$\big((D_0, g_0)\cdots(D_d, g_d)\big)_v = \sum_{j=1}^{\ell} \mathrm{length}_{\mathcal{O}_{X,\eta_j}}(\mathcal{O}_{X,\eta_j}) \Big((D_0, g_0)|_{X_j} \cdots (D_d, g_d)|_{X_j} \Big)_v.$$

Proof. — In the case where $d = 0$, the assertion is obvious. We assume that $d > 0$. By the definition of $\mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})}$ (cf. Section 3.5), if we set

$$b_j = \mathrm{length}_{\mathcal{O}_{X,\eta_j}}(\mathcal{O}_{X,\eta_j}),$$

then one has

$$\int_{X^{\mathrm{an}}} g_d(x)\, \mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})}(\mathrm{d}x) = \sum_{j=1}^{\ell} b_j \int_{X_j^{\mathrm{an}}} g_d(x)\, \mu_{(D_0,g_0)|_{X_j}\cdots(D_{d-1},g_{d-1})|_{X_j}}(\mathrm{d}x).$$

If $\sum_{i=1}^{n} a_i Z_i$ and $\sum_{i=1}^{n} a_{ji} Z_i$ are the cycles associated with D_d and $D_d|_{X_j}$, respectively, then, by (1.3), $a_i = \sum_{j=1}^{\ell} b_j a_{ji}$, so that

$$\begin{aligned}
\sum_{i=1}^{n} a_i \Big((D_0, g_0)|_{Z_i} \cdots (D_{d-1}, g_{d-1})|_{Z_i} \Big)_v
&= \sum_{i=1}^{n}\sum_{j=1}^{\ell} b_j a_{ji} \Big((D_0, g_0)|_{Z_i} \cdots (D_{d-1}, g_{d-1})|_{Z_i} \Big)_v \\
&= \sum_{j=1}^{\ell} b_j \sum_{i=1}^{n} a_{ji} \Big((D_0, g_0)|_{Z_i} \cdots (D_{d-1}, g_{d-1})|_{Z_i} \Big)_v.
\end{aligned}$$

Therefore, since

$$\Big((D_0, g_0)|_{X_j} \cdots (D_d, g_d)|_{X_j} \Big)_v = \sum_{i=1}^{n} a_{ji} \Big((D_0, g_0)|_{Z_i} \cdots (D_{d-1}, g_{d-1})|_{Z_i} \Big)_v + \int_{X_j^{\mathrm{an}}} g_d(x)\, \mu_{(D_0,g_0)|_{X_j}\cdots(D_{d-1},g_{d-1})|_{X_j}}(\mathrm{d}x),$$

one has the desired formula. □

3.6.3. Proposition. — *Let $(D_0, g_0)\ldots,(D_i, g_i),(D'_i, g'_i),\ldots,(D_d, g_d)$ be integrable metrized Cartier divisors on X such that $(D_0,\ldots,D_i,\ldots,D_d)$ and $(D_0,\ldots,D'_i,\ldots,D_d)$ belong to $\mathcal{IP}_X$. Then one has the following:*

(1) *The local intersection pairing is multi-linear, that is,*

$$\begin{cases} \big((D_0,g_0)\cdots(D_i+D'_i,g_i+g'_i)\cdots(D_d,g_d)\big)_v \\ \quad = \big((D_0,g_0)\cdots(D_i,g_i)\cdots(D_d,g_d)\big)_v + \big((D_0,g_0)\cdots(D'_i,g'_i)\cdots(D_d,g_d)\big)_v. \\ \big((D_0,g_0)\cdots(-D_i,-g_i)\cdots(D_d,g_d)\big)_v = -\big((D_0,g_0)\cdots(D_i,g_i)\cdots(D_d,g_d)\big)_v. \end{cases}$$

(2) *We assume that $D_0,\ldots,D_d$ are ample and $g_0,\ldots,g_d$ are plurisubharmonic. For each i, let $(g_{i,n})_{n=1}^{\infty}$ be a sequence of plurisubharmonic Green functions of D_i such that $\lim_{n\to\infty}\|g_i - g_{i,n}\|_{\sup} = 0$. Then*

$$\lim_{n\to\infty}\big((D_0,g_{0,n})\cdots(D_d,g_{d,n})\big)_v = \big((D_0,g_0)\cdots(D_d,g_d)\big)_v$$

(3) *The local intersection pairing is symmetric, that is, for any bijection $\sigma : \{0,\ldots,d\}\to\{0,\ldots,d\}$ one has*

$$\big((D_{\sigma(0)},g_{\sigma(0)})\cdots\cdots(D_{\sigma(d)},g_{\sigma(d)})\big)_v = \big((D_0,g_0)\cdots(D_d,g_d)\big)_v.$$

Proof. — Clearly we may assume that X is integral. We prove (1), (2) and (3) by induction on d. In the case $d = 0$, the assertion is obvious, so that we assume $d > 0$.

(1) If $0 \leqslant i < d$, the assertions follow from the hypothesis of induction and the multi-linearity of the measure $\mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})}$ with respect to (D_0, g_0), …, (D_{d-1}, g_{d-1}), so that we may assume that $i = d$. Let $D_d = a_1Z_1 + \cdots + a_nZ_n$ and $D'_d = a'_1Z_1 + \cdots + a'_nZ_n$ be the decompositions of D_d and D'_d as cycles. Then $D_d + D'_d = (a_1 + a'_1)Z_1 + \cdots + (a_n + a'_n)Z_n$ and $-D_d = (-a_1)Z_1 + \cdots + (-a_n)Z_n$, so that the assertions are obvious.

(2) By (3.14) and the hypothesis of induction, it is sufficient to see

$$\lim_{n\to\infty}\int_{X^{\mathrm{an}}} g_{d,n}\mu_{(D_0,g_{0,n})\cdots(D_{d-1},g_{d-1,n})} = \int_{X^{\mathrm{an}}} g_d\mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})},$$

which follows from [**18**, Corollary (3.6)] and [**11**, Corollaire (5.6.5)].

(3) We may assume that $D_0,\ldots,D_d$ are ample and $g_0,\ldots,g_d$ are plurisubharmonic. By (2) together with regularizations of metrics, we may further assume that metrics $|\cdot|_{g_0},\ldots,|\cdot|_{g_d}$ are smooth. It suffices to prove the assertion in the particular case where σ is a transposition exchanging two indices i and j with $i < j$. If $j < d$, then the assertion follows from the hypothesis of induction, so that we may assume that $j = d$.

If $i < d-1$, then

$$\begin{aligned}\big((D_0,g_0)\cdots(D_i,g_i)\cdots(D_{d-1},g_{d-1})\cdot(D_d,g_d)\big)_v \\ = \big((D_0,g_0)\cdots(D_{d-1},g_{d-1})\cdots(D_i,g_i)\cdot(D_d,g_d)\big)_v.\end{aligned}$$

by the hypothesis of induction. Therefore we may assume that $i = d-1$. Let $D_d = a_1Z_1 + \cdots + a_nZ_n$ and $D_{d-1}|_{Z_i} = a_{i1}Z_{i1} + \cdots + a_{in}Z_{in}$ be the decomposition as cycles. Then

$$\begin{aligned}&\big((D_0,g_0)\cdots(D_{d-1},g_{d-1})\cdot(D_d,g_d)\big)_v \\ &\qquad = \sum_{i,j} a_i a_{ij}\big((D_0,g_0)|_{Z_{ij}}\cdots(D_{d-2},g_{d-2})|_{Z_{ij}}\big)_v \\ &\qquad\qquad + \sum_i a_i \int_{Z_i^{\mathrm{an}}} g_{d-1}(x)\,\mu_{(D_0,g_0)|_{Z_i}\cdots(D_{d-2},g_{d-2})|_{Z_i}}(\mathrm{d}x) \\ &\qquad\qquad\qquad + \int_{X^{\mathrm{an}}} g_d(x)\,\mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})}(\mathrm{d}x).\end{aligned}$$

In the same way, if $D_{d-1} = a'_1Z'_1 + \cdots + a'_nZ'_n$ and $D_d|_{Z'_i} = a'_{i1}Z'_{i1} + \cdots + a'_{in}Z'_{in}$ be the decomposition as cycles, then

$$\begin{aligned}&\big((D_0,g_0)\cdots(D_d,g_d)\cdot(D_{d-1},g_{d-1})\big)_v \\ &\qquad = \sum_{i,j} a'_i a'_{ij}\big((D_0,g_0)|_{Z'_{ij}}\cdots(D_{d-2},g_{d-2})|_{Z'_{ij}}\big)_v \\ &\qquad\qquad + \sum_i a'_i \int_{(Z'_i)^{\mathrm{an}}} g_d(x)\,\mu_{(D_0,g_0)|_{Z'_i}\cdots(D_{d-2},g_{d-2})|_{Z'_i}}(\mathrm{d}x) \\ &\qquad\qquad\qquad + \int_{X^{\mathrm{an}}} g_{d-1}(x)\,\mu_{(D_0,g_0)\cdots(D_{d-2},g_{d-2})\cdot(D_d,g_d)}(\mathrm{d}x).\end{aligned}$$

By [**49**, Proposition 5.2 (2)], one has $\sum_{ij} a_ia_{ij}Z_{ij} = \sum_{ij} a'_ia'_{ij}Z'_{ij}$ as cycles, so that it is sufficient to show that

$$\begin{aligned}&\sum_i a_i \int_{Z_i^{\mathrm{an}}} g_{d-1}(x)\,\mu_{(D_0,g_0)|_{Z_i}\cdots(D_{d-2},g_{d-2})|_{Z_i}}(\mathrm{d}x) + \int_{X^{\mathrm{an}}} g_d(x)\,\mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})}(\mathrm{d}x) \\ &\qquad = \sum_i a'_i \int_{(Z'_i)^{\mathrm{an}}} g_d(x)\,\mu_{(D_0,g_0)|_{Z'_i}\cdots(D_{d-2},g_{d-2})|_{Z'_i}}(\mathrm{d}x) \\ &\qquad\qquad + \int_{X^{\mathrm{an}}} g_{d-1}(x)\,\mu_{(D_0,g_0)\cdots(D_{d-2},g_{d-2})\cdot(D_d,g_d)}(\mathrm{d}x),\end{aligned}$$

which is nothing more than [**49**, Theorem 5.6] for the Archimedean case and [**36**, Proposition 11.5] for the non-Archimedean case. □

3.6.4. Proposition. — *Let $\pi : Y \to X$ be a surjective morphism of integral projective schemes over k. We set $e = \dim X$ and $d = \dim Y$. Let $(D_0,g_0),\ldots,(D_d,g_d)$*

be integrable metrized Cartier divisors on X such that $(\pi^*(D_0),\ldots,\pi^*(D_d)) \in \mathcal{IP}_Y$. Then one has the following:

(1) *If $d > e$, then $(\pi^*(D_0,g_0)\cdots\pi^*(D_d,g_d))_v = 0$.*
(2) *If $d = e$ and $(D_0,\ldots,D_d) \in \mathcal{IP}_X$, then*

$$(\pi^*(D_0,g_0)\cdots\pi^*(D_d,g_d))_v = (\deg\pi)((D_0,g_0)\cdots(D_d,g_d))_v.$$

Proof. — We prove (1) and (2) by induction on e. If $e = 0$, then (2) is obvious. For (1), as $\pi^*(D_0,g_0) = (0,a_0),\ldots,\pi^*(D_d,g_d) = (0,a_d)$ for some $a_0,\ldots,a_d \in \mathbb{R}$, then

$$(\pi^*(D_0,g_0)\cdots\pi^*(D_d,g_d))_v = \int_{X^{\mathrm{an}}} a_d\mu_{(0,a_0)\cdots(0,a_d)} = 0,$$

as desired.

We assume $e > 0$. Let $D_d = a_1Z_1 + \cdots + a_nZ_n$ and $\pi^*(D_d) = b_1Z'_1 + \cdots + b_NZ'_N$ be the decompositions as cycles. By (3.14),

$$\begin{aligned}\big(\pi^*(D_0,g_0)\cdots\pi^*(D_d,g_d)\big)_v = \sum_{j=1}^N b_j\big(\pi^*(D_0,g_0)|_{Z'_j}\cdots\pi^*(D_{d-1},g_{d-1})|_{Z'_j}\big)_v \\ + \int_{Y^{\mathrm{an}}} g_d(\pi^{\mathrm{an}}(y))\,\mu_{\pi^*(D_0,g_0)\cdots\pi^*(D_d,g_d)}(\mathrm{d}y),\end{aligned}$$

Note that if $e < d$, then $\dim\pi(Z'_j) < \dim Z'_j$ and $\pi_*(\mu_{\pi^*(D_0,g_0)\cdots\pi^*(D_d,g_d)}) = 0$ by Proposition 3.5.5, so that one has (1).

Next we assume that $e = d$. For each i, we set $J_i = \{j \in \{1,\ldots,N\} \mid \pi(Z'_j) = Z_i\}$. We set $J_0 = \{1,\ldots,N\} \setminus (J_1 \cup \cdots \cup J_n)$. By the hypothesis of induction for (1), $\big(\pi^*(D_0,g_0)|_{Z'_j}\cdots\pi^*(D_{d-1},g_{d-1})|_{Z'_j}\big)_v = 0$ for all $j \in J_0$, so that, by the hypothesis of induction for (2) and Proposition 3.5.5, the above equation implies

$$\begin{aligned}&\big(\pi^*(D_0,g_0)\cdots\pi^*(D_d,g_d)\big)_v \\ &\quad= \sum_{i=1}^n\sum_{j\in J_i} b_j\big(\pi^*(D_0,g_0)|_{Z'_j}\cdots\pi^*(D_{d-1},g_{d-1})|_{Z'_j}\big)_v \\ &\qquad\qquad + \int_{Y^{\mathrm{an}}} g_d(\pi^{\mathrm{an}}(y))\,\mu_{\pi^*(D_0,g_0)\cdots\pi^*(D_d,g_d)}(\mathrm{d}y) \\ &\quad= \sum_{i=1}^n\big((D_0,g_0)|_{Z_i}\cdots(D_{d-1},g_{d-1})|_{Z_i}\big)_v\sum_{j\in J_i} b_j\deg(\pi|_{Z'_j}) \\ &\qquad\qquad + \deg(\pi)\int_{X^{\mathrm{an}}} g_d(x)\,\mu_{(D_0,g_0)\cdots(D_d,g_d)}(\mathrm{d}x).\end{aligned}$$

Therefore, the assertion follows because $\sum_{j\in J_i} b_j\deg(\pi|_{Z'_j}) = \deg(\pi)a_i$ (cf. [**49**, Lemma 1.12]). □

3.6.5. Proposition. — *Let f be a regular meromorphic function on X and $(D_1,g_1),\ldots,(D_d,g_d)$ be integrable metrized Cartier divisors on X such that*

$(\mathrm{div}(f), D_1, \ldots, D_d) \in \mathcal{IP}_X$. *If we set* $D_1 \cdots D_d = \sum_{x \in X_{(0)}} a_x x$ *as cycle, then*

$$\big(\widehat{\mathrm{div}}(f) \cdot (D_1, g_1) \cdots (D_d, g_d)\big)_v = \sum_{x \in X_{(0)}} a_x\big(- \log |f|(x^{\mathrm{an}})\big), \tag{3.15}$$

where $X_{(0)}$ *is the set of all closed point of* X *and* x^{an} *is the associated absolute value at* x. *Note that in the case where* $\dim(X) = 0$, *the above formula means that*

$$\big(\widehat{\mathrm{div}}(f)\big)_v = 0.$$

Proof. — Let $X = a_1 X_1 + \cdots + a_n X_n$ be the decomposition as cycles. Then

$$\big(\widehat{\mathrm{div}}(f) \cdot (D_1, g_1) \cdots (D_d, g_d)\big)_v = \sum_{i=1}^{n} a_i \big(\widehat{\mathrm{div}}(f)|_{X_i} \cdot (D_1, g_1)|_{X_i} \cdots (D_d, g_d)|_{X_i}\,\big)_v$$

and

$$D_1 \cdots D_d = \sum_{i=1}^{n} a_i \big(\, D_1|_{X_i} \cdots D_d|_{X_i}\,\big),$$

so that we may assume that X is integral.

We prove the equality (3.15) by induction on $d = \dim(X)$. In the case where $\dim(X) = 0$, the assertion is obvious because f is a unit. We assume that $\dim(X) \geqslant 1$. Let $D_d = a_1 Z_1 + \cdots + a_n Z_n$ be the decomposition as cycles. Let $\sum_{x \in X_{(0)}} b_{ix} x$ be the decomposition of $D_1|_{Z_i} \cdots D_{d-1}|_{Z_i} = D_1 \cdots D_{d-1} \cdot Z_i$ as cycles. Then

$$\sum_{i=1}^{n} a_i \sum_{x \in X_{(0)}} b_{ix} x = \sum_{x \in X_{(0)}} a_x x,$$

so that $a_x = \sum_{i=1}^{n} a_i b_{ix}$. On the other hand, by hypothesis of induction,

$$\big(\widehat{\mathrm{div}}(f)|_{Z_i} \cdot (D_1, g_1)|_{Z_i} \cdots (D_{d-1}, g_{d-1})|_{Z_i}\,\big)_v = \sum_{x \in X_{(0)}} b_{ix}(-\log |f|(x^{\mathrm{an}})).$$

Therefore,

$$\begin{aligned}
&\sum_{x \in X_{(0)}} a_x(-\log |f|(x^{\mathrm{an}})) \\
&\qquad = \sum_{x \in X_{(0)}} \Big(\sum_{i=1}^{n} a_i b_{ix}\Big)(-\log |f|(x^{\mathrm{an}})) = \sum_{i=1}^{n} a_i \sum_{x \in X} b_{ix}(-\log |f|(x^{\mathrm{an}})) \\
&\qquad\qquad = \sum_{i=1}^{n} a_i \big(\widehat{\mathrm{div}}(f)|_{Z_i} \cdot (D_1, g_1)|_{Z_i} \cdots (D_{d-1}, g_{d-1})|_{Z_i}\,\big)_v.
\end{aligned}$$

Note that $\mu_{(\widehat{\mathrm{div}}(f) \cdot (D_1, g_1) \cdots (D_d, g_d))} = 0$, and hence the assertion follows by (3.14). $\square$

3.6.6. Proposition. — *Let $(D_0, g_0), \ldots, (D_{d-1}, g_{d-1}), (0, g)$ be integrable metrized Cartier divisors on X with $(D_0, \ldots, D_{d-1}, 0) \in \mathcal{IP}_X$. We assume that $D_0, \ldots, D_{d-1}$ are semiample and $g_0, \ldots, g_{d-1}$ are plurisubharmonic. Then*

$$\big((D_0, g_0) \cdots (D_{d-1}, g_{d-1}) \cdot (0, g)\big)_v = \int_{X^{\mathrm{an}}} g(x) \mu_{(D_0,g_0)\cdots(D_{d-1},g_{d-1})}(\mathrm{d}x).$$

In particular,

$$\begin{aligned} &\min\{g(x) \mid x \in X^{\mathrm{an}}\}(D_0 \cdots D_{d-1}) \\ &\qquad\qquad \leqslant \big((D_0, g_0) \cdots (D_{d-1}, g_{d-1}) \cdot (0, g)\big)_v \\ &\qquad\qquad\qquad\qquad \leqslant \max\{g(x) \mid x \in X^{\mathrm{an}}\}(D_0 \cdots D_{d-1}). \end{aligned}$$

Proof. — This is trivial by the definition. □

3.6.7. Corollary. — *Let $(D_0, g_0), \ldots, (D_d, g_d)$ be integrable arithmetic Cartier divisors on X with $(D_0, \ldots, D_d) \in \mathcal{IP}_X$. We assume that $D_0, \ldots, D_d$ are semiample and $g_0, \ldots, g_d$ are plurisubmarmonic. Let $g'_0, \ldots, g'_d$ be another plurisubharmonic Green functions of $D_0, \ldots, D_d$, respectively. Then one has*

$$\begin{aligned} &\left|\big((D_0, g'_0) \cdots (D_d, g'_d)\big)_v - \big((D_0, g_0) \cdots (D_d, g_d)\big)_v\right| \\ &\qquad\qquad \leqslant \sum_{i=0}^{d} \max\{|g'_i - g_i|(x) \mid x \in X^{\mathrm{an}}\}(D_0 \cdots D_{i-1} \cdot D_{i+1} \cdots D_d). \end{aligned}$$

Proof. — By using Proposition 3.6.3,

$$\begin{aligned} &\big((D_0, g'_0) \cdots (D_d, g'_d)\big) - \big((D_0, g_0) \cdots (D_d, g_d)\big) \\ &\qquad = \sum_{i=0}^{d} \big((D_0, g_0) \cdots (D_{i-1}, g_{i-1}) \cdot (0, g'_i - g_i) \cdot (D_{i+1}, g'_{i+1}) \cdots (D_d, g'_d)\big), \end{aligned}$$

so that the assertion follows from Proposition 3.6.6. □

3.6.8. Proposition. — *We assume that $X = \mathbb{P}^d_k$ and $L = \mathcal{O}_{\mathbb{P}^d}(1)$. Let $\{T_0, \ldots, T_d\}$ be a basis of $H^0(\mathbb{P}^d_k, \mathcal{O}_{\mathbb{P}^d}(1))$ over k. We view $(T_0 : \cdots : T_d)$ as a homogeneous coordinate of $\mathbb{P}^d_k$. Let $\|\cdot\|$ be a norm of $H^0(\mathbb{P}^d_k, \mathcal{O}_{\mathbb{P}^d}(1))$ given by*

$$\|a_0 T_0 + \cdots + a_d T_d\| = \begin{cases} \sqrt{|a_0|^2 + \cdots + |a_d|^2} & \text{if } v \text{ is Archimedean,} \\ \max\{|a_0|, \ldots, |a_d|\} & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

Let φ be the orthogonal quotient metric of $\mathcal{O}_{\mathbb{P}^d}(1)$ given by the surjective homomorphism $H^0(\mathbb{P}^d_k, \mathcal{O}_{\mathbb{P}^d}(1)) \otimes \mathcal{O}_{\mathbb{P}^d} \to \mathcal{O}_{\mathbb{P}^d}(1)$ and the above norm $\|\cdot\|$. We set $H_i = \{T_i = 0\}$ and $h_i = -\log|T_i|_\varphi$. Then

$$((H_0, h_0) \cdots (H_d, h_d))_v = \begin{cases} \widehat{\deg}\Big(\widehat{c}_1(\mathcal{O}_{\mathbb{P}^d_{\mathbb{Z}}}(1), \varphi)^{d+1}\Big) & \text{if } v \text{ is Archimedean,} \\ 0 & \text{if } v \text{ is non-Archimedean,} \end{cases}$$

where $\widehat{\deg}\Big(\widehat{c}_1(\mathcal{O}_{\mathbb{P}^d_{\mathbb{Z}}}(1),\varphi)^{d+1}\Big)$ *is the self-intersection number of the arithmetic first Chern class* $\widehat{c}_1(\mathcal{O}_{\mathbb{P}^d_{\mathbb{Z}}}(1),\varphi)$ *on the d-dimensional projective space* $\mathbb{P}^d_{\mathbb{Z}}$ *over* $\mathbb{Z}$*.*

Proof. — If we set

$$a_m := \int_{\mathbb{P}^m_k} -\log|T_m|_\varphi(x)\,\mu_{(\mathcal{O}_{\mathbb{P}^m}(1),\varphi_{\mathrm{FS}})^m}(\mathrm{d}x)$$

for a positive integer m, then

$$((H_0,h_0)\cdots(H_d,h_d))_v = \sum_{m=1}^d a_m.$$

In the following, we set $x_i = T_i/T_0$.

- **Archimedean case** : The algorithms of the calculation are exactly same as one on $\mathbb{P}^d_{\mathbb{Z}}$, so that we have the assertion.

- **non-Archimedean case** : If we set $|f|_* = \max_{i_1,\ldots,i_m}\{|c_{i_1,\ldots,i_m}|\}$ for

$$f = \sum_{i_1,\ldots,i_m} c_{i_1,\ldots,i_m} x_1^{i_1}\cdots x_m^{i_m} \in k[x_1,\ldots,x_m],$$

then $|\cdot|_*$ extends to an absolute value of $k(x_1,\ldots,x_m)$ (cf. Lemma 2.6.3). We set $U = \{T_m \neq 0\}$. Note that if $\xi \in U^{\mathrm{an}}$, then

$$|T_m|_\varphi(\xi) = \frac{|x_m|_\xi}{\max\{1,|x_1|_\xi,\ldots,|x_m|_\xi\}}.$$

Let $\mathfrak{o}_v$ be the valuation ring of v. Note that φ coincides with the metric of the model $(\mathbb{P}^d_{\mathfrak{o}_v},\mathcal{O}_{\mathbb{P}^d_{\mathfrak{o}_v}}(1))$ by [**13**, Proposition 2.3.12], so that $\mu_{(\mathcal{O}_{\mathbb{P}^m}(1),\varphi)^m} = \delta_{|\cdot|_*}$. Thus

$$a_m = -\log\frac{|x_m|_*}{\max\{1,|x_1|_*,\ldots,|x_m|_*\}} = 0,$$

and hence the assertion follows. □

3.7. Local intersection number over a general field

In this section, we consider the local intersection product and local height formula in the non-necessarily algebraically closed case. We fix in this section a complete valued field $v = (k,|\cdot|)$ such that $|\cdot|$ is not trivial. Let $\mathbb{C}_k$ be the completion of an algebraic closure of k. Note that the absolute value $|\cdot|$ extends naturally to $\mathbb{C}_k$ and the valued field $(\mathbb{C}_k,|\cdot|)$ is both algebraically closed and complete. We denote by v^{ac} the couple $(\mathbb{C}_k,|\cdot|)$. We also fix a projective morphism $\pi : X \to \operatorname{Spec} k$ and we denote by $X_{\mathbb{C}_k}$ the fiber product $X\times_{\operatorname{Spec} k}\operatorname{Spec}\mathbb{C}_k$. Let d be the Krull dimension of X, which is also equal to the Krull dimension of $X_{\mathbb{C}_k}$.

3.7.1. Definition. — Let $(D_0, g_0), \ldots, (D_d, g_d)$ be a family of metrized Cartier divisor on X such that $D_0, \ldots, D_d$ intersect properly and that $g_0, \ldots, g_d$ are integrable Green functions. By Remark 1.3.5, the Cartier divisors $D_{0,\mathbb{C}_k}, \ldots, D_{d,\mathbb{C}_k}$ intersect properly. Moreover, by Remark 3.4.5, the Green functions $g_{0,\mathbb{C}_k}, \ldots, g_{d,\mathbb{C}_k}$ are integrable. We then define the local intersection number of $(D_0, g_0), \ldots, (D_d, g_d)$ as

$$\big((D_0, g_0) \cdots (D_d, g_d)\big)_v := \big((D_{0,\mathbb{C}_k}, g_{0,\mathbb{C}_k}) \cdots (D_{d,\mathbb{C}_k}, g_{d,\mathbb{C}_k})\big)_{v^{\mathrm{ac}}}$$

Several properties of the local intersection number follow directly from the results of the previous section. We gather them below.

3.7.2. Remark. — Recall that $\widehat{\mathrm{Int}}(X)$ denotes the group of integrable metrized Cartier divisors on X. Let $\widehat{\mathcal{IP}}_X$ be the subset of $\widehat{\mathrm{Int}}(X)^{d+1}$ consisting of elements

$$\big((D_0, g_0), \ldots, (D_d, g_d)\big)$$

such that the Cartier divisors $D_0, \ldots, D_d$ intersect properly.

(1) The set $\widehat{\mathcal{IP}}_X$ forms a symmetric multi-linear subset of the group $\widehat{\mathrm{Int}}(X)^{d+1}$. Moreover, the function of local intersection number

$$\big((D_0, g_0) \cdots (D_d, g_d)\big) \longmapsto \big((D_0, g_0) \cdots (D_d, g_d)\big)_v$$

form a symmetric multi-linear map from $\widehat{\mathcal{IP}}_X$ to $\mathbb{R}$. These statements follow from Proposition 3.6.3.

(2) Let $\pi : Y \to X$ be a surjective morphism of geometrically integral projective schemes over k. We set $e = \dim X$ and $d = \dim Y$. Let $(D_0, g_0), \ldots, (D_d, g_d)$ be integrable metrized Cartier divisors on X such that $(\pi^*(D_0), \ldots, \pi^*(D_d)) \in \mathcal{IP}_Y$. Then one has the following:

   (i) If $d > e$, then $(\pi^*(D_0, g_0) \cdots \pi^*(D_d, g_d))_v = 0$.

   (ii) If $d = e$ and $(D_0, \ldots, D_d) \in \mathcal{IP}_X$, then

$$(\pi^*(D_0, g_0) \cdots \pi^*(D_d, g_d))_v = (\deg \pi)((D_0, g_0) \cdots (D_d, g_d))_v.$$

We refer to Proposition 3.6.4 for a proof.

(3) Let f be a regular meromorphic function on X and $(D_1, g_1), \ldots, (D_d, g_d)$ be integrable metrized Cartier divisors on X such that $(\mathrm{div}(f), D_1, \ldots, D_d) \in \mathcal{IP}_X$. Suppose that

$$D_1 \cdots D_d = \sum_{x \in X_{(0)}} a_x x$$

as a cycle, then

$$\big(\widehat{\mathrm{div}}(f) \cdot (D_1, g_1) \cdots (D_d, g_d)\big)_v = \sum_{x \in X_{(0)}} a_x [\kappa(x) : k]_s \big(-\log |f|(x^{\mathrm{an}})\big),$$

where $[\kappa(x) : k]_s$ denotes the separable degree of the residue field $\kappa(x)$ over k. We refer to Proposition 3.6.5 for more details.

(4) Let $\big((D_0,g_0),\ldots,(D_d,g_d)\big)$ be an element of $\widehat{\mathcal{IP}}_X$. We assume that $D_0,\ldots,D_{d-1}$ are semi-ample, $g_0,\ldots,g_{d-1}$ are plurisubharmonic, and $D_d=0$. Then one has

$$\delta \min_{x\in X^{\mathrm{an}}} g_d(x) \leqslant \big((D_0,g_0)\cdots(D_d,g_d)\big)_v \leqslant \delta \max_{x\in X^{\mathrm{an}}} g_d(x),$$

where $\delta=(D_0\cdots D_{d-1})$. See Proposition 3.6.6 for more details.

(5) Let $\big((D_0,g_0),\ldots,(D_d,g_d)\big)$ and $\big((D_0,g_0'),\ldots,(D_d,g_d')\big)$ be two elements of $\widehat{\mathcal{IP}}_X$ having the same family of underlying Cartier divisors. One has

$$\Big|\big((D_0,g_0)\cdots(D_d,g_d)\big)_v - \big((D_0,g_0')\cdots(D_d,g_d')\big)_v\Big| \leqslant \sum_{i=0}^d \max_{x\in X^{\mathrm{an}}} |g_i'-g_i|(x)(D_0\cdots D_{i-1}\cdot D_{i+1}\cdots D_d).$$

See Corollary 3.6.7 for more details.

3.8. Local height

In this section, we fix a complete non-trivial valued field $v=(k,|\cdot|)$ and a projective scheme X over $\operatorname{Spec} k$. Let d be the dimension of X.

3.8.1. Definition. — Let $\overline{L}_i=(L_i,\varphi_i)$, $i\in\{0,\ldots,d\}$ be a family of metrized invertible $\mathcal{O}_X$-modules, where each L_i is an invertible $\mathcal{O}_X$-module, and φ_i is a continuous and integrable metric on L_i. For any $i\in\{0,\ldots,d\}$, we let s_i be a regular meromorphic section of L_i on X. Assume that the Cartier divisors $\operatorname{div}(s_0),\ldots,\operatorname{div}(s_d)$ intersect properly. We define the *local height* of X with respect to the family of metrized invertible $\mathcal{O}_X$-modules $(\overline{L}_i)_{i=0}^d$ and the family of regular meromorphic sections $(s_i)_{i=0}^d$ as the local intersection number (see Definition 3.7.1)

$$h^{s_0,\ldots,s_d}_{\overline{L}_0,\ldots,\overline{L}_d}(X) := \big(\widehat{\operatorname{div}}(s_0)\cdots\widehat{\operatorname{div}}(s_d)\big)_v.$$

3.8.2. Notation. — We often encounter the situation where each $\overline{L}_i$ is the pull-back by a projective morphism $f_i: X\to Y_i$ of a metrized invertible $\mathcal{O}_{Y_i}$-module $\overline{M_i}$ and s_i is the pull-back of a regular meromorphic section t_i. In such a situation, for simplicity of notation, we often use the expressions $h^{t_0,\ldots,t_d}_{\overline{M}_0,\ldots,\overline{M}_d}(X)$ or $h^{t_0,\ldots,t_d}_{\overline{L}_0,\ldots,\overline{L}_d}(X)$ to denote $h^{s_0,\ldots,s_d}_{\overline{L}_0,\ldots,\overline{L}_d}(X)$.

3.8.3. Remark. — We keep the notation of Definition 3.8.1 in assuming that the field k is algebraically closed. Let $X_1,\ldots,X_n$ be irreducible components of X, considered as reduced closed subscheme of X. For any $j\in\{1,\ldots,n\}$, let $\operatorname{mult}_{X_j}(X)$ be the multiplicity of the component X_j, which is by definition the length of the Artinian local ring of $\mathcal{O}_X$ at the generic point of X_j. Then, for any $j\in\{1,\ldots,n\}$, the divisors on X_j associated with the restricted sections $(s_i|_{X_j})_{i=0}^d$ intersect properly on X_j.

Assume firstly that $d=0$. In this case, each X_j consists of a closed point x_j of X, which is actually a rational point since k is supposed to be algebraically closed.

Hence X_j^{an} only contains one point, which we denote by x_j^{an}. Note that s_0 does not vanish at any of the closed points X_j. By definition, $h_{\overline{L}_0}^{s_0}(X)$ is equal to

$$-\sum_{j=1}^{n} \mathrm{mult}_{X_j}(X) \ln |s_0|_{\varphi_0}(x_j^{\mathrm{an}}). \tag{3.16}$$

In the case where $d \geqslant 1$, the induction formula in Definition 3.6.1 for local intersection number leads to the following formula for the local height.

$$\begin{aligned} h_{\overline{L}_0,\ldots,\overline{L}_d}^{s_0,\ldots,s_d}(X) = &\sum_{i=1}^{n} a_i h_{\overline{L}_0,\ldots,\overline{L}_{d-1}}^{s_0,\ldots,s_{d-1}}(Z_i) \\ &- \int_{X^{\mathrm{an}}} \ln |s_d|_{\varphi_d}(x)\, \mu_{(L_0,\varphi_0)\cdots(L_{d-1},\varphi_{d-1})}(\mathrm{d}x), \end{aligned} \tag{3.17}$$

where $\sum_{i=1}^{n} a_i Z_i$ is the cycle associated with $\mathrm{div}(L_d; s_d)$.

3.8.4. Definition. — Let $(E, \|\cdot\|)$ be a finite-dimensional normed vector space over k, and r be the rank of E. We denote by $\|\cdot\|_{\det}$ the norm on the one-dimensional vector space $\det(E) := \Lambda^r(E)$ such that,

$$\forall\, \eta \in \det(E), \quad \|\eta\|_{\det} := \inf_{\substack{(t_1,\ldots,t_r)\in E^r \\ \eta = t_1\wedge\cdots\wedge t_r}} \|t_1\| \cdots \|t_r\|.$$

Note that, if the norm $\|\cdot\|$ is ultrametric or induced by an inner product, for any complete valued extension k' of k, one has (see Definition 3.3.3)

$$\|\cdot\|_{k',\det} = \|\cdot\|_{\det,k'}, \tag{3.18}$$

where we identify $\det(E) \otimes_k k'$ with $\det(E \otimes_k k')$. We refer the readers to [**13**, Proposition 1.3.19] for a proof.

3.8.5. Proposition. — *Let E be a finite-dimensional vector space over k, equipped with a norm $\|\cdot\|$ which is either ultrametric or induced by an inner product, $r = \dim_k(E)$, and $L = \mathcal{O}_E(1)$ be the universal invertible sheaf on $\mathbb{P}(E)$. We equip L with the orthogonal quotient metric φ induced by $\|\cdot\|$ (see Definition 3.3.5). Let $(s_j)_{j=0}^{r}$ be a basis of E over k. If $|\cdot|$ is non-Archimedean, then*

$$h_{\overline{L},\ldots,\overline{L}}^{s_0,\ldots,s_r}(\mathbb{P}(E)) = -\ln \|s_0 \wedge \cdots \wedge s_r\|_{\det};$$

if $|\cdot|$ is Archimedean, then

$$h_{\overline{L},\ldots,\overline{L}}^{s_0,\ldots,s_r}(\mathbb{P}(E)) = -\ln \|s_0 \wedge \cdots \wedge s_r\|_{\det} + \sigma_r,$$

where

$$\sigma_r = \frac{1}{2} \sum_{m=1}^{r} \sum_{\ell=1}^{m} \frac{1}{\ell}$$

is the r-th Stoll number.

Proof. — First, the metric $\varphi_{\mathbb{C}_k}$ identifies with the orthogonal quotient metric induced by $\|\cdot\|_{\mathbb{C}_k}$. Therefore, by (3.18) we may assume without loss of generality that k is algebraically closed.

By Remark 3.5.8, one can find a sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of orthonormally decomposable norms such that

$$\lim_{n\to+\infty} d(\|\cdot\|_n, \|\cdot\|) = 0.$$

By (3.2), if we denote by φ_n the orthogonal quotient metric on L induced by $\|\cdot\|_n$, then one has

$$\lim_{n\to+\infty} d(\varphi_n, \varphi) = 0.$$

By Corollary 3.6.7, one has

$$\lim_{n\to+\infty} h^{s_0,\ldots,s_r}_{(L,\varphi_n),\ldots,(L,\varphi_n)}(\mathbb{P}(E)) = h^{s_0,\ldots,s_r}_{\overline{L},\ldots,\overline{L}}(\mathbb{P}(E)).$$

Moreover, by [**13**, Proposition 1.1.64] one has

$$0 \leqslant d(\|\cdot\|_{n,\det}, \|\cdot\|_{\det}) \leqslant r d(\|\cdot\|_n, \|\cdot\|)$$

and hence

$$\lim_{n\to+\infty} d(\|\cdot\|_{n,\det}, \|\cdot\|_{\det}) = 0.$$

Therefore, without loss of generality, we may assume that the norm $\|\cdot\|$ is orthonormally decomposable.

We reason by induction on r. In the case where $r = 0$, the vector space E is one-dimensional, and s_0 is a non-zero element of E. One has

$$h^{s_0}_{\overline{L}}(\mathbb{P}(E)) = -\ln \|s_0\|.$$

We now assume that $r \geqslant 1$. Let G be the quotient vector space of E by ks_r. Note that the quotient norm $\|\cdot\|_{\mathrm{quot}}$ on G is orthonormally decomposable (see Proposition 3.5.9). For $j \in \{0, \ldots, r-1\}$, let $\overline{s}_j$ be the class of s_j in G. We can also view $\overline{s}_j$ as the restriction of s_j to the closed subscheme $\mathbb{P}(G)$ of $\mathbb{P}(E)$. We apply the induction hypothesis to $(G, \|\cdot\|_{\mathrm{quot}})$ and obtain (see Notation 3.8.2)

$$h^{s_0,\ldots,s_{r-1}}_{\overline{L},\ldots,\overline{L}}(\mathbb{P}(G)) = -\ln \|\overline{s}_0 \wedge \cdots \wedge \overline{s}_{r-1}\|_{\mathrm{quot},\det}$$

when $|\cdot|$ is non-Archimedean and

$$h^{s_0,\ldots,s_{r-1}}_{\overline{L},\ldots,\overline{L}}(\mathbb{P}(G)) = -\ln \|\overline{s}_0 \wedge \cdots \wedge \overline{s}_{r-1}\|_{\mathrm{quot},\det} + \sigma_{r-1}$$

We now compute the integral

$$-\int_{\mathbb{P}(E)^{\mathrm{an}}} \ln |s_r|_\varphi \,\mathrm{d}\mu_{\overline{L}^r}.$$

We first consider the case where $|\cdot|$ is non-Archimedean. By Proposition 3.5.11 one has

$$\int_{\mathbb{P}(E)^{\mathrm{an}}} \ln |s_r|_\varphi \,\mathrm{d}\mu_{\overline{L}^r} = -\ln |s_r|_\varphi(\xi) = -\ln \|s_r\|,$$

where ξ denotes the Gauss point of $\mathbb{P}(E)^{\mathrm{an}}$. Therefore, by [**13**, Proposition 1.2.51] we obtain

$$h^{s_0,\ldots,s_r}_{\overline{L},\ldots,\overline{L}}(\mathbb{P}(E)) = -\ln\|\overline{s}_0\wedge\cdots\wedge\overline{s}_{r-1}\|_{\mathrm{quot,det}} - \ln\|s_r\| = -\ln\|s_0\wedge\cdots\wedge s_r\|_{\mathrm{det}}.$$

In the case where $|\cdot|$ is Archimedean, by [**5**, §1.4.3] Remark (iii), one has

$$-\int_{\mathbb{P}(E)^{\mathrm{an}}} \ln|s_r|_{\varphi_r}\,\mathrm{d}\mu_{\overline{L}^r} = -\ln\|s\| + \frac{1}{2}\sum_{\ell=1}^{r}\frac{1}{\ell}.$$

Therefore

$$\begin{aligned} h^{s_0,\ldots,s_r}_{\overline{L},\ldots,\overline{L}}(\mathbb{P}(E)) &= -\ln\|\overline{s}_0\wedge\cdots\wedge\overline{s}_{r-1}\|_{\mathrm{quot,det}} - \ln\|s_r\| + \frac{1}{2}\sum_{m=1}^{r}\sum_{\ell=1}^{m}\frac{1}{\ell} \\ &= -\ln\|s_0\wedge\cdots\wedge s_r\|_{\mathrm{det}} + \frac{1}{2}\sum_{m=1}^{r}\sum_{\ell=1}^{m}\frac{1}{\ell}. \end{aligned}$$

□

In the remaining of the section, we consider a family

$$(E_i, \|\cdot\|_i), \quad i \in \{0,\ldots,d\}$$

of finite-dimensional vector spaces over k equipped with norms which are either ultrametric or induced by inner products. For each $i\in\{0,\ldots,d\}$, we let $(E_i^\vee, \|\cdot\|_{i,*})$ be the dual normed vector space of $(E_i,\|\cdot\|_i)$, $r_i := \dim_k(E_i)-1$, $(s_{i,j})_{j=0}^{r_i}$ be a basis of E_i over k, and $(\alpha_{i,j})_{j=0}^{r_i}$ be the dual basis of $(s_{i,j})_{j=0}^{r_i}$, namely

$$\alpha_{i,j}(s_{i,j}) = 1 \quad\text{and}\quad \alpha_{i,j}(s_{i,\ell}) = 0 \quad \text{if } j\neq\ell.$$

Let $\check{\mathbb{P}}$ be the product projective space

$$\mathbb{P}(E_0^\vee)\times_k\cdots\times_k\mathbb{P}(E_d^\vee).$$

For any $i\in\{0,\ldots,d\}$, let $\pi_i : \check{\mathbb{P}}\to\mathbb{P}(E_i^\vee)$ be the morphism of projection to the i^{th} coordinate, and $L_i = \pi_i^*(\mathcal{O}_{E_i^\vee}(1))$. We equip L_i with the orthogonal quotient metric induced by $\|\cdot\|_{i,*}$, which we denote by φ_i. Let $(\delta_0,\ldots,\delta_d)$ be an element of $\mathbb{N}^{d+1}$,

$$L = \pi_0^*(\mathcal{O}_{E_0^\vee}(\delta_0))\otimes\cdots\otimes\pi_d^*(\mathcal{O}_{E_d^\vee}(\delta_d)) = L_0^{\otimes\delta_0}\otimes\cdots\otimes L_d^{\otimes\delta_d}.$$

We equip L with the metric

$$\varphi := \varphi_0^{\otimes\delta_0}\otimes\cdots\otimes\varphi_d^{\otimes\delta_d}.$$

Let R be a non-zero element of

$$S^{\delta_0}(E_0^\vee)\otimes_k\cdots\otimes_k S^{\delta_d}(E_d^\vee),$$

which is considered as a global section of L, and also as a multi-homogenous polynomial of multi-degree $(\delta_0,\ldots,\delta_d)$ on $E_0\times\cdots\times E_d$. For any $i\in\{0,\ldots,d\}$, let

$$\overline{\boldsymbol{L}}_i = (\underbrace{\overline{L}_i,\ldots,\overline{L}_i}_{r_i\text{ copies}}), \quad \boldsymbol{\alpha}_i := (\alpha_{i,j})_{j=1}^{r_i}.$$

The purpose of this section is to compute the local height $h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}})$.

3.8.6. Proposition. — *Assume that the sections R and*

$$\alpha_{i,j},\quad i\in\{0,\ldots,d\},\quad j\in\{1,\ldots,r_i\}$$

intersect property on $\check{\mathbb{P}}$. If the absolute value $|\cdot|$ is non-Archimedean, then

$$h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}) = -\ln|R(s_{0,0},\ldots,s_{d,0})| - \sum_{i=0}^d \delta_i \ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det};$$

if the absolute value $|\cdot|$ is Archimedean, then

$$h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}) = -\ln|R(s_{0,0},\ldots,s_{d,0})| - \sum_{i=0}^d \delta_i\big(\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det} - \sigma_{r_i}\big).$$

Proof. — By the same argument as in the beginning of the proof of Proposition 3.8.5, we may assume without loss of generality that k is algebraically closed and that all norms $\|\cdot\|_i$ are orthonormally decomposable.

We reason by induction on $r_0+\cdots+r_d$. Consider first the case where $r_0=\cdots=r_d=0$. One has

$$h^R_{\overline{L}}(\check{\mathbb{P}}) = -\ln|R(s_{0,0},\ldots,s_{d,0})|.$$

In the following, we assume that $r_0+\cdots+r_d>0$. Let i be an element of $\{0,\ldots,d\}$ such that $r_i>0$. We consider the quotient vector space $G_i^\vee = E_i^\vee/k\alpha_{i,r_i}$. For $j\in\{0,\ldots,r_i-1\}$, let $\overline{\alpha}_{i,j}$ be the class of $\alpha_{i,j}$ in G_i. Let $\overline{\boldsymbol{\alpha}}_i := (\overline{\alpha}_{i,j})_{j=1}^{r_i-1}$ and

$$\check{\mathbb{P}}' = \mathbb{P}(E_0)\times_k\cdots\times\mathbb{P}(E_{i-1})\times_k\mathbb{P}(G_i)\times_k\mathbb{P}(E_{i+1})\times_k\cdots\times_k\mathbb{P}(E_d).$$

By the same argument as in Proposition 3.5.11, we obtain that, in the case where the absolute value $|\cdot|$ is non-Archimedean, one has

$$\mu_{\overline{L}\,\overline{L}_0^{r_0}\cdots\overline{L}_{i-1}^{r_{i-1}}\overline{L}_i^{r_i-1}\overline{L}_{i+1}^{r_{i+1}}\cdots\overline{L}_d^{r_d}} = \delta_i\,\mathrm{Dirac}_\xi,$$

where Dirac_ξ denotes the Dirac measure at the Gauss point ξ of $\check{\mathbb{P}}^{\mathrm{an}}$. Hence, by (3.17), one has

$$\begin{aligned}h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}) &= h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_{i-1},\overline{\boldsymbol{\alpha}}_i,\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_{i-1},\overline{\boldsymbol{L}}'_i,\overline{\boldsymbol{L}}_{i+1},\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}') - \delta_i\ln|\alpha_{i,r_i}|_{\varphi_i}(\xi)\\ &= h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_{i-1},\overline{\boldsymbol{\alpha}}_i,\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_{i-1},\overline{\boldsymbol{L}}'_i,\overline{\boldsymbol{L}}_{i+1},\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}') - \delta_i\ln\|\alpha_{i,r_i}\|_{i,*},\end{aligned}$$

where

$$\overline{\boldsymbol{L}}'_i := (\underbrace{\overline{L}_i,\ldots,\overline{L}_i}_{r_i-1\text{ copies}}).$$

By the induction hypothesis, we obtain

$$\begin{aligned} h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}) &= -\ln|R(s_{0,0},\ldots,s_{d,0})| - \sum_{j\in\{0,\ldots,d\}\setminus\{i\}} \delta_j \ln\|\alpha_{j,0}\wedge\cdots\wedge\alpha_{j,r_j}\|_{j,*,\mathrm{det}} \\ &\qquad - \delta_i \ln\|\overline{\alpha}_{i,0}\wedge\cdots\wedge\overline{\alpha}_{i,r_i-1}\|_{i,*,\mathrm{quot},\mathrm{det}} - \delta_i\ln\|\alpha_{i,r_i}\|_{i,*} \\ &= -\ln|R(s_{0,0},\ldots,s_{d,0})| - \sum_{j=0}^{d}\delta_j\ln\|\alpha_{j,0}\wedge\cdots\wedge\alpha_{j,r_j}\|_{j,*,\mathrm{det}}, \end{aligned}$$

where the last equality comes from [**13**, Proposition 1.2.51].

In the case where $|\cdot|$ is Archimedean, by [**5**, §1.4.3] Remark (iii) one has

$$h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}) = h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_{i-1},\overline{\boldsymbol{\alpha}}_i,\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_{i-1},\overline{\boldsymbol{L}}'_i,\overline{\boldsymbol{L}}_{i+1},\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}') - \delta_i\Big(\ln\|\alpha_{i,r_i}\|_{i,*} - \frac{1}{2}\sum_{\ell=1}^{r_i}\frac{1}{\ell}\Big).$$

Thus the induction hypothesis leads to

$$\begin{aligned} h^{R,\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}}) &= -\ln|R(s_{0,0},\ldots,s_{d,0})| - \sum_{\substack{j\in\{0,\ldots,d\}\\ j\neq i}} \delta_j\big(\ln\|\alpha_{j,0}\wedge\cdots\wedge\alpha_{j,r_j}\|_{j,*,\mathrm{det}} - \sigma_{r_j}\big) \\ &\quad - \delta_i\big(\ln\|\overline{\alpha}_{i,0}\wedge\cdots\wedge\overline{\alpha}_{i,r_i-1}\|_{i,*,\mathrm{quot},\mathrm{det}} - \sigma_{r_i-1}\big) - \delta_i\Big(\ln\|\alpha_{i,r_i}\|_{i,*} - \frac{1}{2}\sum_{\ell=1}^{r_i}\frac{1}{\ell}\Big) \\ &= -\ln|R(s_{0,0},\ldots,s_{d,0})| - \sum_{j=0}^{d}\delta_j\big(\ln\|\alpha_{j,0}\wedge\cdots\wedge\alpha_{j,r_j}\| - \sigma_{r_j}\big), \end{aligned}$$

as required. □

3.9. Local height of the resultant

The purpose of this subsection is to relate local heights of a projective variety and its resultant. As in the previous section, $v = (k, |\cdot|)$ denotes a complete valued field such that $|\cdot|$ is not trivial. We fix a projective k-scheme X and we let d be the Krull dimension of X. Let $(E_i)_{i=0}^d$ be a family of finite-dimensional vector spaces over k. For each $i \in \{0,\ldots,d\}$, we denote by $r_i := \dim_k(E_i) - 1$ and let $\|\cdot\|_i$ be a norm on E_i, which is supposed to be either ultrametric or induced by an inner product. Let $f_i : X \to \mathbb{P}(E_i)$ be a closed immersion. We pick elements $s_0,\ldots,s_d$ of $E_0,\ldots,E_d$ respectively, such that

$$\mathrm{div}(s_0|_X),\ldots,\mathrm{div}(s_d|_X)$$

intersect properly on X. For simplicity of notation, we denote by

$$\boldsymbol{s} := (s_0,\ldots,s_d).$$

Let $\check{\mathbb{P}} := \mathbb{P}(E_0^\vee)\times_k\cdots\times_k\mathbb{P}(E_d^\vee)$, and let

$$p : X\times_k\check{\mathbb{P}} \longrightarrow X \quad\text{and}\quad q : X\times_k\check{\mathbb{P}}\longrightarrow\check{\mathbb{P}}$$

be morphisme of projections. For any $i \in \{0, \ldots, d\}$, let $\pi_i : \check{\mathbb{P}} \to \mathbb{P}(E_i^\vee)$ be the projection to the i-th coordinate and let $q_i = \pi_i \circ q$.

For $i \in \{0, \ldots, d\}$, let $\overline{L}_i$ be $L_i = \pi_i^*(\mathcal{O}_{E_i^\vee}(1))$ equipped with the pull-back of the orthogonal quotient metric on $\mathcal{O}_{E_i^\vee}(1)$ associated with $\|\cdot\|_{i,*}$, and let

$$\overline{\boldsymbol{L}}_i := (\underbrace{\overline{L}_i, \ldots, \overline{L}_i}_{r_i \text{ copies}})$$

and

$$(\alpha_{i,0}, \boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,r_i}))$$

be a basis of $E_i^\vee$ such that

$$\alpha_{i,0}(s_i) = 1 \quad \text{and} \quad \alpha_{i,j}(s_i) = 0 \text{ for } j \in \{1, \ldots, r_i\}.$$

For simplicity, we denote by R the resultant

$$R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d}$$

as in Definition 1.6.9, considered as a global section of

$$L = \pi_0^*(L_0)^{\otimes \delta_0} \otimes \cdots \otimes \pi_d^*(L_d)^{\otimes \delta_d},$$

where

$$\delta_i := \deg\big(c_1(L_0) \cdots c_1(L_{i-1}) c_1(L_{i+1}) \cdots c_1(L_d) \cap [X]\big).$$

Note that one has $R(s_0, \ldots, s_d) = 1$. Moreover, the Cartier divisors

$$\operatorname{div}(R), \operatorname{div}(\pi_0^*(\alpha_{0,1})), \ldots, \operatorname{div}(\pi_0^*(\alpha_{0,r_0})), \ldots, \operatorname{div}(\pi_d^*(\alpha_{d,1})), \ldots, \operatorname{div}(\pi_d^*(\alpha_{d,r_d}))$$

intersect properly.

3.9.1. Lemma. — *Assume that the field k is algebraically closed and X is integral. One has*

$$h^{\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}_{\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}\big(\operatorname{div}(R)\big) = h^{q_0^*(\boldsymbol{\alpha}_0),\ldots,q_d^*(\boldsymbol{\alpha}_d)}_{q^*(\overline{\boldsymbol{L}}_0),\ldots,q^*(\overline{\boldsymbol{L}}_d)}(I_X).$$

Proof. — The projection $q : I_X \to \operatorname{div}(R)$ is a birational morphism (see the proof of [**24**, Proposition 3.1]). Hence the equality follows from the induction formula (3.17) and [**50**, Proposition 2.4.11 (4)]. □

3.9.2. Definition. — Assume that the absolute value $|\cdot|$ is non-Archimedean. We equip each symmetric power $S^{\delta_i}(E_i^\vee)$ with the ε-symmetric power norm of $\|\cdot\|_{i,*}$, namely the quotient norm of the ε-tensor power of $\|\cdot\|_{i,*}$ (see Remark 3.3.4 for the definition of the dual norm $\|\cdot\|_{i,*}$). Recall that the ε-tensor power of the norm $\|\cdot\|_{i,*}$ is the norm $\|\cdot\|_{i,*,\varepsilon}$ on $(E_i^\vee)^{\otimes_k \delta_i}$ defined as (see [**13**, Definition 1.1.52])

$$\|T\|_{i,*,\varepsilon} = \sup_{\substack{(t_1,\ldots,t_{\delta_i}) \in E_i^{\delta_i} \\ \forall\, j \in \{1,\ldots,\delta_i\},\ t_j \neq 0}} \frac{|T(t_1, \ldots, t_{\delta_i})|}{\|t_1\|_i \cdots \|t_{\delta_i}\|_i}.$$

We then equip the vector space $S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)$ with the ε-tensor product of the ε-symmetric power norms, which we denote simply by $\|\cdot\|$.

3.9.3. Remark. — Note that, by [**13**, Definition 1.1.58], the norm $\|\cdot\|$ also identifies with the quotient norm by the canonical quotient map

$$(E_0^\vee)^{\otimes_k \delta_0} \otimes_k \cdots \otimes_k (E_d^\vee)^{\otimes_k \delta_d} \longrightarrow S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)$$

of the ε-tensor product of δ_i copies of $\|\cdot\|_{i,*}$, $i \in \{0, \ldots, d\}$. By Propositions 1.3.20 and 1.3.21 of [**13**], we obtain that, for any complete valued extension k' of k, the norm $\|\cdot\|_{k'}$ on

$$\big(S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)\big) \otimes_k k' \cong S^{\delta_0}(E_{0,k'}^\vee) \otimes_{k'} \cdots \otimes_{k'} S^{\delta_d}(E_{d,k'}^\vee)$$

identifies with the ε-tensor product of δ_i copies of $\|\cdot\|_{i,k',*}$, $i \in \{0, \ldots, d\}$.

3.9.4. Lemma. — *In the case where $|\cdot|$ is non-Archimedean and k is algebraically closed, one has*

$$h_{\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}^{\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}(\mathrm{div}(R)) = h_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}^{R,\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}(\check{\mathbb{P}}) + \ln\|R\|. \tag{3.19}$$

Proof. — Let ξ be the Gauss point of $\check{\mathbb{P}}^{\mathrm{an}}$. It suffices to observe that

$$|R|_\varphi(\xi) = \|R\|,$$

where φ is tensor product of orthogonal quotient metrics. In fact, if we consider the Veronese-Segre embedding

$$\check{\mathbb{P}} \longrightarrow \mathbb{P}(S^{\delta_0}(E_0^\vee)) \times_k \cdots \times_k \mathbb{P}(S^{\delta_d}(E_d^\vee)) \longrightarrow \mathbb{P}(S^{\delta_0}(E_0^\vee) \otimes_k \cdots \otimes_k S^{\delta_d}(E_d^\vee)),$$

then the metric φ identifies with the quotient metric induced by $\|\cdot\|$ (see [**13**, Proposition 1.1.58]). Moreover, one has

$$\mu_{\overline{L}_0^{r_0}\cdots\overline{L}_d^{r_d}} = \mathrm{Dirac}_\xi\,.$$

Therefore the equality (3.19) follows from the induction formula (3.17). □

3.9.5. Lemma. — *In the case where $|\cdot|$ is Archimedean and $k = \mathbb{C}$, one has*

$$\begin{aligned} h_{\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}^{\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}(\mathrm{div}(R)) &= h_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}^{R,\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}(\check{\mathbb{P}}) \\ &\quad + \int_{\mathbb{S}_0\times\cdots\times\mathbb{S}_d} \ln|R(z_0,\ldots,z_d)|\,\eta_{\mathbb{S}_0}(\mathrm{d}z_0)\otimes\cdots\otimes\eta_{\mathbb{S}_d}(\mathrm{d}z_d), \end{aligned}$$

where $\mathbb{S}_i$ is the unit sphere of $(E_{i,\mathbb{C}}, \|\cdot\|_{i,\mathbb{C}})$, and $\mu_{\mathbb{S}_i}$ is the $U(E_{i,\mathbb{C}}, \|\cdot\|_{i,\mathbb{C}})$-invariant Borel probability measure on $\mathbb{S}_i$.

Proof. — This is a direct consequence of the induction formula (3.17) and Remark 3.5.12. □

3.9.6. Lemma. — *Assume that the field k is algebraically closed and X is integral. For any $i \in \{0, \ldots, d\}$, we equip $\mathcal{O}_{E_i}(1)$ with the orthogonal quotient metric induced by*

$\|\cdot\|_i$, *and denote by* M_i' *the restriction of* $\mathcal{O}_{E_i}(1)$ *to* X *and equip it with the restricted metric. If* $|\cdot|$ *is non-Archimedean, then one has*

$$h^{s_0,\ldots,s_d}_{\overline{M_0'},\ldots,\overline{M_d'}}(X) = h^{q_0^*(\boldsymbol{\alpha}_0),\ldots,q_d^*(\boldsymbol{\alpha}_d)}_{q^*(\overline{\boldsymbol{L}}_0),\ldots,q^*(\overline{\boldsymbol{L}}_d)}(I_X) + \sum_{i=0}^d \delta_i \ln \|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det};$$

if $|\cdot|$ *is Archimedean, then one has*

$$h^{s_0,\ldots,s_d}_{\overline{M_0'},\ldots,\overline{M_d'}}(X) = h^{q_0^*(\boldsymbol{\alpha}_0),\ldots,q_d^*(\boldsymbol{\alpha}_d)}_{q^*(\overline{\boldsymbol{L}}_0),\ldots,q^*(\overline{\boldsymbol{L}}_d)}(I_X) + \sum_{i=0}^d \delta_i\big(\ln \|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det} - \sigma_{r_i-1}\big),$$

where

$$\sigma_{r_i-1} = \frac{1}{2}\sum_{m=1}^{r_i-1}\sum_{\ell=1}^{m}\frac{1}{\ell}.$$

Proof. — For $i \in \{0,\ldots,d\}$, let t_i be the global section of $\mathcal{O}_{E_i}(1)\boxtimes\mathcal{O}_{E_i^\vee}(1)$ on $\mathbb{P}(E_i)\times_k\mathbb{P}(E_i^\vee)$ defining the incidence subscheme. Then t_i corresponds to the restriction of the trace element of $E_i\otimes_k E_i^\vee$ via the Segre embedding

$$\mathbb{P}(E_i)\times_k\mathbb{P}(E_i^\vee)\longrightarrow\mathbb{P}(E_i\otimes_k E_i^\vee).$$

Let $\boldsymbol{t} = (t_0,\ldots,t_d)$. For any $i\in\{0,\ldots,d\}$, let

$$(s_i, s_{i,1},\ldots,s_{i,r_i})$$

be the dual basis of $(\alpha_{i,j})_{j=0}^{r_i}$. By definition one has

$$t_i = s_i\otimes\alpha_{i,0} + s_{i,1}\otimes\alpha_{i,1}+\cdots+s_{i,r_i}\otimes\alpha_{i,r_i}.$$

For $i\in\{0,\ldots,d\}$, let $L_i := q_i^*(\mathcal{O}_{E_i^\vee}(1))$, $M_i = p^*(\mathcal{O}_{E_i}(1)|_X)$ and $N_i = L_i\otimes M_i$. We use two methods to compute the following local height of $X\times\check{\mathbb{P}}$ (see Notation 3.8.2)

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(X\times_k\check{\mathbb{P}}),$$

where $\overline{\boldsymbol{N}} = (\overline{N}_0,\ldots,\overline{N}_d)$. We will show by induction that

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(X\times_k\check{\mathbb{P}}) = h^{s_0,\ldots,s_d}_{\overline{M_0'},\ldots,\overline{M_d'}}(X) - \sum_{i=0}^d\delta_i\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det} \tag{3.20}$$

if $|\cdot|$ is non-Archimedean, and

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(X\times_k\check{\mathbb{P}}) = h^{s_0,\ldots,s_d}_{\overline{M_0'},\ldots,\overline{M_d'}}(X) - \sum_{i=0}^d\delta_i\big(\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det} - \sigma_{r_i}\big) \tag{3.21}$$

if $|\cdot|$ is Archimedean. Let $i\in\{0,\ldots,d\}$ be such that $r_i>0$. Let $G_i^\vee = E_i^\vee/k\alpha_{i,r_i}$, $\overline{\boldsymbol{\alpha}}_i = (\overline{\alpha}_{i,j})_{j=1}^{r_i-1}$, and

$$\check{\mathbb{P}}' = \mathbb{P}(E_0)\times_k\cdots\times\mathbb{P}(E_{i-1})\times_k\mathbb{P}(G_i)\times_k\mathbb{P}(E_{i+1})\times_k\cdots\times_k\mathbb{P}(E_d).$$

Then, with the notation

$$\overline{\boldsymbol{L}}_i' := (\underbrace{\overline{L}_i,\ldots,\overline{L}_i}_{r_i-1\text{ copies}}),$$

by (3.17) one can write $h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(X\times_k\check{\mathbb{P}})$ as

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_{i-1},\overline{\boldsymbol{\alpha}}_i,\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_{i-1},\overline{\boldsymbol{L}}'_i,\overline{\boldsymbol{L}}_{i+1},\ldots,\overline{\boldsymbol{L}}_d}(X\times\check{\mathbb{P}})-\int_{(X\times\check{\mathbb{P}})^{\mathrm{an}}}\ln|\alpha_{i,r_i}|\,\mathrm{d}\mu_{\overline{N}_0\cdots\overline{N}_d\overline{L}_0^{r_0}\cdots\overline{L}_{i-1}^{r_{i-1}}\overline{L}_i^{r_i-1}\overline{L}_{i+1}^{r_{i+1}}\cdots\overline{L}_d^{r_d}},$$

which is equal to

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_{i-1},\overline{\boldsymbol{\alpha}}_i,\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_{i-1},\overline{\boldsymbol{L}}'_i,\overline{\boldsymbol{L}}_{i+1},\ldots,\overline{\boldsymbol{L}}_d}(X\times\check{\mathbb{P}})-\int_{(X\times\check{\mathbb{P}})^{\mathrm{an}}}\ln|\alpha_{i,r_i}|\,\mathrm{d}\mu_{\overline{M}_0\cdots\overline{M}_{i-1}\overline{M}_{i+1}\cdots\overline{M}_d\overline{L}_0^{r_0}\cdots\overline{L}_d^{r_d}}.$$

If $|\cdot|$ is non-Archimedean, it identifies with

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_{i-1},\overline{\boldsymbol{\alpha}}_i,\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_{i-1},\overline{\boldsymbol{L}}'_i,\overline{\boldsymbol{L}}_{i+1},\ldots,\overline{\boldsymbol{L}}_d}(X\times\check{\mathbb{P}})-\delta_i\ln\|\alpha_{i,r_i}\|_{i,*}.$$

In the case where $|\cdot|$ is Archimedean, it equals

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_{i-1},\overline{\boldsymbol{\alpha}}_i,\boldsymbol{\alpha}_{i+1},\ldots,\boldsymbol{\alpha}_d}_{\overline{N},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_{i-1},\overline{\boldsymbol{L}}'_i,\overline{\boldsymbol{L}}_{i+1},\ldots,\overline{\boldsymbol{L}}_d}(X\times\check{\mathbb{P}})-\delta_i\Big(\ln\|\alpha_{i,r_i}\|_{i,*}-\frac{1}{2}\sum_{\ell=1}^{r_i}\frac{1}{\ell}\Big).$$

Hence by induction we obtain (3.20) and (3.21) according to the nature of $|\cdot|$.

Now let $\boldsymbol{t}'=(t_0,\ldots,t_{d-1})$ and $\overline{\boldsymbol{N}}'=(\overline{N}_0,\ldots,\overline{N}_{d-1})$, still by (3.17) one can write $h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(X\times_k\check{\mathbb{P}})$ as

$$\begin{aligned}&h^{\boldsymbol{t}',\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}}',\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\mathrm{div}(t_d))-\int_{(X\times_k\check{\mathbb{P}})^{\mathrm{an}}}\ln|t_d|\,\mathrm{d}\mu_{\overline{N}_0\cdots\overline{N}_{d-1}\overline{L}_0^{r_0}\cdots\overline{L}_d^{r_d}}\\ =\;&h^{\boldsymbol{t}',\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}}',\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\mathrm{div}(t_d))-\int_{(X\times_k\check{\mathbb{P}})^{\mathrm{an}}}\ln|t_d|\,\mathrm{d}\mu_{\overline{M}_0\cdots\overline{M}_{d-1}\overline{L}_0^{r_0}\cdots\overline{L}_d^{r_d}}\end{aligned}$$

Note that for any element $z\in(X\times_k\check{\mathbb{P}})^{\mathrm{an}}$ represented by

$$(\beta,x_0,\ldots,x_d)\in E^{\vee}_{d,\widehat{\kappa}(z)}\times E_{0,\widehat{\kappa}(z)}\cdots\times E_{d,\widehat{\kappa}(z)}$$

one has

$$\ln|t_d|(z)=\ln\frac{|\beta(x_d)|_z}{\|\beta\|_{d,\widehat{\kappa}(z)}\cdot\|x_d\|_{d,\widehat{\kappa}(z)}}. \tag{3.22}$$

In the case where $|\cdot|$ is non-Archimedean, this leads to

$$\int_{(X\times_k\check{\mathbb{P}})^{\mathrm{an}}}\ln|t_d|\,\mathrm{d}\mu_{\overline{M}_0\cdots\overline{M}_{d-1}\overline{L}_0^{r_0}\cdots\overline{L}_d^{r_d}}=0$$

by using (3.12) and

$$\int_{\mathbb{P}(E^{\vee}_{d,\widehat{\kappa}(z)})^{\mathrm{an}}}\ln|\beta|\,\mathrm{d}\mu_{\overline{\mathcal{O}_{E_d}(d)}^{r_d}}=\ln\|\beta\|_{d,*,\widehat{\kappa}(x)}.$$

In the case where $|\cdot|$ is Archimedean, by [**5**, §1.4.3] Remark (iii), (3.22) leads to

$$-\int_{(X\times_k\check{\mathbb{P}})^{\mathrm{an}}}\ln|t_d|\,\mathrm{d}\mu_{\overline{M}_0\cdots\overline{M}_{d-1}\overline{L}_0^{r_0}\cdots\overline{L}_d^{r_d}}=\frac{\delta_d}{2}\sum_{\ell=1}^{r_i}\frac{1}{\ell}.$$

Then by induction we obtain

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(X\times_k\check{\mathbb{P}})=h^{\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(I_X) \tag{3.23}$$

when $|\cdot|$ is non-Archimedean and

$$h^{\boldsymbol{t},\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{N}},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(X\times_k\check{\mathbb{P}})=h^{\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d}_{\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(I_X)+\frac{1}{2}\delta_i\sum_{i=0}^{d}\sum_{\ell=1}^{r_i}\frac{1}{\ell} \tag{3.24}$$

when $|\cdot|$ is Archimedean. Combining (3.23) with (3.20), and (3.24) with (3.21), we obtain the result. □

3.9.7. Theorem. — *For any $i\in\{0,\ldots,d\}$, we equip $\mathcal{O}_{E_i}(1)$ with the orthogonal quotient metric induced by $\|\cdot\|_i$, and denote by M_i' the restriction of $\mathcal{O}_{E_i}(1)$ to X and equip it with the restricted metric. In the case where $|\cdot|$ is non-Archimedean, one has*

$$h^{s_0,\ldots,s_d}_{\overline{M_0'},\ldots,\overline{M_d'}}(X)=\ln\|R\|,$$

where the norm $\|\cdot\|$ was introduced in Definition 3.9.2. In the case where $|\cdot|$ is Archimedean, one has

$$h^{s_0,\ldots,s_d}_{\overline{M_0'},\ldots,\overline{M_d'}}(X)=\int_{\mathbb{S}_0\times\cdots\times\mathbb{S}_d}\ln|R(z_0,\ldots,z_d)|\,\eta_{\mathbb{S}_0}(\mathrm{d}z_0)\otimes\cdots\otimes\eta_{\mathbb{S}_d}(\mathrm{d}z_d)+\frac{1}{2}\sum_{i=0}^{d}\delta_i\sum_{\ell=1}^{r_i}\frac{1}{\ell},$$

where $\mathbb{S}_i$ is the unit sphere of $(E_{i,\mathbb{C}},\|\cdot\|_{i,\mathbb{C}})$, and $\eta_{\mathbb{S}_i}$ is the $U(E_{i,\mathbb{C}},\|\cdot\|_{i,\mathbb{C}})$-invariant Borel probability measure on $\mathbb{S}_{i,\sigma}$.

Proof. — By Remark 1.6.10,

$$R\otimes 1\in(S^{\delta_0}(E_0^\vee)\otimes_k\cdots\otimes_k S^{\delta_d}(E_d^\vee))\otimes_k\mathbb{C}_k$$

is the resultant of $X_{\mathbb{C}_k}$ with respect to $f_{0,\mathbb{C}_k},\ldots,f_{d,\mathbb{C}_k}$, which takes value 1 at $(s_0,\ldots,s_d)$. Therefore, by extension of scalars, we may assume without loss of generality that k is algebraically closed and X is integral.

We treat first the non-Archimedean case. By Lemma 3.9.6, one has

$$h^{s_0,\ldots,s_d}_{\overline{M_0'},\ldots,\overline{M_d'}}(X)=h^{q_0^*(\boldsymbol{\alpha}_0),\ldots,q_d^*(\boldsymbol{\alpha}_d)}_{q^*(\overline{\boldsymbol{L}}_0),\ldots,q^*(\overline{\boldsymbol{L}}_d)}(I_X)+\sum_{i=0}^{d}\delta_i\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det}.$$

By Lemma 3.9.1, this is also equal to

$$h^{\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}_{\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\mathrm{div}(R))+\sum_{i=0}^{d}\delta_i\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det}.$$

By Lemma 3.9.4, it is equal to

$$h^{R,\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}})+\ln\|R\|+\sum_{i=0}^{d}\delta_i\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det}.$$

By Proposition 3.8.6 and the relation (see Definition 1.6.9)

$$R(s_0,\ldots,s_d)=1,$$

we obtain

$$h^{s_0,\ldots,s_d}_{\overline{M_0'},\ldots,\overline{M_d'}}(X)=\ln\|R\|.$$

The case where $|\cdot|$ is Archimedean is quite similar. We have

$$h^{s_0,\ldots,s_d}_{\overline{M}'_0,\ldots,\overline{M}'_d}(X) = h^{q_0^*(\boldsymbol{\alpha}_0),\ldots,q_d^*(\boldsymbol{\alpha}_d)}_{q^*(\overline{\boldsymbol{L}}_0),\ldots,q^*(\overline{\boldsymbol{L}}_d)}(I_X) + \sum_{i=0}^d \delta_i\big(\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det} - \sigma_{r_i-1}\big)$$

$$= h^{\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}_{\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\mathrm{div}(R)) + \sum_{i=0}^d \delta_i\big(\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det} - \sigma_{r_i-1}\big)$$

$$= h^{R,\pi_0^*(\boldsymbol{\alpha}_0),\ldots,\pi_d^*(\boldsymbol{\alpha}_d)}_{\overline{L},\overline{\boldsymbol{L}}_0,\ldots,\overline{\boldsymbol{L}}_d}(\check{\mathbb{P}})$$

$$+\int_{\mathbb{S}_0\times\cdots\times\mathbb{S}_d}\ln|R(z_0,\ldots,z_d)|\,\eta_{\mathbb{S}_0}(\mathrm{d}z_0)\otimes\cdots\otimes\eta_{\mathbb{S}_d}(\mathrm{d}z_d)$$

$$+\sum_{i=0}^d \delta_i\big(\ln\|\alpha_{i,0}\wedge\cdots\wedge\alpha_{i,r_i}\|_{i,*,\det} - \sigma_{r_i}\big) + \frac{1}{2}\sum_{i=0}^d\delta_i\sum_{\ell=1}^{r_i}\frac{1}{\ell}$$

$$= \int_{\mathbb{S}_0\times\cdots\times\mathbb{S}_d}\ln|R(z_0,\ldots,z_d)|\,\eta_{\mathbb{S}_0}(\mathrm{d}z_0)\otimes\cdots\otimes\eta_{\mathbb{S}_d}(\mathrm{d}z_d) + \frac{1}{2}\sum_{i=0}^d\delta_i\sum_{\ell=1}^{r_i}\frac{1}{\ell},$$

where the first equality comes from Lemma 3.9.6, the second one from Lemma 3.9.1, the third one from Lemma 3.9.5, and the last one from Proposition 3.8.6. □

3.9.8. Remark. — Note that the result of Theorem 3.9.7 does not depend on the choice of the vectors $\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d$. If we are only interested in the equalities in the theorem, we could choose $\boldsymbol{\alpha}_0,\ldots,\boldsymbol{\alpha}_d$ carefully to make the computation simpler. However, the formulae in the lemmas proving the theorem are of their own interest, especially in the computations of height of homogeneous hypersurfaces in multi-projective spaces, and hence are worth to be detailed.

3.9.9. Proposition. — *Assume that the absolute value $|\cdot|$ is non-Archimedean. Let K be an extension of k, on which the absolute value extends. We assume that K is complete with respect to the extended absolute value. Let X be a projective scheme over* $\operatorname{Spec} k$, *d be the dimensional of X, and $\overline{D}_i = (D_i, g_i)$ be a family of integrable metrised Cartier divisors, where $i\in\{0,\ldots,d\}$, such that $D_0,\ldots,D_d$ intersect properly. For each $i\in\{0,\ldots,d\}$, let $\overline{D}_{i,K} := (D_{i,k}, g_{i,k})$. Then the following equality holds:*

$$(\overline{D}_0\cdots\overline{D}_d)_{(k,|\cdot|)} = (\overline{D}_{0,K}\cdots\overline{D}_{d,K})_{(K,|\cdot|)}. \tag{3.25}$$

Proof. — **Step 1**: In this step, we assume that $D_0,\ldots,D_d$ are very ample, and, for each $i\in\{0,\ldots,d\}$, there exist a positive integer m_i and an ultrametric norm $\|\cdot\|_i$ on $E_i = H^0(X,\mathcal{O}_X(m_iD_i))$, such that φ_{g_i} identifies with the quotient metric induced by $\|\cdot\|_i$.

For each $i\in\{0,\ldots,d\}$, let $f_i : X\to\mathbb{P}(E_i)$ be the canonical closed embedding. Note that $\mathcal{O}_X(m_iD_i)\cong f_i^*(\mathcal{O}_{E_i}(1))$. In order to simplify the notation, we let L_i be the line bundle $\mathcal{O}_X(m_iD_i)$ and s_i be the canonical regular meromorphic section of

L_i. Let R be the resultant

$$R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d},$$

which is considered as an element of

$$S^{\delta_0}(E_0^\vee)\otimes_k\cdots\otimes_k S^{\delta_d}(E_d^\vee),$$

and

$$\delta_i=(D_0\cdots D_{i-1}D_{i+1}\cdots D_d).$$

Then, by Theorem 3.9.7, the equality

$$(\overline{D}_0\cdots\overline{D}_d)_{(k,|\cdot|)}=\ln\|R\|$$

holds, where $\|\cdot\|$ denotes the ε-tensor product of ε-tensor powers of $\|\cdot\|_{i,*}$. Similarly, by Remarks 1.6.10 and 3.9.3, one has

$$(\overline{D}_{0,K},\ldots,\overline{D}_{d,K})_{(K,|\cdot|)}=\ln\|R\otimes 1\|_K.$$

By [**13**, Proposition 1.3.1 (1)], one has $\|R\otimes 1\|_K=\|R\|$. Hence the equality (3.25) follows.

Step 2: In this step, we still assume that $D_0,\ldots,D_d$ are very ample. However, the Green functions $g_0,\ldots,g_d$ are only supposed to be plurisubharmonic.

For any $i\in\{0,\ldots,d\}$ and any positive integer m, let $g_i^{(m)}$ be the Green function associated with the quotient metric $\varphi_{g_i}^{(m)}$ as in Definition 3.3.8, and let $\overline{D}_i^{(m)}=(D_i,g_i^{(m)})$. By Proposition 3.3.12, we obtain that, for any $i\in\{0,\ldots,d\}$,

$$\lim_{m\to+\infty}\sup_{x\in X^{\mathrm{an}}}|g_i^{(m)}-g_i|(x)=0, \tag{3.26}$$

Therefore, by Corollary 3.6.7 (see also §3.7), we obtain

$$\lim_{m\to+\infty}(\overline{D}_0^{(m)}\cdots\overline{D}_d^{(m)})_{(k,|\cdot|)}=(\overline{D}_0\cdots\overline{D}_d)_{(k,|\cdot|)}. \tag{3.27}$$

Moreover, (3.26) leads to

$$\lim_{m\to+\infty}\sup_{x\in X_K^{\mathrm{an}}}|g_{i,K}^{(m)}-g_{i,K}|(x)=0.$$

Hence, similarly to (3.27), we have

$$\lim_{m\to+\infty}(\overline{D}_{0,K}^{(m)}\cdots\overline{D}_{d,K}^{(m)})_{(K,|\cdot|)}=(\overline{D}_{0,K}\cdots\overline{D}_{d,K})_{(K,|\cdot|)}$$

Note that, by [**13**, Proposition 1.3.16], $g_{i,K}^{(m)}$ is also the Green function associated with a quotient metric. Therefore, by the result in Step 1, we obtain that

$$(\overline{D}_0^{(m)}\cdots\overline{D}_d^{(m)})_{(k,|\cdot|)}=(\overline{D}_{0,K}^{(m)}\cdots\overline{D}_{d,K}^{(m)})_{(K,|\cdot|)}$$

for any m, so that, by passing to limit when $m\to+\infty$, we obtain (3.25).

Step 3: We now treat the general case. For each $i\in\{-1,0,\ldots,d\}$, we consider the following condition (C_r):

For any $i\in\{0,\ldots,d\}$ such that $1\leqslant i\leqslant r$, the Cartier divisor D_i is very ample and the Green function g_i is plurisubharmonic.

We will show by inverted induction on r that, under the condition (C_r), the equality (3.25) holds. Note that the initial case where $r = d$ is proved in Step 2. We suppose that the equality (3.25) is true under the condition (C_r) and will prove it under the condition (C_{r-1}). Since $\overline{D}_r$ is integrable, there exists very ample Cartier divisors A'_r and A''_r, and plurisubharmonic Green functions h'_r and h''_r of A'_r and A''_r, respectively, such that

$$(D_r, g_r) = (A'_r, h'_r) - (A''_r, h''_r).$$

By Claim 1.3.8 (see also Remark 1.3.9), there exists a very ample Cartier divisor B_r such that

$$(D_0, \dots, D_{r-1}, B_r + A'_r, D_{r+1}, \dots, D_d) \in \mathcal{IP}_X^{(d)}.$$

Since $\mathcal{IP}_X^{(d)}$ is a multilinear subset of $\mathrm{Div}(X)^{n+1}$, we obtain that

$$(D_0, \dots, D_{r-1}, B_r + A''_r, D_{r+1}, \dots, D_d) \in \mathcal{IP}_X^{(d)}.$$

We pick arbitrarily a plurisubharmonic Green function l_r on B_r. Let

$$\overline{D}'_r = (B_r + A'_r, l_r + h'_r), \quad \overline{D}''_r = (B_r + A''_r, l_r + h''_r)$$

Then the induction hypothesis shows that

$$(\overline{D}_0 \cdots \overline{D}_{r-1} \overline{D}'_r \overline{D}_{r+1} \cdots \overline{D}_d)_{(k,|\cdot|)} = (\overline{D}_{0,K} \cdots \overline{D}_{r-1,K} \overline{D}'_{r,K} \overline{D}_{r+1,K} \cdots \overline{D}_{d,K})_{(K,|\cdot|)},$$
$$(\overline{D}_0 \cdots \overline{D}_{r-1} \overline{D}''_r \overline{D}_{r+1} \cdots \overline{D}_d)_{(k,|\cdot|)} = (\overline{D}_{0,K} \cdots \overline{D}_{r-1,K} \overline{D}''_{r,K} \overline{D}_{r+1,K} \cdots \overline{D}_{d,K})_{(K,|\cdot|)}.$$

Taking the difference, we obtain (3.25) □

3.9.10. Remark. — If K is a subfield of $\mathbb{C}_k$, the assertion of Proposition 3.9.9 is obvious by its definition (cf. Definition 3.7.1). In particular, the statement of Proposition 3.9.9 is also true when $|\cdot|$ is Archimedean. Proposition 3.9.9 guarantees the invariance of intersection number under any field extension.

3.10. Trivial valuation case

In this section, we fix a field k and equip it with the trivial absolute value $|\cdot|$, namely $|a| = 1$ for any $a \in k^\times$. Let $K = k(T)$ be the field of rational functions over k, and u be a positive constant such that $u \neq 1$. By Lemma 2.6.3, there exists a non-Archimedean absolute value $|\cdot|_u$ on K which extends the above absolute value $|\cdot|$ on k, such that,

$$\forall\, f = a_0 + a_1 T + \cdots + a_n T^n \in k[T], \quad |f|_u = \max_{i \in \{0,\dots,n\}} |a_i| u^i.$$

Note that $|\cdot|_u$ is not trivial.

3.10.1. Definition. — Let X be a projective scheme of dimension d over $\operatorname{Spec} k$. If $\overline{D}_i = (D_i, g_i)$, $i \in \{0, \dots, d\}$, is a family of integrable metrized Cartier divisors, such that $D_0,\ldots,D_d$ intersect properly. We denote by $(\overline{D}_0\cdots\overline{D}_d)_{(k,|\cdot|)}$ the intersection number

$$((D_{0,K},g_{0,K})\cdots(D_{d,K},g_{d,K}))_{(K,|\cdot|_u)}$$

3.10.2. Notation and assumptions. — Let $((E_i,\|\cdot\|_i))_{i=0}^d$ be a family of finite-dimensional ultrametrically normed vector space over k. For any $i\in\{0,\ldots,d\}$, let $r_i=\dim_k(E_i)-1$, $f_i:X\to\mathbb{P}(E_i)$ be a closed immersion, and s_i be an element of E_i, viewed as a global section of $\mathcal{O}_{E_i}(1)$. We assume that the restriction of s_i to X defines a regular meromorphic section of $L_i:=\mathcal{O}_{E_i}(1)|_X$ and that the Cartier divisors

$$D_i=\operatorname{div}(s_i|_X),\quad i\in\{0,\ldots,d\}$$

intersect properly. We equip each D_i with the Green function associated with the quotient metric induced by $\|\cdot\|_i$. Let R be the resultant

$$R=R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d}\in S^{\delta_0}(E_0^\vee)\otimes_k\cdots\otimes_k S^{\delta_d}(E_d^\vee),$$

where

$$\delta_i=(D_0\cdots D_{i-1}\cdot D_{i+1}\cdots D_d).$$

3.10.3. Proposition. — *Under Notation and assumptions 3.10.2, the following equality holds*

$$(\overline{D}_0\cdots\overline{D}_d)_{(k,|\cdot|)}=\ln\|R\|,\tag{3.28}$$

where $\|\cdot\|$ denotes the ε-tensor product of ε-symmetric power norms of $\|\cdot\|_{i,*}$.

Proof. — Under the isomorphism of K-vector spaces

$$(S^{\delta_0}(E_0^\vee)\otimes_k\cdots\otimes_k S^{\delta_d}(E_d^\vee))\otimes_k K\cong S^{\delta_0}(E_{0,K}^\vee)\otimes_K\cdots\otimes_K S^{\delta_d}(E_{d,K}^\vee),$$

the element $R\otimes 1$ coincides with the resultant (see Remark 1.6.10)

$$R^{X_K,s_0\otimes 1,\ldots,s_d\otimes 1}_{f_{0,K},\ldots,f_{d,K}}.$$

By Theorem 3.9.7 and Remark 3.9.3, one has

$$(\overline{D}_0\cdots\overline{D}_d)_{(k,|\cdot|)}=\ln\|R\otimes 1\|_K.$$

By [**13**, Proposition 1.3.1 (1)], one has $\|R\otimes 1\|_K=\|R\|$. Hence we obtain the equality (3.28). □

3.10.4. Corollary. — *Let X be a projective scheme of dimension d over* $\operatorname{Spec}k$. *If $\overline{D}_i=(D_i,g_i)$, $i\in\{0,\ldots,d\}$, is a family of integrable metrized Cartier divisors, such that $D_0,\ldots,D_d$ intersect properly. Then the intersection number $(\overline{D}_0\cdots\overline{D}_d)_{(k,|\cdot|)}$ does not depend on the choice of u.*

Proof. — By the multi-linearity of the intersection number, it suffices to treat the case where all Cartier divisors D_i are very ample and all g_i are plurisubharmonic. Moreover, by Proposition 3.3.12 and Corollary 3.6.7 we can further reduce the problem to the case of Notation and assumptions 3.10.2. In that case the assertion follows from (3.28). □

3.10.5. Remark. — By using Remark 3.7.2, one has the following properties.

(1) The set $\widehat{\mathcal{IP}}_X$ forms a symmetric multi-linear subset of the group $\widehat{\mathrm{Int}}(X)^{d+1}$. Moreover, the function of local intersection number

$$\big((D_0, g_0) \cdots (D_d, g_d)\big) \longmapsto \big((D_0, g_0) \cdots (D_d, g_d)\big)_v$$

form a symmetric multi-linear map from $\widehat{\mathcal{IP}}_X$ to $\mathbb{R}$.

(2) Let $\pi : Y \to X$ be a surjective morphism of geometrically integral projective schemes over k. We set $e = \dim X$ and $d = \dim Y$. Let $(D_0, g_0), \ldots, (D_d, g_d)$ be integrable metrized Cartier divisors on X such that $(\pi^*(D_0), \ldots, \pi^*(D_d)) \in \mathcal{IP}_Y$. Then one has the following:

(i) If $d > e$, then $(\pi^*(D_0, g_0) \cdots \pi^*(D_d, g_d))_v = 0$.

(ii) If $d = e$ and $(D_0, \ldots, D_d) \in \mathcal{IP}_X$, then

$$(\pi^*(D_0, g_0) \cdots \pi^*(D_d, g_d))_v = (\deg \pi)((D_0, g_0) \cdots (D_d, g_d))_v.$$

(3) Let f be a regular meromorphic function on X and $(D_1, g_1), \ldots, (D_d, g_d)$ be integrable metrized Cartier divisors on X such that $(\mathrm{div}(f), D_1, \ldots, D_d) \in \mathcal{IP}_X$. Then

$$\big(\widehat{\mathrm{div}}(f) \cdot (D_1, g_1) \cdots (D_d, g_d)\big)_v = 0.$$

Note that $-\log |f|(x^{\mathrm{an}}) = 0$ for any $x \in X_{(0)}$ in Remark 3.7.2 because $|\cdot|$ is trivial.

Let $(L_0, \varphi_0), \ldots, (L_d, \varphi_d)$ be a family of integrable metrized invertible $\mathcal{O}_X$-modules. By the property (3), the local intersection number $\big((L_0, \varphi_0) \cdots (L_d, \varphi_d)\big)_v$ is well-defined.

CHAPTER 4

GLOBAL INTERSECTION NUMBER

Let K be a field and $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a adelic curve the underlying field of which is K. For any $\omega \in \Omega$, we denote by K_ω the completion of K with respect to $|\cdot|_\omega$. We assume that, either the σ-algebra $\mathcal{A}$ is discrete, or there exists a countable subfield K_0 of K which is dense in each K_ω, $\omega \in \Omega$. Let X be a d-dimensional projective scheme over K. For any $\omega \in \Omega$, let X_ω be the fiber product $X \times_{\operatorname{Spec} K} \operatorname{Spec} K_\omega$. Note that the morphism $\operatorname{Spec} K_\omega \to \operatorname{Spec} K$ is flat. Hence the morphism of projection $X_\omega \to X$ is also flat (see [**33**, IV_1.(2.1.4)]).

4.1. Reminder on adelic vector bundles

Let E be a finite-dimensional vector space over K. We call *norm family* on E any collection $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$, where $\|\cdot\|_\omega$ is a norm on $E_\omega = E \otimes_K K_\omega$. Note that the dual norms $\xi^\vee := (\|\cdot\|_{\omega,*})_{\omega\in\Omega}$ form a norm family on the dual vector space $E^\vee$. If all norms $\|\cdot\|_\omega$ are either ultrametric or induced by an inner product, we say that the norm family ξ is *Hermitian*.

4.1.1. Example. — Let $\boldsymbol{e} = (e_i)_{i=1}^r$ be a basis of E over K. We denote by $\xi_{\boldsymbol{e}}$ the norm family $(\|\cdot\|_{\boldsymbol{e},\omega})_{\omega\in\Omega}$, where for any $(\lambda_1, \ldots, \lambda_r) \in K_\omega^r$,

$$\|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_{\boldsymbol{e},\omega} = \begin{cases} \max\{|\lambda_1|_\omega, \ldots, |\lambda_r|_\omega\}, & \text{if } |\cdot|_\omega \text{ is non-Archimedean,} \\ (|\lambda_1|_\omega^2, \ldots, |\lambda_r|_\omega^2)^{1/2}, & \text{if } |\cdot|_\omega \text{ is Archimedean.} \end{cases}$$

The norm family $\xi_{\boldsymbol{e}}$ is Hermitian. We call it the *norm family* associated with the basis $\boldsymbol{e}$.

4.1.2. Definition. — Let E be a finite-dimensional vector space over K and $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ be a norm family on E. If for any $s \in E$, the function

$$(\omega \in \Omega) \longmapsto \|s\|_\omega$$

is $\mathcal{A}$-measurable, we say that the norm family ξ is *measurable* (note that under the assumption on S above, this condition also implies that, for any $\alpha \in E^\vee$, the function $(\omega \in \Omega) \mapsto \|\alpha\|_{\omega,*}$ is $\mathcal{A}$-measurable, see [**13**] Proposition 4.1.24). We say that the norm family ξ is *strongly dominated* if there existes an integrable function A on Ω and a basis $\boldsymbol{e}$ of E over K such that

$$\forall\, \omega \in \Omega, \quad d(\|\cdot\|_\omega, \|\cdot\|_{\boldsymbol{e},\omega}) \leqslant A(\omega).$$

If ξ is strongly dominated and measurable, we say that (E, ξ) is a *strongly adelic vector bundle* on S.

4.1.3. Definition. — Let X be a projective K-scheme and L be an invertible $\mathcal{O}_X$-module. For any $\omega \in \Omega$, we denote by L_ω the pull-back of L by the morphism of projection $X_\omega \to X$. We call *metric family* of L and family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$, where each φ_ω is a continuous metric on L_ω (see Definition 3.3.1). Note that the dual metrics $(\varphi_\omega^\vee)_{\omega\in\Omega}$ form a metric family on the dual invertible $\mathcal{O}_X$-module $L^\vee$, which we denote by $\varphi^\vee$. If L_1 and L_2 are invertible $\mathcal{O}_X$-modules, and φ_1 and φ_2 are metric families on L_1 and L_2, respectively, then the metrics $(\varphi_{1,\omega} \otimes \varphi_{2,\omega})_{\omega\in\Omega}$ form a metric family of $L_1 \otimes L_2$, which we denote by $\varphi_1 \otimes \varphi_2$.

If φ and φ' are two metric metrics of the same invertible $\mathcal{O}_X$-module L, we define the *local distance function* between φ and φ' as the function

$$(\omega \in \Omega) \longmapsto d_\omega(\varphi, \varphi') := \sup_{x \in X_\omega^{\mathrm{an}}} \left| \ln \frac{|\cdot|_{\varphi_\omega}(x)}{|\cdot|_{\varphi'_\omega}(x)} \right|$$

4.1.4. Remark. — In the case where X is the spectrum of a finite extension K' of K, an invertible $\mathcal{O}_X$-module L can be considered as a one-dimensional vector space over K', and a metric family on L identifies with a norm family of L if we consider the adelic curve $S \otimes_K K'$.

4.1.5. Definition. — Let $f : Y \to X$ be a projective K-morphism of projective K-schemes. Let L be an invertible $\mathcal{O}_X$-module, equipped with a metric family $\varphi = (\varphi_\omega)_{\omega\in\Omega}$. For any $\omega \in \Omega$, let $f_\omega : Y_\omega \to X_\omega$ be the K_ω-morphism induce by f by extension of scalars. Then, for any $\omega \in \Omega$, the metric φ_ω induces by pull-back a continuous metric $f_\omega^*(\varphi_\omega)$ on $f_\omega^*(L_\omega)$ such that, for any $y \in Y_\omega^{\mathrm{an}}$ and any $\ell \in L_\omega(f^{\mathrm{an}}(y))$, one has

$$|f_\omega^*(\ell)|_{f_\omega^*(\varphi_\omega)}(y) = |\ell|_{\varphi_\omega}(f^{\mathrm{an}}(y)).$$

We denote by $f^*(\varphi)$ the metric family $(f_\omega^*(\varphi_\omega))_{\omega\in\Omega}$ and call it the *pull-back of φ by f*. In the case where f is an immersion, $f^*(\varphi)$ is also called *restriction of φ*.

4.1.6. Example. — A natural example of metric family is the *quotient metric family* induced by a norm family. Denote by $\pi : X \to \operatorname{Spec} K$ the structural morphism. Let E be a finite-dimensional vector space over K and $f : \pi^*(E) \to L^{\otimes n}$ be a surjective homomorphism of $\mathcal{O}_X$-modules, where n is a positive integer. For any $\omega \in \Omega$, the homomorphism f induces by pull-back a surjective homomorphism of $\mathcal{O}_{X_\omega}$-modules

$f_\omega : \pi_{K_\omega}^*(E) \to L_\omega$. Assume given a norm family $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ of E. We denote by φ_ξ the metric family of L consisting of quotient metrics associated with $\|\cdot\|_\omega$ (see Example 3.3.2 (1)), and call it the *quotient metric family* induced by ξ.

Assume that the norm family ξ is Hermitian. For each $\omega \in \Omega$, let $\varphi_{\xi,\omega}^{\mathrm{ort}}$ be the orthogonal quotient metric induced by $\|\cdot\|_\omega$ (see Definition 3.3.5). Note that this metric coincides with $\varphi_{\xi,\omega}$ when $|\cdot|_\omega$ is non-Archimedean or K_ω is complex. The metric family $\varphi_\xi^{\mathrm{ort}}$ is called *orthogonal quotient metric family* induced by ξ.

4.1.7. Example. — Let X be a projective K-scheme, L be an invertible $\mathcal{O}_X$-module, and $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be a metric family on L. Let K'/K be an algebraic extension of the field K, and

$$S \otimes K' = (K', (\Omega', \mathcal{A}', \nu'), \phi')$$

be the corresponding algebraic covering of the adelic curve S (see §2.2). Recall that Ω' is defined as $\Omega \times_{M_K,\phi} M_{K'}$, where M_K and $M_{K'}$ are the sets of all absolute values of K and of K', respectively.

Let X' be the fiber product $X \times_{\operatorname{Spec} K} \operatorname{Spec} K'$ and L' be the pull-back of L on X'. If ω' is an element of Ω' and ω is the image of ω' in Ω by the projection map

$$\Omega' = \Omega \times_{M_K,\phi} M_{K'} \longrightarrow \Omega,$$

then one has

$$X'_{\omega'} := X' \times_{\operatorname{Spec} K'} \operatorname{Spec} K'_{\omega'} \cong (X \times_K K_\omega) \times_{K_\omega} K'_{\omega'}.$$

Moreover, the pull-back of L_ω on $X'_{\omega'}$ identifies with $L'_{\omega'}$. We denote by $p_{\omega'}$ the morphism of projection from $X'_{\omega'}$ to X_ω. Then the map

$$p_{\omega'}^\natural : (X'_{\omega'})^{\mathrm{an}} \longrightarrow X_\omega^{\mathrm{an}},$$

sending any point $x' = (j(x'), |\cdot|_{x'})$ to the pair consisting of the scheme point $p_{\omega'}(j(x'))$ of X_ω and the restriction of $|\cdot|_{x'}$ on the residue field of $p_{\omega'}(j(x'))$, is continuous (see [**13**, Proposition 2.1.17]), where $j : (X'_{\omega'})^{\mathrm{an}} \to X'_{\omega'}$ denotes the map sending a point in the analytic space to its underlying scheme point. Therefore, the continuous metric φ_ω induces by composition with $p^\natural$ a continuous metric $\varphi_{\omega'}$ such that, for any $x' \in (X'_{\omega'})^{\mathrm{an}}$ and any $\ell \in L_\omega(p^\natural(x'))$, one has

$$\forall\, a \in \widehat{\kappa}(x'), \quad |a \otimes \ell|_{\varphi_{\omega'}}(x') = |a|_{x'} \cdot |\ell|_{\varphi_\omega}(x).$$

Therefore, $(\varphi_{\omega'})_{\omega'\in\Omega'}$ forms a metric family of L' which we denote by $\varphi_{K'}$.

4.1.8. Definition. — Let L be an invertible $\mathcal{O}_X$-module and $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be a metric family of L.

(1) We say that φ is *dominated* if there exist invertible $\mathcal{O}_X$-modules L_1 and L_2, respectively equipped with metric families φ_1 and φ_2, which are quotient metric

families associated with dominated norms families, such that $L \cong L_1 \otimes L_2^\vee$ and that the local distance function

$$(\omega \in \Omega) \longmapsto d_\omega(\varphi, \varphi_1 \otimes \varphi_2^\vee)$$

is bounded from above by a ν-integrable function (see [**13**, §6.1.1]);

(2) We say that φ is *measurable* if the following conditions are satisfied (see [**13**, §6.1.4]):

  (2.i) for any closed point P of X, the norm family $P^*(\varphi)$ of $P^*(L)$ is measurable,

  (2.ii) for any $\xi \in X^{\mathrm{an}}$ (where we consider the trivial absolute value on K in the construction of X^{an}) whose associated scheme is of dimension 1 and such that the exponent [(1)] of the absolute value $|\cdot|_\xi$ is rational, and for any $\ell \in L \otimes_{\mathcal{O}_X} \widehat{\kappa}(\xi)$, the function

$$(\omega \in \Omega_0) \longmapsto |\ell|_{\varphi_\omega}(\xi)$$

  is measurable, where Ω_0 is the subset of $\omega \in \Omega$ such that $|\cdot|_\omega$ is trivial, and we consider the restriction of the σ-algebra $\mathcal{A}$ to Ω_0.

If φ is both dominated and measurable, we say that the pair $\overline{L} = (L, \varphi)$ is an *adelic line bundle.*

4.1.9. Proposition. — *Let $\pi : X \to \operatorname{Spec} K$ be a projective scheme over $\operatorname{Spec} K$, L be an invertible $\mathcal{O}_X$-module, φ be a metric family of L, and $E = H^0(X, L)$. We equip E with a norm family $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$. Consider the following norm family $\xi' = (\|\cdot\|'_\omega)_{\omega\in\Omega}$ defined as*

$$\forall\, s \in H^0(X_\omega, L_\omega), \quad \|s\|'_\omega := \max\Big\{ \sup_{x\in X_\omega^{\mathrm{an}}} |s|_{\varphi_\omega}(x), \|s\|_\omega \Big\}.$$

Then one has the following:

(1) *If φ and ξ are both measurable, then ξ' is also measurable.*

(2) *If φ is dominated and ξ is strongly dominated, then ξ' is strongly dominated.*

Proof. — (1) For any $\omega \in \Omega$, we let $\|\cdot\|_{\varphi_\omega}$ be the seminorm on $E \otimes_K K_\omega = H^0(X_\omega, L_\omega)$ defined as

$$\forall\, s \in H^0(X_\omega, L_\omega), \quad \|s\|_{\varphi_\omega} := \sup_{x\in X_\omega^{\mathrm{an}}} |s|_{\varphi_\omega}(x).$$

By [**13**, Propositions 6.1.20 and 6.1.26], for any $s \in H^0(X, L)$, the function

$$(\omega \in \Omega) \longmapsto \|s\|_{\varphi_\omega}$$

1. Since the schematic point associated with ξ is of dimension 1, the absolute value $|\cdot|_\xi$ is discrete and hence is of the form $|\cdot|_\xi = \exp(-t \operatorname{ord}_\xi(\cdot))$, where the (surjective) map $\operatorname{ord}_\xi(\cdot) : \widehat{\kappa}(\xi) \to \mathbb{Z} \cup \{+\infty\}$ is the discrete valuation corresponding to the absolute value $|\cdot|_\xi$. The non-negative real number t is called the *exponent* of the absolute value $|\cdot|_\xi$.

is measurable. Therefore the function

$$(\omega \in \Omega) \longmapsto \|s\|'_\omega = \max\{\|s\|_{\varphi_\omega}, \|s\|_\omega\}$$

is also measurable once the norm family ξ is measurable.

(2) We may assume without loss of generality that there exists a basis $\boldsymbol{e} = (e_i)_{i=1}^r$ of E such that, for any $\omega \in \Omega$

$$\forall\,(\lambda_1, \ldots, \lambda_r) \in K_\omega^r, \quad \|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\omega = \max_{i \in \{1,\ldots,r\}} |\lambda_i|_\omega.$$

By [**13**, Remark 6.1.17], for any $s \in H^0(X, L)$, the function

$$(\omega \in \Omega) \longmapsto \ln \|s\|_{\varphi_\omega}$$

is bounded from above by an integrable function. Let $A : \Omega \to \mathbb{R}_{\geqslant 0}$ be a positive integrable function on Ω such that

$$\forall\,\omega \in \Omega, \quad \max_{i \in \{1,\ldots,r\}} \ln \|e_i\|_{\varphi_\omega} \leqslant A(\omega).$$

For any $\omega \in \Omega \setminus \Omega_\infty$ and any $(\lambda_1, \ldots, \lambda_r) \in K_\omega^r$, one has

$$\ln \|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\omega \leqslant \ln \|\lambda_1 e_1 + \cdots + \lambda_r e_r\|'_\omega \leqslant \max_{i \in \{1,\ldots,r\}} (\ln |\lambda_i|_\omega + \ln \|e_i\|'_\omega).$$

Note that $\|e_i\|_\omega = 1$ and hence

$$\ln \|e_i\|'_\omega = \max\{\ln \|e_i\|_{\varphi_\omega}, \ln(1)\} \leqslant A(\omega).$$

Therefore one has

$$d(\|\cdot\|_\omega, \|\cdot\|'_\omega) \leqslant A(\omega).$$

In the case where $\omega \in \Omega_\infty$, for any $(\lambda_1, \ldots, \lambda_r) \in K_\omega^r$ one has

$$\ln \|\lambda_1 e_1 + \cdots + \lambda_r e_r\|_\omega \leqslant \ln \|\lambda_1 e_1 + \cdots + \lambda_r e_r\|'_\omega \leqslant \max_{i \in \{1,\ldots,r\}} \ln |\lambda_i|_\omega + A(\omega) + \ln(r).$$

Finally we obtain that

$$\forall\,\omega \in \Omega, \quad d_\omega(\xi, \xi') \leqslant A(\omega) + \ln(r)\mathbb{1}_{\Omega_\infty}(\omega).$$

Hence the norm family ξ' is strongly dominated (see [**13**, Proposition 3.1.2] for the fact that $\nu(\Omega_\infty)$ is finite). □

4.1.10. Lemma. — *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve, K' be an algebraic extension of K and $S_{K'} = S \otimes_K K' = (K', (\Omega', \mathcal{A}', \nu'), \phi')$. Let f be a function on Ω. Then one has the following:*

(1) *f is measurable if and only if $f \circ \pi_{K'/K}$ is measurable.*
(2) *f is integrable if and only if $f \circ \pi_{K'/K}$ is integrable.*

Proof. — Clearly we may assume that f is non-negative, so that it is a consequence of [**13**, Proposition 3.4.8 and Proposition 3.4.9]. □

4.2. Integrability of local intersection numbers

In this section, we fix a projective K-scheme X. Let d be the dimension of X.

4.2.1. Definition. — Let D be a Cartier divisor on X. For any $\omega \in \Omega$, let D_ω be the pull-back of D by the morphism of projection $X_\omega \to X$, which is well defined since the morphism of projection $X_\omega \to X$ is flat (see Remark 1.2.13 and Definition 1.2.14). We call *Green function family* of D any family $(g_\omega)_{\omega\in\Omega}$ parametrized by Ω, where each g_ω is a Green function of D_ω. We denote by φ_g the metric family $(|\cdot|_{g_\omega})_{\omega\in\Omega}$ of $\mathcal{O}_X(D)$, where $|\cdot|_{g_\omega}$ is the continuous metric on $\mathcal{O}_{X_\omega}(D_\omega)$ induced by the Green function g_ω (see Remark 3.4.3). If the metric family φ_g is measurable, we say that the Green function family g is *measurable*. If the metric φ_g is dominated, we say that the Green function family g is *dominated*. We refer to Definition 4.1.8 for the dominancy and measurability of metrics. If g is both dominated and measurable, we say that (D, g) is an *adelic Cartier divisor*.

Let D be an invertible $\mathcal{O}_X$-module and g be a Green function family of D. If D is ample and all metrics in the family φ_g are semi-positive, we say that the Green function family g is *semi-positive*. We say that (D, g) is *integrable* if there exist ample Cartier divisors D_1 and D_2, together with semi-positive Green function families g_1 and g_2 of D_1 and D_2 respectively, such that $D = D_1 - D_2$ and $g = g_1 - g_2$. Similarly, we say that an adelic line bundle (L, φ) is *integrable* if there exists ample invertible $\mathcal{O}_X$-modules L_1 and L_2, and metric families consisting of semi-positive metrics φ_1 and φ_2 on L_1 and L_2, respectively, such that $L = L_2 \otimes L_1^\vee$ and $\varphi = \varphi_2 \otimes \varphi_1^\vee$.

Let $D_0, \ldots, D_d$ be a family of Cartier divisors, which intersect properly. For any $i \in \{0, \ldots, d\}$, let g_i be a Green function family of D_i such that (D_i, g_i) is integrable. Then, for any $\omega \in \Omega$, a local intersection number

$$((D_{0,\omega}, g_{0,\omega}), \ldots, (D_{d,\omega}, g_{d,\omega}))_{(K_\omega, |\cdot|_\omega)}$$

has been introduced in Definition 3.6.1, which we denote by

$$(\overline{D}_0 \cdots \overline{D}_d)_\omega$$

for simplicity. Thus the local intersection numbers define a function

$$(\omega \in \Omega) \longrightarrow (\overline{D}_0 \cdots \overline{D}_d)_\omega.$$

4.2.2. Definition. — Let D_1 and D_2 be Cartier divisors on X, and g_1 and g_2 be Green function families of D_1 and D_2, respectively. We say that (D_1, g_1) and (D_2, g_2) are *linearly equivalent* and we note

$$(D_1, g_1) \sim (D_2, g_2)$$

if $\mathcal{O}_X(D_1)$ is isomorphic to $\mathcal{O}_X(D_2)$ and if there exists an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(D_1) \to \mathcal{O}_X(D)$ which identifies the metric φ_{g_1} to φ_{g_2}.

4.2.3. Proposition. — *Assume that, for all Cartier divisors $E_0, \ldots, E_d$ which intersect properly, and measurable (resp. dominated) Green function families $h_0, \ldots, h_d$ of $E_0, \ldots, E_d$ respectively, such that all (E_i, h_i) are integrable and linearly equivalent, the function of local intersection number*

$$(\omega \in \Omega) \longmapsto (\overline{E}_0 \cdots \overline{E}_d)_\omega$$

is measurable (resp. dominated). Then, for all Cartier divisors $D_0, \ldots, D_d$ which intersect properly and measurable (resp. dominated) Green function families $g_0, \ldots, g_d$ of $D_0, \ldots, D_d$ respectively, such that all (D_i, g_i) are integrable (but not necessarily linearly equivalent), the function of local intersection number

$$(\omega \in \Omega) \longmapsto (\overline{D}_0 \cdots \overline{D}_d)_\omega$$

is measurable (resp. dominated).

Proof. — First of all, by Lemma 4.1.10, we may assume that K is algebraically closed. By Lemma 1.3.7, we can choose a matrix

$$(D_{i,j})_{(i,j)\in\{0,\ldots,d\}^2}$$

consisting of Cartier divisors on X such that $(D_{i_0,0}, \ldots, D_{i_d,d}) \in \mathcal{IP}_X$ for any $(i_0, \ldots, i_d) \in \{0, \ldots, d\}^{d+1}$, and that $D_{i,j} \sim D_i$ for all $(i,j) \in \{0, \ldots, d\}^2$. Let $g_{i,j}$ be a family of integrable Green functions of $D_{i,j}$ such that $(D_{i,j}, g_{i,j}) \sim (D_i, g_i)$. By Proposition 1.1.4,

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}(\{0,\ldots,d\})} \big(\overline{D}_{0,\sigma(0)} \cdots \overline{D}_{d,\sigma(d)}\big)_\omega &= \sum_{\sigma\in\mathfrak{S}(\{0,\ldots,d\})} \big(\overline{D}_{\sigma(0),0} \cdots \overline{D}_{\sigma(d),d}\big)_\omega \\ &= \sum_{\varnothing\neq I\subseteq\{0,\ldots,d\}} (-1)^{(d+1)-\mathrm{card}(I)} \bigg(\Big(\sum\nolimits_{i\in I} \overline{D}_{i,0}\Big) \cdots \Big(\sum\nolimits_{i\in I} \overline{D}_{i,d}\Big)\bigg)_\omega,\end{aligned}$$

where $\overline{D}_{i,j} = (D_{i,j}, g_{i,j})$. Note that $\sum_{i\in I} \overline{D}_{i,a} \sim \sum_{i\in I} \overline{D}_{i,b}$, so that

$$(\omega \in \Omega) \mapsto \sum_{\varnothing\neq I\subseteq\{0,\ldots,d\}} (-1)^{(d+1)-\mathrm{card}(I)} \bigg(\Big(\sum\nolimits_{i\in I} \overline{D}_{i,0}\Big) \cdots \Big(\sum\nolimits_{i\in I} \overline{D}_{i,d}\Big)\bigg)_\omega$$

is measurable (resp. dominant) by our assumption. Moreover, by Proposition 3.6.5, for each $\sigma \in \mathfrak{S}(\{0, \ldots, d\})$, there is an integrable function A_σ on Ω such that

$$(\overline{D}_{0,\sigma(0)} \cdots \overline{D}_{d,\sigma(d)})_\omega = (\overline{D}_0 \cdots \overline{D}_d)_\omega + A_\sigma(\omega).$$

Thus the assertion follows. Note that $\displaystyle\int_\Omega A_\sigma(\omega)\,\nu(d\omega) = 0$ if S is proper. □

4.2.4. Theorem. — *Assume that $\Omega_\infty = \varnothing$. Let $(L_i)_{i=0}^d$ be a family of invertible $\mathcal{O}_X$-modules. For each $i \in \{0, \ldots, d\}$, let s_i be a regular meromorphic section of L_i and $D_i = \mathrm{div}(s_i)$. We suppose that $D_0, \ldots, D_d$ intersect properly. For any $i \in \{0, \ldots, d\}$, let $\varphi_i = (\varphi_{i,\omega})_{\omega\in\Omega}$ be a measurable metric family on L_i such that $(L_{i,\omega}, \varphi_{i,\omega})$ is* *integrable, and let $g_i = (g_{i,\omega})_{\omega\in\Omega}$ be the family of Green functions of D_i corresponding to φ_i. Then the function of local intersection numbers*

$$(\omega \in \Omega) \longrightarrow (\overline{D}_0 \cdots \overline{D}_d)_\omega \tag{4.1}$$

is $\mathcal{A}$-measurable.

Proof. — By Lemma 4.1.10, we may assume that K is algebraically closed. By using Proposition 3.6.3, we may further assume that $L_0, \ldots, L_d$ are very ample. For any $i \in \{0, \ldots, d\}$, we denote by δ_i the intersection number

$$\deg(c_1(L_0)\cdots c_1(L_{i-1})c_1(L_{i+1})\cdots c_1(L_d) \cap [X]).$$

We introduce, for each $r \in \{-1, \ldots, d\}$, then following condition (C_r):

> *For each $i \in \{0, \ldots, d\}$ such that $0 \leqslant i \leqslant r$, there exist a positive integer m_i and a measurable Hermitian norm family $\xi_i = (\|\cdot\|_{i,\omega})_{\omega\in\Omega}$ on $H^0(X, L_i^{\otimes m_i})$, such that φ_i identifies with the quotient metric family induced by ξ_i.*

We will prove by inverted induction on r that, under the condition (C_r), the function (4.1) is $\mathcal{A}$-measurable. Note that the condition (C_{-1}) is always true and hence the measurability of (4.1) under (C_{-1}) is just the statement of the theorem. We begin with the case where $r = d$. For any $i \in \{1, \ldots, d\}$, let $E_i = H^0(X, L_i^{\otimes m_i})$ and $f_i : X \to \mathbb{P}(E_i)$ be the canonical closed embedding. Note that $L_i^{\otimes m_i}$ is isomorphic to $f_i^*(\mathcal{O}_{E_i}(1))$. We denote by R the resultant

$$R_{f_0,\ldots,f_d}^{X,s_0^{\otimes m_0},\ldots,s_d^{\otimes m_d}},$$

which is an element of

$$S^{\delta_0 N_0}(E_0^\vee) \otimes_K \cdots \otimes_K S^{\delta_d N_d}(E_d^\vee),$$

where

$$N_i = \frac{m_0 \cdots m_d}{m_i}.$$

We equip this vector space with the family of ε-tensor product of ε-symmetric power norms of $\|\cdot\|_{i,\omega,*}$ (see Definition 3.9.2), which we denote by $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$. By [**13**, Proposition 4.1.24], the norm family ξ is measurable. By Theorem 3.9.7, one has

$$m_0 \cdots m_d(\overline{D}_0 \cdots \overline{D}_d)_\omega = (m_0\overline{D}_0 \cdots m_d\overline{D}_d)_\omega = \ln \|R\|_\omega.$$

Hence the function

$$(\omega \in \Omega) \longmapsto (\overline{D}_0 \cdots \overline{D}_d)_\omega$$

is measurable.

We prove the measurability of (4.1) under (C_{r-1}) in assuming that the measurability of (4.1) is true under (C_r), where $r \in \{0, \ldots, d\}$. For any positive integer m, we let $g_r^{(m)}$ be the Green function family of D_r corresponding to the metric family $\varphi_r^{(m)} = (\varphi_{r,\omega}^{(m)})_{\omega\in\Omega}$ (see Definition 3.3.8). We first show that the function

$$(\omega \in \Omega) \longmapsto (\overline{D}_0 \cdots \overline{D}_{r-1}(D_r, g_r^{(m)})\overline{D}_{r+1} \cdots \overline{D}_d)_\omega$$

is measurable. For this purpose, we choose arbitrarily a measurable norm family $\xi_r = (\|\cdot\|_\omega)_{\omega\in\Omega}$ on the vector space $H^0(X, L^{\otimes m})$ (one can choose $\xi_r = \xi_{\boldsymbol{e}}$, where $\boldsymbol{e}$ is a basis of $H^0(X, L^{\otimes m})$, see Example 4.1.1). For any $a > 0$ and any $\omega \in \Omega$, we let $\varphi_{r,a,\omega}^{(m)}$ be the quotient metric on L_r induced by the norm

$$\|\cdot\|_{a,\omega} := \max\{\|\cdot\|_{m\varphi_r}, a\|\cdot\|_\omega\}$$

on $H^0(X_\omega, L_\omega^{\otimes m})$, and let $g_{r,a}^{(m)}$ be the Green function of D_r corresponding to the metric $\varphi_{r,a,\omega}^{(m)}$. By Proposition 4.1.9, the norm family $\xi_{r,a} := (\|\cdot\|_{a,\omega})_{\omega\in\Omega}$ is measurable. Therefore $\overline{D}_0, \ldots, \overline{D}_{r-1}, (D_r, g_{r,a}^{(m)}), \overline{D}_{r+1} \cdots \overline{D}_d$ satisfy the condition (C_r). By the induction hypothesis, we obtain that the function

$$(\omega \in \Omega) \longmapsto (\overline{D}_0 \cdots \overline{D}_{r-1}(D_r, g_{r,a}^{(m)})\overline{D}_{r+1} \cdots \overline{D}_d)_\omega$$

is measurable. Moreover, by Proposition 3.3.11, we obtain that, for any $\omega \in \Omega$, there exists $a_\omega > 0$ such that $g_{r,a}^{(m)} = g_r^{(m)}$ when $0 < a < a_\omega$. Therefore one has

$$(\overline{D}_0 \cdots \overline{D}_{r-1}(D_r, g_r^{(m)})\overline{D}_{r+1} \cdots \overline{D}_d)_\omega = \lim_{a\in\mathbb{Q},\, a\to 0+} (\overline{D}_0 \cdots \overline{D}_{r-1}(D_r, g_{r,a}^{(m)})\overline{D}_{r+1} \cdots \overline{D}_d)_\omega$$

and hence the function

$$(\omega \in \Omega) \longmapsto (\overline{D}_0 \cdots \overline{D}_{r-1}(D_r, g_r^{(m)})\overline{D}_{r+1} \cdots \overline{D}_d)_\omega$$

is measurable. Finally, by Proposition 3.3.12 and Corollary 3.6.7, one has

$$(\overline{D}_0 \cdots \overline{D}_d)_\omega = \lim_{m\to+\infty} (\overline{D}_0 \cdots \overline{D}_{r-1}(D_r, g_r^{(m)})\overline{D}_{r+1} \cdots \overline{D}_d)_\omega$$

and therefore the function

$$(\omega \in \Omega) \longmapsto (\overline{D}_0 \cdots \overline{D}_d)_\omega$$

is measurable. □

In the following, we study the measurability of the function of local intersection number over Archimedean places. Let us begin with the following lemma.

4.2.5. Lemma. — *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve such that Ω_∞ is not empty. Suppose that $-1 \in K$ admits a square root $\sqrt{-1}$ in K. Then there is a family $(\iota_\omega)_{\omega\in\Omega_\infty}$ of embeddings $K \to \mathbb{C}$ which satisfy the following conditions:*

(1) *for any $\omega \in \Omega_\infty$, $\iota_\omega(\sqrt{-1}) = i$, where $i \in \mathbb{C}$ denotes the usual imaginary unit,*
(2) *for any $\omega \in \Omega_\infty$, $|\cdot|_\omega = |\iota_\omega(\cdot)|$,*
(3) *for any $a \in K$, the function $(\omega \in \Omega_\infty) \mapsto \iota_\omega(a)$ is measurable.*

Proof. — Fix a family $(\sigma_\omega)_{\omega\in\Omega_\infty}$ of embeddings $K \to \mathbb{C}$ such that $|\cdot|_\omega = |\sigma_\omega(\cdot)|$ for all $\omega \in \Omega_\infty$. Note that $\sigma_\omega(\sqrt{-1}) \in \{i, -i\}$ because

$$\sigma_\omega(\sqrt{-1})^2 = \sigma_\omega((\sqrt{-1})^2) = \sigma_\omega(-1) = -1.$$

We define a family $(\iota_\omega)_{\omega\in\Omega_\infty}$ of embeddings by

$$\iota_\omega = \begin{cases} \sigma_\omega & \text{if } \sigma_\omega(\sqrt{-1}) = i, \\ \overline{\sigma_\omega} & \text{if } \sigma_\omega(\sqrt{-1}) = -i, \end{cases}$$

where $\overline{\sigma_w}$ denotes the composition of the complex conjugation with σ_w. Then $\iota_\omega(\sqrt{-1}) = i$ for all $\omega \in \Omega_\infty$. Thus one can see

$$\iota_\omega(a) = \left(|a + (1/2)|_\omega^2 - |a|_\omega^2 - |1/2|_\omega^2\right) + i\left(|a + (\sqrt{-1}/2)|_\omega^2 - |a|_\omega^2 - |\sqrt{-1}/2|_\omega^2\right),$$

as required. □

We assume that $\Omega_\infty = \Omega$. If K contains a square root $\sqrt{-1}$ of -1, then, by Lemma 4.2.5, for each $\omega \in \Omega$, there is an embedding $\sigma_\omega : K \hookrightarrow \mathbb{C}$ with the following properties:

(1) $|\cdot|_\omega = |\sigma_\omega(\cdot)|$ for all $\omega \in \Omega$.
(2) $\sigma_\omega(\sqrt{-1}) = i$, so that $\sigma_\omega(a + \sqrt{-1}b) = a + ib$ for all $a, b \in \mathbb{Q}$, where i is the usual imaginary unit in $\mathbb{C}$.
(3) For $a \in K$, $(\omega \in \Omega) \mapsto \sigma_\omega(a)$ is measurable.

4.2.6. Proposition. — *We assume that $\Omega = \Omega_\infty$ and $\sqrt{-1} \in K$. Let n and d be non-negative integers with $n \geqslant d$ and $\pi : \mathbb{A}_K^n \to \mathbb{A}_K^d$ be the projection given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_d)$. Let U be a non-empty Zariski open set of $\mathbb{A}_K^d$ and X be a reduced closed subscheme of $\pi^{-1}(U)$ such that $\pi|_X : X \to U$ is finite, surjective and étale.*

We assume that either (i) $n = d$ and $X = \pi^{-1}(U)$, or (ii) K is algebraically closed field. Let $f = (f_\omega)_{\omega\in\Omega}$ be a family of functions indexed by Ω, where each f_ω is a C^∞-function on $\pi_\omega^{-1}(U_\omega)$ such that, for any K-rational point $P \in \pi^{-1}(U)(K)$, the function given by $(\omega \in \Omega) \mapsto f_\omega(P_\omega)$ is measurable. If we set $g_\omega = f_\omega|_{X_\omega}$ for $\omega \in \Omega$, then, for any $P \in X(K)$ and $l \in \{1, \ldots, d\}$,

$$(\omega \in \Omega) \mapsto \frac{\partial g_\omega}{\partial z_{l\omega}}(P_\omega) \quad \textit{and} \quad (\omega \in \Omega) \mapsto \frac{\partial g_\omega}{\partial \overline{z}_{l\omega}}(P_\omega)$$

are measurable, where $(z_{1\omega}, \ldots, z_{d\omega})$ denotes the canonical coordinates of $\mathbb{A}^d(\mathbb{C}) \times_{\sigma_\omega} \mathbb{C}$.

Proof. — **Case (i)**: $n = d$ (so that $\pi = \mathrm{id}$) and $X = \pi^{-1}(U)$.

Let $x_{l\omega}$ (resp. $y_{l\omega}$) be the real part (resp. the imaginary part) of $z_{l\omega}$. It is sufficient to show that

$$(\omega \in \Omega) \mapsto \frac{\partial f_\omega}{\partial x_{l\omega}}(P_\omega) \quad \text{and} \quad (\omega \in \Omega) \mapsto \frac{\partial f_\omega}{\partial y_{l\omega}}(P_\omega)$$

are measurable. We set $P_\omega = \sigma_\omega(P) = (a_{1\omega} + ib_{1\omega}, \ldots, a_{n\omega} + ib_{n\omega})$. Then, for $\varepsilon \in \mathbb{Q}^\times$,

$$\begin{cases} (P + \varepsilon e_l)_\omega = \sigma_\omega(P + \varepsilon e_l) = (a_{1\omega} + ib_{1\omega}, \ldots, (a_{l\omega} + \varepsilon) + ib_{l\omega}, \ldots, a_{n\omega} + ib_{n\omega}), \\ (P + \varepsilon i e_l)_\omega = \sigma_\omega(P + \varepsilon i e_l) = (a_{1\omega} + ib_{1\omega}, \ldots, a_{l\omega} + i(b_{l\omega} + \varepsilon), \ldots, a_{n\omega} + ib_{n\omega}), \end{cases}$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of K^n, so that

$$\begin{cases} \displaystyle\lim_{\substack{\varepsilon \in \mathbb{Q}^\times \\ \varepsilon \to 0}} \frac{f_\omega((P + \varepsilon e_l)_\omega) - f_\omega(P_\omega)}{\varepsilon} = \frac{\partial f_\omega}{\partial x_{l\omega}}(P_\omega), \\ \displaystyle\lim_{\substack{\varepsilon \in \mathbb{Q}^\times \\ \varepsilon \to 0}} \frac{f_\omega((P + \varepsilon i e_l)_\omega) - f_\omega(P_\omega)}{\varepsilon} = \frac{\partial f_\omega}{\partial y_{l\omega}}(P_\omega). \end{cases}$$

Note that

$$\begin{cases} \displaystyle(\omega \in \Omega) \mapsto \frac{f_\omega((P + \varepsilon e_l)_\omega) - f_\omega(P_\omega)}{\varepsilon}, \\ \displaystyle(\omega \in \Omega) \mapsto \frac{f_\omega((P + \varepsilon i e_l)_\omega) - f_\omega(P_\omega)}{\varepsilon}. \end{cases}$$

are measurable. Thus the assertion follows.

Case (ii): K is algebraically closed field.

By replacing U and X by $U \setminus \pi(P)$ and $X \setminus P$, we may assume that $P = (0, \ldots, 0)$. If we set $Q = \pi(P)$, then $(\pi|_X)^* : \mathcal{O}^h_{U,Q} \xrightarrow{\sim} \mathcal{O}^h_{X,P}$, where $\mathcal{O}^h_{U,Q}$ and $\mathcal{O}^h_{X,P}$ are the Henselizations of $\mathcal{O}_{U,Q}$ and $\mathcal{O}_{X,P}$, respectively. Thus there are $\varphi_{d+1}, \ldots, \varphi_n \in \mathcal{O}^h_{U,Q}$ such that $(\pi|_X)^*(\varphi_j) = x_j|_X$ for $j \in \{d+1, \ldots, n\}$. We set

$$\varphi_j = \sum_{e_1 \cdots e_d \in \mathbb{Z}_{\geq 0}} a_{j, e_1 \cdots e_d} X_1^{e_1} \cdots X_d^{e_d}$$

as an element of $K[\![X_1, \ldots, X_d]\!]$. Note that if we set

$$\varphi_{j\omega} = \sum_{e_1 \cdots e_d \in \mathbb{Z}_{\geq 0}} \sigma_\omega(a_{j, e_1 \cdots e_d}) X_1^{e_1} \cdots X_d^{e_d},$$

then

$$g_\omega = f_\omega(z_{1\omega}, \ldots, z_{d\omega}, \varphi_{d+1\omega}(z_{1\omega}, \ldots, z_{d\omega}), \ldots, \varphi_{n\omega}(z_{1\omega}, \ldots, z_{d\omega}))$$

as a function on U around Q. Then, for $l \in \{1, \ldots, d\}$,

$$\begin{cases} \displaystyle\frac{\partial g_\omega}{\partial z_{l\omega}}(P_\omega) = \frac{\partial f_\omega}{\partial z_{l\omega}}(0, \ldots, 0) + \sum_{j=d+1}^{n} \frac{\partial f_\omega}{\partial z_{j\omega}}(0, \ldots, 0) \frac{\partial \varphi_{j\omega}}{\partial z_{l\omega}}(0, \ldots, 0), \\ \displaystyle\frac{\partial g_\omega}{\partial \bar{z}_{l\omega}}(P_\omega) = \frac{\partial f_\omega}{\partial \bar{z}_{l\omega}}(0, \ldots, 0). \end{cases}$$

If we denote $a_{j, e_1, \ldots, e_d}$ by $a_{j,l}$ in the case where $e_1 = 0, \ldots, e_l = 1, \ldots, e_d = 0$, then

$$\begin{cases} \displaystyle\frac{\partial g_\omega}{\partial z_{l\omega}}(P_\omega) = \frac{\partial f_\omega}{\partial z_{l\omega}}(0, \ldots, 0) + \sum_{j=d+1}^{n} \frac{\partial f_\omega}{\partial z_{j\omega}}(0, \ldots, 0) \sigma_\omega(a_{j,l}), \\ \displaystyle\frac{\partial g_\omega}{\partial \bar{z}_{l\omega}}(P_\omega) = \frac{\partial f_\omega}{\partial \bar{z}_{l\omega}}(0, \ldots, 0), \end{cases}$$

so that the assertions follow from the case (i). □

*4.2.7. **Proposition.*** — *We assume that $\Omega = \Omega_\infty$ and $\sqrt{-1} \in K$. Let U be a non-empty Zariski open set of $\mathbb{A}^n_K$. Let $h = (h_\omega)_{\omega\in\Omega}$ be a family of functions indexed by Ω such that h_ω is a C^∞-function on U_ω and that, for any K-rational point $P \in U(K)$, the function given by $(\omega \in \Omega) \mapsto h_\omega(P_\omega)$ is measurable. For each $\omega \in \Omega$, let $(z_{1\omega}, \dots, z_{n\omega})$ is the coordinate of $\mathbb{A}^n \otimes_{\sigma_\omega} \mathbb{C}$. If*

$$\int_{U_\omega} \left(\frac{i}{2}\right)^n h_\omega(z_{1\omega}, \dots, z_{n\omega})\, \mathrm{d}z_{1\omega} \wedge \mathrm{d}\bar{z}_{1\omega} \wedge \cdots \wedge \mathrm{d}z_{n\omega} \wedge \mathrm{d}\bar{z}_{n\omega}$$

exists for any $\omega \in \Omega$, then

$$(\omega \in \Omega) \mapsto \int_{U_\omega} \left(\frac{i}{2}\right)^n h_\omega(z_{1\omega}, \dots, z_{n\omega})\, \mathrm{d}z_{1\omega} \wedge \mathrm{d}\bar{z}_{1\omega} \wedge \cdots \wedge \mathrm{d}z_{n\omega} \wedge \mathrm{d}\bar{z}_{n\omega}$$

is measurable.

Proof. — Shrinking U if necessarily, we may assume that $\mathbb{A}^n_K \setminus U$ is defined by $\{F = 0\}$ for some $F \in K[X_1, \dots, X_n] \setminus \{0\}$. We set

$$U_{\omega,N} = \Big\{(z_{1\omega}, \dots, z_{n\omega}) \in \mathbb{C}^n \,\Big|\, \max_{j\in\{1,\dots,n\}} |z_{j\omega}| \leq N \text{ and } |F(z_{1\omega}, \dots, z_{n\omega})| \geqslant 1/N\Big\}.$$

Let $x_{i\omega}$ (resp. $y_{i\omega}$) be the real part (resp. imaginary part) of $z_{i\omega}$. Then

$$\left(\frac{i}{2}\right)^n h_\omega\, \mathrm{d}z_{1\omega} \wedge \mathrm{d}\bar{z}_{1\omega} \wedge \cdots \wedge \mathrm{d}z_{n\omega} \wedge \mathrm{d}\bar{z}_{n\omega} = h_\omega\, \mathrm{d}x_{1\omega} \wedge \mathrm{d}y_{1\omega} \wedge \cdots \wedge \mathrm{d}x_{n\omega} \wedge \mathrm{d}y_{n\omega}.$$

Moreover,

$$\int_{U_{\omega,N}} h_\omega\, \mathrm{d}x_{1\omega} \wedge \mathrm{d}y_{1\omega} \wedge \cdots \wedge \mathrm{d}x_{n\omega} \wedge \mathrm{d}y_{n\omega} = \lim_{m\to\infty} \sum_{\substack{a_1,b_1,\dots,a_n,b_n\in\mathbb{Z} \\ \left(\frac{a_1+ib_1}{m},\dots,\frac{a_n+ib_n}{m}\right)\in U_{\omega,N}}} \frac{1}{m^{2n}} h_\omega\Big(\frac{a_1+ib_1}{m}, \dots, \frac{a_n+ib_n}{m}\Big). \tag{4.2}$$

Note that

$$(\omega \in \Omega_\infty) \longmapsto h_\omega\Big(\frac{a_1+ib_1}{m}, \dots, \frac{a_n+ib_n}{m}\Big)$$

is measurable, so that (4.2) means that

$$(\omega \in \Omega_\infty) \longmapsto \int_{U_{\omega,N}} h_\omega\, \mathrm{d}x_{1\omega} \wedge \mathrm{d}y_{1\omega} \wedge \cdots \wedge \mathrm{d}x_{n\omega} \wedge \mathrm{d}y_{n\omega}$$

is measurable. Therefore, since

$$\lim_{N\to\infty} \int_{U_{\omega,N}} h_\omega\, \mathrm{d}x_{1\omega} \wedge \mathrm{d}y_{1\omega} \wedge \cdots \wedge \mathrm{d}x_{n\omega} \wedge \mathrm{d}y_{n\omega} = \int_{U_\omega} h_\omega\, \mathrm{d}x_{1\omega} \wedge \mathrm{d}y_{1\omega} \wedge \cdots \wedge \mathrm{d}x_{n\omega} \wedge \mathrm{d}y_{n\omega},$$

one has the assertion. □

4.2.8. Theorem. — *We assume that $\Omega = \Omega_\infty$ and K is algebraically closed. Let X be a d-dimensional projective and integral variety over K and L be a very ample invertible $\mathcal{O}_X$-module. Let $(\|\cdot\|_\omega)_{\omega\in\Omega}$ be a measurable family of Hermitian norms on $H^0(X,L)$. Let $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be a family of metrics on L induced by the surjective homomorphism $H^0(X,L)\otimes\mathcal{O}_X \to L$ and $(\|\cdot\|_\omega)_{\omega\in\Omega}$. For $s \in H^0(X,L)\setminus\{0\}$,*

$$(\omega\in\Omega)\mapsto \int_{X_\omega} \log|s|_{\varphi_\omega} c_1(L_\omega,\varphi_\omega)^{\wedge d}$$

is measurable.

Proof. — Let $n = \dim_K H^0(X,L) - 1$ and $X \hookrightarrow \mathbb{P}^n_K$ be the embedding by L. Note that $L = \mathcal{O}_{\mathbb{P}^n_K}(1)\big|_X$. Since $H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(1)) \simeq H^0(X,L)$, one has $t \in H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(1))$ with $t|_X = s$. Let $\psi = (\psi_\omega)_{\omega\in\Omega}$ be a family of metics of $\mathcal{O}_{\mathbb{P}^n_K}(1)$ induced by the surjective homomorphism $H^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(1)) \otimes \mathcal{O}_{\mathbb{P}^n_K} \to \mathcal{O}_{\mathbb{P}^n_K}(1)$ and $(\|\cdot\|_\omega)_{\omega\in\Omega}$. Note that $\psi|_X = \varphi$. By Proposition 1.7.4, we can choose a linear subspace M in $\mathbb{P}^n_K$ such that $\operatorname{codim} M = d+1$, $M\cap X = \varnothing$ and $M \subseteq \{t=0\}$, so that, by Proposition 1.7.4 again, the morphism $\pi : X \to \mathbb{P}^d_K$ induced by the projection $\pi_M : \mathbb{P}^n_K \setminus M \to \mathbb{P}^d_K$ with the center M is finite and surjective. We choose a homogenous coordinate $(T_0 : \ldots : T_n)$ on $\mathbb{P}^n_K$ such that

$$t = T_0 \quad\text{and}\quad M = \{T_0 = \cdots = T_d = 0\}.$$

Then π_M is given by $(T_0 : \cdots : T_n) \mapsto (T_0 : \cdots : T_d)$. Let U be a non-empty open of $\mathbb{P}^d_K$ such that $\pi : X \to \mathbb{P}^d_K$ is étale over U. We may assume that $U \subseteq \{T_0 \neq 0\}$. We set $X_j = T_j/T_0$ $(j = 1,\ldots,n)$. Then

$$\begin{cases}\mathbb{P}^n_K \setminus \{T_0 = 0\} = \operatorname{Spec}(K[X_1,\ldots,X_n]) = \mathbb{A}^n_K,\\ \mathbb{P}^d_K \setminus \{T_0 = 0\} = \operatorname{Spec}(K[X_1,\ldots,X_d]) = \mathbb{A}^d_K\end{cases}$$

and π_M on $\mathbb{P}^n_K \setminus \{T_0=0\}$ is given by $(X_1,\ldots,X_n)\mapsto(X_1,\ldots,X_d)$. Let

$$(z_{1\omega},\ldots,z_{n\omega}) \quad\text{and}\quad (z_{1\omega},\ldots,z_{d\omega})$$

be the coordinates of $\mathbb{A}^n_K \otimes_{\sigma_\omega} \mathbb{C}$ and $\mathbb{A}^d_K \otimes_{\sigma_\omega} \mathbb{C}$, respectively. Note that $f_\omega := \log|t|_{\psi_\omega}$ is C^∞ on $\mathbb{A}^n_K \otimes_{\sigma_\omega} \mathbb{C}$. Then, by Proposition 4.2.6, if we set

$$f_\omega|_{X_\omega} c_1(L_\omega,\varphi_\omega)^{\wedge d} = i^d h_\omega (dz_{1\omega}\wedge d\bar{z}_{1\omega})\wedge\cdots\wedge(dz_{d\omega}\wedge d\bar{z}_{d\omega})$$

on $\pi_\omega^{-1}(U_\omega)$, then, for $P \in \pi^{-1}(U)$, $(\omega\in\Omega)\mapsto h(P_\omega)$ is measurable. Note that

$$\begin{aligned}\int_{X_\omega} \log|s|_{\varphi_\omega} c_1(L_\omega,\varphi_\omega)^{\wedge d} &= \int_{\pi_\omega^{-1}(U_\omega)} f_\omega|_{X_\omega} c_1(L_\omega,\varphi_\omega)^{\wedge d}\\ &= \int_{\pi_\omega^{-1}(U_\omega)} i^d h_\omega (dz_{1\omega}\wedge d\bar{z}_{1\omega})\wedge\cdots\wedge(dz_{d\omega}\wedge d\bar{z}_{d\omega})\\ &= \int_{U_\omega} i^d (\pi_\omega)_*(h_\omega)(z_{1\omega}\wedge d\bar{z}_{1\omega})\wedge\cdots\wedge(dz_{d\omega}\wedge d\bar{z}_{d\omega}).\end{aligned}$$

Moreover, $(\pi_\omega)_*(h_\omega)$ is C^∞ over U_ω. Further, for $P \in U(K)$, if we set $\pi^{-1}(P) = \{Q_1, \ldots, Q_r\}$, then

$$(\pi_\omega)_*(h_\omega)(P_\omega) = \sum_{i=1}^r h_\omega(Q_{i\omega}),$$

so that $(\omega \in \Omega) \mapsto (\pi_\omega)_*(h_\omega)(P_\omega)$ is measurable. Therefore, by Proposition 4.2.7,

$$(\omega \in \Omega) \mapsto \int_{U_\omega} i^d (\pi_\omega)_*(h_\omega)(dz_{1\omega} \wedge d\bar{z}_{1\omega}) \wedge \cdots \wedge (dz_{d\omega} \wedge d\bar{z}_{d\omega})$$

is measurable. Thus the assertion follows. □

4.2.9. Theorem. — *We assume that $\Omega = \Omega_\infty$. Let X be a projective scheme over K and L be an ample invertible $\mathcal{O}_X$-module. Let $\varphi = (\varphi_\omega)_{\omega\in\Omega}$ be a measurable family of semipositive metrics. Then, for $s \in H^0(X, L) \setminus \{0\}$,*

$$(\omega \in \Omega) \longmapsto \int_{X_\omega} \log |s|_{\varphi_\omega} c_1(L_\omega, \varphi_\omega)^d$$

is measurable.

Proof. — By Lemma 4.1.10, we may assume that K is algebraically closed. We choose a positive integer N such that $L^{\otimes n}$ is very ample for for all $n \geqslant N$. Let $\varphi_n = (\varphi_{n,\omega})_{\omega\in\Omega}$ be the quotient metric family of $L^{\otimes n}$ induced by $H^0(X, L^{\otimes n}) \otimes \mathcal{O}_X \to L^{\otimes n}$ and $\xi_{n\varphi} = (\|\cdot\|_{n\varphi_\omega})_{\omega\in\Omega}$. Moreover, by [**13**, Theorem 4.1.26], there is a measurable Hermitian norm family $\xi_n^H = (\|\cdot\|_{n,\omega}^H)_{\omega\in\Omega}$ on $H^0(X, L^{\otimes n})$ such that

$$\|\cdot\|_{n\varphi_\omega} \leqslant \|\cdot\|_{n,\omega}^H \leqslant (h^0(L^{\otimes n}) + 1)^{1/2} \|\cdot\|_{n\varphi_\omega}$$

for $\omega \in \Omega$. Let $\varphi_{n,\omega}^H$ be the quotient metric family of $L^{\otimes n}$ induced by $H^0(X, L^{\otimes n}) \otimes \mathcal{O}_X \to L^{\otimes n}$ and ξ_n^H. Note that

$$d_\omega\left(\frac{1}{n}\varphi_n, \frac{1}{n}\varphi_n^H\right) \leqslant \frac{d(\|\cdot\|_{n\varphi_\omega}, \|\cdot\|_{n,\omega}^H)}{n} \leqslant \frac{\ln(h^0(L^{\otimes n}) + 1)}{2n}.$$

Therefore, if we set $\psi_{n,\omega} = (1/n)\varphi_{n,\omega}^H$, then $\lim_{n\to\infty} d_\omega(\varphi, \psi_n) = 0$ for all $\omega \in \Omega$ because $\lim_{n\to\infty} d_\omega(\varphi, (1/n)\varphi_n) = 0$. By Theorem 4.2.8,

$$(\omega \in \Omega_\infty) \mapsto \int_{X_\omega} \log |s|_{\psi_{n,\omega}} c_1(L_\omega, \psi_{n,\omega})^d = \frac{1}{n^{d+1}} \int_{X_\omega} \log |s^n|_{\varphi_{n,\omega}^H} c_1(nL_\omega, \varphi_{n,\omega}^H)^d$$

is measurable. Further, by [**18**, Corollary 3.6],

$$\lim_{n\to\infty} \int_{X_\omega} \log |s|_{\psi_{n,\omega}} c_1(L_\omega, \psi_{n,\omega})^d = \int_{X_\omega} \log |s|_{\varphi_n} c_1(L_\omega, \varphi_\omega)^d.$$

Therefore, the assertion follows. □

Combining Theorems 4.2.4 and 4.2.9, we obtain the following result.

4.2.10. Theorem. — *Let $X \to \operatorname{Spec} K$ be a projective scheme over K and d be the dimension of X. Let $D_0, \ldots, D_d$ be Cartier divisors on X, which intersect properly. We equip each D_i with a measurable Green function family g_i such that (D_i, g_i) is integrable. Then the local intersection function*

$$(\omega \in \Omega) \longmapsto ((D_0, g_0) \cdots (D_d, g_d))_\omega$$

is $\mathcal{A}$-measurable.

Proof. — The measurability over $\Omega \setminus \Omega_\infty$ follows directly from Theorem 4.2.4. Moreover, in view of Theorem 4.2.9, the measurability over Ω_∞ follows from Proposition 3.6.6 and the multi-linearity of the local intersection measure. □

4.2.11. Theorem. — *Let $X \to \operatorname{Spec} K$ be a projective scheme over K and d be the dimension of X. Let $D_0, \ldots, D_d$ be Cartier divisors on X, which intersect properly. We equip each D_i with a dominated Green function family g_i such that (D_i, g_i) is integrable. Then the local intersection function*

$$(\omega \in \Omega) \longmapsto ((D_0, g_0) \cdots (D_d, g_d))_\omega \tag{4.3}$$

is dominated.

Proof. — By Lemma 4.1.10, we may assume that K is algebraically closed. By using Proposition 3.6.3, we may further assume that $D_0, \ldots, D_d$ are very ample. Moreover, by Proposition 4.2.3, we may assume without loss of generality that there are an integrable adelic line bundle (L, φ) and non-zero rational sections $s_0, \ldots, s_d$ of L such that $\mathcal{O}_X(D_i) = L$ and $g_i = -\log |s_i|_\varphi$ for $i \in \{0, \ldots, d\}$. Note that L is very ample. Thus, by Proposition 1.7.4, there is a finite and surjective morphism $\pi : X \to \mathbb{P}^d_K$ such that $L = \pi^*(\mathcal{O}_{\mathbb{P}^d}(1))$. Let $(T_0 : \cdots : T_d)$ be a homogeneous coordinate of $\mathbb{P}^d_K$. We consider $(T_i)_{i=0}^n$ as a basis of $H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$. Let φ_{FS} be the quotient metric on $\mathcal{O}_{\mathbb{P}^d}(1)$ induced by the universal quotient homomorphism

$$H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1)) \otimes_K \mathcal{O}_{\mathbb{P}^d} \longrightarrow \mathcal{O}_{\mathbb{P}^d}(1)$$

and the norm family associated with the basis $(T_i)_{i=0}^d$ (see Example 4.1.1, see also Proposition 3.6.8). Moreover, we set $h_i = -\log |T_i|_{\varphi_{\mathrm{FS}}}$.

First we assume that $\varphi = \pi^*(\varphi_{\mathrm{FS}})$. If $D_i = \pi^*(H_i)$ for $i \in \{0, \ldots, d\}$, then the dominancy of (4.3) follows from Proposition 3.6.4 and Proposition 3.6.8. In general, there are non-zero rational functions $f_0, \ldots, f_d$ on X such that $D_i = \pi^*(H_i) + (f_i)$ for $i \in \{0, \ldots, d\}$. Then, by Proposition 3.6.5, there is an integrable function θ on Ω such that

$$((D_0, g_0) \cdots (D_d, g_d))_\omega = ((\pi^*(H_0), \pi^*(h_0)) \cdots (\pi^*(H_d), \pi^*(h_d)))_\omega + \theta(\omega).$$

Thus one has the dominancy of (4.3).

In general, there is a family g of integrable continuous functions such that $\varphi = \exp(g)\pi^*(\varphi_{\mathrm{FS}})$. In this case, the dominancy of (4.3) follows from Corollary 3.6.7. □

Finally, we obtain the following integrability theorem.

4.2.12. Theorem. — *Let X be a projective K-scheme of dimension d, and $\overline{D}_0,\ldots,\overline{D}_d$ be a family of integrable adelic Cartier divisors. Assume that the underlying Cartier divisors $D_0,\ldots,D_d$ intersect properly. Then the function of local intersection numbers*

$$(\omega\in\Omega)\longmapsto(\overline{D}_0\cdots\overline{D}_d)_\omega \tag{4.4}$$

is integrable on the measure space $(\Omega,\mathcal{A},\nu)$.

4.2.13. Definition. — Let X be a projective K-scheme of dimension d, and $\overline{D}_0,\ldots,\overline{D}_d$ be a family of integrable adelic Cartier divisors, such that $D_0,\ldots,D_d$ intersect properly. We define the global intersection number of $\overline{D}_0,\ldots,\overline{D}_d$ as

$$(\overline{D}_0\cdots\overline{D}_d)_S:=\int_{\omega\in\Omega}(\overline{D}_0\cdots\overline{D}_d)_\omega\,\nu(\mathrm{d}\omega).$$

4.2.14. Remark. — Let X be a projective K-scheme of dimension d. For any $i\in\{0,\ldots,d\}$, let

$$(E_i,\xi_i=(\|\cdot\|_{i,\omega})_{\omega\in\Omega})$$

be a Hermitian adelic vector bundle on S, and $f_i:X\to\mathbb{P}(E_i)$ be a closed embedding. Let L_i be the restriction of $\mathcal{O}_{E_i}(1)$ to X, which is equipped with the orthogonal quotient metric family φ_i induced by ξ_i. We choose a global section s_i of L_i such that $s_0,\ldots,s_d$ intersect properly. For each $i\in\{0,\ldots,d\}$, let D_i be the Cartier divisor $\mathrm{div}(s_i)$ and g_i be the Green function family of D_i corresponding to φ_i. By Theorem 3.9.7, if we denote by R the resultant

$$R^{X,s_0,\ldots,s_d}_{f_0,\ldots,f_d}\in S^{\delta_0}(E_0^\vee)\otimes_K\cdots\otimes_K S^{\delta_d}(E_d^\vee),$$

where $\delta_i=(D_0\cdots D_{i-1}D_{i+1}\cdots D_d)$, then the following equality holds

$$\begin{aligned}(\overline{D}_0\cdots\overline{D}_d)=&\int_{\omega\in\Omega\setminus\Omega_\infty}\ln\|R\|_\omega\,\nu(\mathrm{d}\omega)\\&+\int_{\sigma\in\Omega_\infty}\nu(\mathrm{d}\sigma)\int_{\mathbb{S}_{0,\sigma}\times\cdots\times\mathbb{S}_{d,\sigma}}\ln|R_\sigma(z_0,\ldots,z_d)|\,\eta_{\mathbb{S}_{0,\sigma}}(\mathrm{d}z_0)\otimes\cdots\otimes\eta_{\mathbb{S}_{d,\sigma}}(dz_d)\\&+\nu(\Omega_\infty)\frac{1}{2}\sum_{i=0}^d\delta_i\sum_{\ell=1}^{r_i}\frac{1}{\ell},\end{aligned}$$

where

(1) $\|\cdot\|_\omega$ is the ε-tensor product of δ_i-th ε-symmetric tensor power of $\|\cdot\|_{i,\omega,*}$,
(2) R_σ is the element of

$$S^{\delta_0}(E^\vee_{0,\mathbb{C}_\sigma})\otimes_{\mathbb{C}_\sigma}\cdots\otimes_{\mathbb{C}_\sigma}S^{\delta_{d,\mathbb{C}_\sigma}}(E^\vee_{d,\mathbb{C}_\sigma})$$

indued by R,
(3) $\mathbb{S}_{i,\sigma}$ is the unique sphere of $(E_{i,\mathbb{C}_\sigma},\|\cdot\|_{i,\sigma,\mathbb{C}_\sigma})$,
(4) $\eta_{\mathbb{S}_{i,\sigma}}$ is the $U(E_{i,\mathbb{C}_\sigma},\|\cdot\|_{i,\mathbb{C}_\sigma})$-invariant Borel probaility measure on $\mathbb{S}_{i,\sigma}$.

4.3. Invariance of intersection number by coverings

Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve. Consider a covering

$$\alpha = (\alpha^{\#}, \alpha_{\#}, I_\alpha)$$

from another adelic curve $S' = (K', (\Omega', \mathcal{A}', \nu'), \phi')$ to S (see Definition 2.1.2). We assume that, either both σ-algebra $\mathcal{A}$ and $\mathcal{A}'$ are discrete, or there exist countable subfields K_0 and K_0' of K and K' respectively, such that K_0 is dense in each K_ω with $\omega \in \Omega$, and K_0' is dense in each $K'_{\omega'}$ with $\omega' \in \Omega'$. Recall that $\alpha^{\#} : K \longrightarrow K'$ is a field homomorphism,

$$\alpha_{\#} : (\Omega', \mathcal{A}') \to (\Omega, \mathcal{A})$$

is a measurable map, and

$$I_\alpha : \mathscr{L}^1(\Omega', \mathcal{A}', \nu') \longrightarrow \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$$

is a disintegration kernel of ν' over ν such that, for any $g \in \mathscr{L}^1(\Omega, \mathcal{A}, \nu)$, one has $g \circ \alpha_{\#} \in \mathscr{L}^1(\Omega', \mathcal{A}', \nu')$ and $I_\alpha(g \circ \alpha_{\#}) = g$. In this section, we consider a projective scheme X of dimension d over $\operatorname{Spec} K$ and a family

$$\overline{D}_0 = (D_0, g_0), \ldots, \overline{D}_d = (D_d, g_d)$$

of adelic Cartier divisors, such that $D_0, \ldots, D_d$ intersect properly. The purpose of this section is to define the extension of scalars $\overline{D}_{i,\alpha}$ of each adelic Cartier divisor $\overline{D}_i$ by α and show the following equality

$$(\overline{D}_{0,\alpha} \cdots \overline{D}_{d,\alpha})_{S'} = (\overline{D}_0 \cdots \overline{D}_d)_S.$$

4.3.1. Definition. — Let D be a Cartier divisor on X and $g = (g_\omega)_{\omega \in \Omega}$ be a Green function family of D (see Definition 4.2.1). Let X_α be the fiber product

$$X \times_{\operatorname{Spec} K, \alpha^{\#}} \operatorname{Spec} K'$$

and D_α be the pull-back of D by the morphism of projection $X_\alpha \to X$. If ω' is an element of Ω' and $\omega = \alpha_{\#}(\omega')$, then the Cartier divisor $D_{\alpha,\omega'}$ identifies with the pull-back of D_ω by the morphism of projection

$$X_{\alpha,\omega'} = X_\alpha \times_{\operatorname{Spec} K'} \operatorname{Spec} K'_{\omega'} \cong X_\omega \times_{\operatorname{Spec} K_\omega} \operatorname{Spec} K'_{\omega'} \longrightarrow X_\omega.$$

We denote by $g_{\alpha,\omega'}$ the Green function $g_{\omega,K'_{\omega'}}$ (see Remark 3.4.5). Then the family $g_\alpha := (g_{\alpha,\omega'})_{\omega' \in \Omega'}$ forms a Green function family of the Cartier divisor D_α.

Let L be an invertible $\mathcal{O}_X$-module and $\varphi = (\varphi_\omega)_{\omega \in \Omega}$ be a metric family on L. We denote by L_α the pull-back of L by the morphism of projection $X_\alpha \to X$. If ω' is an element of Ω' and $\omega = \alpha_{\#}(\omega')$, then the invertible sheaf $L_{\alpha,\omega'}$ identifies with the pull-back of L_ω by the morphism of projection $X_{K',\omega'} \to X_\omega$. We denote by $\varphi_{\alpha,\omega'}$ the continuous metric $\varphi_{\omega,K'_{\omega'}}$ (see Example 3.3.2 (5)) on $L_{\alpha,\omega'}$. Then the family $\varphi_\alpha := (\varphi_{\alpha,\omega'})_{\omega' \in \Omega'}$ forms a metric family of L_α. Note that, if s is a regular meromorphic section of L, $D = \operatorname{div}(s)$ and $g = (g_\omega)_{\omega \in \Omega}$ is the Green function family of D corresponding to the metric family φ, then g_α is the Green function family of D_α corresponding to φ_α.

*4.3.2. **Proposition.*** — *Let $\pi : X \to \operatorname{Spec} K$ be a projective K-scheme.*

(1) *Let L be an invertible $\mathcal{O}_X$-module and φ be a metric family on L. If φ is dominated, then φ_α is also dominated.*
(2) *Let D be a Cartier divisor on X and g is a Green function family of g. If g is dominated, then g_α is also dominated.*

Proof. — It suffices to prove the first statement. Assume that ψ is another metric family on L. If ω' is an element of Ω' and if $\omega = \alpha_\#(\omega')$, then by (3.4) one has

$$d_{\omega'}(\varphi_\alpha, \psi_\alpha) = d_\omega(\varphi, \psi).$$

Therefore, if the function $(\omega \in \Omega) \mapsto d_\omega(\varphi, \psi)$ is dominated, so is the function $(\omega' \in \Omega') \mapsto d_{\omega'}(\varphi_\alpha, \psi_\alpha)$. To prove that the metric family φ is dominated, we can assume without loss of generality that there exist a finite-dimensional vector space over K, a strongly dominated norm family $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ on E, a positive integer n and a surjective homomorphism $f : \pi^*(E) \to L^{\otimes n}$ such that φ identifies with the orthogonal quotient metric family induced by ξ (see Definition 3.3.5). We may assume further that ξ is Hermitian and E admits a basis $\boldsymbol{e}$ which is orthonormal with respect to all norms $\|\cdot\|_\omega$.

For any $\omega' \in \Omega'$, let $\|\cdot\|_{\omega'}$ be the norm $\|\cdot\|_{\omega,K'_{\omega'}}$, where $\omega = \alpha_\#(\omega')$. Then $\xi_\alpha^H = (\|\cdot\|_{\omega'})_{\omega'\in\Omega'}$ is a norm family on $E_{K'}$. Moreover, if we view $\boldsymbol{e}$ as a basis of $E_{K'}$ over K', then it is orthonormal with respect to all norms $\|\cdot\|_{\omega'}$. In particular, the norm family ξ_α^H is strongly dominated. Since φ_α coincides with the orthogonal quotient metric family induced by ξ_α^H, we deduce that the metric family φ_α is also dominated. □

*4.3.3. **Definition.*** — Let E be a finite-dimensional vector space over K and $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ be a norm family on E. We define $\xi_\alpha = (\|\cdot\|_{\omega'})_{\omega'\in\Omega'}$ as the following norm family on $E_\alpha := E \otimes_{K,\alpha^\#} K'$. In the case where ω' is non-Archimedean, the norm $\|\cdot\|_{\omega'}$ is the ε-extension of scalars of $\|\cdot\|_\omega$, where $\omega = \alpha_\#(\omega')$; in other words, one has

$$\forall\, s \in E_{\alpha,K'_{\omega'}}, \quad \|s\|_{\omega'} = \sup_{f\in E^\vee_{K_\omega}\setminus\{0\}} \frac{|f(s)|_{\omega'}}{\|f\|_{\omega,*}}.$$

In the case where ω' is Archimedean, the norm $\|\cdot\|_{\omega'}$ is the π-extension of scalars of $\|\cdot\|_\omega$, in other words, one has

$$\forall\, s \in E_{\alpha,K'_{\omega'}}, \quad \|s\|_{\omega'} = \inf\left\{ |\lambda_1|_{\omega'}\cdot\|s_1\|_\omega + \cdots + |\lambda_N|_{\omega'}\cdot\|s_N\|_\omega \;\middle|\; \begin{array}{l} N\in\mathbb{N},\, N\geqslant 1 \\ (\lambda_1,\ldots,\lambda_N)\in(K'_{\omega'})^N \\ (s_1,\ldots,s_N)\in E_\omega^N \\ s=\lambda_1 s_1+\cdots+\lambda_N s_N \end{array}\right\}.$$

Similarly, we define $\xi_{\alpha,\varepsilon}$ the norm family on E_α consisting of ε-extension of scalars (for both non-Archimedean and Archimedean absolute values).

4.3.4. Lemma. — *Let E be a finite-dimension vector space over K and $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$ be a measurable norm family on E. Then the norm families $\xi_{\alpha,\varepsilon}$ and ξ_α defined above are also measurable.*

Proof. — The proof is very similar to that of [**13**, Proposition 4.1.24 (1.c)]. The case where $\mathcal{A}$ and $\mathcal{A}'$ are discrete is trivial. In the following, we will treat the case where K and K' admit countable subfields K_0 and K_0' such that K_0 is dense in each K_ω with $\omega \in \Omega$, and K_0' is dense in each $K'_{\omega'}$ with $\omega' \in \Omega'$, respectively. We first check the measurability of $\xi_{\alpha,\varepsilon}$. For any $\omega' \in \Omega'$, let $\|\cdot\|_{\omega',\varepsilon}$ be the norm indexed by ω' in the family $\xi_{\alpha,\varepsilon}$. Let H_0 be a finite-dimensional K_0-vector subspace of $E^\vee$ which generates $E^\vee$ as a vector space over K. Then $H_0 \setminus \{0\}$ is dense in $E^\vee_{K_\omega} \setminus \{0\}$ for any $\omega \in \Omega$. If s is an element of E_α, then for any $\omega' \in \Omega'$,

$$\|s\|_{\omega',\varepsilon} = \sup_{f\in H_0\setminus\{0\}} \frac{|f(s)|_{\omega'}}{\|f\|_{\omega,*}}.$$

Hence it is the supremum of a countable family of $\mathcal{A}'$-measurable function in ω'. As for the second statement, it suffices to apply the first statement to $\xi^\vee$ to obtain the measurability of $(\xi^\vee)_{\alpha,\varepsilon}$. Since ξ_α is the dual norm family of $(\xi^\vee)_{\alpha,\varepsilon}$ (see [**13**, Proposition 1.3.20]), by [**13**, Proposition 4.1.24 (1.c)] we obtain the measurability of ξ_α. □

4.3.5. Proposition. — *Let X be a projective scheme over* $\operatorname{Spec} K$.

(1) *Let L be an invertible $\mathcal{O}_X$-module and φ be a metric family on L. We assume that L is ample and all metrics in the family φ are semi-positive If φ is measurable, then φ_α is also measurable.*
(2) *Let D be a Cartier divisor on X and g be a Green function family of g. Assume that D is ample and g is semi-positive. If g is measurable, then g_α is also measurable.*

Proof. — It suffices to prove the first statement. Similarly to the proof of Theorem 4.2.4, for any $m \in \mathbb{N}_{\geqslant 1}$ such that $L^{\otimes m}$ is very ample we choose a norm family $\xi_m = (\|\cdot\|_\omega^{(m)})_{\omega\in\Omega}$ on $H^0(X, L^{\otimes m})$ such that $H^0(X, L^{\otimes m})$ admet a basis which is orthonormal with respect to each norm $\|\cdot\|_\omega^{(m)}$. This norm family is clearly measurable. For any $b > 0$ and any $\omega \in \Omega$, let $\varphi_{b,\omega}^{(m)}$ the quotient metric on L induced by the norm

$$\|\cdot\|_{b,\omega}^{(m)} = \max\{\|\cdot\|_{m\varphi}, b\|\cdot\|_\omega^{(m)}\}$$

on $H^0(X_\omega, L_\omega^{\otimes m})$. By Proposition 4.1.9, the norm family $\xi_b^{(m)} := (\|\cdot\|_{b,\omega}^{(m)})_{\omega\in\Omega}$ is measurable. By Lemma 4.3.4, we deduce that the norm family $\xi_{b,\alpha}^{(m)}$ of $H^0(X_\alpha, L_\alpha^{\otimes m})$ is $\mathcal{A}'$-measurable.

Let $\varphi_b^{(m)}$ be the quotient metric family on L induced by $\xi_b^{(m)}$. By [**13**, Remark 2.2.14], the metric $\varphi_{b,\alpha}^{(m)}$ identifies with the quotient metric family on L_α induced by $\xi_{b,\alpha}^{(m)}$. Since the norm family $\xi_{b,\alpha}^{(m)}$ is measurable, by [**13**, Proposition 6.1.30], the metric family $\varphi_{b,\alpha}^{(m)}$ is measurable. By Proposition 3.3.11, for any fixed $\omega' \in \Omega'$ and $\omega = \alpha_\#(\omega')$, for sufficiently small b one has $\varphi_{b,\omega}^{(m)} = \varphi_\omega^{(m)}$ and hence $\varphi_{b,\alpha,\omega'}^{(m)} = \varphi_{\alpha,\omega'}^{(m)}$. Therefore, by [**13**, Proposition 6.1.29] we obtain that $\varphi_\alpha^{(m)}$ is measurable. By (3.4), for any $\omega' \in \Omega'$ and $\omega = \alpha_\#(\omega)$, one has

$$d_{\omega'}(\varphi_\alpha^{(m)}, \varphi_\alpha) \leqslant d_\omega(\varphi^{(m)}, \varphi).$$

Since the metric family φ is semi-positive, by Proposition 3.3.12, we deduce that, for any $\omega' \in \Omega'$, one has

$$\lim_{m\to+\infty} d_{\omega'}(\varphi_\alpha^{(m)}, \varphi_\alpha) = 0.$$

Still by [**13**, Proposition 6.1.29], we obtain that the metric family φ is measurable. □

4.3.6. Theorem. — *Let X be a projective scheme over* $\operatorname{Spec} K$ *and d be the dimension of X. Let $D_0, \ldots, D_d$ be Cartier divisors on X which intersects properly. We assume that each Cartier divisor D_i is equipped with an integrable Green function family g_i. The the following equality holds*

$$((D_{0,\alpha}, g_{0,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{\omega'} = ((D_0, g_0) \cdots (D_d, g_d))_{\alpha_\#(\omega')}.$$

In particular, if all Green function family g_i are dominated (resp. measurable), then the function

$$(\omega' \in \Omega') \longmapsto ((D_{0,\alpha}, g_{0,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{\omega'}$$

is dominated (resp. measurable). If all (D_i, g_i) are adelic Cartier divisors, then the following equality holds

$$((D_0, g_0) \cdots (D_d, g_d))_S = ((D_{0,\alpha}, g_{d,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{S'}.$$

Proof. — For any $\omega' \in \Omega'$ and $\omega = \alpha_\#(\omega')$, the equality

$$((D_{0,\alpha}, g_{0,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{\omega'} = ((D_0, g_0) \cdots (D_d, g_d))_\omega$$

follows from Proposition 3.9.9 (see also Remark 3.9.10).

If $g_0, \ldots, g_d$ are measurable, by Theorem 4.2.10, the function

$$(\omega \in \Omega) \longmapsto ((D_0, g_0) \cdots (D_d, g_d))_\omega$$

is $\mathcal{A}$-measurable. Since $\alpha_\#$ is a measurable map, we deduce that the function

$$(\omega' \in \Omega') \longmapsto ((D_{0,\alpha}, g_{0,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha})_{\omega'}$$

is $\mathcal{A}'$-measurable.

Assume that the Green function families $g_0, \ldots, g_d$ are dominated. By Theorem 4.2.11, there exists an integrable function F on the measure space $(\Omega, \mathcal{A}, \nu)$ such that

$$\forall\, \omega \in \Omega, \quad |((D_0, g_0) \cdots (D_d, g_d))_\omega| \leqslant F(\omega).$$

Hence

$$\forall\, \omega' \in \Omega, \quad |((D_{0,\alpha}, g_{0,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{\omega'}| \leqslant F(\alpha_\#(\omega')).$$

Hence the function

$$(\omega' \in \Omega') \longmapsto ((D_{0,\alpha}, g_{0,\alpha}), \cdots, (D_{d,\alpha}, g_{d,\alpha}))_{\omega'}$$

is dominated. Finally, if the function

$$(\omega \in \Omega) \longmapsto ((D_0, g_0) \cdots (D_d, g_d))_{\omega}$$

is integrable, then also is the function

$$(\omega' \in \Omega') \longmapsto ((D_{0,\alpha}, g_{0,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{\omega'} = ((D_0, g_0) \cdots (D_d, g_d))_{\alpha_{\#}(\omega')}$$

is also integrable, and one has

$$\begin{aligned} ((D_{0,\alpha}, g_{d,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{S'} &= \int_{\Omega'} ((D_{0,\alpha}, g_{0,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{\omega'} \, \nu'(\mathrm{d}\omega') \\ &= \int_{\Omega} I_{\alpha}(\omega' \longmapsto ((D_{0,\alpha}, g_{0,\alpha}) \cdots (D_{d,\alpha}, g_{d,\alpha}))_{\omega'}) \, \nu(\mathrm{d}\omega) \\ &= \int_{\Omega} ((D_0, g_0) \cdots (D_d, g_d))_{\omega} \, \nu(\mathrm{d}\omega) = ((D_0, g_0) \cdots (D_d, g_d))_S. \end{aligned}$$

□

4.4. Multi-heights

From now on, we assume that the adelic curve S is proper.

4.4.1. Definition. — Let X be a projective scheme over $\operatorname{Spec} K$. If f is a regular meromorphic function on X, we denote by $\widehat{\operatorname{div}}(f)$ the following adelic Cartier divisor

$$(\operatorname{div}(f), (-\ln |f|_{\omega})_{\omega \in \Omega}).$$

If $\overline{L} = (L, \varphi)$ is an adelic line bundle on X and if s is a regular meromorphic section of L on X, we denote by $\widehat{\operatorname{div}}(s)$ the following adelic Cartier divisor

$$(\operatorname{div}(s), (-\ln |s|_{\varphi_{\omega}})_{\omega \in \Omega}).$$

4.4.2. Proposition. — *Let X be a projective K-scheme of dimension d, and $\overline{D}_0, \ldots, \overline{D}_d$ and $\overline{D}'_0, \ldots, \overline{D}'_d$ be families of integrable adelic Cartier divisors, such that $D_0, \ldots, D_d$ and $D'_0, \ldots, D'_d$ intersect properly. If there is a family of regular meromorphic functions $f_0, \ldots, f_d$ on X such that $\overline{D}_i = \overline{D}'_i + \widehat{\operatorname{div}}(f_i)$ for $i \in \{0, \ldots, d\}$. Then*

$$(\overline{D}_0 \cdots \overline{D}_d)_S = (\overline{D}'_0 \cdots \overline{D}'_d)_S.$$

Proof. — It is sufficient to prove that if f is a regular meromorphic function on X and $\overline{D}_1, \ldots, \overline{D}_d$ are integrable adelic Cartier divisors such that $\operatorname{div}(f), D_1, \ldots, D_d$ intersect properly, then $(\widehat{\operatorname{div}}(f) \cdot \overline{D}_1 \cdots \overline{D}_d)_S = 0$. Clearly we may assume that K is algebraically closed, so that the assertion follows from Proposition 3.6.5 and the product formula. □

4.4.3. Definition. — Let $\overline{L}_0 = (L_0, \varphi_0), \ldots, \overline{L}_d = (L_d, \varphi_d)$ be a family of integrable adelic line bundles. Let $s_0, \ldots, s_d$ be regular meromorphic sections of $L_0, \ldots, L_d$, respectively such that $\operatorname{div}(s_0), \ldots, \operatorname{div}(s_d)$ intersect properly. Then, by Proposition 4.4.2, the global intersection number

$$(\widehat{\operatorname{div}}(s_0) \cdots \widehat{\operatorname{div}}(s_d))_S$$

does not depend on the choice of $s_0, \ldots, s_d$. The *global intersection number*

$$(\overline{L}_0 \cdots \overline{L}_d)_S$$

of $\overline{L}_0 \cdots \overline{L}_d$ over S is then defined as

$$(\widehat{\operatorname{div}}(s_0) \cdots \widehat{\operatorname{div}}(s_d))_S.$$

This number is also called the *multi-height* of X with respect to $\overline{L}_0, \ldots, \overline{L}_d$ and is denoted by

$$h_{\overline{L}_0, \ldots \overline{L}_d}(X).$$

In the particular case where $\overline{L}_0, \ldots \overline{L}_d$ are all equal to the same integrable adelic line bundle $\overline{L}$, the number $h_{\overline{L}, \ldots, \overline{L}}(X)$ is denoted by $h_{\overline{L}}(X)$ in abbreviation, and is called the *height* of X with respect to $\overline{L}$.

4.4.4. Proposition. — (1) *The global intersection pairing is a symmetric bilinear form on the group consisting of integrable adelic line bundle.*

(2) *Let $X_1, \ldots, X_\ell$ be irreducible components of X and $\eta_1, \ldots, \eta_\ell$ be the generic points of $X_1, \ldots, X_\ell$, respectively. Then*

$$\big(\overline{L}_0 \cdots \overline{L}_d\big)_S = \sum_{j=1}^{\ell} \operatorname{length}_{\mathcal{O}_{X,\eta_j}}(\mathcal{O}_{X,\eta_j})\big(\, \overline{L}_0\big|_{X_j} \cdots \overline{L}_d\big|_{X_j}\,\big)_S.$$

(3) *Let s_d be a regular meromorphic section of L_d and $\operatorname{div}(s_d) = a_1 Z_1 + \cdots + a_n Z_n$ be the decomposition as cycles. Then*

$$\begin{aligned}(\overline{L}_0 \cdots \overline{L}_d)_S = &\int_\Omega \left(\int_{X_\omega^{\mathrm{an}}} -\log |s_d|_{\varphi_\omega}(x)\, \mu_{(L_{0,\omega},\varphi_{0,\omega}),\cdots(L_{d-1,\omega},\varphi_{d-1,\omega})}(\mathrm{d}x) \right) \nu(\mathrm{d}\omega) \\ &+ \sum_{i=1}^{n} a_i (\overline{L}_0\big|_{Z_i} \cdots \overline{L}_{d-1}\big|_{Z_i})_S.\end{aligned}$$

Proof. — They follows from (3.14) and Proposition 3.6.3. □

Finally let us consider the projection formula for our intersection theory. For this purpose, we need three lemmas.

4.4.5. Lemma. — *Let $(A, \mathfrak{m})$ be a local Artinian ring and B be an A-algebra such that B is finitely generated as an A-module. Let M be a finitely generated B-module. Then*

$$\operatorname{length}_A(M) = \sum_{\mathfrak{n} \in \operatorname{Spec}(B)} [B/\mathfrak{n} : A/\mathfrak{m}] \operatorname{length}_{B_\mathfrak{n}}(M_\mathfrak{n}).$$

In particular, if B is flat over A, then

$$\operatorname{rk}_A(B)\operatorname{length}_A(A) = \sum_{\mathfrak{n}\in\operatorname{Spec}(B)} [B/\mathfrak{n} : A/\mathfrak{m}]\operatorname{length}_{B_\mathfrak{n}}(B_\mathfrak{n}).$$

Proof. — Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated B-modules. Then, both sides of the above first equation are additive with respect to the exact sequence. Therefore, we may assume that $M = B/\mathfrak{n}$ for some $\mathfrak{n} \in \operatorname{Spec}(B)$. In this case, it is obvious. □

4.4.6. Lemma. — *Let A be an integral domain and B be a flat A-algebra. If we denote the structure homomorphism $A \to B$ by ϕ, then $\phi^{-1}(P) = \{0\}$ for any $P \in \operatorname{Ass}_B(B)$.*

Proof. — We set $P = \operatorname{ann}(b)$ for some $b \in B \setminus \{0\}$. If there is $a \in \phi^{-1}(P) \setminus \{0\}$, then $\phi(a)b = 0$. Since B is flat over A, $\phi(a)$ is regular, so that $b = 0$. This is a contradiction. □

4.4.7. Lemma. — *Let $f : Y \to X$ be a proper and surjective morphism of integral scheme of finite type over a field k such that $\dim X = \dim Y$. For an extension filed k' of k, if $X' := X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k')$, $Y' := Y \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k')$ and $f' : X' \to Y'$ is the induced morphism, then*

$$f'_*([X']) = [k(Y) : k(X)][Y'].$$

Proof. — By Lemma 4.4.6, any irreducible component of X' (resp. Y') maps surjectively to X (resp. Y) by $X' \to X$ (resp. $Y' \to Y$). Moreover, we can find a non-empty Zariski open set U of X such that $f^{-1}(U) \to U$ is finite and flat. Note that if we set $U' := U \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k')$, then $f'^{-1}(U') = f^{-1}(U) \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k')$ and $f'^{-1}(U') \to U'$ is finite and flat. Therefore, we may assume that f is finite and flat, so that the assertion is a consequence of the second formula in Lemma 4.4.5. □

4.4.8. Definition. — Let $Z = a_1 Z_1 + \cdots + a_r Z_r$ be an l-dimensional cycle on X and $\overline{L}_0, \ldots, \overline{L}_l$ be integrable adelic line bundles. Then $(\overline{L}_0 \cdots \overline{L}_l \mid Z)_S$ is defined to be

$$(\overline{L}_0 \cdots \overline{L}_l \mid Z)_S := \sum_{j=1}^r a_j \left(\overline{L}_0\big|_{Z_j} \cdots \overline{L}_l\big|_{Z_j} \right)_S.$$

In the case where $\overline{L}_0, \ldots, \overline{L}_l$ are all equal to the same adelic line bundle $\overline{L}$, we call it the *height* of the cycle Z with respect to $\overline{L}$, and denote it by $h_{\overline{L}}(Z)$.

***4.4.9. Theorem* (Projection formula).** — *Let $f : Y \to X$ be a morphism of projective schemes over K and $\overline{L}_0, \ldots, \overline{L}_l$ be integrable adelic line bundles on X. For an l-cycle Z on Y,*

$$(f^*(\overline{L}_0) \cdots f^*(\overline{L}_l) \mid Z)_S = (\overline{L}_0 \cdots \overline{L}_l \mid f_*(Z))_S.$$

Proof. — First let us see the following:

4.4.10. Claim. — *If f is a surjective morphism of projective integral schemes over K, then*

$$(f^*(\overline{L}_0)\cdots f^*(\overline{L}_l))_S = \begin{cases} \deg(f)(\overline{L}_0\cdots\overline{L}_l)_S & \textit{if } \dim X = \dim Y, \\ 0 & \textit{if } \dim X < \dim Y. \end{cases}$$

In other words,

$$(f^*(\overline{L}_0)\cdots f^*(\overline{L}_l) \mid Y)_S = (\overline{L}_0\cdots\overline{L}_l \mid f_*(Y))_S.$$

Proof. — We choose rational sections $s_0, \ldots, s_d$ of $L_0, \ldots, L_d$, respectively such that $\mathrm{div}(s_0), \ldots, \mathrm{div}(s_d)$ intersect properly on X and $f^*(\mathrm{div}(s_0)), \ldots, f^*(\mathrm{div}(s_d))$ intersect properly on Y. Let K_ω be the completion of K with respect to $\omega \in \Omega$, $X_\omega := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K_\omega)$, $Y_\omega := Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K_\omega)$ and $f_\omega : Y_\omega \to X_\omega$ be the induced morphism. Further let $\pi_{X,\omega} : X_\omega \to X$ and $\pi_{Y,\omega} : Y_\omega \to Y$ be the projections. Then the following diagram is commutative.

$$\begin{tikzcd} Y_\omega \arrow[r, "f_\omega"] \arrow[d, "\pi_{Y,\omega}"'] & X_\omega \arrow[d, "\pi_{X,\omega}"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Since X and Y are integral, $f^*(\mathrm{div}(s_i))$ is well defined as a Cartier divisor. Moreover, $\pi_{Y,\omega}^*(f^*(\mathrm{div}(s_i)))$ and $\mathrm{div}(s_i)_\omega := \pi_{X,\omega}^*(\mathrm{div}(s_i))$ are defined because $\pi_{Y,\omega}$ and $\pi_{X,\omega}$ are flat. Therefore, $f_\omega^*(\mathrm{div}(s_i)_\omega)$ is defined as a Cartier divisor on Y_ω for each $i = 0, \ldots, d$. Let $Y_{\omega,1}, \ldots, Y_{\omega,m_\omega}$ (resp. $X_{\omega,1}, \ldots, X_{\omega,n_\omega}$) be irreducible components of Y_ω (resp. X_ω).

First we assume that $\dim X < \dim Y$. Then, by Proposition 3.6.4,

$$\left(f_\omega^*(\mathrm{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega})|_{Y_{\omega,j}} \cdots f_\omega^*(\mathrm{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})|_{Y_{\omega,j}} \right)_\omega = 0$$

for all $j = 1, \ldots, m_\omega$. Therefore,

$$\left(f_\omega^*(\mathrm{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega}) \cdots f_\omega^*(\mathrm{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})\right)_\omega = 0,$$

and hence the assertion follows.

Next we assume that $\dim X = \dim Y$. For each $i \in \{1, \ldots, n_\omega\}$, let

$$J_{\omega,i} := \{j \in \{1, \ldots, m_\omega\} \mid f_\omega(Y_{\omega,j}) = X_{\omega,i}\}$$

and

$$J_{\omega,0} := \{1, \ldots, n_\omega\} \setminus (J_{\omega,1} \cup \cdots \cup J_{\omega,n_\omega}).$$

By Proposition 3.6.4, if $j \in J_{\omega,i}$ ($i \in \{1, \ldots, n_\omega\}$), then

$$\begin{aligned} &\left(f_\omega^*(\mathrm{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega})|_{Y_{\omega,j}} \cdots f_\omega^*(\mathrm{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})|_{Y_{\omega,j}} \right)_\omega \\ &\quad = \deg(f_\omega|_{Y_{\omega,j}})\left((\mathrm{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega})|_{X_{\omega,i}} \cdots (\mathrm{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})|_{X_{\omega,i}} \right)_\omega. \end{aligned}$$

Moreover, if $j \in J_{\omega,0}$, then

$$\left(f_\omega^*(\operatorname{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega})|_{Y_{\omega,j}} \cdots f_\omega^*(\operatorname{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})|_{Y_{\omega,j}} \right)_\omega = 0.$$

Thus, by Lemma 4.4.7, one has

$$\begin{aligned}&\Big(f_\omega^*(\operatorname{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega}) \cdots f_\omega^*(\operatorname{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})\Big)_\omega \\ &\qquad = \deg(f)\Big((\operatorname{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega}) \cdots (\operatorname{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})\Big)_\omega.\end{aligned}$$

Therefore,

$$\begin{aligned}&(f^*(\overline{L}_0)\cdots f^*(\overline{L}_l))_S \\ &\qquad = \int_\Omega \Big(f_\omega^*(\operatorname{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega}) \cdots f_\omega^*(\operatorname{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})\Big)_\omega \nu(\mathrm{d}\omega) \\ &\qquad = \deg(f)\int_\Omega \Big((\operatorname{div}(s_0)_\omega, -\log|s_0|_{\varphi_\omega}) \cdots (\operatorname{div}(s_d)_\omega, -\log|s_d|_{\varphi_\omega})\Big)_\omega \nu(\mathrm{d}\omega) \\ &\qquad = \deg(f)(\overline{L}_0\cdots\overline{L}_l)_S.\end{aligned}$$

as required. □

In general, if we set $Z = a_1Z_1 + \cdots + a_rZ_r$, then, by Claim 4.4.10,

$$\begin{aligned}(f^*(\overline{L}_0)\cdots f^*(\overline{L}_l) \mid Z)_S &= \sum_{j=1}^r a_j(f^*(\overline{L}_0)\cdots f^*(\overline{L}_l) \mid Z_j)_S \\ &= \sum_{j=1}^r a_j(\overline{L}_0\cdots\overline{L}_l \mid f_*(Z_j))_S = (\overline{L}_0\cdots\overline{L}_l \mid f_*(Z))_S.\end{aligned}$$

□

4.5. Polarized adelic structure case

Let K be a finitely generated field over $\mathbb{Q}$ and n be the transcendental degree of K over $\mathbb{Q}$. Let $(\mathscr{B}; \overline{\mathscr{H}}_1, \ldots, \overline{\mathscr{H}}_n)$ be a polarization of K and $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be the polarized adelic structure by $(\mathscr{B}; \overline{\mathscr{H}}_1, \ldots, \overline{\mathscr{H}}_n)$ (for details, see Section 2.8).

Let X be a d-dimensional projective and integral scheme over K. We choose a projective arithmetic variety $\mathscr{X}$ and a morphism $\pi : \mathscr{X} \to \mathscr{B}$ such that the generic fiber of $\mathscr{X} \to \mathscr{B}$ is X. Let $L_0, \ldots, L_d$ be invertible $\mathcal{O}_X$-modules. We assume that there are C^∞-metrized invertible $\mathcal{O}_{\mathscr{X}}$-modules $\overline{\mathscr{L}}_0 = (\mathscr{L}, h_0), \ldots, \overline{\mathscr{L}}_d = (\mathscr{L}_d, h_d)$ in the usual sense on arithmetic varieties such that $\mathscr{L}_0, \ldots, \mathscr{L}_d$ coincides with $L_0, \ldots, L_d$ on X. Note that, for each $\omega \in \Omega$, $\overline{\mathscr{L}}_i$ yields a smooth metric $\varphi_{i,\omega}$ of $L_{i,\omega}$, that is, if $\omega \in \Omega_\infty$, then $\varphi_{i,\omega} = h_i|_{\pi^{-1}(\omega)}$; if $\omega \in \Omega \setminus \Omega_\infty$, then $\varphi_{i,\omega}$ is the model metric induced by the model $(\mathscr{X}, \mathscr{L}_i)$. We denote $\{(L_{i,\omega}, \varphi_{i,\omega})\}_{\omega\in\Omega}$ by $\overline{L}_i$.

4.5.1. Proposition. — $(\overline{L}_0\cdots\overline{L}_d)_S = (\overline{\mathscr{L}}_0\cdots\overline{\mathscr{L}}_d \cdot \pi^*(\overline{\mathscr{H}}_1)\cdots\pi^*(\overline{\mathscr{H}}_n))$

Proof. — We prove the assertion by induction on d. Clearly we may assume that $\mathscr{X}$ is normal. If $d = 0$, that is, $\dim \mathscr{X} = n+1$, then it is an easy consequence of [**49**, Lemma 1.12, Lemma 1.15, Proposition 5.3, Lemma 5.15 and Theorem 5.20].

We assume that $d > 0$. Let us choose a non-zero rational section s_0 of $\mathscr{L}_0$. Let $\operatorname{div}(s_0) = a_1\mathcal{Z}_1 + \cdots + a_r\mathcal{Z}_r$ be the decomposition as a cycle. Then one has

$$
\begin{aligned}
&(\overline{\mathscr{L}}_0 \cdots \overline{\mathscr{L}}_d \cdot \pi^*(\overline{\mathscr{H}}_1) \cdots \pi^*(\overline{\mathscr{H}}_n)) \\
&\quad = \sum_{i=1}^{r} a_i (\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_d \cdot \pi^*(\overline{\mathscr{H}}_1) \cdots \pi^*(\overline{\mathscr{H}}_n) \cdot (\mathcal{Z}_i, 0)) \\
&\qquad\quad + \int_{\mathscr{X}(\mathbb{C})} -\log |s_0|_{h_0} c_1(\overline{\mathscr{L}_1}) \wedge \cdots \wedge c_1(\overline{\mathscr{L}_d}) \wedge c_1(\pi^*\overline{\mathscr{H}_1}) \wedge \cdots \wedge c_1(\pi^*\overline{\mathscr{H}_n}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{\mathscr{X}(\mathbb{C})} -\log |s_0|_{h_0} c_1(\overline{\mathscr{L}_1}) \wedge \cdots \wedge c_1(\overline{\mathscr{L}_d}) \wedge c_1(\pi^*\overline{\mathscr{H}_1}) \wedge \cdots \wedge c_1(\pi^*\overline{\mathscr{H}_n}) \\
&= \int_{\mathscr{B}(\mathbb{C})} \left(\int_{\mathscr{X}(\mathbb{C})/\mathscr{B}(\mathbb{C})} -\log |s_0|_{h_0} c_1(\overline{\mathscr{L}_1}) \wedge \cdots \wedge c_1(\overline{\mathscr{L}_d}) \right) c_1(\overline{\mathscr{H}_1}) \wedge \cdots \wedge c_1(\overline{\mathscr{H}_n}).
\end{aligned}
$$

Here we consider the following claim:

4.5.2. Claim. — *Let $\psi : \mathscr{Y} \to \mathscr{C}$ be a surjective morphism of projective arithmetic varieties. Let $\overline{\mathscr{M}}_1, \ldots, \overline{\mathscr{M}}_d$ (resp. $\overline{\mathscr{D}}_1, \ldots, \overline{\mathscr{D}}_n$) be metrized integrable invertible $\mathcal{O}_{\mathscr{Y}}$-modules (resp. $\mathcal{O}_{\mathscr{C}}$-modules) such that $d + n = \dim \mathscr{Y}$. Let $\mathscr{Y}_\eta$ be the generic fiber of $\psi : \mathscr{Y} \to \mathscr{C}$. Then*

$$
\begin{aligned}
&(\overline{\mathscr{M}}_1 \cdots \overline{\mathscr{M}}_d \cdot \pi^*\overline{\mathscr{D}}_1 \cdots \pi^*\overline{\mathscr{D}}_n) \\
&\qquad = \begin{cases} \big(\mathscr{M}_1|_{\mathscr{Y}_\eta} \cdots \mathscr{M}_d|_{\mathscr{Y}_\eta} \big)(\overline{\mathscr{D}}_1 \cdots \overline{\mathscr{D}}_n), & \text{if } d = \dim \mathscr{Y}_\eta, \\ 0, & \text{if } d < \dim \mathscr{Y}_\eta. \end{cases}
\end{aligned}
$$

Proof. — This is a consequence of the projection formula (cf. [**49**, Theorem 5.20]). □

By the above claim, if $\mathcal{Z}$ is a prime divisor on $\mathscr{X}$ with $\pi(\mathcal{Z}) \neq \mathscr{B}$, then

$$
\begin{aligned}
&\big(\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_n \cdot \pi^*(\overline{\mathscr{H}}_1) \cdots \pi^*(\overline{\mathscr{H}}_d) \cdot (\mathcal{Z}, 0)\big) \\
&\qquad = \begin{cases} \big(\mathscr{L}_1|_{\mathcal{Z}_\eta} \cdots \mathscr{L}_n|_{\mathcal{Z}_\eta} \big)(\overline{\mathscr{H}}_1 \cdots \overline{\mathscr{H}}_d \cdot (\pi(\mathcal{Z}), 0)), & \text{if } \operatorname{codim}(\pi(\mathcal{Z}); \mathscr{B}) = 1, \\ 0, & \text{if } \operatorname{codim}(\pi(\mathcal{Z}); \mathscr{B}) \geqslant 2, \end{cases}
\end{aligned}
$$

where $\mathcal{Z}_\eta$ is the generic fiber of $\mathcal{Z} \to \pi(\mathcal{Z})$. Therefore, if we set

$$
\begin{cases} I_h := \{ i \in \{1, \ldots, r\} \mid \pi(\mathcal{Z}_i) = \mathscr{B} \}, \\ I_\Gamma := \{ i \in \{1, \ldots, r\} \mid \pi(\mathcal{Z}_i) = \Gamma \} \end{cases}
$$

for $\Gamma \in \Omega \setminus \Omega_\infty$, and denote

$$\sum_{i=1}^{r} a_i(\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_d \cdot \pi^*(\overline{\mathscr{H}}_1) \cdots \pi^*(\overline{\mathscr{H}}_n) \cdot (\mathscr{Z}_i, 0))$$

by T, then, by Example 3.5.2 and hypothesis of induction on d, one has

$$\begin{aligned} T &= \sum_{i \in I_h} a_i(\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_d \cdot \pi^*(\overline{\mathscr{H}}_1) \cdots \pi^*(\overline{\mathscr{H}}_n) \cdot (\mathscr{Z}_i, 0)) \\ &\qquad + \sum_{\Gamma \in \Omega \setminus \Omega_\infty} \sum_{i \in I_\Gamma} a_i(\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_d \cdot \pi^*(\overline{\mathscr{H}}_1) \cdots \pi^*(\overline{\mathscr{H}}_n) \cdot (\mathscr{Z}_i, 0)) \\ &= \sum_{i \in I_h} a_i\big(\, \overline{L}_1\big|_{Z_i} \cdots \overline{L}_d\big|_{Z_i}\,\big)_S \\ &\qquad + \sum_{\Gamma \in \Omega \setminus \Omega_\infty} (\overline{\mathscr{H}}_1 \cdots \overline{\mathscr{H}}_d \cdot (\Gamma, 0)) \int_{X_\Gamma^{\mathrm{an}}} -\log|s_0|_{\varphi_{0,\Gamma}} c_1(L_1, \varphi_{1,\Gamma}) \cdots c_1(L_d, \varphi_{d,\Gamma}), \end{aligned}$$

where Z_i is the generic fiber of $\mathscr{Z}_i \to \mathscr{B}$ for $i \in I_h$. Thus, by (3.14),

$$\begin{aligned} (\overline{\mathscr{L}}_0 \cdots \overline{\mathscr{L}}_d \cdot \pi^*(\overline{\mathscr{H}}_1) \cdots \pi^*(\overline{\mathscr{H}}_n)) &= \sum_{i \in I_h} a_i\big(\, \overline{L}_1\big|_{Z_i} \cdots \overline{L}_d\big|_{Z_i}\,\big)_S \\ + \int_\Omega \left(\int_{X_\omega^{\mathrm{an}}} -\log|s_0|_{\varphi_{0,\omega}} c_1(L_1, \varphi_{1,\omega}) \cdots c_1(L_d, \varphi_{d,\omega}) \right) \nu(\mathrm{d}\omega) &= (\overline{L}_1 \cdots \overline{L}_d)_S, \end{aligned}$$

as required. $\square$

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