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Okounkov bodies of filtered linear series

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Abstract

We associate to certain filtrations of a graded linear series of a big line bundle a concave function on its Okounkov body, whose law with respect to the Lebesgue measure describes the asymptotic distribution of the jumps of the filtration. As a consequence, we obtain a Fujita-type approximation theorem in this general filtered setting. We then specialize these results to the filtrations by minima in the usual context of Arakelov geometry (and for more general adelically normed graded linear series), thereby obtaining in a simple way a natural construction of an arithmetic Okounkov body, the existence of the arithmetic volume as a limit and an arithmetic Fujita approximation theorem for adelically normed graded linear series. We also obtain an easy proof of the existence of the sectional capacity previously obtained by Lau, Rumely and Varley.

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Introduction

Okounkov bodies

Let X be an n-dimensional projective variety defined over an arbitrary field K and let L be a *big* line bundle on X/K. Its *Okounkov body* $\Delta(L) \subset \mathbb{R}^n$ is a compact convex set designed to study the asymptotic behavior of $H^0(kL)$ as $k \to \infty$ by generalizing to some extent the usual picture in toric geometry. Okounkov bodies were introduced and studied by Lazarsfeld and Mustaţă in [LM09] and independently by Kaveh and Khovanskii in [KK08, KK09], both building on ideas of Okounkov [Ok96] (himself relying on former results of Khovanskii [Kho93]). They have the crucial property that

$$\operatorname{vol} \Delta(L) = \lim_{k\to\infty} \frac{\dim H^0(kL)}{k^n}.$$

Note that such a statement implicitly contains the *existence* of the right-hand limit, a basic birational-geometric invariant of the big line bundle L known (after multiplication by $n!$) as its *volume* $\operatorname{vol}(L)$.

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It was more generally shown in [LM09] that one can attach to a *graded linear series* $V_\bullet$ of L (i.e. a graded subalgebra of $R(L) := \bigoplus_{k\geqslant 0} H^0(kL)$) a convex body $\Delta(V_\bullet) \subset \mathbb{R}^n$ such that

$$\mathrm{vol}\,\Delta(V_\bullet) = \lim_{k\to\infty} \frac{\dim V_k}{k^n}$$

as soon as $V_\bullet$ *contains an ample series* (cf. Definition 1.1 below). The right-hand side is here again known (after multiplication by $n!$) as the *volume* $\mathrm{vol}(V_\bullet)$ of $V_\bullet$, and a general version of Fujita's approximation theorem in this setting was also obtained in [LM09], to the effect that the volume of $V_\bullet$ can be approximated by that of its *finitely generated* graded subseries.

Arakelov-geometric analogues

Assume now that K is a number field, and consider the following *Arakelov-geometric* setting: let $\mathcal{X}$ be a flat projective model of X over the ring of integers $\mathcal{O}_K$, $\mathcal{L}$ be a model of L on $\mathcal{X}$ and assume also that we are given a conjugation-invariant continuous Hermitian metric on $L_\mathbb{C}$ over

$$X(\mathbb{C}) = \coprod_{\sigma: K\hookrightarrow\mathbb{C}} X_\sigma(\mathbb{C}),$$

the whole data being summarized as $\overline{L}$. Given a finite set S, we set

$$\widehat{\dim}_K S := \frac{1}{[K:\mathbb{Q}]} \log \#S.$$

One then replaces $H^0(kL)$ with the finite set of *small sections*

$$\widehat{H}^0(k\overline{L}) := \left\{ s \in H^0(\mathcal{X}, k\mathcal{L}),\ \sup_{X(\mathbb{C})} |s| \leqslant 1 \right\},$$

and it is a basic problem to construct the analogue of Okounkov bodies in this arithmetic setting (cf. for instance [LM09, p. 51, Question 7.7]). This was largely accomplished by Yuan in [Yua09b], at least in the case of complete linear series. Indeed, Yuan was able to construct a family of compact convex sets $\Delta_p(\overline{L}) \subset \mathbb{R}^{n+1}$ indexed by an infinite family of prime numbers p in such a way that

$$\lim_{p\to\infty} \mathrm{vol}(\Delta_p(\overline{L})) \log p = \limsup_{k\to\infty} \frac{\widehat{\dim}_K \widehat{H}^0(k\overline{L})}{k^{n+1}}.$$

The right-hand side (multiplied by $(n+1)!$) is known as the *arithmetic volume* $\widehat{\mathrm{vol}}(\overline{L})$ of $\overline{L}$, and Yuan inferred from the above result a Fujita-type approximation theorem for $\widehat{\mathrm{vol}}(\overline{L})$. Such a result reduces for many purposes the study of arithmetic volumes to the *arithmetically ample* case (where $\mathcal{L}$ is relatively ample and ϕ is smooth and strictly positively curved), which has been well understood since the work of Gillet–Soulé [GS92] (see also [AB95]) and Zhang [Zha95a], since it can be expressed as an arithmetic intersection number.

A Fujita-type approximation theorem was also independently obtained by the second-named author in [Che10a] using a different approach.

The general filtered setting

Our goal in the present article is to unify in a natural and elementary way the above results. Let $V_\bullet$ be as above a graded linear series of L containing an ample series and consider the finite set

$$\widehat{V}_k := V_k \cap \widehat{H}^0(k\overline{L})$$

of small sections in V_k. We construct a (single) compact convex set $\widehat{\Delta}(\overline{V}_\bullet) \subset \mathbb{R}^{n+1}$ such that

$$\operatorname{vol} \widehat{\Delta}(\overline{V}_\bullet) = \lim_{k\to\infty} \frac{\widehat{\dim}_K \widehat{V}_k}{k^{n+1}} \tag{0.1}$$

and show that such an arithmetic volume can be approximated by that of finitely generated graded subseries.

More specifically, consider the *filtration by minima* $\mathcal{F} = \mathcal{F}_{\min}$ of each K-vector space V_k, defined by letting, for each $t \in \mathbb{R}$,

$$\mathcal{F}^t V_k := \operatorname{Vect}_K \left\{ s \in V_k \cap H^0(\mathcal{X}, k\mathcal{L}),\ \sup_{X(\mathbb{C})} |s| \leqslant e^{-t} \right\}.$$

The *jumping numbers*

$$e_j(V_k, \mathcal{F}) := \sup\{t \in \mathbb{R}, \dim \mathcal{F}^t V_k \geqslant j\} \quad (j = 1, \ldots, \dim V_k)$$

of this filtration are then essentially equal to the classical *successive minima* of $\overline{V}_k$ and it thus follows from Gillet–Soulé's work [GS91] that

$$\sum_{\substack{1 \leqslant j \leqslant \dim V_k \\ e_j(V_k, \mathcal{F}) > 0}} e_j(V_k, \mathcal{F}) = \widehat{\dim}_K \widehat{V}_k + o(k^{n+1}). \tag{0.2}$$

On the other hand, the filtration induced by $\mathcal{F}$ on $V_\bullet$ is *multiplicative* in the sense that

$$(\mathcal{F}^t V_k)(\mathcal{F}^s V_m) \subset \mathcal{F}^{t+s} V_{k+m},$$

it is *pointwise bounded below*, i.e. for each k, we have

$$\mathcal{F}^{-t} V_k = V_k \quad \text{for } t \gg 1,$$

and it is *linearly bounded above* in the sense that there exists $C > 0$ such that

$$\mathcal{F}^t V_k = 0 \quad \text{for } t \geqslant Ck$$

(cf. Proposition 2.6). We can then consider the Okounkov body

$$\Delta(V_\bullet^t) \subset \Delta(V_\bullet)$$

of the graded subseries $V_\bullet^t$ defined by $V_k^t := \mathcal{F}^{kt} V_k$ and introduce

$$G_\mathcal{F} : \Delta(V_\bullet) \to [-\infty, C]$$

as the incidence function

$$G_\mathcal{F}(x) := \sup\{t \in \mathbb{R}, x \in \Delta(V_\bullet^t)\}$$

of the filtration $t \mapsto \Delta(V_\bullet^t)$. We show that $G_\mathcal{F}$ is concave, upper semicontinuous and finite valued on the interior $\Delta(V_\bullet)^\circ$. Our main result is then the following.

Theorem A. *Let K be an arbitrary field, X be a projective K-variety and L be a big line bundle on X/K. Let $V_\bullet$ be a graded linear series of L containing an ample series and let $\mathcal{F}$ be a decreasing $\mathbb{R}$-filtration of $V_\bullet$ which is furthermore multiplicative, pointwise bounded below and linearly bounded above. Then the scaled jumping numbers $k^{-1}e_j(V_k, \mathcal{F})$ of the restriction of $\mathcal{F}$ to V_k equidistribute as $k \to \infty$ according to the push-forward by $G_\mathcal{F}$ of the Lebesgue measure.*

In other words, this result says that

$$\lim_{k\to\infty} k^{-n} \sum_j f(k^{-1} e_j(V_k, \mathcal{F})) = \int_{\Delta(V_\bullet)^\circ} (f \circ G_\mathcal{F})\, d\lambda$$

for every bounded continuous function f on $\mathbb{R}$, with λ denoting standard Lebesgue measure on $\mathbb{R}^n$. This result extends in particular the second-named author's result [Che10a, Proposition 4.6], expressing furthermore the limit measure as the push-forward of λ by $G_{\mathcal{F}}$.

We next define the *filtered Okounkov body* of $(V_\bullet, \mathcal{F})$ as the compact convex subset

$$\widehat{\Delta}(V_\bullet, \mathcal{F}) := \{(x,t) \in \Delta(V_\bullet) \times \mathbb{R}, 0 \leqslant t \leqslant G_{\mathcal{F}}(x)\} \subset \mathbb{R}^{n+1},$$

and set in the Arakelov-geometric case $\widehat{\Delta}(\overline{V}_\bullet) := \widehat{\Delta}(V_\bullet, \mathcal{F}_{\min})$ with $\mathcal{F}_{\min}$ standing for the filtration by minima. Gillet–Soulé's result (0.2) combined with Theorem A will then easily be seen to imply (0.1).

As a consequence of Theorem A, we will also obtain the following filtered analogue of Lazarsfeld–Mustaţă's Fujita-type approximation theorem.

Theorem B. *Let X, L and $\mathcal{F}$ be as in Theorem A. For each $\varepsilon > 0$, there exists a finitely generated subseries $W_\bullet$ of $V_\bullet$ such that*

$$\operatorname{vol} \widehat{\Delta}(W_\bullet, \mathcal{F}) \geqslant \operatorname{vol} \widehat{\Delta}(V_\bullet, \mathcal{F}) - \varepsilon.$$

The arithmetic applications of these results will be more generally obtained for *adelically normed* graded linear series satisfying a rather mild finiteness condition. We will also show that Theorem A enables us to recover in a reasonably general special case the existence of the *sectional capacity*, first obtained by Rumely *et al.* in [RLV00].

Relations to other works

We have already mentioned above that Theorems A and B yield in particular simpler proofs of the main results of [Che10a, Mor09a, Yua09b], and it is therefore conversely clear that these works have had a strong influence on the present article. The work of Witt Nyström [WN09] was also influential as far as the idea of constructing a concave function on the Okounkov body is concerned, even though the final outcome of our construction is not a Chebyshev-type function.

While the present work was being written, Yuan introduced in [Yua09a] another kind of concave function $c[\overline{L}]$ on the Okounkov body of a line bundle endowed with Arakelov-geometric data. His construction is closer in spirit to that of [WN09], since it consists in summing up the analogues of Witt Nyström's Chebyshev functions at all places of K. It is also closely related to [RLV00], and indeed Yuan's goal is to show that the mean value of $c[\overline{L}]$ coincides with the quantity he denotes by $\operatorname{vol}_\chi(\overline{L})$ and which is equal by definition to $-\log$ of the sectional capacity from [RLV00].

We shall discuss in more detail the relation between [Yua09a] and the present work in § 4. We will show in particular in an example that the two constructions do not coincide in general.

Organization of the paper

Let us briefly describe the structure of our article.

- Section 1.1 contains the necessary definitions and results on Okounkov bodies extracted from [LM09], whereas § 1.2 introduces some terminology related to filtrations.
- In § 1.3, we define the concave function attached to a filtered graded linear series and prove our main results in this general setting. Theorem A corresponds to Theorem 1.11, whereas Theorem B is Theorem 1.14.
- We then show in § 2 how to relate the Arakelov-geometric setting (and more generally the adelic setting) to the filtered case using filtrations by minima.

- The existence of the sectional capacity is obtained in § 3.
- In the final § 4, we discuss the relation between the present work and [Yua09a].

1. The concave transform of a filtered graded linear series

In this section, K denotes an arbitrary field and K^{a} an algebraic closure of K.

1.1 Okounkov bodies

The original idea of Okounkov bodies [Ok96] was systematically developed in [LM09] and independently in [KK08] in order to study the asymptotic behavior of graded linear series.

Let X be a projective K-variety (i.e. a geometrically integral projective K-scheme) of dimension n and fix a system of parameters $z=(z_1,\dots,z_n)$ centered at a regular closed point $p\in X(K^{\mathrm{a}})$. We then get a rank-n valuation

$$\mathrm{ord}_z:\mathcal{O}_X\backslash\{0\}\to\mathbb{N}^n$$

centered at p as follows: expand a given $f\in\mathcal{O}_{X,p}$ as a formal power series

$$f=\sum_{\alpha\in\mathbb{N}^n}a_\alpha z^\alpha$$

and set

$$\mathrm{ord}_z(f)=\min_{\mathrm{lex}}\{\alpha\in\mathbb{N}^n,a_\alpha\neq 0\},$$

where the minimum is taken with respect to the lexicographic order on $\mathbb{N}^n$. Note that ord_z only depends on the choice of uniformizing parameters z *via* the flag they induce on the tangent space of $X_{K^{\mathrm{a}}}$ at p (compare [LM09, § 5.2]).

Given a line bundle L on X, the function ord_z induces a valuation-like function on $H^0(L)_{K^{\mathrm{a}}}\backslash\{0\}$ by composing it with the evaluation operator, with the basic property that each graded piece has

$$\dim_{K^{\mathrm{a}}}(\{s\in W,\mathrm{ord}_z(s)\geqslant_{\mathrm{lex}}\alpha\}/\{s\in W,\mathrm{ord}_z(s)>_{\mathrm{lex}}\alpha\})\leqslant 1 \tag{1.1}$$

for each subspace $W\subset H^0(L)_{K^{\mathrm{a}}}$. Indeed, given s_1,s_2 with $\mathrm{ord}_z(s_j)=\alpha$, $i=1,2$, we have $s_j=c_jz^\alpha+(\text{higher order terms})$ with $c_j\in K^{\mathrm{a}^*}$, and it immediately follows that s_1 and s_2 are K^{a}-linearly dependent modulo $\{\mathrm{ord}_z>\alpha\}$ (see also [LM09, Lemma 1.3]).

Note that (1.1) implies in particular that the set $\mathrm{ord}_z(W)\subset\mathbb{N}^n$ has cardinality equal to $\dim_{K^{\mathrm{a}}}W$.

Now let $V_\bullet$ be a *graded linear series* of L, i.e. a graded K-subalgebra of

$$R(L):=\bigoplus_{k\geqslant 0}H^0(kL).$$

One associates to $V_\bullet$ the semigroup

$$\Gamma(V_\bullet):=\{(k,\alpha)\in\mathbb{N}^{n+1},\alpha=\mathrm{ord}_z(s)\text{ for some non-zero }s\in V_k\otimes K^{\mathrm{a}}\}$$

whose slice over k

$$\Gamma_k:=\Gamma(V_\bullet)\cap(\{k\}\times\mathbb{N}^n)$$

has cardinality equal to $\dim_K V_k$ by (1.1). The closed convex cone $\Sigma(V_\bullet) \subset \mathbb{R}^{n+1}$ generated by $\Gamma(V_\bullet)$ has a compact convex basis

$$\Delta(V_\bullet) := \Sigma(V_\bullet) \cap (\{1\} \times \mathbb{R}^n)$$

(cf. [LM09, p. 18]). By [LM09, Proposition 2.1], it is actually a convex body (i.e. it has non-empty interior) and its Euclidian volume satisfies

$$\operatorname{vol} \Delta(V_\bullet) = \lim_{k\to\infty} k^{-n} \dim V_k \tag{1.2}$$

as soon as $\Gamma(V_\bullet)$ generates $\mathbb{Z}^{n+1}$ as a group. As shown in [LM09, Lemma 2.12], this is in particular the case when $V_\bullet$ contains an ample series in the following sense.

Definition 1.1. Let $V_\bullet$ be a graded linear series of L. We will say that $V_\bullet$ *contains an ample series* if:

(i) $V_k \neq 0$ for all $k \gg 1$;

(ii) there exist a Kodaira-type decomposition $L = A + E$ into $\mathbb{Q}$-divisors with A ample and E effective such that

$$H^0(kA) \subset V_k \subset H^0(kL)$$

for all sufficiently large and divisible k.

This corresponds exactly to condition (C) in [LM09, p. 20], as explained in Remark 2.10 thereof.

1.2 Filtered vector spaces and algebras

All decreasing $\mathbb{R}$-filtrations $t \mapsto \mathcal{F}^t V$ of a finite-dimensional vector space V considered in this article will be *left-continuous* in the sense that for each $t \in \mathbb{R}$ we have $\mathcal{F}^t V = \mathcal{F}^{t-\varepsilon} V$ for $0 < \varepsilon \ll 1$. Given such a filtered vector space $(V, \mathcal{F})$, we set

$$e_{\min}(V, \mathcal{F}) := \inf\{t \in \mathbb{R}, \mathcal{F}^t V \neq V\}$$

and

$$e_{\max}(V, \mathcal{F}) := \sup\{t \in \mathbb{R}, \mathcal{F}^t V \neq 0\}$$

and we shall say that $\mathcal{F}$ is *bounded below* (respectively *bounded above*) if $e_{\min}(V, \mathcal{F}) > -\infty$ (respectively $e_{\max}(V, \mathcal{F}) < +\infty$). We shall simply say that $\mathcal{F}$ is *bounded* if $\mathcal{F}$ is bounded above and below.

Definition 1.2. Let $\mathcal{F}$ be a bounded filtration of an N-dimensional K-vector space V, $N \geqslant 1$.

(i) The *jumping numbers*

$$e_{\min}(V, \mathcal{F}) = e_N(V, \mathcal{F}) \leqslant \cdots \leqslant e_1(V, \mathcal{F}) = e_{\max}(V, \mathcal{F})$$

of $(V, \mathcal{F})$ are defined by

$$e_j(V, \mathcal{F}) := \sup\{t \in \mathbb{R}, \dim \mathcal{F}^t V \geqslant j\}.$$

(ii) The *mass* of $(V, \mathcal{F})$ is defined by

$$\operatorname{mass}(V, \mathcal{F}) := \sum_j e_j(V, \mathcal{F}).$$

(iii) The *positive mass* of $(V, \mathcal{F})$ is defined as

$$\mathrm{mass}_+(V, \mathcal{F}) := \sum_{e_j(V,\mathcal{F})>0} e_j(V, \mathcal{F})$$

(or 0 if $e_{\max}(V, \mathcal{F}) \leqslant 0$).

The non-increasing left-continuous step function $t \mapsto \dim \mathcal{F}^t V$ therefore satisfies

$$\dim \mathcal{F}^t V = j \iff t \in \left]e_{j+1}(V, \mathcal{F}), e_j(V, \mathcal{F})\right]$$

(with $e_{N+1}(V, \mathcal{F}) = -\infty$ and $e_0(V, \mathcal{F}) = +\infty$ by convention), which implies that

$$\frac{d}{dt} \dim \mathcal{F}^t V = -\sum\nolimits_{j=1}^{N} \delta_{e_j(V,\mathcal{F})} \tag{1.3}$$

holds in the sense of distributions.

In what follows, a graded K-algebra $V_\bullet = \bigoplus_{k\in\mathbb{N}} V_k$ will always be indexed by $\mathbb{N}$ and with finite-dimensional pieces such that $V_0 = K$.

Definition 1.3. Let $\mathcal{F}$ be a decreasing, left-continuous $\mathbb{R}$-filtration of a graded K-algebra $V_\bullet$. We shall say that:

(i) $\mathcal{F}$ is *multiplicative* if

$$(\mathcal{F}^t V_k)(\mathcal{F}^s V_m) \subset \mathcal{F}^{t+s} V_{k+m}$$

holds for all $k, m \in \mathbb{N}$ and $s, t \in \mathbb{R}$;

(ii) $\mathcal{F}$ is *pointwise bounded below (respectively pointwise bounded above)* if $(V_k, \mathcal{F})$ is bounded below (respectively bounded above) for each k;

(iii) $\mathcal{F}$ is *linearly bounded below (respectively linearly bounded above)* if there exists $C > 0$ such that $e_{\min}(V_k, \mathcal{F}) \geqslant -Ck$ (respectively $e_{\max}(V_k, \mathcal{F}) \leqslant Ck$).

If $\mathcal{F}$ is a multiplicative filtration on $V_\bullet$, then setting

$$V_k^t := \mathcal{F}^{kt} V_k$$

defines a graded subalgebra $V_\bullet^t := \bigoplus_k V_k^t$ of $V_\bullet$.

We introduce

$$e_{\min}(V_\bullet, \mathcal{F}) := \liminf_{k\to\infty} \frac{e_{\min}(V_k, \mathcal{F})}{k} \tag{1.4}$$

and

$$e_{\max}(V_\bullet, \mathcal{F}) := \limsup_{k\to\infty} \frac{e_{\max}(V_k, \mathcal{F})}{k}. \tag{1.5}$$

Note that, when $V_k \neq 0$ for $k \gg 1$, $\mathcal{F}$ is linearly bounded below (respectively linearly bounded above) if and only if $e_{\min}(V_\bullet, \mathcal{F})$ (respectively $e_{\max}(V_\bullet, \mathcal{F})$) is finite.

Lemma 1.4. *Assume that the graded algebra $V_\bullet$ is integral and satisfies $V_k \neq 0$ for all $k \gg 1$. Then we have*

$$e_{\max}(V_\bullet, \mathcal{F}) = \lim_{k\to\infty} \frac{e_{\max}(V_k, \mathcal{F})}{k} = \sup_{k\geqslant 1} \frac{e_{\max}(V_k, \mathcal{F})}{k}.$$

Proof. It is easily checked under the standing assumptions that $e_{\max}(V_k, \mathcal{F})$ is super-additive in k for $k \gg 1$, and the result follows from the standard fact that $\lim_{k\to\infty} a_k/k = \sup_{k\geqslant 1} a_k/k$ for every super-additive sequence (a_k). □

Lemma 1.5. *If the graded algebra $V_\bullet$ is finitely generated, then any multiplicative filtration on $V_\bullet$ which is pointwise bounded below is linearly bounded below.*

Proof. Let p be such that $V_\bullet$ is generated by $V_1 + \cdots + V_p$ and choose t such that $\mathcal{F}^{jt} V_j = V_j$ for $j = 1, \ldots, p$. Writing a given $v \in V_k$ as a homogeneous polynomial in elements of $V_1, \ldots, V_p$ and using the multiplicative property of $\mathcal{F}$, it is then easily seen that x lies in $\mathcal{F}^{kt} V_k$. □

1.3 The concave transform of a filtration

In what follows, we fix a projective K-variety X of dimension n, a set of uniformizing parameters z at a given regular K^{a}-point and a big bundle L on X/K.

Lemma 1.6. *Let $\mathcal{F}$ be a multiplicative filtration on a graded linear series $V_\bullet$ of L containing an ample series. Then for each $t < e_{\max}(V_\bullet, \mathcal{F})$ the graded linear series $V_\bullet^t$ also contains an ample series.*

This gives in particular an elementary proof of [Che10a, Proposition 4.9].

Proof. Given $t < e_{\max}(V_\bullet, \mathcal{F})$, we have $kt < e_{\max}(V_k, \mathcal{F})$ for all $k \gg 1$; thus, $V_k^t = \mathcal{F}^{kt} V_k \neq 0$ by definition, and we see that Definition 1.1(i) is satisfied.

Let us now turn to (ii). By Definition 1.1, there exist an integer m, an ample line bundle A and a non-zero section $s \in H^0(mL - A)$ such that the image W_k of the map $H^0(kA) \to H^0(kmL)$ given by multiplication by s^k is contained in V_{km} for all $k \geqslant 1$. Since $W_\bullet$ is finitely generated, there exists $a \in \mathbb{R}$ such that $W_k \subset V_{km}^a$ for all k by Lemma 1.5.

On the other hand, let $\varepsilon > 0$ be such that $t + \varepsilon < e_{\max}(V_\bullet, \mathcal{F})$. We may then choose $p \gg 1$ such that there exists a non-zero element $v_p \in \mathcal{F}^{p(t+\varepsilon)} V_p$, and such that

$$s_p := \frac{akm + (t+\varepsilon)kp}{km + kp} > t.$$

The image of the map $H^0(kA) \to V_{k(m+p)}$ given by multiplication by $(sv_p)^k$ then lies in

$$W_k \cdot V_{kp}^{t+\varepsilon} \subset V_{km}^a \cdot V_{kp}^{t+\varepsilon} \subset V_{k(m+p)}^{s_p} \subset V_{k(m+p)}^t,$$

as desired. □

The Okounkov bodies $\Delta(V_\bullet^t)$ make up a non-increasing family of compact convex subsets of $\Delta(V_\bullet)$. Since $\mathcal{F}$ is multiplicative, it is straightforward to check that the following *convexity property* holds:

$$\Delta(V_\bullet^{ts_1 + (1-t)s_2}) \supset t\Delta(V_\bullet^{s_1}) + (1-t)\Delta(V_\bullet^{s_2}) \tag{1.6}$$

for all $s_1, s_2 \in \mathbb{R}$ and $0 \leqslant t \leqslant 1$. The interiors of the $\Delta(V_\bullet^t)$'s fill in that of $\Delta(V_\bullet)$ in the following circumstance.

Lemma 1.7. *Let $\mathcal{F}$ be a multiplicative filtration on a graded linear series $V_\bullet$ of L that contains an ample series. If $\mathcal{F}$ is furthermore pointwise bounded below, then we have*

$$\Delta(V_\bullet)^\circ = \bigcup_{t \in \mathbb{R}} \Delta(V_\bullet^t)^\circ.$$

Proof. Since $\mathcal{F}$ is pointwise bounded below, we have $V_k = \bigcup_{t \in \mathbb{R}} V_k^t$ for each k, which implies that

$$\Gamma(V_\bullet) = \bigcup_{t \in \mathbb{R}} \Gamma(V_\bullet^t).$$

On the other hand, $V_\bullet^t$ contains an ample series for each $t < e_{\max}(V_\bullet, \mathcal{F})$ by Lemma 1.6, and it follows by [LM09, Lemma 2.12] that $\Gamma(V_\bullet^t)$ spans $\mathbb{Z}^{n+1}$. The result now follows in view of [LM09, Lemma A.3]. □

DEFINITION 1.8. Let $\mathcal{F}$ be a multiplicative filtration of a graded linear series $V_\bullet$ of L containing an ample line bundle, and assume that $\mathcal{F}$ is pointwise bounded below and linearly bounded above. The *concave transform* of $\mathcal{F}$ is the concave function

$$G_{\mathcal{F}} : \Delta(V_\bullet) \to [-\infty, +\infty[$$

defined by

$$G_{\mathcal{F}}(x) := \sup\{t \in \mathbb{R}, x \in \Delta(V_\bullet^t)\}.$$

The function $G_{\mathcal{F}}$ is indeed concave by (1.6). It takes finite values on the interior $\Delta(V_\bullet)^\circ$ by Lemma 1.7 and is thus continuous there, since it is concave. It is furthermore upper semicontinuous on the whole of $\Delta(V_\bullet)$, since we have for each $t \in \mathbb{R}$

$$\{G_{\mathcal{F}} \geqslant t\} = \bigcap_{s<t} \Delta(V_\bullet^s). \tag{1.7}$$

Note also that

$$e_{\min}(V_\bullet, \mathcal{F}) \leqslant G_{\mathcal{F}} \leqslant e_{\max}(V_\bullet, \mathcal{F}) \tag{1.8}$$

on $\Delta(V_\bullet)$.

DEFINITION 1.9. With $V_\bullet$, $\mathcal{F}$ as in Definition 1.8, we define the *filtered Okounkov body* as the compact convex subset of $\mathbb{R}^{n+1}$ defined by

$$\widehat{\Delta}(V_\bullet, \mathcal{F}) := \{(x, t) \in \Delta(V_\bullet) \times \mathbb{R}, 0 \leqslant t \leqslant G_{\mathcal{F}}(x)\}.$$

Remark 1.10. $G_{\mathcal{F}}$ can also be obtained as the *concave envelope* in the sense of [WN09, § 3] of the super-additive function $g : \Gamma(V_\bullet) \to \mathbb{R}$ defined by

$$g(k, \alpha) := \sup\{t \in \mathbb{R}, \alpha = \mathrm{ord}_z(s) \text{ for some non-zero } s \in \mathcal{F}^t V_k\}. \tag{1.9}$$

We are now in a position to state our main result, which corresponds to Theorem A in the introduction.

THEOREM 1.11. *Let $\mathcal{F}$ be a pointwise bounded below and linearly bounded above multiplicative filtration of a graded linear series $V_\bullet$ of L containing an ample series. Then the discrete measures on $\mathbb{R}$ defined by*

$$\mu_k := k^{-n} \sum_j \delta_{k^{-1} e_j(V_k, \mathcal{F})}$$

converge weakly to $(G_{\mathcal{F}})_* \lambda$ as $k \to \infty$, where λ denotes the restriction of the Lebesgue measure of $\mathbb{R}^n$ to $\Delta(V_\bullet)^\circ$.

Proof. By (1.3), the measure $-\mu_k$ is the distributional derivative of the non-increasing left-continuous step function

$$g_k(t) := k^{-n} \dim \mathcal{F}^{kt} V_k = k^{-n} \dim V_k^t.$$

Since $V_\bullet^t$ contains an ample series for $t < e_{\max}(V_\bullet, \mathcal{F})$, we may apply Lazarsfeld and Mustaţă's result (1.2) to get

$$\lim_{k\to\infty} g_k(t) = g(t) := \mathrm{vol}\, \Delta(V_\bullet^t)$$

for each $t < e_{\max}(V_\bullet, \mathcal{F})$. Since

$$0 \leqslant g_k(t) \leqslant k^{-n} \dim H^0(kL)$$

is uniformly bounded, it follows by dominated convergence that $g_k \to g$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ and thus $-\mu_k = g_k' \to g'$ as distributions. We claim on the other hand that $g' = -\nu$, which will conclude the proof. Indeed, (1.7) yields

$$\lim_{s \to t_-} g(s) = \lambda(\{G_{\mathcal{F}} \geqslant t\}) =: h(t).$$

Since the discontinuity locus of the non-increasing function g is at most countable, it follows in particular that $g = h$ as distributions. In terms of the finite measure $\nu := (G_{\mathcal{F}})_* \lambda$ on $\mathbb{R}$, we have $h(t) = \nu(\{x \in \mathbb{R}, x \geqslant t\})$, which indeed implies that $h' = -\nu$ by basic integration theory. □

Remark 1.12. (i) The convergence in Theorem 1.11 holds in the weak topology, i.e. against all continuous functions with compact support. But the total mass of μ_k is equal to $k^{-n} \dim V_k$, which converges to $\mathrm{vol}\, \Delta(\Gamma)$, the total mass of $(G_{\mathcal{F}})_* \lambda$. Indeed, this follows from (1.2) since $V_\bullet$ is assumed to contain an ample series. By a standard result in integration theory that the convergence holds as well against all *bounded* continuous functions on $\mathbb{R}$, and the result can be reformulated in more probabilistic terms by saying that the scaled jumping numbers

$$k^{-1} e_1(V_k, \mathcal{F}) \geqslant \cdots \geqslant k^{-1} e_{N_k}(V_k, \mathcal{F})$$

of the sequence of filtered vector spaces $(V_k, \mathcal{F})$ equidistribute as $k \to \infty$ according to the law of $G_{\mathcal{F}}$ (with respect to the Lebesgue measure on $\Delta(V_\bullet)$ scaled to mass 1).

(ii) We also remark that Theorem 1.11 shows in particular that $(G_{\mathcal{F}})_* \lambda$ does not depend on the choice of the regular system of parameters z, even though the Okounkov body $\Delta(V_\bullet)$ and hence the function $G_{\mathcal{F}}$ will depend on z (or rather on the infinitesimal flag it defines) in general.

Corollary 1.13. *With the same assumptions as in Theorem 1.11, the euclidian volume of the filtered Okounkov body satisfies*

$$\mathrm{vol}\, \widehat{\Delta}(V_\bullet, \mathcal{F}) = \int_{t=0}^{+\infty} \mathrm{vol}\, \Delta(V_\bullet^t)\, dt = \lim_{k \to \infty} \frac{\mathrm{mass}_+(V_k, \mathcal{F})}{k^{n+1}}.$$

Proof. Let $e := e_{\max}(V_\bullet, \mathcal{F})$. In the notation of Theorem 1.11, we have by definition

$$\frac{\mathrm{mass}_+(V_k, \mathcal{F})}{k^{n+1}} = \int_0^{+\infty} t \mu_k$$

and

$$\mathrm{vol}(\widehat{\Delta}(V_\bullet, \mathcal{F})) = \int_0^{+\infty} t (G_{\mathcal{F}})_* \lambda.$$

Now both μ_k and $(G_{\mathcal{F}})_* \lambda$ are supported on $]-\infty, e]$ by Lemma 1.4 and the upper bound $G_{\mathcal{F}} \leqslant e$. The result thus follows by applying Theorem 1.11 to a continuous function with compact support and coinciding with $\max(t, 0)$ on $]-\infty, e]$. □

We finally get a filtered version of the Lazarsfeld–Mustaţă approximation theorem.

Theorem 1.14. *Let $\mathcal{F}$ and $V_\bullet$ be as in Theorem 1.11. Then, for each $\varepsilon > 0$, there exists a finitely generated graded subalgebra $W_\bullet$ of $V_\bullet$ such that*

$$\operatorname{vol}\widehat{\Delta}(W_\bullet, \mathcal{F}) \geqslant \operatorname{vol}\widehat{\Delta}(V_\bullet, \mathcal{F}) - \varepsilon.$$

Proof. For each p, let $S^\bullet V_{\leqslant p}$ be the graded subalgebra of $V_\bullet$ generated by $V_1 + \cdots + V_p$. For each $t < e_{\max}(V_\bullet, \mathcal{F})$, the graded linear series $V_\bullet^t$ contains an ample series by Lemma 1.6; thus, [LM09, Theorem 3.5] implies that

$$\lim_{p\to\infty} \operatorname{vol}(S^\bullet V_{\leqslant p}^t) = \operatorname{vol}(V_\bullet^t)$$

for each $t < e_{\max}(V_\bullet, \mathcal{F})$. Since we also have $\operatorname{vol}(S^\bullet V_{\leqslant p}^t) \leqslant \operatorname{vol}(V_\bullet^t)$, it follows from Corollary 1.13 and dominated convergence that

$$\lim_{p\to\infty} \operatorname{vol}\widehat{\Delta}(S^\bullet V_{\leqslant p}, \mathcal{F}) = \operatorname{vol}\widehat{\Delta}(V_\bullet, \mathcal{F})$$

and the result follows. □

This result gives in particular a simple proof of [Che10a, Theorem 5.1].

2. Applications to the study of arithmetic volumes

In this section, K will stand for a number field, $\mathcal{O}_K$ for its ring of integers and $\mathbb{A}_K$ for its ring of adèles. We denote by K^{a} an algebraic closure of K. The set of finite (respectively infinite) places of K will be denoted by Σ_f (respectively Σ_∞), and we write $\Sigma := \Sigma_f \cup \Sigma_\infty$. For each finite place v, we denote by $\mathcal{O}_{K_v}$ the discrete valuation ring of the completion K_v of K. For each $v \in \Sigma$, let $|\cdot|_v$ be the associated absolute value on K, normalized so that the product formula reads

$$\sum_{v\in\Sigma} [K_v : \mathbb{Q}_v] \log |x|_v = 0$$

for all $x \in K^*$. Denote by $\mathbb{C}_v$ the completion of an algebraic closure of K_v. Given a finite set S, we will write

$$\widehat{\dim}_K S = \frac{1}{[K:\mathbb{Q}]} \log \# S.$$

As an (admittedly loose) justification for this notation, we note that

$$\widehat{\dim}_K S = (\dim_K F)\,\widehat{\dim}_F S$$

for any finite extension F of K.

2.1 Pure v-adic norms and lattices

Let v be a finite place of K and V be a finite-dimensional K_v-vector space. Any (ultra-metric) v-adic norm $\|\cdot\|$ on V admits an *orthogonal basis*, i.e. a basis $(e_i)_{i=1}^N$ of V such that

$$\|a_1 e_1 + \cdots + a_N e_N\| = \max_i |a_i|_v \|e_i\|$$

for all $(a_1, \ldots, a_N) \in K^N$, and it is clear that $\|\cdot\|$ admits an *orthonormal basis*, i.e. an orthogonal basis (e_i) such that $\|e_i\| = 1$, if and only if the values of $\|\cdot\|$ on V are equal to those $|\cdot|_v$ on K_v. Such a norm is said to be *pure* in the terminology of [Gau09, Définition 2.2].

Pure norms admit the following alternative characterization in terms of lattices. Recall that an $\mathcal{O}_{K_v}$-submodule $\mathcal{V}$ of V is said to be a *K_v-lattice* of V if there exists a K_v-basis $e_1, \dots, e_N$ of V such that

$$\mathcal{V} = \mathcal{O}_{K_v} e_1 + \cdots + \mathcal{O}_{K_v} e_N. \tag{2.1}$$

By [BG06, Proposition C.2.2], $\mathcal{V}$ is a K_v-lattice if and only if it is finitely generated and spans V over K_v, and also if and only if $\mathcal{V}$ is compact and open in V. A K_v-lattice $\mathcal{V}$ of V determines an ultra-metric norm $\|\cdot\|_{\mathcal{V}}$ on V by setting

$$\|s\|_{\mathcal{V}} := \inf\{|a|_v, a \in K_v, a^{-1}s \in \mathcal{V}\},$$

and $\mathcal{V}$ is then uniquely determined as the unit ball of $\|\cdot\|_{\mathcal{V}}$. We claim that all pure norms are of the form $\|\cdot\|_{\mathcal{V}}$ for some K_v-lattice $\mathcal{V}$. Indeed, the unit ball $\mathcal{V}$ of any v-adic norm $\|\cdot\|$ is open and compact and hence a K_v-lattice, and we have

$$\|s\|_{\mathcal{V}} = \min\{t \in |K_v|, t \geqslant \|s\|\}$$

for each $s \in V$, since $|\cdot|_v$ is discrete. If follows that $\|\cdot\| \leqslant \|\cdot\|_{\mathcal{V}}$, with equality if and only if $\|\cdot\|$ is pure. We set $\|\cdot\|_{\mathrm{pur}} := \|\cdot\|_{\mathcal{V}}$ and call it the *purification* of $\|\cdot\|$.

2.2 Adelically normed vector spaces and their filtration by minima

Definition 2.1. An *adelically normed* K-vector space $\overline{V}$ is a finite-dimensional K-vector space V endowed for each place v with a norm $\|\cdot\|_v$ on $V_v := V \otimes_K K_v$ (non-archimedean when v is finite), such that the following *local finiteness* condition holds:

$$\text{for each } x \in V \text{ we have } \|x\|_v \leqslant 1 \text{ for all but finitely many places } v. \tag{2.2}$$

The local finiteness condition in Definition 2.1 corresponds to [RLV00, p. 3, Assumption (A2)] (except that we do not require the v-adic norm $\|\cdot\|_v$ to be defined on $V \otimes_K \mathbb{C}_v$ and invariant under the continuous Galois group of $\mathbb{C}_v/K_v$).

The associated $\mathcal{O}_K$-module of $\overline{V}$ is defined as

$$\mathcal{V} := \{x \in V, \|x\|_v \leqslant 1 \text{ for all } v \in \Sigma_f\}$$

and the set of *small elements* is defined by

$$\widehat{V} := \{x \in V, \|x\|_v \leqslant 1 \text{ for all } v \in \Sigma\}.$$

The local finiteness condition (2.2) is equivalent to the fact that the associated $\mathcal{O}_K$-submodule $\mathcal{V}$ spans V over K, and implies that for each finite place v the closure $\mathcal{V}_v = \mathcal{O}_{K_v}\mathcal{V}$ of $\mathcal{V}$ in V_v spans V_v over K_v. Note that $\mathcal{V}_v$ coincides with the unit ball

$$\{x \in V_v, \|x\|_v \leqslant 1\},$$

and is thus a K_v-lattice of V_v.

Definition 2.2. An adelically normed K-vector space $\overline{V}$ is said to be *generically trivial* if there exists a K-basis $e_1, \dots, e_N$ of V which is $\|\cdot\|_v$-orthonormal for all but finitely many $v \in \Sigma_f$.

A generically trivial adelically normed K-vector space $\overline{V}$ is the same thing as a normed K-vector space in the sense of [Zha95b, § 1.6] (or rather its immediate extension from the case $K = \mathbb{Q}$). It is also coincides with the notion of an *adelic vector bundle over* Spec K in the sense of [Gau08, Définition 3.1] (except, here again, that the norms considered by Gaudron are defined as Galois-invariant norms on $V \otimes_K \mathbb{C}_v$).

The *purification* $\overline{V}_{\rm pur}$ of $\overline{V}$ is defined by setting $\|\cdot\|_{{\rm pur},v} = \|\cdot\|_v$ when v is infinite and letting $\|\cdot\|_{{\rm pur},v} \geqslant \|\cdot\|_v$ be the purification of $\|\cdot\|_v$ when v is finite. Note that $\overline{V}_{\rm pur}$ and $\overline{V}$ share the same associated $\mathcal{O}_K$-module $\mathcal{V}$ and set of small elements $\widehat{V}$.

Definition 2.3. Let $\overline{V}$ be an adelically normed K-vector space. The *filtration by minima* is the (left-continuous) decreasing $\mathbb{R}$-filtration $\mathcal{F}_{\rm min}$ of V defined by

$$\mathcal{F}^t_{\rm min}V := {\rm Vect}_K\Big\{x \in \mathcal{V},\ \max_{v\in\Sigma_\infty} \|x\|_v \leqslant e^{-t}\Big\}.$$

Note that the filtrations by minima induced by $\overline{V}$ and its purification $\overline{V}_{\rm pur}$ coincide. The filtration by minima is trivially bounded below, but not necessarily bounded above, as we shall now see.

Recall that an $\mathcal{O}_K$-submodule $\mathcal{V}$ of V is said to be a *K-lattice* if $\mathcal{V}$ is finitely generated and spans V over K.

On the other hand, a *normed vector bundle over* Spec $\mathcal{O}_K$ is given by a finitely generated projective (or equivalently torsion-free) $\mathcal{O}_K$-module $\mathcal{V}$ together with a family of norms $\|\cdot\|_v$, $v \in \Sigma_\infty$ on $V = \mathcal{V} \otimes_{\mathcal{O}_K} K$. It is thus the same data as [BG06, p. 610, C.2.8]. It induces an adelically normed K-vector space structure $\overline{V}$ by letting $\|\cdot\|_v$ be the unique pure v-adic norm on V_v with unit ball $\mathcal{V}_v := \mathcal{O}_{K_v}\mathcal{V}$ for each $v \in \Sigma_f$. Since $\mathcal{V}$ is then recovered as the associated $\mathcal{O}_K$-module of $\overline{V}$, we shall simply say that $\overline{V}$ itself is a normed vector bundle over Spec $\mathcal{O}_K$.

Proposition 2.4. *Let $\overline{V}$ be an adelically normed K-vector space. The following conditions are equivalent.*

(i) *The filtration by minima $\mathcal{F}_{\rm min}$ of $\overline{V}$ is bounded above.*

(ii) *The set of small elements $\widehat{V}$ is finite.*

(iii) *The associated $\mathcal{O}_K$-module $\mathcal{V}$ of $\overline{V}$ is a K-lattice of V.*

(iv) *$\overline{V}_{\rm pur}$ is a normed vector bundle over* Spec $\mathcal{O}_K$.

Proof. Since $\mathcal{V}$ spans V over K by the local finiteness condition, we see that $\mathcal{V}$ is a K-lattice if and only if $\mathcal{V}$ is finitely generated as an $\mathcal{O}_K$-module or, equivalently, as an abelian group. Now (ii) means that the image of $\mathcal{V}$ in $V \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{v\in\Sigma_\infty} V_v$ meets the unit ball of the sup-norm $\max_{v\in\Sigma_\infty} \|\cdot\|_v$ in a finite set. This holds if and only if the image of $\mathcal{V}$ in $V \otimes_{\mathbb{Q}} \mathbb{R}$ is a discrete subgroup, and is thus equivalent to (i). Since each discrete subgroup of $V \otimes_{\mathbb{Q}} \mathbb{R}$ is finitely generated, it then follows that $\mathcal{V}$ is a K-lattice. Conversely, if $\mathcal{V}$ is a K-lattice then its image in $V \otimes_{\mathbb{Q}} \mathbb{R}$ is a (usual) lattice by [BG06, Corollary C.2.7], and is thus discrete in particular. Finally, the equivalence of (iii) and (iv) follows from the fact that $\mathcal{O}_{K_v}\mathcal{V}$ is the unit ball of the v-adic norm of $\overline{V}$. □

2.3 Gillet–Soulé's theorem

Let $\overline{V}$ be an adelically normed K-vector space of dimension N. If its filtration by minima $\mathcal{F}_{\rm min}$ is bounded, then we may consider its jumping numbers of $\mathcal{F}_{\rm min}$

$$e_1(V, \mathcal{F}_{\rm min}) \geqslant \cdots \geqslant e_N(V, \mathcal{F}_{\rm min}).$$

By definition, these jumping numbers are obtained by applying $-\log$ to the *successive minima* in the sense of [BG06, Definition C.2.9].

The key point for us is now [GS91, Proposition 6], which asserts that the set of small elements $\widehat{V}$ relates to the positive mass

$$\mathrm{mass}_+(V, \mathcal{F}_{\min}) = \sum_{e_j > 0} e_j(V, \mathcal{F}_{\min})$$

of the filtration by minima by

$$\widehat{\dim}_K \widehat{V} = \mathrm{mass}_+(V, \mathcal{F}_{\min}) + O(N \log N), \tag{2.3}$$

where the constant in the O only depends on K. Indeed, both sides of (2.3) only depend on $\overline{V}_{\mathrm{pur}}$, which is a normed vector bundle over $\operatorname{Spec} \mathcal{O}_K$ by Proposition 2.4. The result can then be deduced from [GS91, Proposition 6] just as the adelic version of Minkowski's second theorem is deduced from its classical version in [BG06, p. 614, Appendix C.2.18].

2.4 Adelically normed graded linear series

Definition 2.5. Let L be a line bundle over a projective K-variety X.

(i) An *adelically normed graded linear series* $\overline{V}_\bullet$ of L is a graded linear series $V_\bullet \subset R(L)$ such that each graded piece V_k is adelically normed with norms $\|\cdot\|_{v,k}$ satisfying

$$\|st\|_{v,k+m} \leqslant \|s\|_{v,k}\|t\|_{v,m}$$

for each $s \in V_k$, $t \in V_m$ and each $v \in \Sigma$.

(ii) The *arithmetic volume* of $\overline{V}_\bullet$ is then defined by

$$\widehat{\mathrm{vol}}(\overline{V}_\bullet) := \limsup_{k\to\infty} \frac{(n+1)!}{k^{n+1}} \widehat{\dim}_K \overline{V}_k$$

with $n := \dim X$ (compare [Mor09b]).

Our definition of an adelically normed graded linear series is a natural extension of conditions (A1) and (A2) in [RLV00, p. 3].

The filtrations by minima of the graded pieces $\overline{V}_k$ induce a *multiplicative filtration* $\mathcal{F}_{\min}$ on $V_\bullet$, the *filtration by minima of* $\overline{V}_\bullet$. This filtration is always pointwise bounded below, but not necessarily pointwise bounded above in general (see Proposition 2.4).

In concrete terms, the filtration by minima of $\overline{V}_\bullet$ is linearly bounded below if and only if the associated $\mathcal{O}_K$-module $\mathcal{V}_k$ contains a finite set of generators of V_k whose norms grow at most exponentially fast with k, whereas it is linearly bounded above if and only if the minimal norm of a non-zero vector in $\mathcal{V}_k$ decays at most exponentially fast.

The main example of adelically normed graded linear series arises in the usual Arakelov-geometric setting. Let $\mathcal{L} \to \mathcal{X}$ be a projective flat model of $L \to X$ over $\mathcal{O}_K$ and assume that the line bundle $L_{\mathbb{C}}$ over

$$X(\mathbb{C}) = \coprod_{\sigma: K \hookrightarrow \mathbb{C}} X_\sigma(\mathbb{C})$$

is endowed with a conjugation-invariant continuous Hermitian metric, the whole data being summarized by $\overline{L}$. Then for each k we get a structure of a normed vector bundle over $\operatorname{Spec} \mathcal{O}_K$, denoted by $\overline{H^0(X, kL)}^{\sup}$, by taking $H^0(\mathcal{X}, k\mathcal{L})$ as its associated $\mathcal{O}_K$-module and the sup-norms over $X(\mathbb{C})$ as the norms at infinity. We denote by $\overline{R(L)}^{\sup}$ the corresponding adelically normed graded linear series.

Denote by $h_{\overline{L}}$ the induced (normalized) height function on $X(K^{\mathrm{a}})$, and recall that the *essential minimum* of the height function is defined by

$$\text{ess-min}\, h_{\overline{L}} := \sup\left\{\inf_{U(K^{\mathrm{a}})} h_{\overline{L}}, U \subset X \text{ non-empty Zariski open}\right\}.$$

Proposition 2.6. *The filtration by minima $\mathcal{F}_{\min}$ of $\overline{R(L)}^{\sup}$ (and hence of any graded subseries) induced by the above Arakelov-geometric data is linearly bounded above. In fact, we have*

$$e_{\max}(\overline{R(L)}^{\sup}, \mathcal{F}_{\min}) \leqslant \text{ess-min}\, h_{\overline{L}} < +\infty, \tag{2.4}$$

and moreover ess-min $h_{\overline{L}} > -\infty$ *as soon as* $H^0(kL) \neq 0$ *for some* k.

These facts are well known and follow in particular from the deep results of [Zha95a]. We however take the opportunity to present here an elementary proof, which has kindly been suggested to us by Antoine Chambert-Loir.

Proof. Given a non-zero section $\sigma \in H^0(\mathcal{X}, k\mathcal{L})$ and $x \in X(K^{\mathrm{a}})$ such that $\sigma(x) \neq 0$, the height $h_{\overline{L}}(x)$ is no less than the mean value of the function $-1/k \log |\sigma|$ along the Galois orbit of x, where $|\sigma|$ denotes the length of σ with respect to the given metric on $L_{\mathbb{C}}$. The first inequality of (2.4) follows immediately.

Let us now prove that ess-min $h_{\overline{L}} < +\infty$. Changing the Arakelov data only affects $h_{\overline{L}}$ by a bounded term. Its class modulo $O(1)$, denoted by h_L, fits into Weil's height machine. The assertion that ess-min $h_L < +\infty$ means that there exists $C > 0$ such that

$$\{x \in X(K^{\mathrm{a}}), h_L(x) \leqslant C\}$$

is Zariski dense in X. We may find two very ample line bundles A, B such that $L = A - B$; since h_B is bounded below, we may thus assume that $L = A$ is itself very ample. But in that case there exists a finite morphism $\pi : X \to \mathbb{P}^n$ such that $\pi^*\mathcal{O}(1) = L$ and hence $h_L = h_{\mathcal{O}(1)} \circ \pi + O(1)$ by [HS00, Theorem B.3.6], and we get a Zariski dense subset of $X(K^{\mathrm{a}})$ with bounded height by considering for instance the inverse image by π of the set of points in $\mathbb{P}^n(K^{\mathrm{a}})$ with roots of unity as homogeneous coordinates.

Finally, the assertion that ess-min $h_L > -\infty$ means that there exists a non-empty Zariski open subset U of X such that h_L is bounded below on $U(K^{\mathrm{a}})$, which is true with U the complement of the base locus of kL by [HS00, Theorem B.3.6]. □

Definition 2.7. Let L be a big line bundle on X/K and let $\overline{V}_\bullet$ be an adelically normed graded linear series of L such that:

(i) $V_\bullet$ contains an ample series;
(ii) the induced filtration by minima on $V_\bullet$ is linearly bounded above.

Given the choice of a uniformizing system of parameters $z = (z_1, \ldots, z_n)$ at a regular point of $X(K^{\mathrm{a}})$, we define the *arithmetic Okoukov body* of $\overline{V}_\bullet$ (with respect to z) as

$$\widehat{\Delta}(\overline{V}_\bullet) := \{(x, t) \in \Delta(V_\bullet) \times \mathbb{R}, 0 \leqslant t \leqslant G_{\mathcal{F}_{\min}}(x)\},$$

where $G_{\mathcal{F}_{\min}}$ is the concave transform of the filtration by minima $\mathcal{F}_{\min}$ given by Definition 1.8.

Note that $\widehat{\Delta}(\overline{V}_\bullet)$ is a compact convex subset of $\mathbb{R}^{n+1}$ (but possibly of empty interior), since

$$G_{\mathcal{F}_{\min}} : \Delta(V_\bullet) \to [-\infty, +\infty[$$

is upper semicontinuous. As a consequence of Theorem 1.11, we shall prove the following.

Theorem 2.8. *Let $\overline{V}_\bullet$ be an adelically normed graded linear series which contains an ample series and whose filtration by minima is linearly bounded above. Then we have*

$$\operatorname{vol}\widehat{\Delta}(\overline{V}_\bullet) = \lim_{k\to\infty}\frac{\widehat{\dim}_K \widehat{V}_k}{k^{n+1}} = (n+1)!\,\widehat{\operatorname{vol}}(\overline{V}_\bullet).$$

Proof. By Corollary 1.13, we have

$$\operatorname{vol}\widehat{\Delta}(\overline{V}_\bullet) = \lim_{k\to\infty}\frac{\operatorname{mass}_+(V_k,\mathcal{F}_{\min})}{k^{n+1}},$$

where

$$\operatorname{mass}_+(V_k,\mathcal{F}) = \widehat{\dim}_K \widehat{V}_k + O(N_k\log N_k)$$

by Gillet–Soulé's result (2.3). Since $N_k = \dim V_k = O(k^n)$, we have

$$N_k\log N_k = O(k^n\log k) = o(k^{n+1})$$

and the result follows. □

2.5 Fujita approximation and log-concavity

Let us first spell out Theorem 1.14 in the adelic case.

Theorem 2.9. *Let $\overline{V}_\bullet$ be an adelically normed graded linear series which contains an ample series and whose filtration by minima is linearly bounded above. Then for every $\varepsilon > 0$ there exists a finitely generated subseries $W_\bullet$ of $V_\bullet$ whose arithmetic volume satisfies*

$$\widehat{\operatorname{vol}}(\overline{W}_\bullet) \geqslant \widehat{\operatorname{vol}}(\overline{V}_\bullet) - \varepsilon.$$

Indeed, this follows directly from Theorem 1.14 in view of Theorem 2.8.

We next get as usual the log-concavity of arithmetic volumes.

Proposition 2.10. *Let L and M be two big line bundles on X/K. Assume that $U_\bullet$, $V_\bullet$ and $W_\bullet$ are adelically normed graded linear series of L, M and $L+M$ respectively such that each of them contains an ample series and has a linearly bounded above filtration by minima. Assume furthermore that:*

(i) *$U_k\cdot V_k\subset W_k$ for each k;*
(ii) *for any $v\in\Sigma$ and all $s\in U_k$, $s'\in V_k$, one has $\|s\cdot s'\|_v\leqslant\|s\|_v\|s'\|_v$.*

Then one has

$$\widehat{\operatorname{vol}}(\overline{W}_\bullet)^{1/(n+1)} \geqslant \widehat{\operatorname{vol}}(\overline{U}_\bullet)^{1/(n+1)} + \widehat{\operatorname{vol}}(\overline{V}_\bullet)^{1/(n+1)}. \tag{2.5}$$

Proof. The assumptions easily imply that

$$\mathcal{F}^s_{\min}U_k\cdot\mathcal{F}^t_{\min}V_k\subset\mathcal{F}^{s+t}_{\min}W_k$$

for all k and all $s,t\in\mathbb{R}$, which yields in turn

$$\widehat{\Delta}(\overline{U}_\bullet)+\widehat{\Delta}(\overline{V}_\bullet)\subset\widehat{\Delta}(\overline{W}_\bullet)$$

and the result follows by the Brunn–Minkowski inequality. □

3. Application to the sectional capacity and the Arakelov degree

3.1 Existence of the sectional capacity

Given an adelically normed N-dimensional K-vector space $\overline{V}$, we consider as in [Gau08, RLV00, Yua09a, Zha95a] its (normalized) *adelic Euler characteristic*

$$\chi(\overline{V}) := \frac{1}{[K:\mathbb{Q}]} \log \operatorname{vol} \mathbb{B}(\overline{V}) \in \,]-\infty, +\infty], \tag{3.1}$$

where $\mathbb{B}(\overline{V}) \subset V \otimes_K \mathbb{A}_K$ denotes the adelic unit ball induced by the family of norms $\|\cdot\|_v$ of $\overline{V}$ and vol is the Haar measure of $V \otimes_K \mathbb{A}_K$ normalized by

$$\operatorname{vol}(V \otimes_K \mathbb{A}_K / V) = 1$$

(compare this normalization to [BG06, Proposition C.1.10]). Since $\|\cdot\|_v$ and $\|\cdot\|_{\mathrm{pur},v}$ share the same unit ball $\mathcal{V}_v$ in V_v for all places v, we see that $\mathbb{B}(\overline{V}) = \mathbb{B}(\overline{V}_{\mathrm{pur}})$.

Assume now that the filtration by minima $\mathcal{F}_{\min}$ of $\overline{V}$ is bounded. According to Proposition 2.4, $\overline{V}_{\mathrm{pur}}$ is then a normed vector bundle over $\operatorname{Spec} \mathcal{O}_K$, and it follows in particular that $\chi(\overline{V}) = \chi(\overline{V}_{\mathrm{pur}})$ is finite. Moreover, the adelic version of Minkowski's second theorem applies to $\overline{V}_{\mathrm{pur}}$ (cf. [BG06, Theorem C.2.11]) and yields

$$\chi(\overline{V}) = \operatorname{mass}(V, \mathcal{F}_{\min}) + O(N \log N), \tag{3.2}$$

where the constant in O only depends on K. As a consequence, we will show similarly to Theorem 2.8 the following.

Theorem 3.1. *Let $\overline{V}_\bullet$ be an adelically normed graded linear series of L that contains an ample series and whose filtration by minima $\mathcal{F}_{\min}$ is linearly bounded above. If the filtration by minima is furthermore linearly bounded below (e.g. if $V_\bullet$ is finitely generated), then the concave transform $G_{\mathcal{F}_{\min}}$ satisfies*

$$\int_{\Delta(V_\bullet)} G_{\mathcal{F}_{\min}} \, d\lambda = \lim_{k\to\infty} \frac{\chi(\overline{V}_k)}{k^{n+1}}.$$

In particular, the right-hand limit exists in $\mathbb{R}$.

If L is ample and $V_k = H^0(kL)$ (so that $V_\bullet$ is indeed finitely generated), then it was shown in [RLV00] that $\exp(-(n+1)!/k^{n+1}\chi(\overline{V}_k))$ admits a limit in $[0, +\infty[$, called the *sectional capacity*. This result was proved without assuming that $\mathcal{F}_{\min}$ is linearly bounded above (but recall that this condition always holds in the Arakelov-geometric setting). On the other hand, the proof of Theorem 3.1 we now present is substantially shorter than in [RLV00].

Proof. Set as in Theorem 1.11

$$\mu_k := k^{-n} \sum_j \delta_{k^{-1} e_j(V_k, \mathcal{F}_{\min})}.$$

By Minkowski's second theorem (3.2), we have

$$\frac{\chi(\overline{V}_k)}{k^{n+1}} = \int_{\mathbb{R}} t \, \mu_k(dt) + o(1)$$

as $k \to \infty$. On the other hand, since $\mathcal{F}_{\min}$ is assumed to be linearly bounded above and below, the support of μ_k stays in a fixed compact set of $\mathbb{R}$ independently of k and Theorem 1.11 therefore yields

$$\lim_{k\to\infty}\int_{\mathbb{R}} t\mu_k(dt) = \int_{\Delta(V_\bullet)} G_{\mathcal{F}_{\min}}\, d\lambda,$$

which proves the result. □

3.2 Arakelov degree

The aim of this section is to discuss a *tentative* definition of Arakelov degree for adelically normed vector spaces and to introduce a *filtration by height*, which differs from the filtration by minima and is better suited to the study of the asymptotic behavior of Arakelov degrees in the non-pure case.

All adelically normed vector spaces $\overline{V}$ considered in this section will be assumed to be *generically trivial*, and we shall moreover assume that each norm $\|\cdot\|_v$, $v\in\Sigma$, is defined on $V\otimes_K\mathbb{C}_v$ and invariant under the continuous Galois group $\mathrm{Gal}(\mathbb{C}_v/K_v)$, in line with [Gau08, RLV00].

As in [Gau08, Définition 4.11], we introduce the following definition.

Definition 3.2. Let $\overline{V}$ be a (generically trivial) adelically normed K-vector space. The (normalized) projective height of $s\in V_{K^{\mathrm{a}}}-\{0\}$ is defined by

$$h_{\overline{V}}(s) := \sum_{v\in\Sigma_F}\frac{[F_v:\mathbb{Q}_v]}{[F:\mathbb{Q}]}\log\|s\|_v,$$

where F is a field of definition of s.

The height of s only depends on $[s]\in\mathbb{P}(V_{K^{\mathrm{a}}})$ by the usual arguments relying on the product formula. When V itself is a line, we simply write $h(\overline{V}) := h_{\overline{V}}(s)$ for any non-zero $s\in V$. When $\overline{V}$ is *pure*, [Gau08, Lemme 4.4] yields

$$h_{\overline{V}}(s) = \chi(\overline{F}) - \chi(\overline{Fs})$$

for every field of definition F of s, where $\overline{F}$ is adelically normed in the trivial way and $\overline{Fs}$ is of course endowed with the adelic structure induced by that of $\overline{V}$. This relation is no longer true in general when $\overline{V}$ is not pure. What remains true in general is the relation

$$h_{\overline{V}}(s) = \inf_F h_{\overline{V}_{F,\mathrm{pur}}}(s) = \inf_F(\chi(\overline{F}) - \chi(\overline{Fs})), \tag{3.3}$$

where F runs over all fields of definition of s. Note that $h_{\overline{V}_{F,\mathrm{pur}}}(s)$ decreases when F increases, so that the infimum in (3.3) is actually a limit.

The *normalized Arakelov degree* $\widehat{\deg}(\overline{V})$ of a Hermitian vector bundle $\overline{V}$ over $\operatorname{Spec}\mathcal{O}_K$ (i.e. such that each norm $\|\cdot\|_v$ at infinity is induced by a Galois-invariant Hermitian scalar product on $V\otimes_K\mathbb{C}_v$), as defined in [Bos96, § A.2], satisfies

$$\widehat{\deg}(\overline{V}) = -h(\det\overline{V}) = \chi(\overline{V}) - \chi(\overline{K^N}),$$

where the determinant line $\det V := \bigwedge^{\dim V} V$ is endowed with the induced adelic norm (cf. [Cha02, (5.5.2)] and [Gau08, Lemme 4.5]).

In [Gau08, Définition 4.1], Gaudron used the right-hand side of this formula as the *definition* of $\widehat{\deg}(\overline{V})$ when $\overline{V}$ is pure but not necessarily Hermitian (i.e. for a normed vector bundle over $\operatorname{Spec}\mathcal{O}_K$ by Proposition 2.4). In the general non-pure case, we arrive at the following tentative definition in view of (3.3).

DEFINITION 3.3. Let $\overline{V}$ be a (generically trivial) adelically normed K-vector space V. We define its *Arakelov degree* as

$$\widehat{\deg}(\overline{V}) := \sup_F(\chi(\overline{V_F}) - \chi(\overline{F^N})) = \sup_F \widehat{\deg}(\overline{V}_{F,\mathrm{pur}}) \in\,]-\infty, +\infty],$$

where F runs over all finite extensions of K.

A word of warning is in order. If $\overline{V}$ is *Hermitian*, then $F \mapsto \widehat{\deg}(\overline{V}_{F,\mathrm{pur}})$ is easily seen to be non-decreasing, but this property becomes unclear (and even doubtful) in the non-Hermitian case (as pointed out to us by Gaël Rémond). As a consequence, Definition 3.3 is probably the 'right one' when $\overline{V}$ is Hermitian, but less clearly so in general.

When V is one dimensional, we have at any rate

$$\widehat{\deg}(\overline{V}) = -h(\overline{V}) \tag{3.4}$$

by (3.3).

For each fixed finite extension F, a computation shows that

$$\chi(\overline{F^N}) = O(N \log N). \tag{3.5}$$

In the pure case, we thus have

$$\widehat{\deg}(\overline{V}) = \widehat{\deg}(\overline{V}_{\mathrm{pur}}) = \chi(\overline{V}) + O(N \log N),$$

so that Theorem 3.1 applies to describe the asymptotic behavior of the Arakelov degree of a purely adelically normed graded linear series. In the general case, this simple connection with the Euler characteristic is lost since (3.5) is not uniform with respect to F.

We introduce a new filtration that will be used to describe the asymptotic behavior of the Arakelov degree in the general non-pure case.

DEFINITION 3.4. Let $\overline{V}$ be a (generically trivial) adelically normed K-vector space. The *filtration by height* of $V_{K^{\mathrm{a}}}$ is defined by

$$\mathcal{F}^t_{\mathrm{he}} V_{K^{\mathrm{a}}} := \mathrm{Vect}_{K^{\mathrm{a}}}\{s \in V_{K^{\mathrm{a}}} - \{0\}, h_{\overline{V}}(s) \leqslant -t\}.$$

The filtration by height is trivially bounded below and left continuous. It is bounded above if and only if $h_{\overline{V}}$ is bounded below on $\mathbb{P}(V_{K^{\mathrm{a}}})$. This is for instance the case when $\overline{V}$ is pure (compare the proof of [Gau08, Proposition 4.12]).

We shall now deduce from [Gau08] the following Siegel lemma.

PROPOSITION 3.5. *Let $\overline{V}$ be a (generically trivial) adelically normed K-vector space.*

(i) *The Arakelov degree of $\overline{V}$ satisfies*

$$\widehat{\deg}(\overline{V}) = -\inf_{(s_j)}\left(\sum_{j=1}^N h_{\overline{V}}(s_j)\right) + O(N \log N),$$

where (s_j) runs over all K^{a}-bases of $V_{K^{\mathrm{a}}}$ and the constant in the O is a universal numerical constant.

(ii) *If the filtration by height of $\overline{V}$ is bounded (i.e. if $h_{\overline{V}}$ is bounded below on $\mathbb{P}(V_{K^{\mathrm{a}}})$), then we have*

$$\mathrm{mass}(V_{K^{\mathrm{a}}}, \mathcal{F}_{\mathrm{he}}) = -\inf_{(s_j)}\left(\sum_{j=1}^N h_{\overline{V}}(s_j)\right),$$

where (s_j) runs as above over all K^{a}-bases of $V_{K^{\mathrm{a}}}$. In particular, the Arakelov degree $\widehat{\deg}(\overline{V})$ is finite in that case.

Proof. Let $s_1, \ldots, s_N$ be a given K^{a}-basis of $V_{K^{\mathrm{a}}}$. Let F be a field of definition of the vectors s_j. Since (s_j) is an F-basis of V_F, applying [Gau08, Proposition 4.13] to the *pure* adelically normed F-vector space $\overline{V}_{F,\mathrm{pur}}$ yields

$$\widehat{\deg}(\overline{V}) \geqslant \widehat{\deg}(\overline{V}_{F,\mathrm{pur}}) \geqslant -\sum_{j=1}^{N} h_{\overline{V}_{F,\mathrm{pur}}}(s_j) - \frac{N}{[F:\mathbb{Q}]} \log \Delta(\overline{V}_{F,\mathrm{pur}}).$$

But, [Gau08, (27), p. 21] shows that $\log \Delta(\overline{V}_{F,\mathrm{pur}}) \leqslant [F:\mathbb{Q}]/2 \log(2N)$, so we get

$$\widehat{\deg}(\overline{V}) \geqslant -\sum_{j=1}^{N} h_{\overline{V}_{F,\mathrm{pur}}}(s_j) - CN \log N,$$

where C is a universal numerical constant. Since the infimum in (3.3) is a limit, we infer that

$$\widehat{\deg}(\overline{V}) \geqslant -\sum_{j=1}^{N} h_{\overline{V}}(s_j) - CN \log N.$$

We have thus established that

$$\widehat{\deg}(\overline{V}) \geqslant -\inf_{(s_j)} \left(\sum_{j=1}^{N} h_{\overline{V}}(s_j) \right) - CN \log N. \tag{3.6}$$

On the other hand, let F be a given finite extension of K. The absolute Siegel lemma [Gau08, Theorem 4.14] applied to the pure adelically normed F-vector space $\overline{V}_{F,\mathrm{pur}}$ shows that

$$\widehat{\deg}(\overline{V}_{F,\mathrm{pur}}) \leqslant -\inf_{(s_j)} \left(\sum_{j=1}^{N} h_{\overline{V}}(s_j) \right) + \frac{1}{2} N \log N + \frac{N}{[F:\mathbb{Q}]} \log \mathrm{vr}(\overline{V}_{F,\mathrm{pur}}).$$

By [Gau08, Définition 4.7], we have

$$\log \mathrm{vr}(\overline{V}_{F,\mathrm{pur}}) = \sum_{v|\infty} [F_v : \mathbb{Q}_v] \log \mathrm{vr}(V_v),$$

where the sum is over all infinite places of F. Since $\log \mathrm{vr}(V_v) \leqslant \frac{1}{2} \log N$ for each infinite place v by [Gau08, p. 8], it thus follows that

$$\frac{N}{[F:\mathbb{Q}]} \log \mathrm{vr}(\overline{V}_F) \leqslant \frac{1}{2} N \log N$$

and we get

$$\widehat{\deg}(\overline{V}_{F,\mathrm{pur}}) \leqslant -\inf_{(s_j)} \left(\sum_{j=1}^{N} h_{\overline{V}}(s_j) \right) + N \log N,$$

so that

$$\widehat{\deg}(\overline{V}) \leqslant -\inf_{(s_j)} \left(\sum_{j=1}^{N} h_{\overline{V}}(s_j) \right) + N \log N$$

by Definition 3.3. Combining this estimate with (3.6) completes the proof of (i).

The proof of (ii) easily follows from Definition 1.2 and is left to the reader. □

Many results of [Gau08] may similarly be extended to the more general non-pure setting. For instance, [Gau08, Proposition 4.22] immediately implies the following lemma.

Lemma 3.6. *Let $\overline{W} \subset \overline{V}$ be two adelically normed K-vector spaces. Then we have*

$$\widehat{\deg}(\overline{V}) = \widehat{\deg}(\overline{W}) + \widehat{\deg}(\overline{V/W}) + O(N \log N),$$

where the constant in the O is a universal numerical constant.

As a consequence of Proposition 3.5, we have obtained the analogue of (3.2) for the Arakelov degree and the filtration by height

$$\widehat{\deg}(\overline{V}) = \operatorname{mass}(V_{K^{\mathrm{a}}}, \mathcal{F}_{\mathrm{he}}) + O(N \log N) \tag{3.7}$$

as soon as the filtration by height is bounded. We may thus directly adapt the proof of Theorem 3.1 to get the following.

Theorem 3.7. *Let $\overline{V}_\bullet$ be an adelically normed graded linear series of L which contains an ample series and such that each graded piece $\overline{V}_k$ is generically trivial. Assume that the filtration by height of $\overline{V}_\bullet$ is linearly bounded above and below (the latter condition being automatic if $V_\bullet$ is finitely generated). Then its concave transform $G_{\mathcal{F}_{\mathrm{he}}}$ satisfies*

$$\int_{\Delta(V_\bullet)} G_{\mathcal{F}_{\mathrm{he}}}\, d\lambda = \lim_{k\to\infty} \frac{\widehat{\deg}(\overline{V}_k)}{k^{n+1}}.$$

In particular, the right-hand limit exists in $\mathbb{R}$.

Remark 3.8. Let $\overline{V}_\bullet$ be an adelically normed graded linear series of L.

(i) The condition that $\mathcal{F}_{\mathrm{he}}$ is linearly bounded above is similar to the increasing speed condition of [RLV00, Theorem A].

(ii) It is immediate to see that the filtration by minima and the filtration by height of an adelically normed vector space $\overline{V}$ satisfy $\mathcal{F}^t_{\mathrm{min}} \subset \mathcal{F}^t_{\mathrm{he}}$, and thus $G_{\mathcal{F}_{\mathrm{min}}} \leqslant G_{\mathcal{F}_{\mathrm{he}}}$ for an adelically normed graded linear series $\overline{V}_\bullet$.
In the pure case, i.e. when each $\overline{V}_k$ is pure, one can actually prove that $G_{\mathcal{F}_{\mathrm{min}}} = G_{\mathcal{F}_{\mathrm{he}}}$. More precisely, there exists in that case a sequence $C_k = O(\log N_k) = o(k)$ such that

$$\mathcal{F}^t_{\mathrm{he}} V_k \subset (\mathcal{F}^{t-C_k}_{\mathrm{min}} V_k)_{K^{\mathrm{a}}}$$

for all k and all $t \in \mathbb{R}$ (compare with Proposition 3.6 in the arXiv version of [Che10a]). Details will appear elsewhere.

4. Comparison with other results

4.1 Asymptotic measures

We first compare our results with some of the second author's previous results. In [Che10b, Che10a], the second author constructed the *asymptotic measure* of a big line bundle L endowed with Arakelov-geometric data $\overline{L}$ as above. This measure describes the asymptotic distribution of the jumping numbers of the *Harder–Narasimhan filtration* (see [Che10b, § 2.2]). Now a comparison of the latter filtration with the filtration by minima as in [Che10a] shows that the asymptotic measure coincides with the limit measure we obtain in Theorem 1.11, i.e. with the direct image of λ by $G_{\mathcal{F}_{\mathrm{min}}}$. A special case of this general phenomenon appears in the example computed in [Che10b, § 4.1.5].

4.2 Yuan's construction

As mentioned in the introduction, given a big line bundle L endowed with Arakelov-geometric data $\overline{L}$, Yuan constructed in his recent paper [Yua09a] a concave function on the Okounkov body $\Delta(L)$, which we shall denote by $Y_{\overline{L}}$, and whose mean value (under adequate assumptions) computes the sectional capacity. His construction consists in summing up the analogues of Witt Nyström's Chebyshev-type transforms over all places of K. We would like here to give an alternative description of his construction. We will show in the next section that Yuan's function does *not* coincide in general with the concave transform of either the filtration by minima or the filtration by height.

Let L be a big line bundle on X/K and let $\overline{V}_\bullet$ be an adelically normed graded linear series of L. Assume as in § 3.2 that each $\overline{V}_k$ is generically finite.

Given a system of parameters $z=(z_1,\dots,z_n)$ at a given regular point $p\in X(K^{\mathrm{a}})$, the valuation ord_z induces an $\mathbb{N}^n$-indexed filtration of each V_k. Let $\mathcal{G}_{k,\alpha}$, $\alpha\in\mathbb{N}^n$, be the associated graded pieces, i.e.

$$\mathcal{G}_{k,\alpha}:=\{s\in V_{k,K^{\mathrm{a}}},\mathrm{ord}_z(s)\geqslant\alpha\}/\{s\in V_{k,K^{\mathrm{a}}},\mathrm{ord}_z(s)>\alpha\},$$

and endow them with the induced adelically normed vector space structure $\overline{\mathcal{G}}_{k,\alpha}$. Each $\mathcal{G}_{k,\alpha}$ is at most one dimensional over K^{a} by the basic property (1.1) of the valuation ord_z, and we have $\mathcal{G}_{k,\alpha}\neq 0$ if and only if $(\alpha,k)\in\Gamma(V_\bullet)$. Let $y:\Gamma(V_\bullet)\to\mathbb{R}$ be defined by

$$y(k,\alpha):=\widehat{\deg}(\overline{\mathcal{G}}_{k,\alpha}), \tag{4.1}$$

where $\widehat{\deg}$ denotes the Arakelov degree (cf. Definition 3.3). Since $\mathcal{G}_{k,\alpha}$ is one dimensional, we have

$$\widehat{\deg}(\overline{\mathcal{G}}_{k,\alpha})=-h(\overline{\mathcal{G}}_{k,\alpha}) \tag{4.2}$$

for any non-zero $s\in\mathcal{G}_{k,\alpha}$ by (3.4). One infers from this that y is super-additive, whereas Lemma 3.6 yields

$$\widehat{\deg}(\overline{V}_k)=\sum_{\alpha\in\Gamma_k}y(k,\alpha)+O(N_k\log N_k) \tag{4.3}$$

with $\Gamma_k:=\Gamma(V_\bullet)\cap(\{k\}\times\mathbb{N}^n)$.

Now assume that $V_\bullet$ contains an ample series, so that the semigroup $\Gamma(V_\bullet)$ spans $\mathbb{Z}^{n+1}$ as a group. Assume also that there exists $C>0$ such that

$$-Ck\leqslant y(k,\alpha)\leqslant C(k+|\alpha|) \tag{4.4}$$

for all $(k,\alpha)\in\Gamma(V_\bullet)$. By [WN09, § 3], the upper bound in (4.4) enables us to consider the concave envelope

$$Y_{\overline{V}_\bullet}:\Delta(V_\bullet)\to\mathbb{R}$$

of the super-additive function y (compare Remark 1.10 above), which satisfies

$$Y_{\overline{V}_\bullet}(\alpha)=\lim_{k\to\infty}\frac{1}{k}y(k,\alpha_k) \tag{4.5}$$

for any sequence $(k,\alpha_k)\in\Gamma(V_\bullet)$ such that $\alpha_k/k\to\alpha\in\Delta(V_\bullet)^\circ$, and is in particular bounded above and below by (4.4). One then shows exactly as in [WN09, § 8.2] that (4.3) implies that

$$\int_{\Delta(V_\bullet)}Y_{\overline{V}_\bullet}\,d\lambda=\lim_{k\to\infty}\frac{\widehat{\deg}(\overline{V}_k)}{k^{n+1}}. \tag{4.6}$$

Let us now assume that $V_\bullet = R(L)$ is endowed with the adelic norms $\overline{R(L)}^{\sup}$ coming from the Arakelov-geometric setting. This is essentially the setting considered in [Yua09a]. In that case, Yuan's $F[m\overline{L}]$ satisfies

$$F[m\overline{L}](\alpha) = -y(m, \alpha)$$

for any $(m, \alpha) \in \Gamma(L)$, as one easily sees using (4.2). The upper bound in (4.4) is then shown to hold true in [Yua09a, Lemma 2.3], and is the counterpart in this setting to Proposition 2.6. On the other hand, the lower bound in (4.4) is *assumed* to be true in [Yua09a, Theorem 1.2], and (4.6) is then equivalent to Yuan's result, since $c[\overline{L}] = -Y_{\overline{L}}$ while $\widehat{\deg}(\overline{V}_k) = \chi(\overline{V}_k) + O(k^n \log k)$ since we are dealing with pure adelic norms. Yuan conjectures that the lower bound of (4.4) always holds (in the Arakelov-geometric setting), and notes that it is the case when $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$ [Yua09a, p. 18].

4.3 A counterexample

Recall that the concave transforms $G_{\mathcal{F}_{\min}}$ and $G_{\mathcal{F}_{\mathrm{he}}}$ are actually equal when the $\overline{V}_k$ are pure (and generically trivial) (cf. Remark 3.8(ii)). We will now show on the other hand in a very simple example in the Arakelov-geometric setting that they do not coincide with Yuan's function $Y_{\overline{V}_\bullet}$ in general.

Let $K := \mathbb{Q}$, $X := \mathbb{P}^1$ and $L := \mathcal{O}(1)$, endowed with the Arakelov-geometric data $\overline{L}$ given by the standard model $(\mathcal{X}, \mathcal{L})$ of (X, L) over $\mathbb{Z}$ and the Fubiny–Study metric on $L_{\mathbb{C}}$.

Given a point $p \in X(\mathbb{Q})$, we may then consider the valuation ord_p, which does not depend on the specific choice of a coordinate in this one-dimensional situation. The Okounkov body $\Delta = \Delta(L)$ of L is then equal to the unit segment $[0, 1] \subset \mathbb{R}$ for any choice of p (cf. [LM09, Example 1.13]).

Since $\overline{L}$ is arithmetically ample, we have on the one hand $e_{\min}(\overline{L}) \geqslant 0$ and hence $G_{\mathcal{F}_{\min}} \geqslant 0$ by (1.8). On the other hand, pick $m \in \mathbb{Z}$ and choose $p = [m : 1]$ in homogeneous coordinates. Under the standard identification $H^0(\mathcal{X}, k\mathcal{L}) = \mathbb{Z}[X, Y]_k$ with homogeneous polynomials of degree k, we have

$$\{s \in H^0(kL), \mathrm{ord}_p(s) \geqslant \alpha\} = (X - mY)^\alpha \mathbb{Z}[X, Y]_{k-\alpha}$$

and hence the function $y : \Gamma(\overline{L}) \to \mathbb{R}$ defined by (4.1) satisfies

$$\begin{aligned} y(k, k) &= \widehat{\deg}(\overline{\mathbb{Q} \cdot (X - mY)^k}) = -\log \|(X - mY)^k\|_{\sup} \\ &= -\log \sup_{(x,y) \neq (0,0)} \frac{(x - my)^k}{(x^2 + y^2)^{k/2}} = -\frac{k}{2} \log(1 + m^2). \end{aligned}$$

By (4.5), Yuan's function $Y_{\overline{L}} : [0, 1] \to \mathbb{R}$ satisfies

$$Y_{\overline{L}}(1) = -\tfrac{1}{2} \log(1 + m^2).$$

We thus conclude that the functions corresponding to any point $p := [m : 1]$ with $m \neq 0$ satisfy

$$\inf_{\Delta(L)} Y_{\overline{L}} < 0 \leqslant \inf_{\Delta(L)} G_{\mathcal{F}_{\min}},$$

which shows indeed that they cannot be equal.

4.4 The toric case

Let us briefly discuss the toric case. Let X be a smooth toric variety and L be a toric big line bundle with associated polytope Δ. Note that (X, L) is automatically defined over $\mathbb{Z}$.

The toric line bundle $L_{\mathbb{C}}$ is canonically trivialized on the torus $(\mathbb{C}^*)^n \subset X(\mathbb{C})$, and a continuous invariant metric on L therefore defines a function on $(\mathbb{C}^*)^n$ of the form $g(\log|z_1|, \ldots, \log|z_n|)$. The Legendre transform

$$g^*(t) := \sup_{s\in\mathbb{R}^n} (\langle s, t\rangle - g(s))$$

is a continuous convex function on Δ (known as the *symplectic potential* when ϕ is smooth and positively curved). If one uses a toric system of parameters z at a toric point p of X, then the Okounkov body of L coincides with the polytope Δ (cf. [LM09, § 6.1]), and one can then check in this specific case that both our concave transform $G_{\mathcal{F}_{\min}}$ and Yuan's function $Y_{\overline{L}}$ coincide with $-g^*$. Note that a similar construction appears also in the recent work of Burgos Gil *et al.* [BPS09].

We are not going to prove this, but only indicate the key points. The first idea is to replace as in [WN09, § 9.3] the sup-norm by the L^2-norm with respect to a volume form invariant under the compact torus. These norms are not submultiplicative any more, but one can still consider the functions g and y defined by (1.9) and (4.1), whose concave envelope will still compute $G_{\mathcal{F}_{\min}}$ and $Y_{\overline{L}}$ by the usual argument that the distortion between the sup-norm and the L^2-norm has subexponential growth. Now the decomposition

$$H^0(kL) = \bigoplus_{\alpha\in k\Delta} \mathbb{Q}s_\alpha$$

in monomials s_α is both orthogonal with respect to the L^2-scalar product and defined over $\mathbb{Z}$, and this enables us to perform the computation of the functions g and y explicitly. One then concludes exactly as in [WN09, Lemma 9.2].

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