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Algebraicity of formal varieties and positivity of vector bundles

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Abstract We propose a positivity condition for vector bundles on a projective variety and prove an algebraicity criterion for formal schemes. Then we apply the algebraicity criterion to the study of formal principle in algebraic geometry.

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1 Introduction

Let k be a field and X be an integral projective scheme over k. Suppose given an integral closed subscheme Y of X and a formal closed subscheme $\widehat{V}$ of $\widehat{X}_Y$, and assume that the scheme of definition of $\widehat{V}$ is Y. The formal subscheme $\widehat{V}$ is said to be *algebraic* if the dimension of $\widehat{V}$ equals that of the Zariski closure of $\widehat{V}$ in X. The algebraicity of formal schemes plays an important role in Grothendieck's formal existence theorem, and has many applications in arithmetic geometry (when the field k is a number field), such as the algebraicity of formal leaves of foliations [3], rationality of formal germs of functions on algebraic curves [5], Grothendieck-Katz conjecture (see [6]), etc.

An algebraic criterion has been proposed by Bost [3], asserting that the formal subscheme $\widehat{V}$ is algebraic when Y and $\widehat{V}$ are smooth, Y has dimension $\geqslant 1$ and the normal bundle $N_Y\widehat{V}$ is *ample*. The proof relies on a result of Hartshorne [18] concerning the transcendence degree of the function field of a formal scheme. Note that Bost's algebraicity criterion can be compared to a previous result of Gieseker [10] on *formal principle* in algebraic geometry, where Hartshorne's work is also an important element in the proof. Let (W, A) be a pair of projective algebraic varieties over

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Spec k, where A is a closed subvariety of W of dimension $\geqslant 1$. We say that the formal principle holds for the pair (W, A) if for any pair of algebraic varieties (W', A), the existence of an isomorphism between the formal schemes $\widehat{W}_A$ and $\widehat{W}'_A$ (we require that the isomorphism extends the identity map of A) implies that A has a common étale neighbourhood in W and in W'. Gieseker has proved that if A and W are smooth and if the normal bundle $N_A W$ is ample, then the formal principle holds for (W, A).

The analogue of the ampleness in the complex analytic setting is Grauert positivity, namely the zero section of the dual bundle is exceptional. Comparing Gieseker's work to similar results in complex analytic and algebraic geometry, notably those of Commichau and Grauert [8], Hirschowitz [22], and Bădescu and Schneider [2], suggests that the ampleness condition in Bost's criterion may be largely weaken.

In this work, we study the algebraicity of formal schemes in algebraic geometry by using ideas from Arakelov geometry. In the following, we assume that the scheme Y is of dimension $\geqslant 1$ and is locally of complete intersection in $\widehat{V}$. Note that vanishing of the highest cohomology group is a crucial argument in [18], which can also be interpreted as vanishing of global sections of certain vector bundle by Serre duality. Inspired by this observation, we establish the following result.

Theorem A *Assume the following condition $P_3(Y, N)$:*

> *there exist an ample line bundle L on Y and $\lambda > 0$ such that, for all integers n and D with $n/\lambda > D > \lambda$, one has $H^0(Y, S^n(N^\vee) \otimes L^{\otimes D}) = 0$.*

The formal scheme $\widehat{V}$ is algebraic.

The above positivity condition P_3, which may appear technical at first sight, is closely related to the classical method of "auxiliary polynomials" in Diophantine approximation, or its avatar the slope method in Arakelov geometry. We refer the readers to Bost [4, §2.3] for the relation between positivity conditions and his slope method, and to Chen [7, §4.2.4] for an analogue of the condition P_3 in Arakelov geometry and its relation with the arithmetic invariants.

Note that the condition $P_3(Y, N)$ is satisfied when the vector bundle N is ample, or more generally, is ample along a generic curve in Y, and can be easily verified in diverse geometric situations. In positive characteristic case, the condition $P_3(Y, N)$ does not follow directly from the ampleness of N by Serre duality: in general $S^n(N^\vee)$ is not isomorphic to $S^n(N)^\vee$ when n exceeds the characteristic of k. An essential step turns out to be a generic hyperplane section argument (see Proposition 3). This argument permits to reduce the problem to the case of curves, where Hartshorne's method is valid, even in positive characteristic case. Moreover, we show by an example (see Proposition 13) that the condition P_3 is in general much weaker than the amplitude. In the case of $k = \mathbb{C}$, P_3 can also be compared with conditions in a differential geometric flavor. Namely, in the case where N is a line bundle, the condition $P_3(Y, N)$ is satisfied once the Levi form of N equipped with a hermitian structure has at leat one positive eigenvalue at every point in $Y(\mathbb{C})$ (see Proposition 14).

The algebraicity criterion given in Theorem A can be applied to the study of formal principle in algebraic geometry, and leads to the following result.

Theorem B *Let* (W, A) *be a pair of smooth projective varieties where* A *has dimension* $\geqslant 1$. *Assume that the normal bundle* $N_A W$ *verifies the condition* P_3*, then the formal principle holds for the pair* (W, A).

The comparison of the condition P_3 to various positivity conditions shows that Theorem B generalizes the result of Gieseker [10]. Moreover, Theorem B can also be compared to the approach of Bădescu and Schneider [2] (see Remark 4).

The article is organized as follows. In the second section, we establish an abstract algebraicity criterion of formal schemes and prove Theorem A. In the third section, we define and compare several positivity conditions. Finally in the fourth section, we apply the algebraicity criterion to the study of formal principle.

2 An abstract algebraicity criterion

In this section, we give an abstract algebraicity criterion of formal subschemes. This generalizes a result of Bost [4, Lemma 2.4] to higher dimensional case. The idea consists of interpreting the dimension of the formal subscheme by the asymptotic behaviour of the Hilbert function of an ample line bundle on it. We begin by introducing the notation and some assumptions.

1. The expression k denotes a field of arbitrary characteristic.
2. We fix a projective k-scheme $q : X \to \operatorname{Spec} k$ and an integral closed subscheme Y of X, and assume that $\dim(Y) \geqslant 1$. We also fix a closed formal subscheme $\widehat{V}$ of $\widehat{X}_Y$, which admits Y as the scheme of definition.
3. We assume
   (1) the formal scheme $\widehat{V}$ is Zariski dense in X,
   (2) the scheme Y is locally of complete intersection in $\widehat{V}$.
   Note that in the study of algebraicity of $\widehat{V}$, the assumption (1) is not essential. In fact, if Z is the Zariski closure of $\widehat{V}$ in X, then $\widehat{V}$ is also a closed formal subscheme of $\widehat{Z}_Y$. If we replace X by Z, then we reduce our problem to the case where the assumption (1) is fulfilled.
4. For $i \in \mathbb{N}$, denote by V_i the ith infinitesimal neighbourhood of Y in $\widehat{V}$. One has successive closed immersions
$$Y = V_0 \subset V_1 \subset V_2 \subset \cdots$$
   and the locally ringed space $\widehat{V}$ identifies with the inductive limit $\varinjlim V_i$. Denote by $\varphi_i : V_i \to X$ the immersion and $\varphi : \widehat{V} \to X$ the morphism of locally ringed spaces induced by $(\varphi_i)_{i \geqslant 0}$.
5. We say that the formal subscheme $\widehat{V}$ is *algebraic* if $\dim(\widehat{V})$ equals the dimension of the Zariski closure of $\widehat{V}$ in X (which equals $\dim(X)$ by the assumption ((1) above). Remind that $\dim(\widehat{V})$ is defined as the supremum of Krull dimensions of all local rings of $\widehat{V}$.
6. Denote by N the normal bundle of Y in $\widehat{V}$. Note that the rank of N is equal to $\dim(\widehat{V}) - \dim(Y)$.
7. We fix an ample line bundle L on X. For any integer $D \geqslant 1$, denote by $E_D := H^0(X, L^{\otimes D})$.

8. For all integers i and D such that $i \geqslant 0$ and $D \geqslant 1$, let η_D^i be the evaluation map from E_D to $H^0(|Y|, \varphi_i^* L^{\otimes D})$, where $\varphi_i : V_i \to X$ is the closed immersion. Let E_D^i be the kernel of η_D^{i-1} $(i \geqslant 1)$. It is the space of global sections of $L^{\otimes D}$ vanishing at V_i. Define $E_D^0 = E_D$ by convention.
9. Denote by $\mathfrak{I}$ the ideal sheaf of Y in $\widehat{V}$. By definition, the ideal sheaf of V_i in $\widehat{V}$ is $\mathfrak{I}^{i+1}$. The exact sequence

$$0 \longrightarrow \varphi^* L^{\otimes D} \otimes_{\mathcal{O}_{\widehat{V}}} \left(\mathfrak{I}^i / \mathfrak{I}^{i+1}\right) \longrightarrow \varphi_i^* L^{\otimes D} \longrightarrow \varphi_{i-1}^* L^{\otimes D} \longrightarrow 0$$

induces (by identifying $\varphi^* L^{\otimes D} \otimes_{\mathcal{O}_{\widehat{V}}} S^i N^\vee$ with $L|_Y^{\otimes D} \otimes_{\mathcal{O}_Y} S^i N^\vee$) a commutative diagram whose line is exact:

$$\begin{array}{ccccc} 0 \longrightarrow H^0\left(|Y|, L|_Y^{\otimes D} \otimes S^i N^\vee\right) & \longrightarrow & H^0\left(|Y|, \varphi_i^* L^{\otimes D}\right) & \longrightarrow & H^0\left(|Y|, \varphi_{i-1}^* L^{\otimes D}\right) \\ & & \uparrow {\scriptstyle \eta_D^i} & \nearrow {\scriptstyle \eta_D^{i-1}} & \\ & & E_D & & \end{array}$$

This induces an injective homomorphism of vector spaces over k:

$$\gamma_D^i : E_D^i / E_D^{i+1} \longrightarrow H^0\left(|Y|, L|_Y^{\otimes D} \otimes S^i N^\vee\right).$$

Lemma 1 *With the notation 8, there exists a constant $c > 0$ such that, for all integers $i \geqslant 0$ and $D \geqslant 1$, one has $\mathrm{rk}(E_D^i / E_D^{i+1}) \leqslant c(i + D)^{d-1}$ where d is the dimension of $\widehat{V}$.*

Proof Since there is an injective k-linear map (see notation 9)

$$E_D^i / E_D^{i+1} \longrightarrow H^0\left(|Y|, L|_Y^{\otimes D} \otimes S^i N^\vee\right),$$

the rank of E_D^i / E_D^{i+1} is bounded from above by that of

$$H^0\left(|Y|, L|_Y^{\otimes D} \otimes S^i N^\vee\right) \cong H^0\left(\mathbb{P}(N^\vee), \pi^* L|_Y^{\otimes D} \otimes \mathcal{O}_{N^\vee}(i)\right),$$

where $\pi : \mathbb{P}(N^\vee) \to Y$ is the projection. Since the dimension of $\mathbb{P}(N^\vee)$ is $d - 1$, we obtain the result.

Proposition 1 *If*

$$\liminf_{D \to \infty} \frac{\sum_{i \geqslant 0} (i/D) \,\mathrm{rk}\left(E_D^i / E_D^{i+1}\right)}{\sum_{i \geqslant 0} \mathrm{rk}\left(E_D^i / E_D^{i+1}\right)} < +\infty, \tag{1}$$

then the formal scheme $\widehat{V}$ is algebraic.

Proof Let $\lambda > 0$ be an integer. By the previous lemma, there exists a constant $c > 0$ such that $\mathrm{rk}(E_D^i/E_D^{i+1}) \leqslant c(i+D)^{d-1}$ holds for integers $D \geqslant 1$ and $i \geqslant 0$, where d is the dimension of $\widehat{V}$. Hence

$$\begin{aligned}\sum_{i\geqslant 0}\frac{i}{D}\,\mathrm{rk}\left(E_D^i/E_D^{i+1}\right) &\geqslant \sum_{i\geqslant\lambda D}\frac{i}{D}\,\mathrm{rk}\left(E_D^i/E_D^{i+1}\right) \geqslant \lambda\sum_{i\geqslant\lambda D}\mathrm{rk}\left(E_D^i/E_D^{i+1}\right)\\ &= \lambda\left[\sum_{i\geqslant 0}\mathrm{rk}\left(E_D^i/E_D^{i+1}\right) - \sum_{0\leqslant i<\lambda D}\mathrm{rk}\left(E_D^i/E_D^{i+1}\right)\right]\\ &\geqslant \lambda\sum_{i\geqslant 0}\mathrm{rk}\left(E_D^i/E_D^{i+1}\right) - c\lambda\sum_{0\leqslant i<\lambda D}(i+D)^{d-1}\\ &\geqslant \lambda\sum_{i\geqslant 0}\mathrm{rk}\left(E_D^i/E_D^{i+1}\right) - c\lambda^2(\lambda+1)^{d-1}D^d.\end{aligned}$$

We have assumed that $\widehat{V}$ is dense in X, which implies $\bigcap_{i\geqslant 0}E_D^i = 0$. Hence by the asymptotic Riemann–Roch theorem (see for example [25, 1.2.19]),

$$\sum_{i\geqslant 0}\mathrm{rk}\left(E_D^i/E_D^{i+1}\right) = \mathrm{rk}(E_D) \sim \frac{\deg_L(X)}{\dim(X)!}D^{\dim X}\quad (D\to+\infty)\,.$$

Therefore we obtain

$$\liminf_{D\to\infty}\frac{\sum_{i\geqslant 0}(i/D)\,\mathrm{rk}\left(E_D^i/E_D^{i+1}\right)}{\sum_{i\geqslant 0}\mathrm{rk}\left(E_D^i/E_D^{i+1}\right)} \geqslant \lambda - \limsup_{D\to\infty}\frac{\dim(X)!c\lambda^2(\lambda+1)^{d-1}D^d}{\deg_L(X)D^{\dim X}}.$$

Suppose that $\widehat{V}$ is not algebraic, that is $\dim X > d$. Then

$$\limsup_{D\to\infty}\frac{\dim(X)!c\lambda^2(\lambda+1)^{d-1}D^d}{\deg_L(X)D^{\dim X}} = 0.$$

Therefore

$$\liminf_{D\to\infty}\frac{\sum_{i\geqslant 0}(i/D)\,\mathrm{rk}\left(E_D^i/E_D^{i+1}\right)}{\sum_{i\geqslant 0}\mathrm{rk}_K\left(E_D^i/E_D^{i+1}\right)} \geqslant \lambda.$$

Since λ is arbitrary, we obtain

$$\liminf_{D\to\infty}\frac{\sum_{i\geqslant 0}(i/D)\,\mathrm{rk}\left(E_D^i/E_D^{i+1}\right)}{\sum_{i\geqslant 0}\mathrm{rk}\left(E_D^i/E_D^{i+1}\right)} = +\infty.$$

The above result leads to the following criterion of the algebraicity of $\widehat{V}$, which was mentioned in Sect. 1 as Theorem A.

Theorem 1 *Assume that* (Y, N) *satisfies the following condition:*

> *there exists* $\lambda > 0$ *such that, for any integer* $D > \lambda$ *and any integer* $i > \lambda d$*, one has* $H^0(Y, S^i N^\vee \otimes L|_Y^{\otimes D}) = 0$,

then the formal subscheme $\widehat{V}$ *is algebraic.*

Proof Since each subquotient E_D^i/E_D^{i+1} identifies via γ_D^i with a vector subspace of $H^0(Y, L|_Y^{\otimes D} \otimes S^i N^\vee)$ (see notation 9), for each integer $D > \lambda$,

$$\begin{aligned}\sum_{i \geqslant 0} \frac{i}{D} \operatorname{rk}_K \left(E_D^i/E_D^{i+1}\right) &= \sum_{0 \leqslant i \leqslant \lambda D} \frac{i}{D} \operatorname{rk}_K \left(E_D^i/E_D^{i+1}\right) \\ &\leqslant \lambda \sum_{0 \leqslant i \leqslant \lambda D} \operatorname{rk}_K \left(E_D^i/E_D^{i+1}\right) \\ &\leqslant \lambda \sum_{i \geqslant 0} \operatorname{rk}_K \left(E_D^i/E_D^{i+1}\right).\end{aligned}$$

The assertion then follows from Proposition 1.

3 Positivity conditions

The results obtained in the previous section, notably Theorem 1, suggest that a suitable condition on the normal bundle $N_Y\widehat{V}$ implies the algebraicity of $\widehat{V}$. Motivated by this observation, we propose the following condition for vector bundles on projective schemes.

Definition 1 Let Z be a projective scheme of dimension $\geqslant 1$ defined over k and E be a non-zero vector bundle on Z. We say that the pair (Z, E) satisfies the condition P_3 if, for any line bundle L on Z, there exists $\lambda > 0$ such that, for all integers d and n with $d > \lambda$ and $n > \lambda d$, one has $H^0(Z, S^n E^\vee \otimes L^{\otimes d}) = 0$.

With this notation, the result of Theorem 1 can be written as

$$P_3(Y, N_Y\widehat{V}) \Longrightarrow \widehat{V} \textit{ is algebraic}.$$

The main purpose of this section and the next one is to compare the condition P_3 to some classical conditions and show that it is a very weak positivity condition for vector bundles. In order to simplify the presentation, we introduce the following notation.

Definition 2 Let Z be a projective scheme of dimension $\geqslant 1$ over $\operatorname{Spec} k$ and E be a non-zero vector bundle on Z. If E is ample, we say that (Z, E) satisfies the condition P_1.

Recall that a non-zero vector bundle E on Z is said to be *ample* if for any coherent $\mathcal{O}_Z$-module $\mathcal{F}$, there exists an integer $n_0 > 0$ such that, for any integer $n \geqslant n_0$, the sheaf $S^n E \otimes \mathcal{F}$ is generated by its global sections over Z. Any non-zero quotient bundle of an ample vector bundle is still ample.

We shall prove $P_1(Z, E) \Rightarrow P_3(Z, E)$ when Z is an integral projective scheme (see Theorem 2 and the remark afterwards). In the particular case where E is a line bundle, this is a consequence of Serre duality. In fact, one has

$$H^0\left(Z, E^{\vee\otimes n} \otimes L^{\otimes d}\right) = H^m\left(Z, E^{\otimes n} \otimes L^{\vee\otimes d} \otimes \omega_Z\right),$$

where m is the dimension of Z, and ω_Z denotes the dualizing sheaf of Z. Then the relation $P_1(Z, E) \Rightarrow P_3(Z, E)$ results from Serre's vanishing theorem (see for example [25, Theorem 1.2.6]). A similar argument also works for vector bundles in the case where the characteristic of k is zero, where we use [26, Theorem 6.1.10]. When the characteristic p of k is positive, the situation becomes more subtle: the relation $S^n(E^\vee) \cong (S^n E)^\vee$ does not hold in general. Our main contribution here is to introduce a "*generic curve*" argument (Proposition 3), which permits to reduce the problem to the particular case of projective curves, where the implication $P_1 \Rightarrow P_3$ follows from a result of Hartshorne [18, Lemma 6.1]. More precisely, we shall introduce the following auxiliary condition P_2, which is clearly weaker than P_1, and we establish the implication $P_2 \Rightarrow P_3$.

Definition 3 Let Z be an integral projective scheme of dimension $\geqslant 1$ over $\operatorname{Spec} k$ and E be a non-zero vector bundle on Z. We say that (Z, E) satisfies the condition P_2 if there exist an integral projective scheme W of dimension $\geqslant 1$ defined over an extension (possibly transcendental) k' of k, together with a dominant k-morphism $h : W \to Z$ such that h^*E is an ample vector bundle on W.

3.1 Some properties of conditions P_1 and P_2

We begin with a reminder on several elementary properties of ampleness and refer to the works of Hartshorne [17,19] for details.

One has

$$P_1(Z, E) \Longleftrightarrow P_1(\mathbb{P}(E), \mathcal{O}_E(1)), \tag{2}$$

where $\mathcal{O}_E(1)$ is the universal line bundle on $\mathbb{P}(E)$. Moreover, if W is another projective scheme over $\operatorname{Spec} k$ and $f : W \to Z$ is a finite surjective k-morphism, then

$$P_1(Z, E) \Longleftrightarrow P_1(W, f^*E). \tag{3}$$

Lemma 2 *Let $f : W \to Z$ be a quasi-compact morphism of schemes, Z' be an affine scheme, $g : Z' \to Z$ be a morphism of schemes and E be a vector bundle on W. Let $W' = W \times_Z Z'$ and $\pi : W' \to W$ be the first projection. If W is quasi-compact and quasi-separated and if E is ample, then $E' = \pi^*(E)$ is ample on W'.*

Proof By [17, 3.2], it suffices to verify that the universal line bundle on $\mathbb{P}(E')$ is ample. By [13, II.4.1.3], we obtain the following commutative diagram whose squares are cartesian:

$$\begin{array}{ccccc} \mathbb{P}(E') & \longrightarrow & W' & \longrightarrow & Z' \\ {\scriptstyle p}\downarrow & \square & {\scriptstyle \pi}\downarrow & \square & \downarrow{\scriptstyle g} \\ \mathbb{P}(E) & \longrightarrow & W & \xrightarrow[f]{} & Z \end{array}$$

where $p : \mathbb{P}(E') \to \mathbb{P}(E)$ is the projection. Moreover, one has $\mathcal{O}_{E'}(1) \cong p^*(\mathcal{O}_E(1))$. Since E is ample on W, $\mathcal{O}_E(1)$ is ample on $\mathbb{P}(E)$, hence Z-ample. By [13, II.4.6.13], $\mathcal{O}_{E'}(1)$ is Z'-ample. Since Z' is affine, $\mathcal{O}_{E'}(1)$ is ample by [13, II.4.6.6].

Remark 1 By Lemma 2, if Z is a projective scheme of dimension $\geqslant 1$ defined over a field k, E is a non-zero vector bundle on Z, and k' is an extension of k, then

$$P_1(Z, E) \Longrightarrow P_1(Z_{k'}, E_{k'}) .$$

The converse is also true, which is a consequence of the cohomological criterion of ampleness (see [26, Theoreme 6.1.10]) and the base change formula for cohomological groups.

The following proposition for the condition P_2 is analogous to the relation (3).

Proposition 2 *Let W and Z be two integral projective schemes of dimension $\geqslant 1$ over* Spec k *and $f : W \to Z$ be a surjective k-morphism, then*

$$P_2(W, f^*E) \Longrightarrow P_2(Z, E).$$

If in addition f is finite, then

$$P_2(Z, E) \Longrightarrow P_2(W, f^*E).$$

Proof "$P_2(W, f^*E) \Longrightarrow P_2(Z, E)$": Let V be an integral projective scheme of dimension $\geqslant 1$ defined on an extension k' of k and $h : V \to W$ be a dominant k-morphism such that $h^*(f^*(E))$ is ample. Then the k-morphism $fh : V \to Z$ is dominant and $(fh)^*E = h^*(f^*E)$ is ample.

"$P_2(Z, E) \Longrightarrow P_2(W, f^*E)$": Let V be an integral projective scheme of dimension $\geqslant 1$ defined over an extension k' of k and $h : V \to Z$ be a dominant k-morphism such that h^*E is ample. Let $V' = V \times_Z W$ and $p : V' \to W$, $q : V' \to V$ be the two projections:

$$\begin{array}{ccc} V' & \xrightarrow{q} & V \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle h} \\ W & \xrightarrow[f]{} & Z \end{array}$$

Since f is surjective, also is q (cf. [16] I.3.6.1), hence $fp = hq$ is dominant (because h is). Let w_0 be the generic point of W and z_0 be the image of w_0 in Z. Since f is finite and surjective, the point z_0 is the only generalization of itself (hence is the generic point of Z). Therefore z_0 lies in the image of h since h is dominant. Moreover, w_0 is the only point in W whose image in Z is z_0. So w_0 is in the image of p and thus the morphism p is dominant.

Let V_1' be an irreducible component of V' which maps surjectively to V. The morphism q being finite and surjective, the integral scheme V_1' has dimension $\geqslant 1$ and w_0 lies in the image of V_1' by p. Let p_1 be the restriction of p on V_1'. The morphism p_1 is dominant, and the vector bundle $p_1^*(f^*E) = (q^*h^*E)|_{V_1'}$ is ample since q is finite and surjective and h^*E is ample. The assertion is thus proved.

Given an integral projective scheme Z of dimension $n \geqslant 1$ over $\operatorname{Spec} k$, for any integer $m \in \{1, \dots, n\}$, there exists a "generic m-fold W in Z" such that the pull-back of any ample vector bundle on Z is ample on W (see the proposition below). This result will be an essential step in the comparison of the positivity conditions P_1 and P_3, established in Theorem 2. The main idea is to intersect Z by a pencil of hyperplanes and then extend the base field to the function field of the parameter space of the pencil. This procedure permits to reduce the dimension of the variety in keeping the information of vanishing of global sections (see Lemma 4).

Proposition 3 *Let Z be an integral projective scheme of dimension $n \geqslant 1$ defined over a field k. For each integer m such that $0 < m < n$, there exist a pure transcendental extension k'/k, an integral projective scheme W of dimension m over $\operatorname{Spec} k'$, and a dominant k-morphism from W to Z such that for any ample vector bundle E on Z, the vector bundle $h^*(E_{k'})$ is ample on W.*

Proof By induction we reduce the proposition to the case where $m = n - 1$. Let M be a vector space of dimension $n + 1$ over k. Having chosen a basis of M over k, we can identify the scheme $P = \mathbb{P}_k^n$ with either the projective bundle $\mathbb{P}(M)$ of M over $\operatorname{Spec} k$, or the projective bundle $\mathbb{P}(M^\vee)$ of $M^\vee$ in considering the dual basis. By Noether normalization [9, Lemma 13.2], there exists a finite and surjective morphism g from Z onto $P = \mathbb{P}(M)$. Denote by $\check{P} := \mathbb{P}(M^\vee)$. Let I be the incident subscheme of $P \times_k \check{P}$, namely the closed subscheme of $P \times_k \check{P}$ defined by the relation $f(x) = 0$ ($f \in M^\vee$, $x \in M$). It is a subscheme of codimension 1 in $P \times_k \check{P}$. Let $T = Z \times_P I$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
T & \xrightarrow{j} & Z \times_k \check{P} & \xrightarrow{\pi} & Z \\
\downarrow & \square & \downarrow{\scriptstyle g_{\check{P}}} & \square & \downarrow{\scriptstyle g} \\
I & \xrightarrow[i]{} & P \times_k \check{P} & \xrightarrow{\mathrm{pr}_1} & P \\
 & & {\scriptstyle \mathrm{pr}_2}\downarrow & \square & \downarrow \\
 & & \check{P} & \longrightarrow & \operatorname{Spec} k
\end{array}$$

where π is the first projection, $i : I \to P \times_k \check{P}$ are $j : T \to Z \times_k P$ are canonical closed immersions. Let k' be the field of rational functions on $\check{P}$. It is a pure transcendental extension of k and its transcendence degree over k is n. Let $P_{k'} := P \otimes_k k'$, $Z_{k'} := Z \otimes_k k'$. Define $T_{k'}$ by the following cartesian diagram

$$\begin{array}{ccc} T_{k'} & \xrightarrow{\;u\;} & T \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_2 \circ g_{\check{P}} \circ j} \\ \operatorname{Spec} k' & \longrightarrow & \check{P} \end{array}$$

Consider the following commutative diagram

$$\begin{array}{ccccc} T_{k'} & \xrightarrow{\;u\;} & T & & \\ {\scriptstyle j_{k'}}\big\downarrow & \square & \big\downarrow{\scriptstyle j} & & \\ Z_{k'} & \longrightarrow & Z \times_k \check{P} & \xrightarrow{\;\pi\;} & Z \\ {\scriptstyle g_{k'}}\big\downarrow & \square & \big\downarrow{\scriptstyle g_{\check{P}}} & \square & \big\downarrow{\scriptstyle g} \\ P_{k'} & \longrightarrow & P \times_k \check{P} & \xrightarrow{\;\mathrm{pr}_1\;} & P \\ \big\downarrow & \square & {\scriptstyle \mathrm{pr}_2}\big\downarrow & \square & \big\downarrow \\ \operatorname{Spec} k' & \longrightarrow & \check{P} & \longrightarrow & \operatorname{Spec} k \end{array}$$

The schemes $Z_{k'}$ and $T_{k'}$ are projective over $\operatorname{Spec} k'$. Moreover, as fields are universally catenary (see [9, corollary 13.5]), one has

$$\dim T_{k'} = \dim T - \dim \check{P} = \dim I - \dim \check{P} = 2n - 1 - n = n - 1.$$

Since $\mathrm{pr}_1 i : I \to P$ is surjective, also is $\pi j : T \to Z$. Furthermore, locally for the Zariski topology, the morphism $\mathrm{pr}_1 i : I \to P$ is a fibration with fibre $\mathbb{P}^{n-1}$. By base change, also is $\pi j : T \to Z$. This shows that the scheme T is integral. The canonical projection $u : T_{k'} \to T$ is thus dominant. Therefore, the morphism $\pi j u : T_{k'} \to Z$ is also dominant. If E is an ample vector bundle on Z, by Lemma 2, $E_{k'}$ is ample on $Z_{k'}$. Since $j_{k'}$ is a closed immersion, $j_{k'}^*(E_{k'})$ is ample on $T_{k'}$. Finally, by taking an irreducible component of $(T_{k'})_{\mathrm{red}}$, we obtain an integral projective scheme W of dimension $n - 1$ over $\operatorname{Spec} k'$ together with a morphism $h : W \to Z_{k'}$ which is the composition of $j_{k'}$ with the closed immersion of W in $T_{k'}$ so that the morphism from W to Z is dominant and that, for any ample vector bundle E on Z, $h^*(E_{k'})$ is ample on W.

Remark 2 (1) An interesting by-product of the above proof is that, the field k' can be chosen such that its transcendence degree over k is

$$\sum_{m<i\leqslant n} i = \frac{1}{2}(n^2+n-m^2-m).$$

(2) A particular case (where $m=1$) of Proposition 3 shows that, for any integral projective scheme Z of dimension $\geqslant 1$ over $\operatorname{Spec} k$ and any non-zero vector bundle on Z, the condition $P_2(Z,E)$ is equivalent to:

there exists an integral projective curve C defined over an extension k' of k together with a dominant k-morphism $h : C \to Z$ such that h^*E is an ample vector bundle on C.

3.2 Properties of the condition P_3

In this subsection, we discuss several basic properties of the condition P_3.

Lemma 3 *Let Z be a projective scheme on* $\operatorname{Spec} k$*, L be an ample line bundle on Z and $\mathcal{F}$ be a torsion-free coherent $\mathcal{O}_Z$-module. There exist two integers $a, m > 0$ and an injective homomorphism from $\mathcal{F}$ into $(L^{\otimes m})^{\oplus a}$.*

Proof Since $\mathcal{F}$ is torsion-free, the canonical homomorphism

$$\theta_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$$

is injective. Since L is ample, it exists an integer $m > 0$ such that $L^{\otimes m} \otimes \mathcal{F}^{\vee}$ is generated by its global sections, i.e., there exists an integer a and a surjective homomorphism

$$\varphi : \mathcal{O}_Z^{\oplus a} \longrightarrow L^{\otimes m} \otimes \mathcal{F}^{\vee}.$$

By passing to dual modules we obtain an injective homomorphism

$$L^{\vee\otimes m} \otimes \mathcal{F} \xrightarrow{1\otimes\theta_{\mathcal{F}}} L^{\vee\otimes m} \otimes \mathcal{F}^{\vee\vee} \xrightarrow{\varphi^{\vee}} \mathcal{O}_Z^{\oplus a}$$

which induces an injective homomorphism $\mathcal{F} \to (L^{\otimes m})^{\oplus a}$.

Proposition 4 *Let Z be a projective scheme of dimension $\geqslant 1$ and E be a vector bundle on Z. Then the condition $P_3(Z,E)$ is equivalent to each of the following conditions:*

(1) *for any line bundle L on Z and any torsion-free coherent $\mathcal{O}_Z$-module $\mathcal{F}$, there exists $\lambda > 0$ such that, for all integers d and n satisfying $n/\lambda > d > \lambda$, one has*

$$H^0(Z, S^n E^{\vee} \otimes L^{\otimes d} \otimes \mathcal{F}) = 0.$$

(2) *there exists an ample line bundle L on Z such that, for any torsion-free coherent $\mathcal{O}_Z$-module $\mathcal{F}$, there exists $\lambda > 0$ such that, for all integers d and n satisfying $n/\lambda > d > \lambda$, one has*

$$H^0\left(Z, S^n E^\vee \otimes L^{\otimes d} \otimes \mathcal{F}\right) = 0.$$

(3) *there exist an ample line bundle L on Z and a real number $\lambda > 0$ such that, for all integers d and n satisfying $n/\lambda > d > \lambda$, one has*

$$H^0\left(Z, S^n E^\vee \otimes L^{\otimes d}\right) = 0.$$

Proof "(1) $\Longrightarrow$ (2) $\Longrightarrow$ (3)" are trivial. "$P_3(Z, E) \Longrightarrow$ 1)": Let L_1 be an ample line bundle on Z such that $L_2 := L_1 \otimes L^\vee$ is very ample. By Lemma 3, there exist two integers m and a together with an injective homomorphism from $\mathcal{F}$ into $(L_1^{\otimes m})^{\oplus a}$. It induces an injective homomorphism

$$S^n E^\vee \otimes L^{\otimes d} \otimes \mathcal{F} \longrightarrow S^n E^\vee \otimes \left(L_1^{\otimes(d+m)}\right)^{\oplus a} \otimes L_2^{\vee\otimes d}.$$

Since $L_2^{\otimes d}$ is very ample, it has a non-zero global section, which induces an injective homomorphism from $L_2^{\vee\otimes d}$ to $\mathcal{O}_X$. Therefore, we obtain an injective homomorphism

$$S^n E^\vee \otimes \left(L_1^{\otimes(d+m)}\right)^{\oplus a} \otimes L_2^{\vee\otimes d} \longrightarrow \left(S^n E^\vee \otimes L_1^{\otimes(d+m)}\right)^{\oplus a}.$$

If (Z, E) verifies the condition P_3, there exists $\lambda > 0$ such that, for all integers d and n satisfying $n/\lambda > d > \lambda$, one has $H^0(Z, S^n E^\vee \otimes L_1^{\otimes(d+m)}) = 0$. Hence $H^0(Z, S^n E^\vee \otimes L^{\otimes d} \otimes \mathcal{F}) = 0$ since it is a subgroup of $H^0(Z, S^n E^\vee \otimes L_1^{\otimes(d+m)})^{\oplus a}$.

"(3) $\Longrightarrow P_3(Z, E)$": Let L' be a line bundle on Z. There exists an integer $m > 0$ such that $L^{\otimes m} \otimes L'^\vee$ is very ample. Hence there exists an injective homomorphism $\mathcal{O}_Z \to L^{\otimes m} \otimes L'^\vee$ which induces an injective homomorphism $L' \to L^{\otimes m}$. Therefore, for all integers $n > 0$ and $d > 0$, one has an injective homomorphism $S^n E^\vee \otimes L'^{\otimes d} \longrightarrow S^n E^\vee \otimes L^{\otimes md}$ which induces an injective homomorphism of global sections:

$$H^0\left(Z, S^n E^\vee \otimes L'^{\otimes d}\right) \longrightarrow H^0(Z, S^n E^\vee \otimes L^{\otimes md}).$$

The implication "(3) $\Longrightarrow P_3(Z, E)$" then follows from this relation.

Proposition 5 *Let Z be a projective scheme over* $\operatorname{Spec} k$, *E be a non-zero vector bundle on Z and F be a non-zero quotient bundle of E. Then*

$$P_3(Z, E) \Longrightarrow P_3(Z, F).$$

Proof Let $\varphi : E \to F$ be the canonical surjective homomorphism which induces an injective homomorphism $\varphi^\vee : F^\vee \to E^\vee$. Thus, for any line bundle L on Z and all positive integers n and d, one has an injective homomorphism

$$S^n\left(F^\vee\right) \otimes L^d \longrightarrow S^n\left(E^\vee\right) \otimes L^d.$$

Therefore

$$H^0\left(Z, S^n\left(E^\vee\right) \otimes L^d\right) = 0 \Longrightarrow H^0\left(Z, S^n\left(F^\vee\right) \otimes L^d\right) = 0.$$

Lemma 4 *Let $h : W \to Z$ be a dominant morphism of integral schemes. If F is a vector bundle on Z such that $H^0(W, h^*(F)) = 0$, then $H^0(Z, F) = 0$.*

Proof Let w and z be the generic point of W and Z respectively. Since h is dominant, one has $h(w) = z$. If $s \in H^0(Z, F)$ is non-zero, then $s(z) \neq 0$, so its image in $H^0(W, h^*(F))$ is non-zero since the canonical homomorphism $F_z \longrightarrow h^*(F)_w$ is injective.

Proposition 6 *Let Z be an integral projective scheme over* $\operatorname{Spec} k$*. Let k'/k be a field extension and W be an integral projective scheme over* $\operatorname{Spec} k'$*. Suppose given a dominant k-morphism from W to Z. Then for any non-zero vector bundle E on Z,*

$$P_3(W, h^*E) \Longrightarrow P_3(Z, E).$$

Proof Let L be an arbitrary line bundle on Z. If $P_3(W, h^*E)$ holds, then there exists $\lambda > 0$ such that, for all integers n and d satisfying $n/\lambda > d > \lambda$, one has

$$H^0\left(W, S^n(h^*E)^\vee \otimes (h^*L)^{\otimes d}\right) = H^0\left(W, h^*\left(S^n(E^\vee) \otimes L^{\otimes d}\right)\right) = 0.$$

Therefore, by Lemma 4, one obtains $H^0(Z, S^n(E^\vee) \otimes L^{\otimes d}) = 0$. Since L is arbitrary, the condition $P_3(Z, E)$ holds.

Proposition 7 *Let Z be a projective scheme over* $\operatorname{Spec} k$ *and E be a non-zero vector bundle on Z. If k'/k is a field extension, then*

$$P_3(Z_{k'}, E_{k'}) \Longleftrightarrow P_3(Z, E).$$

Proof For any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$, one has (by [14, III.1.4.15], see also [15, IV.1.7.21])

$$H^0(Z_{k'}, \mathcal{F}_{k'}) \cong H^0(Z, \mathcal{F}) \otimes_k k'.$$

Moreover, if L is an ample line bundle on Z, then $L_{k'}$ is ample on $Z_{k'}$. Hence the proposition follows by Proposition 4 (3).

Lemma 5 *Let $\pi : W \to Z$ be a surjective morphism of projective schemes over* $\operatorname{Spec} k$*. If L is a line bundle on W which is relatively ample with respect to π, then there exists an ample line bundle M on Z such that $L \otimes \pi^*M$ is ample.*

Proof Since L is relatively ample with respect to π, there exist an integer $n > 0$, a non-zero vector bundle E on Z and a closed immersion $f : W \to \mathbb{P}(E)$ which is compatible with π and such that $L^{\otimes n} \cong f^*(\mathcal{O}_E(1))$. Denote by $p : \mathbb{P}(E) \to Z$ the projection. One has the following commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\ f\ } & \mathbb{P}(E) \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle p} & \\ Z & & \end{array}$$

Since Z is projective, there exists an ample line bundle M on Z such that $E \otimes M^{\otimes n}$ is an ample vector bundle. One has $\mathbb{P}(E) \cong \mathbb{P}\left(E \otimes M^{\otimes n}\right)$, and

$$\mathcal{O}_{E\otimes M^{\otimes n}}(1) \cong \mathcal{O}_E(1) \otimes p^* M^{\otimes n}.$$

Since f is a closed immersion and $E \otimes M^{\otimes n}$ is ample, one obtains that

$$\begin{aligned} \left(L \otimes \pi^* M\right)^{\otimes n} &= f^*\left(\mathcal{O}_E(1)\right) \otimes f^*(p^*M)^{\otimes n} \\ &= f^*\left(\mathcal{O}_E(1) \otimes p^* M^{\otimes n}\right) \cong f^*\left(\mathcal{O}_{E\otimes M^{\otimes n}}(1)\right) \end{aligned}$$

is ample.

Proposition 8 *Let $\pi : W \to Z$ be a surjective morphism of projective schemes over k. If F is a non-zero vector bundle on W which is ample relatively to π, then there exists an ample line bundle M on Z such that $F \otimes \pi^* M$ is ample.*

Proof Let $q : \mathbb{P}(F) \to W$ be the canonical morphism and $L := \mathcal{O}_F(1)$. Since F is ample relatively to π, L is ample relatively to $f = \pi q$. We apply Lemma 5 to f and L, and obtain that there exists an ample line bundle M on Z such that $L \otimes f^* M$ is ample. As $L \otimes f^* M = L \otimes q^*(\pi^* M)$ is isomorphic to $\mathcal{O}_{F\otimes \pi^* M}(1)$, the vector bundle $F \otimes \pi^* M$ is ample.

Proposition 9 *Let Z be a projective scheme of dimension $\geqslant 1$ over* $\operatorname{Spec} k$ *and E be a non-zero vector bundle on Z. Then*

$$P_3\left(\mathbb{P}(E^\vee), \mathcal{O}_{E^\vee}(-1)\right) \Longleftrightarrow P_3(Z, E).$$

Proof Denote by π the projection morphism from $\mathbb{P}(E^\vee)$ to Z.

"$\Longrightarrow$": Let L be a line bundle on Z. Since $(\mathbb{P}(E^\vee), \mathcal{O}_{E^\vee}(-1))$ verifies the condition P_3, there exists $\lambda > 0$ such that, for all integers d and n satisfying $n/\lambda > d > \lambda$,

$$H^0\left(\mathbb{P}(E^\vee), \mathcal{O}_{E^\vee}(n) \otimes \pi^* L^{\otimes d}\right) = 0.$$

By the canonical isomorphism

$$\pi_*\left(\mathcal{O}_{E^\vee}(n) \otimes \pi^* L^{\otimes d}\right) \cong \pi_*\left(\mathcal{O}_{E^\vee}(n)\right) \otimes L^{\otimes d} \cong S^n E^\vee \otimes L^{\otimes d},$$

one obtains that, for $n > \lambda d$,

$$H^0\left(Z, S^n E^\vee \otimes L^{\otimes d}\right) = H^0\left(\mathbb{P}\left(E^\vee\right), \mathcal{O}(n) \otimes \pi^* L^{\otimes d}\right) = 0.$$

"$\Longleftarrow$": By Lemma 5, there exists an ample line bundle M on Z such that $L = \pi^* M \otimes \mathcal{O}_{E^\vee}(1)$ is ample on $\mathbb{P}(E^\vee)$. If the condition $P_3(Z, E)$ is fulfilled, there exists a constant $\lambda > 0$ such that, for all integers d and n satisfying $n/\lambda > d > \lambda$, one has $H^0(Z, S^n E^\vee \otimes M^{\otimes d}) = 0$. Hence

$$H^0\left(\mathbb{P}\left(E^\vee\right), \mathcal{O}_{E^\vee}(n) \otimes L^{\otimes d}\right) \cong H^0\left(Z, S^{n+d} E^\vee \otimes M^{\otimes d}\right)$$

vanishes if $n > (\lambda - 1)d$. Since L is ample on $\mathbb{P}(E^\vee)$, the pair $\left(\mathbb{P}\left(E^\vee\right), \mathcal{O}_{E^\vee}(-1)\right)$ satisfies the condition P_3.

3.3 The case of curves

If Z is a scheme and E is a vector bundle on Z, the expression $\Gamma^n(E)$ denotes $S^n(E^\vee)^\vee$. If $n!$ is invertible on Z, then $\Gamma^n(E)$ is canonically isomorphic to $S^n E$. This does not hold in general. We recall below a result of Hartshorne, which will be used to compare the conditions P_1 and P_3 in the case of curves (Proposition 10). See [18, Lemma 6.1] for the proof of the Lemma.

Lemma 6 (Hartshorne) *Let C be an integral projective curve over* Spec k, *E be a vector bundle of rank $r > 0$ on C, L be an ample line bundle on C, and $\mathcal{F}$ be a coherent $\mathcal{O}_C$-module. If E is ample, then there exists an integer $\lambda > 0$ such that, for all integers n and d satsifying $n/\lambda > d > \lambda$,*

$$H^1\left(C, \Gamma^n(E) \otimes L^{\otimes -d} \otimes \mathcal{F}\right) = 0.$$

Proposition 10 *Let C be an integral projective curve over* Spec k *and E be a non-zero vector bundle on C. Then*

$$P_1(C, E) \Longrightarrow P_3(C, E).$$

Proof By Serre duality, (C, E) verifies the condition P_3 if and only if there exists an integer $\lambda > 0$ such that, for all integers n and d satisfying $n/\lambda > d > \lambda$, one has

$$\dim H^1\left(C, \Gamma^n(E) \otimes L^{\otimes -d} \otimes \omega_C\right) = 0,$$

where ω_C is the dualizing sheaf of C. So Lemma 6 implies that, if E is ample, then (C, E) satisfies the condition P_3.

The following Propositions 11 and 12 show that the conditions P_1 and P_3 are actually equivalent for line bundles on curves and for vector bundles on curves defined over a field of characteristic 0.

Proposition 11 *Let C be a regular projective curve over* Spec k *and L be a line bundle on C. Then*

$$P_1(C, L) \Longleftrightarrow P_3(C, L).$$

Proof By Proposition 10, $P_3(C, L)$ is a consequence of $P_1(C, L)$. In the following, we prove the converse implication. It suffices to prove $\deg L > 0$. Let L' be an ample line bundle on C such that $\deg(L') > g$, where g is the genus of C. By the condition $P_3(C, L)$, there exists an integer $n > 0$ such that

$$\dim H^0(C, L^{\otimes -n} \otimes L') = 0.$$

By Riemann–Roch theorem, we obtain

$$\begin{aligned} &\dim H^0\left(C, L^{\otimes -n} \otimes L'\right) - \dim H^0\left(C, \omega_C \otimes L^{\otimes n} \otimes L'^{\vee}\right) \\ &= 1 - g - n\deg(L) + \deg(L'). \end{aligned}$$

Hence

$$n\deg(L) \geqslant 1 - g + \deg(L') > 0.$$

Proposition 12 *Let C be a regular projective curve over* Spec k *and E a non-zero vector bundle on C. Assume that the characteristic of k is* 0. *Then*

$$P_1(C, E) \Longleftrightarrow P_3(C, E).$$

Proof We only need to prove $P_3(C, E) \Rightarrow P_1(C, E)$. Since C is a curve, it suffices to prove that, for any coherent $\mathcal{O}_C$-module $\mathcal{F}$, the cohomology group $H^1(C, S^n E \otimes \mathcal{F})$ vanishes for sufficiently large n. Let L be an ample line bundle on C. There exists an integer d such that $L^{\otimes d} \otimes \mathcal{F} \otimes \omega_C^{\vee}$ is generated by its global sections on C. Hence there exist an integer $a > 0$ and a surjective homomorphism $\varphi : (L^{\otimes -d} \otimes \omega_C)^{\oplus a} \to \mathcal{F}$. Since (C, E) satisfies the condition P_3, there exists $n_0 > 0$ such that $H^0\left(C, S^n(E^{\vee}) \otimes L^{\otimes d}\right) = 0$ for any $n \geqslant n_0$. By Serre duality, this is equivalent to

$$H^1\left(C, S^n(E) \otimes L^{\otimes -d} \otimes \omega_C\right) = 0$$

since $S^n(E) \cong \Gamma^n(E)$. As C is a curve, φ induces a surjective homomorphism (since the second cohomology group vanishes)

$$H^1\left(C, (S^n(E) \otimes L^{\otimes -d} \otimes \omega_C)^{\oplus a}\right) \longrightarrow H^1\left(C, S^n(E) \otimes \mathcal{F}\right).$$

Therefore $H^1\left(C, S^n(E) \otimes \mathcal{F}\right) = 0$.

3.4 Comparison of positivity conditions in general case

In this subsection, we prove $P_2 \Longrightarrow P_3$. We also show by a counter example that the condition P_2 is in general strictly weaker than the ampleness, even in the case of line bundles (on a projective variety of dimension $\geqslant 2$).

Theorem 2 *Let Z be an integral projective scheme of dimension $\geqslant 1$ over $\operatorname{Spec} k$ and E be a non-zero vector bundle on Z. Then*

$$P_1(Z, E) \Longrightarrow P_2(Z, E) \Longrightarrow P_3(Z, E).$$

Proof "$P_1(Z, E) \Longrightarrow P_2(Z, E)$" is trivial.

"$P_2(Z, E) \Longrightarrow P_3(Z, E)$": By Remark 2, there exist an integral curve C over an extension of k and a dominant morphism $h : C \to Z$ such that h^*E is ample on C. By Proposition 10, (C, h^*E) verifies the condition P_3. Finally, since h is dominant, by Proposition 6 the condition $P_3(Z, E)$ holds.

Remark 3 Let Z be a projective scheme over $\operatorname{Spec} k$ which is reduced, and E be a vector bundle on Z. Assume that each irreducible component of Z has dimension $\geqslant 1$. If for any irreducible component Z_i of Z, the pair $\left(Z_i, E|_{Z_i}\right)$ satisfies the condition P_2. Then by the above theorem, the condition $P_3(Z_i, E|_{Z_i})$ holds for each i and hence $P_3(Z, E)$ holds. As the restriction of an ample vector bundle on Z to each irreducible component of Z is ample, we obtain in particular $P_1(Z, E) \Rightarrow P_3(Z, E)$ for any non-zero vector bundle E on Z.

We now show an example where the condition P_2 holds but P_1 fails.

Proposition 13 *Let Z and Z' be two integral projective schemes of dimension $\geqslant 1$ over k. Let E and E' be respectively a non-zero vector bundle on Z and Z'. Denote by W the fibre product $Z \times_k Z'$. Let $\mathrm{pr}_1 : W \to Z$ and $\mathrm{pr}_2 : W \to Z'$ be the two projections, and $F = \mathrm{pr}_1^* E \otimes \mathrm{pr}_2^* E'$. If one of the vector bundles E and E' is ample, then $P_2(W, F)$ holds.*

Proof We may assume that E is ample on Z. Let k' be the field of rational functions on Z'. Then $W_{k'} = Z_{k'}$ and $F_{k'} = E_{k'}^{\oplus \mathrm{rk}(E')}$. Hence $F_{k'}$ is ample on $W_{k'}$.

In the following (Proposition 14), we compare P_3 to a positivity condition in the context of complex analytic geometry.

Lemma 7 *Let Z be a compact complex analytic variety and (L, h) be a smooth hermitian line bundle on Z. If L has a holomorphic section which does not vanish identically on Z, then there is a point $z \in Z$ such that the Levi form $\Theta(L, h)_z$ on z is semi-positive.*

Proof Let e be a holomorphic section of L which does not vanish identically on Z. Let z_0 be a point of Z where the function $\|e\|$ attains the maximum. The existence of such point results from the compactness of Z. By Lelong-Poincaré theorem, in an open neighbourhood of z_0 one has

$$\Theta(L, h) = -\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log \|e\|^2.$$

Let U be an open set of $\mathbb{C}$ and $j : U \to X$ be a holomorphic map whose image contains z_0 and such that the pull-back of L on $j(U)$ is trivialized by the section e. One has

$$j^*\Theta(L,h) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|e\circ j\|^2 = -\frac{1}{4\pi}\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)\log\|e\circ j\|^2 dx\wedge dy,$$

where (x, y) is the natural coordinate of $\mathbb{C}$. Since j is arbitrary, we obtain that $\Theta(L,h)_{x_0}$ is semi-positive.

Proposition 14 *Let Z be a smooth projective variety over* $\operatorname{Spec}\mathbb{C}$ *and $\overline{L}$ be a hermitian line bundle on $Z^{\mathrm{an}}(\mathbb{C})$. If for any $z \in Z$, the Levi form $\Theta(\overline{L})$ is not semi-negative on z, then (Z, L) satisfies the condition P_3.*

Proof Suppose that $P_3(Z, L)$ is not verified. Assume that $\overline{L}'$ is a hermitian line bundle on Z and $(n_i, d_i)_{i\geqslant 1}$ is a sequence in $\mathbb{N}^2_{>0}$ such that $\lim_{i\to+\infty} n_i/d_i = +\infty$ and that

$$H^0\left(Z, L^{\vee\otimes n_i}\otimes L'^{\otimes d_i}\right)\neq 0$$

for any $i \geqslant 1$. By Lemma 7, for any integer $i \geqslant 1$, there exists a point $z_i \in Z(\mathbb{C})$ such that the hermitian forme $-n_i\Theta(\overline{L})_{z_i} + d_i\Theta(\overline{L}')_{z_i}$ is semi-positive. In other words, the hermitian form $\Theta(\overline{L})_{z_i} - (d_i/n_i)\Theta(\overline{L}')_{z_i}$ is semi-negative. Since $Z^{\mathrm{an}}(\mathbb{C})$ is compact, there exists a subsequence of $(z_i)_{i\geqslant 1}$ which converges to some point $z_0 \in Z^{\mathrm{an}}(\mathbb{C})$. Since d_i/n_i tends to 0 when $i \to +\infty$, by passing to limit, we obtain that $\Theta(\overline{L})_{z_0}$ is semi-negative. This is absurd.

4 Application to formal principle

Formal principle has been originally considered in complex analytic setting by Grauert [11] (see also the related work [27]) and studied in papers such as [12,20,21,23,24]. Given a complex analytic space M and a compact complex analytic subspace A (of dimension $\geqslant$1) of M. We say that the formal principle holds for the pair (M, A) if for any other pair (M', A') of complex analytic space M' and a compact complex analytic subspace A' of M', the formal neighbourhoods of A and A' being isomorphic implies the existence of a biholomorphism from an open neighbourhood of A in X onto an open neighbourhood of A' in X'. The formal principle does not hold in general. In fact, Arnold [1] has constructed a torus embedded in a complex manifold M of complex dimension two whose normal bundle is trivial. The formal neighbourhood of the torus is isomorphic to that of the zero section in its normal bundle, but there is not any biholomorphism between an open neighbourhood of the torus in M and that of the zero section in the normal bundle.

Under suitable conditions on the normal bundle N_AM of A in M (here we assume that A is locally a complete intersection in M), one can establish the formal principle for the pair (M, A). Curiously, the formal principle holds when N_AM verifies either a negativity condition [11,20] or a positivity condition [8,12,22,27]. For example, Grauert [11], Hironaka and Rossi [20] have proved the formal principle under the hypothesis that the subvariety A is strongly exceptional in X, that is, A is contractible to a point. This condition holds when the normal bundle $N_A M$ is negative (in the sense of Grauert, namely its zero section admits a strongly pseudoconvex neighbourhood, or equivalently its zero section is contractible to one point). On the other hand, Nirenberg and Spencer [27], Hironaka [21], Griffiths [12] have proved that the formal principle holds if $N_A M$ is positive. Later Commichau and Grauert [8] and Hirschowitz [22] have proved the formal principle under a weaker positivity condition for the normal bundle.

The formal principle in algebraic geometry has been studied by Gieseker [10] (see Theorem 4.2 *loc. cit.*). His result can be considered as a counterpart of Griffiths' work [12] in algebraic geometry. The main purpose of this section is to establish a criterion of formal principle (Theorem 3) which generalizes Gieseker's work, by using the algebraicity condition (Theorem 1). We begin by introducing several notions concerning formal principle in algebraic geometry.

(1) We call (noetherian) *scheme pair* any pair (W, Z) consisting of a noetherian scheme W with a reduced closed subscheme Z of W.
(2) Let (W, Z) and (W', Z') be two scheme pairs. We call *morphism* of scheme pairs from (W, Z) to (W', Z') any morphism of finite type from W to W' whose restriction on Z defines a morphism from Z to Z'.
(3) Let (W, Z) be a scheme pair. We call an *étale neighbourhood* of (W, Z) a scheme pair (U, Z) together with a morphism of scheme pairs from (U, Z) to (W, Z) whose restriction on Z is the identity morphism on Z and which is étale on any point of Z.
(4) Two scheme pairs (W, Z) and (W', Z) are said to be *equivalent in a formal neighbourhood* if there exists an isomorphism of formal scheme $\varphi : \widehat{W}_Z \to \widehat{W}'_Z$ whose restriction on Z is the identity morphism.
(5) Two scheme pairs (W, Z) and (W', Z) are said to be *equivalent in an étale neighbourhood* if there exist a scheme pair (U, Z) and two morphisms $f : (U, Z) \to (W, Z)$ and $g : (U, Z) \to (W', Z)$ which define an étale neighbourhood of (W, Z) and of (W', Z), respectively.
(6) Let (W, Z) be a scheme pair. We say that the *formal principle* holds for (W, Z) if for any scheme pair (W', Z), the equivalence of (W, Z) and (W', Z) in a formal neighbourhood implies their equivalence in an étale neighbourhood.

Gieseker has proved that a pair (X, Y) of smooth projective varieties with $\dim(Y) \geqslant 1$ satisfies the formal principle once the normal bundle $N_Y X$ is ample (see [10, Theorem 4.2]). The following theorem (mentioned in Sect. 1 as Theorem B) extends his result to the case where $N_Y X$ verifies the condition P_3.

Theorem 3 *Let (X, Y) and (W, Y) be two pairs of smooth projective varieties over a field k. Assume that $\dim(Y) \geqslant 1$ and that the normal bundle of Y in X satisfies the condition P_3. Then the pairs (X, Y) and (W, Y) are equivalent in a formal neighbourhood if and only if they are equivalent in an étale neighbourhood. In other words, the formal principle holds for the scheme pair (X, Y).*

Proof "$\Longleftarrow$" is a direct consequence of [10, Lemma 4.5].

"$\Longrightarrow$": Let $\Delta : Y \xrightarrow{(\mathrm{Id},\mathrm{Id})} Y \times_k Y \longrightarrow X \times_k W$ be the diagonal morphism from Y into $X \times_k W$. Assume that $\varphi : \widehat{X}_Y \to \widehat{W}_Y$ is an isomorphism of formal scheme whose restriction on Y is the identity morphism. Consider the "formal graph" of φ

$$\Gamma_\varphi : \widehat{X}_Y \longrightarrow (\widehat{X \times_k W})_{\Delta(Y)}$$

which identifies $\widehat{X}_Y$ with a formal subscheme of $(\widehat{X \times_k W})_{\Delta(Y)}$. Since $N_Y\widehat{X}_Y \cong N_Y X$ satisfies the condition P_3, $\Gamma_\varphi(\widehat{X}_Y)$ is algebraic in $(\widehat{X \times_k W})_{\Delta(Y)}$, i.e., there exists a closed subscheme Z of $X \times_k W$ containing $\Delta(Y)$ such that $\dim Z = \dim \widehat{X}_Y$ ($= \dim X = \dim W$) and that $\widehat{Z}_{\Delta(Y)} \supset \Gamma_\varphi(\widehat{X}_Y)$. Let $\upsilon : \widetilde{Z} \to Z$ be the normalization of Z. It induces a morphism $\widehat{\widetilde{Z}}_{\upsilon^{-1}(Y)} \to \widehat{Z}_Y$, whose restriction on $\Gamma_\varphi(\widehat{X}_Y)$ (which is formally smooth) is an isomorphism. Hence we obtain the existence of an integral subscheme $\widetilde{Y}$ in $\widetilde{Z}$ such that $\upsilon|_{\widetilde{Y}} : \widetilde{Y} \to Y$ is an isomorphism and that $\widehat{\upsilon}$ induces an isomorphism $\widehat{\upsilon} : \widehat{\widetilde{Z}}_{\widetilde{Y}} \to \Gamma_\varphi(\widehat{X}_Y)$. Therefore, the morphisms of formal schemes induced by the morphisms of pairs $\mathrm{pr}_1 \circ \upsilon : (\widetilde{Z}, \widetilde{Y}) \to (X, Y)$ and $\mathrm{pr}_2 \circ \upsilon : (\widetilde{Z}, \widetilde{Y}) \to (W, Y)$ are isomorphisms, where $\mathrm{pr}_1 : Z \to X$ and $\mathrm{pr}_2 : Z \to W$ are projections. By [10, Lemma 4.5], $(\widetilde{Z}, \widetilde{Y})$ define a common étale neighbourhood of (X, Y) and (W, Y).

Remark 4 The comparison between the ampleness and the condition P_3, namely Theorem 3.22 and Proposition 3.24, shows that the above theorem generalizes the algebraic counterpart of a result of Griffiths [12] on formal principle (see [10, p.1150]).

Theorem 3 can also be compared with an approach of Bădescu and Schneider [2]. In fact, they proved that, given a pair (X, Y) of a smooth projective variety X and a smooth closed subvariety Y of dimension d, if the normal bundle $N_Y X$ is $(d-1)$-ample in the sense of Sommese [28], then Y is G2 in X (in the terminology of Hironaka and Matsumura, see [19, p. 190]), namely the field of formal rational functions on $\widehat{X}_Y$ is a finite extension of $k(X)$. Thus by [10, Theorem 4.2], the formal principle holds for (X, Y) once $N_Y X$ is $(d-1)$-ample.

Recall that a vector bundle E on a projective variety Y is said to be k-*ample* if there exists some integer $r > 0$ such that $\mathcal{O}_E(r)$ is generated by global sections and if for any coherent sheaf $\mathcal{F}$ on Y there exists an integer $N(\mathcal{F}) > 0$ such that $H^j(Y, S^n E \otimes \mathcal{F})$ vanishes for all integers j and n such that $j > k$ and $n \geqslant N(\mathcal{F})$. In particular, the 0-ampleness is just the classical ampleness condition. In view of Serre duality, if the characteristic of k is zero, then the vanishing condition in $(\dim Y - 1)$-ampleness of E is quite similar to the condition $P_3(Y, E)$. However, in the condition P_3 we do not require the freeness of $\mathcal{O}_E(r)$ (which actually implies that $\mathcal{O}_E(1)$ is nef). In particular, Proposition 13 provides an example where the condition P_3 holds where the vector bundle could contain a very negative part.

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