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Differentiability of the arithmetic volume function

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ABSTRACT

We introduce the positive intersection product in Arakelov geometry and prove that the arithmetic volume function is differentiable.

1. *Introduction*

Let k be a field. If Y is a projective variety of dimension d over $\operatorname{Spec} k$ and if L is a line bundle on Y, then the volume of L is defined as

$$\operatorname{vol}(L) := \limsup_{n\to\infty} \frac{\operatorname{rk}_k H^0(Y, L^{\otimes n})}{n^d/d!}.$$

The line bundle L is said to be *big* if its volume is positive. In [7], Boucksom, Favre and Jonsson have proved that the function $\operatorname{vol}(L)$ is continuously differentiable on the big cone. The same result has also been independently obtained by Lazarsfeld and Mustaţă [16] by using Okounkov bodies. The differential of vol contains the positive intersection product, initially defined in [6] in the analytic-geometrical framework, and redefined algebraically in [7].

In Arakelov geometry, the analogue of the volume function is defined for Hermitian line bundles. Let K be a number field, $\mathcal{O}_K$ be its integer ring and $\pi : X \to \operatorname{Spec}\mathcal{O}_K$ be a projective arithmetic variety (that is, π is a projective and flat morphism) of relative dimension d. The *arithmetic volume* of a Hermitian line bundle $\bar{L}$ (with continuous metrics) on X is by definition

$$\widehat{\operatorname{vol}}(\bar{L}) := \limsup_{n\to\infty} \frac{\hat{h}^0(X, \bar{L}^{\otimes n})}{n^{d+1}/(d+1)!}, \tag{1.1}$$

where

$$\hat{h}^0(X, \bar{L}^{\otimes n}) = \log \# \left\{ s \in \pi_*(L^{\otimes n}) \;\middle|\; \sup_{\sigma: K\to\mathbb{C}} \|s\|_{\sigma,\sup} \leqslant 1 \right\}.$$

In this article, we introduce an analogue in Arakelov geometry of the positive intersection product in [7], and prove that the arithmetic volume function $\widehat{\operatorname{vol}}$ is continuously differentiable on $\widehat{\operatorname{Pic}}(X)$. We shall establish the following result.

THEOREM A. *Let $\bar{L}$ and $\bar{M}$ be two continuous Hermitian line bundles on X. Assume that $\bar{L}$ is big. Then*

$$D_{\bar{M}}\widehat{\operatorname{vol}}(\bar{L}) := \lim_{n\to+\infty} \frac{\widehat{\operatorname{vol}}(\bar{L}^{\otimes n}\otimes\bar{M}) - \widehat{\operatorname{vol}}(\bar{L}^{\otimes n})}{n^d}$$

exists in $\mathbb{R}$, and the function $\bar{M} \mapsto D_{\bar{M}}\widehat{\operatorname{vol}}(\bar{L})$ is additive on $\widehat{\operatorname{Pic}}(X)$. Furthermore, one has

$$D_{\bar{M}}\widehat{\operatorname{vol}}(\bar{L}) = (d+1)\langle \hat{c}_1(\bar{L})^d\rangle \cdot \hat{c}_1(\bar{M}).$$

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Here the *positive intersection product* $\langle \hat{c}_1(\bar{L})^d \rangle$ is defined as the least upper bound of self-intersections of ample Hermitian line bundles dominated by $\bar{L}$ (see § 3.3). In particular, one has $\langle \hat{c}_1(\bar{L})^d \rangle \cdot \hat{c}_1(\bar{L}) = \langle \hat{c}_1(\bar{L})^{d+1} \rangle = \widehat{\mathrm{vol}}(\bar{L})$.

The existence of $D\widehat{\mathrm{vol}}(\bar{L})$ comes from the log-concavity of the arithmetic volume function (see [**27**, Theorem B]), which asserts that the function $\widehat{\mathrm{vol}}^{1/(d+1)}$ is super-additive. The additivity of $\bar{M} \mapsto D_{\bar{M}}\widehat{\mathrm{vol}}(\bar{L})$ on $\widehat{\mathrm{Pic}}(X)$ relies on the observation that if a super-additive (homogeneous) function on $\widehat{\mathrm{Pic}}(X)$ is bounded from below by an additive function, then they are equal (see Proposition 4.1). This convexity argument permits us to reduce the proof of the theorem to establishing the inequality $D\widehat{\mathrm{vol}}(\bar{L}) \geqslant (d+1)\langle \hat{c}_1(\bar{L})^d \rangle$, which is a consequence of Siu's inequality in Arakelov geometry (see [**26**, § 1.3]) and the arithmetic Fujita approximation theorem (cf. [**11**, Theorem 4.3; **27**, Theorem C]).

Compared with the proof of the differentiability of the geometric volume function in [**7**], our proof relies on similar properties of the volume function (Fujita approximation and Siu's inequality). However, the convexity argument as above permits us to clarify the role that these properties played in the differentiability of the volume function and to make the proof more clear. The differentiability criterion in Proposition 4.1 combined with [**7**, Corollary 3.4] also gives a shorter proof of the differentiability of the geometric volume function.

As an application of the differentiability of the arithmetic volume function, we calculate the distribution function of the asymptotic measure (see [**10**, **11**]) of a generically big Hermitian line bundle in terms of positive intersection numbers. Let $\bar{L}$ be a Hermitian line bundle on X such that L_K is big. The asymptotic measure $\nu_{\bar{L}}$ is the vague limit (when n goes to infinity) of Borel probability measures whose distribution functions are determined by the filtration of $H^0(X_K, L_K^{\otimes n})$ by successive minima (see (5.1) *infra*). Several asymptotic invariants can be obtained by integration with respect to $\nu_{\bar{L}}$. We shall calculate the distribution function of $\nu_{\bar{L}}$ in Proposition 5.2.

Another application consists of interpreting the variational principle in the Arakelov geometry approach of the equidistribution problem by the differentiability of certain arithmetic invariants. Here we use a variant of the additivity criterion mentioned above, which asserts that if a super-additive homogeneous function is bounded from below by a differentiable function and if the two functions share the same value on some point, then the former function is also differentiable at this point (see Proposition 5.4).

The article is organized as follows. In § 2, we recall some positivity conditions for Hermitian line bundles and discuss their properties. In § 3, we define the positive intersection product in Arakelov geometry. It is in § 4 that we establish the differentiability of the arithmetic volume function. Finally, in § 5, we compare our result to some known results on the differentiability of arithmetic invariants and discuss some applications.

2. *Preliminaries*

In this article, we fix a number field K and denote by $\mathcal{O}_K$ its integer ring. Let $\bar{K}$ be an algebraic closure of K. Let $\pi : X \to \operatorname{Spec} \mathcal{O}_K$ be a projective and flat morphism and d be the relative dimension of π. Denote by $\widehat{\mathrm{Pic}}(X)$ the group of isomorphism classes of (continuous) Hermitian line bundles on X. If $\bar{L}$ is a Hermitian line bundle on X, then we denote by $\pi_*(\bar{L})$ the $\mathcal{O}_K$-module $\pi_*(L)$ equipped with sup norms.

In the following, we recall several notions about Hermitian line bundles. The references are [**5**, **15**, **18**, **28**].

(1) Assume that $x \in X(\bar{K})$ is an algebraic point of X. Denote by K_x the field of definition of x and by $\mathcal{O}_x$ its integer ring. The morphism $x : \operatorname{Spec} \bar{K} \to X$ gives rise to a point P_x of X valued in $\mathcal{O}_x$. The pull-back of $\bar{L}$ by P_x is a Hermitian line bundle on $\operatorname{Spec} \mathcal{O}_x$. We denote by

$h_{\bar{L}}(x)$ its normalized Arakelov degree, called the *height* of x. Note that the height function is additive with respect to $\bar{L}$.

(2) Let $\bar{L}$ be a Hermitian line bundle on X. We say that a section $s \in \pi_*(L)$ is *effective* or *strictly effective* if, for any $\sigma : K \to \mathbb{C}$, one has $\|s\|_{\sigma,\sup} \leqslant 1$ or $\|s\|_{\sigma,\sup} < 1$, respectively. We say that the Hermitian line bundle $\bar{L}$ is *effective* if it admits a non-zero effective section.

(3) Let $\bar{L}_1$ and $\bar{L}_2$ be two Hermitian line bundles on X. We say that $\bar{L}_1$ is *smaller* than $\bar{L}_2$ and we define $\bar{L}_1 \leqslant \bar{L}_2$ if the Hermitian line bundle $\bar{L}_1^\vee \otimes \bar{L}_2$ is effective.

(4) We say that a Hermitian line bundle $\bar{A}$ is ample if A is *ample*, $c_1(\bar{A})$ is semipositive in the sense of current on $X(\mathbb{C})$ and $\hat{c}_1(\bar{A}|_Y)^{\dim Y} > 0$ for any integral sub-scheme Y of X which is flat over $\operatorname{Spec} \mathcal{O}_K$. Here the intersection number $\hat{c}_1(\bar{A}|_Y)^{\dim Y}$ is defined in the sense of [**28**] (see [**28**, Lemma 6.5; see also **29**]). Note that there always exists an ample Hermitian line bundle on X. In fact, since X is projective, it can be embedded in a projective space $\mathbb{P}^N$. Then the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ with Fubini–Study metrics on X is ample. Note that the Hermitian line bundle $\bar{A}$ thus constructed has positive *smooth* metrics. Thus, if $\bar{M}$ is an arbitrary Hermitian line bundle with smooth metrics on X, then, for sufficiently large n, $\bar{M} \otimes \bar{A}^{\otimes n}$ is still ample.

(5) We say that a Hermitian line bundle $\bar{N}$ is *vertically nef* if the restriction of N on each fibre of π is nef and $c_1(\bar{N})$ is semipositive in the sense of current on $X(\mathbb{C})$. We say that $\bar{N}$ is *nef* if it is vertically nef and $\hat{c}_1(\bar{N}|_Y)^{\dim Y} \geqslant 0$ for any integral sub-scheme Y of X which is flat over $\operatorname{Spec} \mathcal{O}_K$. By definition, an ample Hermitian line bundle is always nef. Furthermore, if $\bar{A}$ is an ample Hermitian line bundle and if $\bar{N}$ is a Hermitian line bundle such that $\bar{N}^{\otimes n} \otimes \bar{A}$ is ample for any integer $n \geqslant 1$, then $\bar{N}$ is nef. We denote by $\widehat{\mathrm{Nef}}(X)$ the sub-semigroup of $\widehat{\mathrm{Pic}}(X)$ consisting of nef Hermitian line bundles.

(6) If $f : X(\mathbb{C}) \to \mathbb{R}$ is a continuous function which is invariant by the complex conjugation, then we denote by $\bar{\mathcal{O}}(f)$ the Hermitian line bundle on X whose underlying line bundle is trivial, and such that the norm of the unit section $\mathbf{1}$ at $x \in X(\mathbb{C})$ is $e^{-f(x)}$. Note that if f is non-negative, then $\bar{\mathcal{O}}(f)$ is effective. Moreover, for any $a \in \mathbb{R}$, $\bar{\mathcal{O}}(a)$ is nef if and only if $a \geqslant 0$. If $\bar{L}$ is a Hermitian line bundle on X, then we shall use the notation $\bar{L}(f)$ to denote $\bar{L} \otimes \bar{\mathcal{O}}(f)$.

(7) We say that a Hermitian line bundle $\bar{L}$ is *big* if its arithmetic volume $\widehat{\mathrm{vol}}(\bar{L})$ is positive. By [**26**, Corollary 2.4], $\bar{L}$ is big if and only if a positive tensor power of $\bar{L}$ can be written as the tensor product of an ample Hermitian line bundle with an effective one. Furthermore, the analogue of Fujita's approximation theorem holds for big Hermitian line bundles; cf. [**11**, **27**].

(8) The arithmetic volume function $\widehat{\mathrm{vol}}$ is actually a limit (cf. [**11**, Remark 3.9]): one has

$$\widehat{\mathrm{vol}}(\bar{L}) = \lim_{n\to\infty} \frac{\hat{h}^0(X, \bar{L}^{\otimes n})}{n^{d+1}/(d+1)!}.$$

It is (positively) homogeneous of degree $d+1$; namely, for any integer $n \geqslant 1$, one has

$$\widehat{\mathrm{vol}}(\bar{L}^{\otimes n}) = n^{d+1}\widehat{\mathrm{vol}}(\bar{L}).$$

Moreover, it is a birational invariant which is continuous on $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$, and can be continuously extended to $\widehat{\mathrm{Pic}}(X)_{\mathbb{R}}$; cf. [**19**, **20**]. The analogue of Siu's inequality and the log-concavity hold for $\widehat{\mathrm{vol}}$ (cf. [**26**, **27**]); namely,

(a) if $\bar{L}$ and $\bar{M}$ are two Hermitian line bundles on X which are nef, then

$$\widehat{\mathrm{vol}}(\bar{L} \otimes \bar{M}^\vee) \geqslant \hat{c}_1(\bar{L})^{d+1} - (d+1)\hat{c}_1(\bar{L})^d \cdot \hat{c}_1(\bar{M});$$

(b) if $\bar{L}$ and $\bar{M}$ are two Hermitian line bundles on X, then

$$\widehat{\mathrm{vol}}(\bar{L} \otimes \bar{M})^{1/(d+1)} \geqslant \widehat{\mathrm{vol}}(\bar{L})^{1/(d+1)} + \widehat{\mathrm{vol}}(\bar{M})^{1/(d+1)}.$$

REMARK 1. (1) In [**18**, **28**], the notions of ample and nef line bundles were reserved for line bundles with smooth metrics, which is not the case here.

(2) Note that there exists another (non-equivalent) definition of arithmetic volume function in the literature. See [**2**, § 10.1; **9**, § 5] where the 'arithmetic volume' of a Hermitian line bundle $\bar{L}$ was defined as the following number:

$$S(\bar{L}) := \limsup_{n\to+\infty} \frac{\chi(\pi_*(\bar{L}^{\otimes n}))}{n^{d+1}/(d+1)!} \in [-\infty, +\infty[, \tag{2.1}$$

which is also called (logarithmic) *sectional capacity* following the terminology of [**23**]. However, in the analogy between Arakelov geometry and relative algebraic geometry over a regular curve, it is (1.1) that corresponds to the geometric volume function. Note that one always has (see [**26**, § 2.2, Remark 2]) $\widehat{\mathrm{vol}}(\bar{L}) \geqslant S(\bar{L})$, and the equality holds when $\bar{L}$ is nef. In fact, under the assumption of nefness, both quantities are equal to the intersection number $\hat{c}_1(\bar{L})^{d+1}$; see [**22**, Théorème A and Corollaire 3.11; **19**, Corollary 5.5].

In the following, we present some properties of nef line bundles. Note that Propositions 2.1 and 2.2 have been proved in [**18**, § 2] for Hermitian line bundles with smooth metrics. Here we adapt these results to the continuous metric case by using the continuity of intersection numbers.

PROPOSITION 2.1. *Let $\bar{N}$ be a Hermitian line bundle on X which is vertically nef. Assume that, for any $x \in X(\bar{K})$, one has $h_{\bar{N}}(x) \geqslant 0$; then the Hermitian line bundle $\bar{N}$ is nef.*

Proof. Choose an ample Hermitian line bundle $\bar{A}$ on X such that $h_{\bar{A}} : X(\bar{K}) \to \mathbb{R}_+$ has a positive lower bound ε. For any integer $n \geqslant 1$, let $\bar{L}_n := (L_n, (\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}})$ be the tensor product $\bar{N}^{\otimes n} \otimes \bar{A}$. Since $h_{\bar{L}_n} = nh_{\bar{L}} + h_{\bar{A}} \geqslant h_{\bar{A}}$, the height function $h_{\bar{L}_n}$ is bounded from below by the same positive number ε. Note that the metrics of $\bar{L}_n$ are semipositive. By [**17**, Theorem 4.6.1] (see also [**22**, § 3.9]), there exists a sequence of smooth positive metric families $(\alpha_m)_{m\geqslant 1}$ with $\alpha_m = (\|\cdot\|_{\sigma,m})_{\sigma:K\to\mathbb{C}}$, such that $\|\cdot\|_{\sigma,m}$ converges uniformly to $\|\cdot\|_\sigma$ when m tends to infinity. Define $\bar{L}_{n,m} = (L_n, \alpha_m)$. For sufficiently large m, $h_{\bar{L}_{n,m}}$ is bounded from below by $\varepsilon_n/2$. Thus, [**28**, Corollary 5.7] implies that, for any integral sub-scheme Y of X which is flat over $\operatorname{Spec}\mathcal{O}_K$, one has $n^{-\dim Y}\hat{c}_1(\bar{L}_{n,m}|_Y)^{\dim Y} \geqslant 0$. By taking the limit when m and n tend to infinity successively, one obtains $\hat{c}_1(\bar{N}|_Y)^{\dim Y} \geqslant 0$. Therefore, $\bar{N}$ is nef. □

We say that a Hermitian line bundle $\bar{L}$ on X is *integrable* if there exist two ample Hermitian line bundles $\bar{A}_1$ and $\bar{A}_2$ such that $\bar{L} = \bar{A}_1 \otimes \bar{A}_2^\vee$. Denote by $\widehat{\mathrm{Int}}(X)$ the subgroup of $\widehat{\mathrm{Pic}}(X)$ formed by all integrable Hermitian line bundles. If $(\bar{L}_i)_{i=0}^d$ is a family of integrable Hermitian line bundles on X, then the intersection number

$$\hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_d)$$

is defined (see [**17**, § 5, **28**, Lemma 6.5 and **29**, § 1]). Furthermore, it is a symmetric multilinear form which is continuous in each $\bar{L}_i$; namely, for any family $(\bar{M}_i)_{i=0}^d$ of integrable Hermitian line bundles, one has

$$\lim_{n\to+\infty} n^{-d-1}\hat{c}_1(\bar{L}_0^{\otimes n} \otimes \bar{M}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_d^{\otimes n} \otimes \bar{M}_d) = \hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_d).$$

PROPOSITION 2.2. *Let $(\bar{L}_i)_{i=0}^{d-1}$ be a family of nef Hermitian line bundles on X, and let $\bar{M}$ be an integrable Hermitian line bundle on X which is effective. Then*

$$\hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_{d-1}) \cdot \hat{c}_1(\bar{M}) \geqslant 0. \tag{2.2}$$

Proof. Choose an ample Hermitian line bundle $\bar{A}$ on X such that $h_{\bar{A}}$ is bounded from below by a positive number. By virtue of the proof of Proposition 2.1, for any $i \in \{0, \ldots, d-1\}$ and any integer $n \geqslant 1$, there exists a sequence of nef Hermitian line bundles with smooth metrics $(\bar{L}_{i,n}^{(m)})_{m \geqslant 1}$ whose underlying line bundle is $L_i^{\otimes n} \otimes A$ and whose metrics converge uniformly to that of $\bar{L}_i^{\otimes n} \otimes \bar{A}$. By [18, Proposition 2.3], one has

$$\hat{c}_1(\bar{L}_0^{\otimes n} \otimes \bar{A}) \cdot \ldots \cdot \hat{c}_1(\bar{L}_{d-1}^{\otimes n} \otimes \bar{A}) \cdot \hat{c}_1(\bar{M}) \geqslant 0.$$

By taking the limit when n tends to infinity, one obtains (2.2). □

Remark 2. Using the same method, we can prove that if $(\bar{L}_i)_{i=0}^d$ is a family of nef Hermitian line bundles on X, then

$$\hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_d) \geqslant 0. \tag{2.3}$$

Proposition 2.3. *Let $\bar{L}$ be a Hermitian line bundle on X such that $c_1(\bar{L})$ is semipositive in the sense of current on $X(\mathbb{C})$. Assume that there exists an integer $n > 0$ such that $L^{\otimes n}$ is generated by its effective sections. Then the Hermitian line bundle $\bar{L}$ is nef.*

Proof. Since $L^{\otimes n}$ is generated by its sections, the line bundle L is nef relatively to π. By Proposition 2.1, it suffices to verify that, for any $x \in X(\bar{K})$, one has $h_{\bar{L}}(x) \geqslant 0$. For any integer $m \geqslant 1$ let $B_m = \pi_*(L^{\otimes m})$ and let $B_m^{[0]}$ be the sub-$\mathcal{O}_K$-module of B_m generated by effective sections. Since $L^{\otimes n}$ is generated by its effective sections, so is $L^{\otimes np}$ for any integer $p \geqslant 1$. In particular, one has surjective homomorphisms $x^*\pi^*B_{pn,K_x}^{[0]} \to x^*L_{K_x}^{\otimes np}$. By slope inequality (see [4, Appendix A]), one has $nph_{\bar{L}}(x) \geqslant \hat{\mu}_{\min}(\bar{B}_{np}^{[0]})$. By passing to the limit, one obtains $h_{\bar{L}}(x) \geqslant 0$. □

Definition 1. We say that a Hermitian line bundle $\bar{L}$ on X is *free* if $c_1(\bar{L})$ is semipositive in the sense of current on $X(\mathbb{C})$ and if some positive tensor power of L is generated by effective global sections. We denote by $\widehat{\mathrm{Fr}}(X)$ the sub-semigroup of $\widehat{\mathrm{Pic}}(X)$ consisting of free Hermitian line bundles.

Remark 3. (1) By Proposition 2.3, one has $\widehat{\mathrm{Fr}}(X) \subset \widehat{\mathrm{Nef}}(X)$.

(2) Unlike the ampleness, the properties of being big, nef or free are all invariant by birational modifications; that is, if $\nu : X' \to X$ is a birational projective morphism, and if $\bar{L}$ is a Hermitian line bundle on X which is big, nef or free, then so is $\nu^*(\bar{L})$.

3. *Positive intersection product*

In this section, we shall define the positive intersection product for big (not-necessarily integrable) Hermitian line bundles. When all Hermitian line bundles involved are nef, the positive intersection product coincides with the usual intersection product. Furthermore, the highest positive auto-intersection number is just the arithmetic volume of the Hermitian line bundle. We shall use the positive intersection product to interpret the differential of the arithmetic volume function.

3.1. *Admissible decompositions*

Definition 2. Let $\bar{L}$ be a big Hermitian line bundle on X. We define an *admissible decomposition* of $\bar{L}$ as any triple $(\nu, \bar{N}, p)$, where
(1) $\nu : X' \to X$ is a birational projective morphism;
(2) $\bar{N}$ is a free Hermitian line bundle on X';
(3) $p \geqslant 1$ is an integer such that $\nu^*(\bar{L}^{\otimes p}) \otimes \bar{N}^\vee$ is effective.
Denote by $\Theta(\bar{L})$ the set of all admissible decompositions of $\bar{L}$.

We introduce an order relation on the set $\Theta(\bar{L})$. Let $D_i = (\nu_i : X_i \to X, \bar{N}_i, p_i)$ $(i = 1, 2)$ be two admissible decompositions of $\bar{L}$. We say that D_1 is *superior* to D_2, denoted by $D_1 \succ D_2$, if p_2 divides p_1 and if there exists a projective birational morphism $\eta : X_1 \to X_2$ such that $\nu_2\eta = \nu_1$ and $\bar{N}_1 \otimes (\eta^*\bar{N}_2)^{\vee\otimes(p_1/p_2)}$ is effective.

Remark 4. (1) Assume that $D = (\nu : X' \to X, \bar{N}, p)$ is an admissible decomposition of $\bar{L}$. Then, for any birational projective morphism $\eta : X'' \to X'$, the triplet $\eta^*D := (\nu\eta, \eta^*\bar{N}, p)$ is also an admissible decomposition of $\bar{L}$, and one has $\eta^*D \succ D$.

(2) Assume that $D_1 = (\nu, \bar{N}_1, p)$ and $D_2 = (\nu, \bar{N}_2, q)$ are two admissible decompositions of $\bar{L}$ whose underlying birational projective morphisms are the same. Then $D_1 \otimes D_2 := (\nu, \bar{N}_1 \otimes \bar{N}_2, p + q)$ is an admissible decomposition of $\bar{L}$.

(3) Assume that $D = (\nu, \bar{N}, p)$ is an admissible decomposition of $\bar{L}$. Then, for any integer $n \geqslant 1$, one has $D^{\otimes n} \succ D$.

(4) Assume that $\bar{M}$ is an effective Hermitian line bundle on X. By definition, any admissible decomposition of $\bar{L}$ is also an admissible decomposition of $\bar{L} \otimes \bar{M}$.

In the following proposition, we show that the set $\Theta(\bar{L})$ is filtered with respect to the order $\succ$.

Proposition 3.1. *If D_1 and D_2 are two admissible decompositions of $\bar{L}$, then there exists an admissible decomposition D of $\bar{L}$ such that $D \succ D_1$ and $D \succ D_2$.*

Proof. By Remark 4, we may assume that the first and the third components of D_1 and D_2 are the same. Assume that $D_1 = (\nu, \bar{N}_1, p)$ and $D_2 = (\nu, \bar{N}_2, p)$, where $\nu : X' \to X$ is a birational projective morphism. The main idea consists in constructing a suitable birational modification of X' such that the pull-back of the morphisms $\bar{N}_i \to \nu^*\bar{L}^{\otimes p}$ factors through a common free Hermitian line bundle $\bar{N}$. Let $\bar{M}_i = \nu^*\bar{L}^{\otimes p} \otimes \bar{N}_i^\vee$ $(i = 1, 2)$. Since $\bar{M}_1$ and $\bar{M}_2$ are effective, there exist homomorphisms $u_i : M_i^\vee \to \mathcal{O}_{X'}$ corresponding to effective sections $s_i : \mathcal{O}_{X'} \to M_i$ $(i = 1, 2)$. Let $\eta : X'' \to X'$ be the blow-up of the ideal sheaf $\mathrm{Im}(u_1 \oplus u_2)$. Let M be the exceptional line bundle and $s : \mathcal{O}_{X''} \to M$ be the section that trivializes M outside the exceptional divisor. The canonical surjective homomorphism $\eta^*(M_1^\vee \oplus M_2^\vee) \to M^\vee$ induces by duality an injective homomorphism $\varphi : M \to \eta^*(M_1 \oplus M_2)$. We equip $M_1 \oplus M_2$ with metrics $(\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}}$ such that, for any $x \in X''_\sigma(\mathbb{C})$ and any section (u, v) of $M_{1,\sigma} \oplus M_{2,\sigma}$ over a neighbourhood of x, one has $\|(u, v)\|_\sigma(x) = \max\{\|u\|_{\sigma,1}(x), \|v\|_{\sigma,2}(x)\}$. As $\varphi s = (\eta^*s_1, \eta^*s_2)$, and the sections s_1 and s_2 are effective, one obtains that the section s is also effective. Let $\bar{N} = (\nu\eta)^*\bar{L}^{\otimes p} \otimes \bar{M}^\vee$. One has a natural surjective homomorphism

$$\psi : \eta^*N_1 \oplus \eta^*N_2 \longrightarrow N.$$

Furthermore, if we equip $\eta^*N_1 \oplus \eta^*N_2$ with metrics $(\|\cdot\|_\sigma)_{\sigma:K\to\mathbb{C}}$ such that, for any $x \in X''_\sigma(x)$, $\|(u, v)\|_\sigma(x) = \|u\|_\sigma(x) + \|v\|_\sigma(x)$, then the metrics on N are just the quotient metrics by the surjective homomorphism ψ, which are semipositive since the metrics of $\eta^*\bar{N}_1$ and of $\eta^*\bar{N}_2$ are.

As both Hermitian line bundles N_1 and N_2 are generated by effective global sections, so also is $\bar{N}$. Therefore, $(\nu\eta, \bar{N}, p)$ is an admissible decomposition of $\bar{L}$, which is superior to both D_1 and D_2. □

3.2. *Intersection of admissible decompositions*

Let $(\bar{L}_i)_{i=0}^d$ be a family of Hermitian line bundles on X. Let $m \in \{0, \dots, d\}$. Assume that $\bar{L}_i$ is big for $i \in \{0, \dots, m\}$ and is integrable for $i \in \{m+1, \dots, d\}$. For any $i \in \{0, \dots, m\}$, let $D_i = (\nu_i : X_i \to X, \bar{N}_i, p_i)$ be an admissible decomposition of $\bar{L}_i$. Choose a birational projective morphism $\nu : X' \to X$ which factorizes through ν_i for any $i \in \{0, \dots, m\}$. Denote by $\eta_i : X_i \to X$ the projective birational morphism such that $\nu = \nu_i \eta_i$ $(0 \leqslant i \leqslant m)$. Define $(D_0 \cdot \dots \cdot D_m) \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\bar{L}_d)$ as the normalized intersection product

$$\hat{c}_1(\eta_0^* \bar{N}_0) \cdot \dots \cdot \hat{c}_1(\eta_m^* \bar{N}_m) \cdot \hat{c}_1(\nu^* \bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\nu^* \bar{L}_d) \prod_{i=0}^m p_i^{-1}.$$

This definition does not depend on the choice of ν.

PROPOSITION 3.2. *Let $(\bar{L}_i)_{0 \leqslant i \leqslant d}$ be a family of Hermitian line bundles on X. Let $m \in \{0, \dots, d\}$. Assume that $\bar{L}_i$ is big for $i \in \{0, \dots, m\}$, and is nef for $i \in \{m+1, \dots, d\}$. For any $i \in \{0, \dots, m\}$ let D_i and D_i' be two admissible decompositions of $\bar{L}_i$ such that $D_i \succ D_i'$. Then*

$$(D_0 \cdot \dots \cdot D_m) \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\bar{L}_d) \geqslant (D_0' \cdot \dots \cdot D_m') \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\bar{L}_d). \tag{3.1}$$

Proof. By substituting progressively D_i with D_i', it suffices to prove the following particular case:

$$(D_0 \cdot D_1 \cdot \dots \cdot D_m) \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\bar{L}_d) \geqslant (D_0' \cdot D_1 \cdot \dots \cdot D_m) \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\bar{L}_d),$$

which is a consequence of Proposition 2.2. □

COROLLARY 3.3. *With the notation and the assumptions of Proposition* 3.2, *the supremum*

$$\sup\{(D_0 \cdot \dots \cdot D_m) \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\bar{L}_d) \,|\, 0 \leqslant i \leqslant m, D_i \in \Theta(\bar{L}_i)\} \tag{3.2}$$

exists in $\mathbb{R}_{\geqslant 0}$.

Proof. For any $i \in \{0, \dots, m\}$ let $\bar{A}_i$ be an arithmetically ample Hermitian line bundle on X such that $\bar{A}_i \otimes \bar{L}_i^\vee$ is effective. Then all numbers in the set (3.2) are bounded from above by $\hat{c}_1(\bar{A}_0) \cdot \dots \cdot \hat{c}_1(\bar{A}_m) \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\bar{L}_d)$. □

3.3. *Positive intersection product*

Let $(\bar{L}_i)_{i=0}^m$ be a family of big Hermitian line bundles on X, where $0 \leqslant m \leqslant d$. Denote by $\langle \hat{c}_1(\bar{L}_0) \cdot \dots \cdot \hat{c}_1(\bar{L}_m) \rangle$ the function on $\widehat{\mathrm{Nef}}(X)^{d-m}$ which sends a family of nef Hermitian line bundles $(\bar{L}_j)_{j=m+1}^d$ to the supremum

$$\sup\{(D_0 \cdot \dots \cdot D_m) \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \dots \cdot \hat{c}_1(\bar{L}_d) \,|\, 0 \leqslant i \leqslant m,\ D_i \in \Theta(\bar{L}_i)\}.$$

Since all admissible decomposition sets $\Theta(\bar{L}_i)$ are filtered, this function is additive in each $\bar{L}_j$ $(m+1 \leqslant j \leqslant d)$. Thus, it extends naturally to a multilinear function on $\widehat{\mathrm{Int}}(X)^{d-m}$ which we still denote by $\langle \hat{c}_1(\bar{L}_0) \cdot \dots \cdot \hat{c}_1(\bar{L}_m) \rangle$, called the *positive intersection product* of $(\bar{L}_i)_{i=0}^m$.

REMARK 5. (1) Note that the tensor product of a nef Hermitian line bundle with an ample one is always ample, and a tensor power of an ample Hermitian line bundle with sufficiently large exponent is free (see [**28**, Theorem 4.2]). Therefore, by the continuity of the intersection product, if all Hermitian line bundles $(\bar{L}_i)_{i=0}^m$ are nef, then the positive intersection product of them coincides with the usual intersection product. A similar argument shows that if in the definition of admissible decompositions we replace '*free*' by '*nef*',† the corresponding positive intersection product is the same as the function defined above.

(2) The positive intersection product is homogeneous in each variable $\bar{L}_i$ $(0 \leqslant i \leqslant m)$. However, in general it is not additive in each variable. If we consider it as a function on $\widehat{\mathrm{Nef}}(X)^{d-m}$, then it is super-additive in each variable.

(3) Assume that all Hermitian line bundles $(\bar{L}_i)_{i=0}^m$ are the same, that is,

$$\bar{L}_0 = \ldots = \bar{L}_m = \bar{L}.$$

We use the expression $\langle \hat{c}_1(\bar{L})^{m+1} \rangle$ to denote the positive intersection product

$$\Big\langle \underbrace{\hat{c}_1(\bar{L}) \cdot \ldots \cdot \hat{c}_1(\bar{L})}_{m+1 \text{ copies}} \Big\rangle.$$

With this notation, for any $(\bar{L}_j)_{j=m+1}^d \in \widehat{\mathrm{Nef}}(X)^{d-m}$, one has

$$\langle \hat{c}_1(\bar{L})^m \rangle \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \ldots \cdot \hat{c}_1(\bar{L}_d) = \sup_{D \in \Theta(\bar{L})} (D \cdot \ldots \cdot D) \cdot \hat{c}_1(\bar{L}_{m+1}) \cdot \ldots \cdot \hat{c}_1(\bar{L}_d).$$

This equality comes from the fact that the ordered set $\Theta(\bar{L})$ is filtered (Proposition 3.1) and from the comparison (3.1).

The following result shows that the positive auto-intersection product of a big Hermitian line bundle coincides with the arithmetic volume function.

THEOREM 3.4. *Let $\bar{L}$ be a big Hermitian line bundle on X. One has*

$$\langle \hat{c}_1(\bar{L})^{d+1} \rangle = \widehat{\mathrm{vol}}(\bar{L}).$$

Proof. By definition, one has

$$\langle \hat{c}_1(\bar{L})^{d+1} \rangle = \sup_{D \in \Theta(\bar{L})} \Big(\underbrace{D \cdot \ldots \cdot D}_{d+1 \text{ copies}} \Big).$$

For any admissible decomposition $D = (\nu, \bar{N}, p)$ of $\bar{L}$, one has

$$\Big(\underbrace{D \cdot \ldots \cdot D}_{d+1 \text{ copies}} \Big) = p^{-d-1} \hat{c}_1(\nu^* \bar{N})^{d+1} = p^{-d-1} \widehat{\mathrm{vol}}(\bar{N}).$$

By Fujita's approximation theorem in Arakelov geometry (see [**11**, **27**]), we obtain the result. □

REMARK 6. The above result could be considered as a reformulation of the arithmetic Fujita approximation theorem. We establish further in Corollary 4.4 that one has $\langle \hat{c}_1(\bar{L})^d \rangle \cdot \hat{c}_1(\bar{L}) =$

†Which is the usual choice in the geometric setting. Here in the arithmetic setting, our choice is for technical considerations, notably for the simplification of several proofs.

$\widehat{\mathrm{vol}}(\bar{L})$. Although apparently similar to Theorem 3.4, this inequality should be interpreted as an asymptotic orthogonality of Fujita approximation to Zariski decomposition in the arithmetic setting. Its proof relies on the differentiability of the arithmetic volume function, discussed in the next section.

LEMMA 3.5. *Let $(\bar{L}_i)_{i=0}^m$ be a family of big Hermitian line bundles on X, where $m \in \{0, \ldots, d\}$. For any $i \in \{0, \ldots, m\}$, let $\bar{M}_i$ be an effective Hermitian line bundle on X and let $\bar{N}_i = \bar{L}_i \otimes \bar{M}_i$. Then one has*

$$\langle \hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m) \rangle \leqslant \langle \hat{c}_1(\bar{N}_0) \cdot \ldots \cdot \hat{c}_1(\bar{N}_m) \rangle, \tag{3.3}$$

where we have considered the positive intersection products as functions on $\widehat{\mathrm{Nef}}(X)^{d-m}$.

Proof. By Remark 4(4), if D_i is an admissible decomposition of $\bar{L}_i$, then it is also an admissible decomposition of $\bar{N}_i$. Hence, by the definition of positive intersection product, the inequality (3.3) is true. □

The following proposition shows that the positive intersection product is continuous in each variable.

PROPOSITION 3.6. *Let $(\bar{L}_i)_{0 \leqslant i \leqslant m}$ be a family of big Hermitian line bundles on X, where $m \in \{0, \ldots, d\}$. Let $(\bar{M}_i)_{0 \leqslant i \leqslant m}$ be a family of Hermitian line bundles on $\mathscr{X}$. Then one has*

$$\lim_{n \to \infty} n^{-m} \langle \hat{c}_1(\bar{L}_0^{\otimes n} \otimes \bar{M}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m^{\otimes n} \otimes \bar{M}_m) \rangle = \langle \hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m) \rangle \tag{3.4}$$

as functions on $\widehat{\mathrm{Int}}(X)^{d-m}$.

Proof. We consider first both positive intersection products as functions on $\widehat{\mathrm{Nef}}(X)$. Let $\alpha_n = \langle \hat{c}_1(\bar{L}_0^{\otimes n} \otimes \bar{M}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m^{\otimes n} \otimes \bar{M}_m) \rangle$. Since $\bar{L}_i$ is big, there exists an integer $q \geqslant 1$ such that the Hermitian line bundles $\bar{L}_i^{\otimes q} \otimes \bar{M}_i$ and $\bar{L}_i^{\otimes q} \otimes \bar{M}_i^\vee$ are both effective. Thus, Lemma 3.5 implies that

$$\begin{aligned}
\alpha_n &\geqslant \langle \hat{c}_1(\bar{L}_0^{\otimes (n-q)}) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m^{\otimes (n-q)}) \rangle = (n-q)^m \langle \hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m) \rangle, \\
\alpha_n &\leqslant \langle \hat{c}_1(\bar{L}_0^{\otimes (n+q)}) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m^{\otimes (n+q)}) \rangle = (n+q)^m \langle \hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m) \rangle.
\end{aligned}$$

By passing to the limit, we obtain (3.4) as an equality of functions on $\widehat{\mathrm{Nef}}(X)^{d-m}$. The general case follows from the multilinearity. □

REMARK 7. Proposition 3.6 implies in particular that if $(f_n^{(i)})_{n \geqslant 1}$ $(i = 0, 1, \ldots, m)$ are families of continuous functions on $X(\mathbb{C})$ which converge uniformly to zero, then one has

$$\lim_{n \to +\infty} \langle \hat{c}_1(\bar{L}_0(f_n^{(0)})) \cdot \ldots \cdot \hat{c}_m(\bar{L}_m(f_n^{(m)})) \rangle = \langle \hat{c}_1(\bar{L}_0) \cdot \ldots \cdot \hat{c}_m(\bar{L}_m) \rangle.$$

In particular, the mapping

$$t \longmapsto \langle \hat{c}_1(\bar{L}_0(t)) \cdot \ldots \cdot \hat{c}_1(\bar{L}_m(t)) \rangle$$

is continuous on the (open) interval on which it is well defined.

Proposition 3.7. *Let $(\bar{L}_i)_{i=0}^{d-1}$ be a family of big Hermitian line bundles on X. If $\bar{M}$ is an effective integrable Hermitian line bundle on X, then*

$$\langle \hat{c}_1(\bar{L}_0)\cdot \ldots \cdot \hat{c}_1(\bar{L}_{d-1})\rangle \cdot \hat{c}_1(\bar{M}) \geqslant 0.$$

Proof. This is a direct consequence of Proposition 2.2. □

Remark 8. Proposition 3.7 permits us to extend the function

$$\langle \hat{c}_1(\bar{L}_0)\cdot \ldots \cdot \hat{c}_1(\bar{L}_{d-1})\rangle$$

on $\widehat{\mathrm{Pic}}(X)$. Let $\bar{M}$ be an arbitrary Hermitian line bundle on X. By the Weierstrass–Stone theorem, there exists a sequence $(f_n)_{n\geqslant 1}$ of continuous functions on $X(\mathbb{C})$ which converges uniformly to 0, and such that $\bar{M}(f_n)$ has smooth metrics for any n. Thus, $\bar{M}(f_n)$ is integrable and $a_n = \langle \hat{c}_1(\bar{L}_0)\cdot \ldots \cdot \hat{c}_1(\bar{L}_{d-1})\rangle \cdot \hat{c}_1(\bar{M}(f_n))$ is well defined. Let $\varepsilon_{n,m} = \|f_n - f_m\|_{\sup}$. Choose an ample Hermitian line bundle $\bar{A}$ such that $\bar{A}\otimes \bar{L}_i^\vee$ is effective for any $i\in\{0,\ldots,d-1\}$. Note that

$$\begin{aligned} a_n - a_m &= \langle \hat{c}_1(\bar{L}_0)\cdot \ldots \cdot \hat{c}_1(\bar{L}_{d-1})\rangle \cdot \hat{c}_1(\bar{\mathcal{O}}(f_n - f_m)) \\ &\leqslant \langle \hat{c}_1(\bar{L}_0)\cdot \ldots \cdot \hat{c}_1(\bar{L}_{d-1})\rangle \cdot \hat{c}_1(\bar{\mathcal{O}}(\varepsilon_{n,m})) \\ &\leqslant \langle \hat{c}_1(\bar{A})^d\rangle \cdot \hat{c}_1(\bar{\mathcal{O}}(\varepsilon_{n,m})) = \varepsilon_{n,m} c_1(A_K)^d. \end{aligned} \tag{3.5}$$

By interchanging the roles of n and m in (3.5) and then combining the two inequalities, one obtains $|a_n - a_m| \leqslant \varepsilon_{n,m} c_1(A_K)^d$. Therefore, $(a_n)_{n\geqslant 1}$ is a Cauchy sequence that converges to a real number which we denote by $\langle \hat{c}_1(\bar{L}_0)\cdot \ldots \cdot \hat{c}_1(\bar{L}_{d-1})\rangle \cdot \hat{c}_1(\bar{M})$. By an argument similar to the inequality (3.5), this definition does not depend on the choice of the sequence $(f_n)_{n\geqslant 1}$. The extended function is additive on $\widehat{\mathrm{Pic}}(X)$, non-negative on the subgroup of effective Hermitian line bundles and satisfies the equality (3.4) (of functions on $\widehat{\mathrm{Pic}}(X)$).

4. *Differentiability of the arithmetic volume function*

We establish the differentiability of the arithmetic volume function. In this section, X denotes an arithmetic variety of relative dimension d over $\operatorname{Spec}\mathcal{O}_K$.

4.1. *Differentiability criterion*

We begin with introducing the notion of differentiability of homogeneous functions defined on the semigroup of big Hermitian line bundles. For the consideration of lucidity, we choose to work in the setting of general semigroups.

Notation. Let G be a commutative group whose composition law is written additively. Let C be a sub-semigroup of G and let H be a subgroup of G.

(1) We say that the sub-semigroup C is *open* with respect to H if, for any $x\in C$ and any $v\in H$, there exists an integer $n_0>0$ such that $n_0x+v\in C$ (note that this implies in particular that $nx+v\in C$ for any integer $n\geqslant n_0$).

(2) Let $\delta\geqslant 1$ be an integer. We say that a function $f: C\to\mathbb{R}$ is (positively) *homogeneous* of degree δ if, for any $x\in C$ and any integer $n\geqslant 1$, one has $f(nx)=n^\delta f(x)$. In particular, a homogeneous function of degree 1 is simply said to be *homogeneous*.

(3) Assume that the sub-semigroup C is open with respect to H. Let $x\in C$ and $v\in H$. We say that a homogeneous function $f: C\to\mathbb{R}$ of degree δ is *differentiable* at x along v if the

sequence†

$$\frac{f(nx+v)-f(nx)}{n^{\delta-1}}, \quad n \text{ sufficiently large},$$

converges in $\mathbb{R}$. In this case, we denote the limit by $Df(x)(v)$ or $D_v f(x)$. We say that the function f is differentiable at x with respect to H if it is differentiable at x along any element in H and if the function $Df(x): H \to \mathbb{R}$ is a morphism of groups.

Example 1. Note that the set $\widehat{\mathrm{Pic}}(X)$ together with the tensor product law forms a group. The subset $\widehat{\mathrm{Big}}(X)$ of big Hermitian line bundles is a sub-semigroup of $\widehat{\mathrm{Pic}}(X)$. Moreover, it is open with respect to $\widehat{\mathrm{Pic}}(X)$. This is a consequence of [**26**, Corollary 2.4].

Remark 9. Let G be a commutative group, H be a subgroup of G and C be a sub-semigroup of G, assumed to be open with respect to H.

(1) If $f: C \to \mathbb{R}$ is a non-negative homogeneous function of degree δ, where δ is a positive integer, then $f^{1/\delta}$ is homogeneous of degree 1. In particular, the function $\widehat{\mathrm{vol}}^{1/(d+1)}$ is homogeneous of degree 1 on $\widehat{\mathrm{Big}}(X)$.

(2) Assume that $f: C \to \mathbb{R}$ is a homogeneous (degree 1) function which is super-additive; namely, for all $x, y \in C$, one has $f(x+y) \geqslant f(x) + f(y)$. Then, for any $x \in C$ and any $v \in H$, the sequence $n \mapsto f(nx+v) - f(nx)$ is increasing. Hence, it converges to an element in $\mathbb{R} \cup \{+\infty\}$, denoted by $D_v f(x)$. In particular, by the log-concavity of the arithmetic volume function (cf. [**27**, Theorem B]), the function $\widehat{\mathrm{vol}}^{1/(d+1)}$ is super-additive on $\widehat{\mathrm{Big}}(X)$ and hence $D_{\bar{M}} \widehat{\mathrm{vol}}^{1/(d+1)}(\bar{L})$ is well defined in $\mathbb{R} \cup \{+\infty\}$ for any $\bar{L} \in \widehat{\mathrm{Big}}(X)$ and any $\bar{M} \in \widehat{\mathrm{Pic}}(X)$.

(3) Assume that $f: C \to \mathbb{R}$ is a homogeneous function (of degree 1) which is differentiable at $x \in C$ with respect to H. Let $\delta \geqslant 1$ be an integer and $g = f^\delta$. Then the function g is also differentiable at x with respect to H. In fact, for any $v \in H$, one has

$$\begin{aligned}
&\lim_{n\to\infty} \frac{f(nx+v)^\delta - f(nx)^\delta}{n^{\delta-1}} \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{\delta} \frac{f(nx+v)^i f(nx)^{\delta-i} - f(nx+v)^{i-1} f(nx)^{\delta-i+1}}{n^{\delta-1}} \\
&\quad = \sum_{i=1}^{\delta} \left(\lim_{n\to\infty} \frac{f(nx+v)^{i-1} f(nx)^{\delta-i}}{n^{\delta-1}} \right) \left(\lim_{n\to\infty} f(nx+v) - f(nx) \right) \\
&\quad = \delta f(x)^{\delta-1} D_v f(x).
\end{aligned}$$

This formula also shows that $Dg(x) = \delta f(x)^{\delta-1} Df(x)$.

(4) The same argument as in (3) shows that if $f: C \to \mathbb{R}$ is a homogeneous function that is non-negative and super-additive and if $g = f^\delta$ (δ being a positive integer), then, for any $x \in C$ and any $v \in H$, the limit

$$D_v g(x) := \lim_{n\to\infty} \frac{g(nx+v) - g(nx)}{n^{\delta-1}}$$

exists in $\mathbb{R} \cup \{+\infty\}$, and one has $Dg(x) = \delta f(x)^{\delta-1} Df(x)$. Here we use the fact that

$$\lim_{n\to\infty} \frac{f(nx+v)}{n} = f(x),$$

†This definition is 'justified' by the following formal equality (here we use the homogeneousness of f):

$$(f(nx+v) - f(nx))/n^{\delta-1} = (f(x+n^{-1}v) - f(x))/n^{-1}.$$

if $D_v f(x) < +\infty$. In particular, the function $D\widehat{\mathrm{vol}}(\bar{L}) : \widehat{\mathrm{Pic}}(X) \to \mathbb{R} \cup \{+\infty\}$ is well defined since $\widehat{\mathrm{vol}}^{1/(d+1)}$ is super-additive.

The following proposition is a differentiability criterion.

PROPOSITION 4.1. *Let G be a commutative group, H be a subgroup of G and C be a sub-semigroup of G that is open with respect to H.*

(1) *If $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a super-additive function which is not identically $+\infty$ and if $\psi : H \to \mathbb{R}$ is a morphism of groups such that $\varphi \geqslant \psi$, then one has $\varphi = \psi$.*

(2) *Let $f : C \to \mathbb{R}$ be a positive, super-additive and homogeneous function, and $g = f^\delta$ where δ is a positive integer. Let x be an element of $C \cap H$. Assume that the function $Dg(x) : H \to \mathbb{R} \cup \{+\infty\}$ is bounded from below by a group homomorphism from H to $\mathbb{R}$; then the function g is differentiable at x with respect to H.*

Proof. (1) Let $\eta = \varphi - \psi$. The function $\eta : H \to \mathbb{R} \cup \{+\infty\}$ is non-negative and super-additive. Denote by θ the neutral element of H. Assume that $x \in H$ is an element such that $\eta(x) > 0$. Then $\eta(\theta) = \eta(x + (-x)) \geqslant \eta(x) + \eta(-x) > 0$. Moreover, the function η is not identically infinite. There exists $y \in H$ such that $\eta(y) < +\infty$. Since $\eta(y) = \eta(y + \theta) \geqslant \eta(y) + \eta(\theta)$, one obtains $\eta(\theta) \leqslant 0$, which leads to a contradiction.

(2) Since the function f is super-additive, so is $Df(x)$. In fact, if v and w are two elements in H, then, for sufficiently positive integer n, one has

$$f(2nx + v + w) - f(2nx) \geqslant f(nx + v) - f(nx) + f(nx + w) - f(nx).$$

By taking the limit when n tends to infinity, we obtain $D_{v+w} f(x) \geqslant D_v f(x) + D_w f(x)$. By definition, one has $D_x f(x) = f(x) < +\infty$. By Remark 9(4), one has $Dg(x) = \delta f(x)^{\delta-1} Df(x)$. So $Dg(x)$ is super-additive and $D_x g(x) < +\infty$. Hence, by (1) one obtains that $Dg(x)$ is a group homomorphism. □

4.2. *Proof of the main theorem*

We shall prove Theorem A by using the differentiability criterion above. The main idea is to establish a suitable lower bound of $D\widehat{\mathrm{vol}}(\bar{L})$. We begin with the following lemma, which is analogous to [7, Corollary 3.4].

LEMMA 4.2. *Let $\bar{L}$ and $\bar{N}$ be two nef Hermitian line bundles on X. Let $\bar{M}$ be an integrable Hermitian line bundle on X. Assume that $\bar{M} \otimes \bar{N}$ and $\bar{M}^\vee \otimes \bar{N}$ are nef and that $\bar{L}^\vee \otimes \bar{N}$ is effective. Then there exists a constant $C > 0$ depending only on d such that*

$$\widehat{\mathrm{vol}}(\bar{L}^{\otimes n} \otimes \bar{M}) \geqslant n^{d+1} \widehat{\mathrm{vol}}(\bar{L}) + (d+1) n^d \hat{c}_1(\bar{L})^d \cdot \hat{c}_1(\bar{M}) - C \widehat{\mathrm{vol}}(\bar{N}) n^{d-1}.$$

Proof. Recall that (see [26, Theorem 2.2; see also 19, Theorem 5.6]) if $\bar{A}$ and $\bar{B}$ are two nef Hermitian line bundles on X, then

$$\widehat{\mathrm{vol}}(\bar{B} \otimes \bar{A}^\vee) \geqslant \hat{c}_1(\bar{B})^{d+1} - (d+1) \hat{c}_1(\bar{B})^d \cdot \hat{c}_1(\bar{A}). \tag{4.1}$$

Let $\bar{B} = \bar{L}^{\otimes n} \otimes \bar{M} \otimes \bar{N}$. It is a nef Hermitian line bundle on X. If one applies (4.1) to $\bar{B}$ and to $\bar{A} = \bar{N}$, then one obtains

$$\begin{aligned} \widehat{\mathrm{vol}}(\bar{L}^{\otimes n} \otimes \bar{M}) &= \widehat{\mathrm{vol}}(\bar{B} \otimes \bar{N}^\vee) \geqslant \hat{c}_1(\bar{B})^{d+1} - (d+1) \hat{c}_1(\bar{B})^d \cdot \hat{c}_1(\bar{N}) \\ &= n^{d+1} \hat{c}_1(\bar{L})^{d+1} + (d+1) n^d \hat{c}_1(\bar{L})^d \cdot \hat{c}_1(\bar{M}) + O(n^{d-1}), \end{aligned}$$

where the implied constant is a linear combination of intersection numbers of Hermitian line bundles of the form $\bar{L}$ or $\bar{M} \otimes \bar{N}$, and hence can be bounded from above by a multiple of $\hat{c}_1(\bar{N})^{d+1} = \widehat{\mathrm{vol}}(\bar{N})$, according to Proposition 2.2. □

THEOREM 4.3. *The function* $\widehat{\mathrm{vol}}$ *is differentiable at any point of* $\widehat{\mathrm{Big}}(X)$ *with respect to* $\widehat{\mathrm{Pic}}(X)$. *Moreover, for any* $\bar{L} \in \widehat{\mathrm{Big}}(X)$ *and any* $\bar{M} \in \widehat{\mathrm{Pic}}(X)$, *one has*

$$D_{\bar{M}}\widehat{\mathrm{vol}}(\bar{L}) = (d+1)\langle \hat{c}_1(\bar{L})^d\rangle \cdot \hat{c}_1(\bar{M}). \tag{4.2}$$

Proof. By Proposition 4.1(2) (applied to $f = \widehat{\mathrm{vol}}^{1/(d+1)}$ and $g = \widehat{\mathrm{vol}} = f^{d+1}$), it suffices to prove that the function $D\widehat{\mathrm{vol}}(\bar{L})$ is bounded from below by the group homomorphism

$$\bar{M} \longmapsto (d+1)\langle \hat{c}_1(\bar{L})^d\rangle \cdot \hat{c}_1(\bar{M}).$$

Let $\bar{L}$ be a big Hermitian line bundle and $\bar{M}$ be an integrable Hermitian line bundle on X. Choose a nef Hermitian line bundle $\bar{N}$ on X such that $\bar{M} \otimes \bar{N}$ and $\bar{M}^\vee \otimes \bar{N}$ are nef and $\bar{L}^\vee \otimes \bar{N}$ is effective. Note that, for any admissible decomposition $D = (\nu, \bar{A}, p)$ of $\bar{L}$, the Hermitian line bundles $\nu^*\bar{M}^{\otimes p} \otimes \nu^*\bar{N}^{\otimes p}$ and $\nu^*\bar{M}^{\vee\otimes p} \otimes \nu^*\bar{N}^{\otimes p}$ are nef, and $\bar{A}^\vee \otimes \nu^*\bar{N}^{\otimes p}$ is effective. Lemma 4.2 applied to $\bar{A}$, $\bar{N}^{\otimes p}$ and $\bar{M}^{\otimes p}$ shows that

$$\begin{aligned}\widehat{\mathrm{vol}}(\bar{L}^{\otimes n} \otimes \bar{M}) &\geqslant p^{-(d+1)}\widehat{\mathrm{vol}}(\bar{A}^{\otimes n} \otimes \bar{M}^{\otimes p})\\ &\geqslant (n/p)^{d+1}\widehat{\mathrm{vol}}(\bar{A}) + (d+1)(n/p)^d(D \cdot \ldots \cdot D) \cdot \hat{c}_1(\bar{M}) - C\widehat{\mathrm{vol}}(\bar{N})n^{d-1}.\end{aligned}$$

Since D is arbitrary, one obtains

$$\widehat{\mathrm{vol}}(\bar{L}^{\otimes n} \otimes \bar{M}) \geqslant n^{d+1}\widehat{\mathrm{vol}}(\bar{L}) + (d+1)\langle \hat{c}_1(\bar{L})^d\rangle \cdot \hat{c}_1(\bar{M})n^d - C\widehat{\mathrm{vol}}(\bar{N})n^{d-1},$$

$$\frac{\widehat{\mathrm{vol}}(\bar{L}^{\otimes n} \otimes \bar{M}) - \widehat{\mathrm{vol}}(\bar{L}^{\otimes n})}{n^d} \geqslant (d+1)\langle \hat{c}_1(\bar{L})^d\rangle \cdot \hat{c}_1(\bar{M}) - C\widehat{\mathrm{vol}}(\bar{N})n^{-1}.$$

By taking the limit, we obtain the lower bound

$$D_{\bar{M}}\widehat{\mathrm{vol}}(\bar{L}) \geqslant (d+1)\langle \hat{c}_1(\bar{L})^d\rangle \cdot \hat{c}_1(\bar{M}).$$

In the case where $\bar{M}$ is a general Hermitian line bundle, we choose a sequence of continuous non-negative functions $(\varphi_m)_{m\geqslant 1}$ on $X(\mathbb{C})$ which converges uniformly to 0 and such that each Hermitian line bundle $\bar{M}(-\varphi_m)$ is integrable.† Thus, by the lower bound established above, we obtain

$$D\widehat{\mathrm{vol}}(\bar{L})(\bar{M}) \geqslant D\widehat{\mathrm{vol}}(\bar{L})(\bar{M}(-\varphi_m)) \geqslant (d+1)\langle \hat{c}_1(\bar{L})^d\rangle \cdot \hat{c}_1(\bar{M}(-\varphi_m)).$$

By passing to the limit, we obtain the required lower bound. □

A direct consequence of Theorem 4.3 is the asymptotic orthogonality of arithmetic Fujita approximation.

COROLLARY 4.4. *Assume that* $\bar{L}$ *is a big Hermitian line bundle on* X. *One has*

$$\langle \hat{c}_1(\bar{L})^d\rangle \cdot \hat{c}_1(\bar{L}) = \widehat{\mathrm{vol}}(\bar{L}). \tag{4.3}$$

†We can arrange, for example, that each Hermitian line bundle $\bar{M}(-\varphi_m)$ has smooth metrics.

Proof. By definition

$$D_{\bar{L}}\widehat{\mathrm{vol}}(\bar{L}) = \lim_{n\to+\infty}\frac{\widehat{\mathrm{vol}}(\bar{L}^{\otimes(n+1)}) - \widehat{\mathrm{vol}}(\bar{L}^{\otimes n})}{n^d}$$
$$= \widehat{\mathrm{vol}}(\bar{L})\lim_{n\to+\infty}\frac{(n+1)^{d+1}-n^{d+1}}{n^d} = (d+1)\widehat{\mathrm{vol}}(\bar{L}).$$

So (4.3) follows from Theorem 4.3. □

Remark 10. As mentioned in Section 1, the differentiability of the geometric volume function can be obtained by using the method of Okounkov bodies developed in [**21**]. See [**16**] for a proof of this result and other interesting results concerning geometric volume functions. Recently, Boucksom and the current author have proposed an analogue of Lazarsfeld and Musţată's construction for filtered linear series (compare with a previous construction of Yuan [**27**]). Hopefully, this will lead to an alternative proof of the differentiability of the arithmetic volume function.

4.3. *Lower bound of the positive intersection product*

Our differentiability result permits us to obtain a lower bound for positive intersection products of the form $\langle\hat{c}_1(\bar{L})^d\rangle\cdot\hat{c}_1(\bar{M})$, where $\bar{L}$ is a big Hermitian line bundle on X and $\bar{M}$ is an effective Hermitian line bundle on X, by using the log-concavity of the arithmetic volume function proved in [**27**].

Proposition 4.5. *Let $\bar{L}$ and $\bar{M}$ be two Hermitian line bundles on X. Assume that $\bar{L}$ is big and $\bar{M}$ is effective. Then*

$$\langle\hat{c}_1(\bar{L})^d\rangle\cdot\hat{c}_1(\bar{M}) \geqslant \mathrm{vol}(\bar{L})^{d/(d+1)}\mathrm{vol}(\bar{M})^{1/(d+1)}. \tag{4.4}$$

Proof. Theorem 4.3 shows that

$$\lim_{n\to+\infty}\frac{\mathrm{vol}(\bar{L}^{\otimes n}\otimes\bar{M}) - \mathrm{vol}(\bar{M})}{n^d} = (d+1)\langle\hat{c}_1(\bar{L})^d\rangle\cdot\hat{c}_1(\bar{M}).$$

By [**27**, Theorem B], one has

$$\widehat{\mathrm{vol}}(\bar{L}^{\otimes n}\otimes\bar{M}) \geqslant (\widehat{\mathrm{vol}}(\bar{L}^{\otimes n})^{1/(d+1)} + \widehat{\mathrm{vol}}(\bar{M})^{1/(d+1)})^{d+1}.$$

By taking the limit, we obtain the required inequality. □

Remark 11. The inequality (4.4) could be considered as an analogue in Arakelov geometry (suggested by Bertrand [**3**]) of the *isoperimetric inequality* proved by Federer [**12**, 3.2.43]. See [**13**, § 5.4] for an interpretation in terms of intersection theory, and [**3**, § 1.2] for an analogue in the geometry of numbers.

4.4. *Comparison to other differentiability results*

The differentiability of the arithmetic volume function can be compared with several results in the literature which can be interpreted as the differentiability of arithmetic invariants.

Intersection number. Recall that the self-intersection number $\hat{c}_1(\bar{L})^{d+1}$ is well defined for integrable Hermitian line bundles $\bar{L}$; see [**14**, **28**, **29**]. Furthermore, it is a polynomial function.

Therefore, for any integrable Hermitian line bundles $\bar{L}$ and $\bar{M}$, one has

$$\lim_{n\to+\infty}\frac{\hat{c}_1(\bar{L}^{\otimes n}\otimes\bar{M})^{d+1}-\hat{c}_1(\bar{L}^{\otimes n})}{n^d}=(d+1)\hat{c}_1(\bar{L})^d\cdot\hat{c}_1(\bar{M}).$$

This formula shows that the intersection number is differentiable at $\bar{L}$ along all directions in $\widehat{\mathrm{Int}}(X)$.

Sectional capacity. By using the analogue of Siu's inequality in Arakelov geometry, Yuan [**26**] has actually proved that the sectional capacity S (see Remark 1) is differentiable at any Hermitian line bundle $\bar{L}$ such that L is ample and the metrics of $\bar{L}$ are semipositive. Furthermore, for such $\bar{L}$, one has

$$D_{\bar{M}}S(\bar{L}):=\lim_{n\to+\infty}\frac{S(\bar{L}^{\otimes n}\otimes\bar{M})-S(\bar{L}^{\otimes n})}{n^d}=(d+1)\hat{c}_1(\bar{L})^d\cdot\hat{c}_1(\bar{M}),$$

where $\bar{M}$ is an arbitrary integrable Hermitian line bundle. This result has been established by Autissier [**1**] in the case where $d=1$. Recently, Berman and Boucksom [**2**] have proved a general differentiability result for sectional capacity. They have proved that the function S is differentiable along the directions defined by continuous functions on $X(\mathbb{C})$ on the cone of generically big Hermitian line bundles; namely, for any continuous function f on $X(\mathbb{C})$ and any Hermitian line bundle $\bar{L}$ on X, such that L_K is big and $S(\bar{L})$ is finite, the limit

$$\lim_{n\to+\infty}\frac{S(\bar{L}^{\otimes n}(f))-S(\bar{L}^{\otimes n})}{n^d}$$

exists. They have also computed explicitly the differential in terms of the Monge–Ampère measure of $\bar{L}$ (see [**2**, Theorem 5.7 and Remark 5.8]).

5. *Applications*

In this section, we discuss two applications of the differentiability of the arithmetic volume function: the computation of the asymptotic measure and a conceptual interpretation of the variational principle in equidistribution problems.

5.1. *Asymptotic measure*

Let $\bar{L}$ be a Hermitian line bundle on X such that L_K is big. The *asymptotic measure* of $\bar{L}$ is the vague limit in the space of Borel probability measures:

$$\nu_{\bar{L}}:=-\lim_{n\to+\infty}\frac{d}{dt}\frac{\mathrm{rk}(\mathrm{Vect}_K(\{s\in\pi_*L^{\otimes n}\mid\max_\sigma\|s\|_{\sigma,\sup}\leqslant e^{-nt}\}))}{\mathrm{rk}(\pi_*L^{\otimes n})},\tag{5.1}$$

where the derivative is taken in the sense of distribution.

Note that the support of the probability measure $\nu_{\bar{L}}$ is contained in $]-\infty,\hat{\mu}^\pi_{\max}(\bar{L})]$, where $\hat{\mu}^\pi_{\max}(\bar{L})$ is the limit of maximal slopes (see [**10**, Theorem 4.1.8]):

$$\hat{\mu}^\pi_{\max}(\bar{L}):=\lim_{n\to+\infty}\frac{\hat{\mu}_{\max}(\pi_*\bar{L}^{\otimes n})}{n}.$$

By definition, one has $\hat{\mu}^\pi_{\max}(\bar{L}(a))=\hat{\mu}^\pi_{\max}(\bar{L})+a$ for any $a\in\mathbb{R}$. Moreover, $\hat{\mu}^\pi_{\max}(\bar{L})>0$ if and only if $\bar{L}$ is big (see [**11**, Proposition 3.11]). Therefore, $\hat{\mu}^\pi_{\max}(\bar{L})$ is also the infimum of all real numbers ε such that $\bar{L}(-\varepsilon)$ is big.

Several arithmetic invariants of $\bar{L}$ can be represented as integrals with respect to $\nu_{\bar{L}}$. In the following, we discuss some examples. The *asymptotic positive slope* of $\bar{L}$ is defined as

$$\hat{\mu}^\pi_+(\bar{L}):=\frac{1}{[K:\mathbb{Q}]}\frac{\widehat{\mathrm{vol}}(\bar{L})}{(d+1)\mathrm{vol}(L_K)}.$$

It has the following integral form (see [11, Theorem 3.8]):

$$\hat{\mu}_+^\pi(\bar{L}) = \int_{\mathbb{R}} \max(x,0)\,\nu_{\bar{L}}(dx). \tag{5.2}$$

More generally, for any $a \in \mathbb{R}$, one has

$$\int_{\mathbb{R}} \max(x-a,0)\nu_{\bar{L}}(dx) = \hat{\mu}_+^\pi(\bar{L}(-a)). \tag{5.3}$$

Another example is the *asymptotic slope* of $\bar{L}$, which is

$$\hat{\mu}^\pi(\bar{L}) := \frac{1}{[K:\mathbb{Q}]}\frac{S(\bar{L})}{(d+1)\mathrm{vol}(L_K)} \in [-\infty,+\infty[,$$

where $S(\bar{L})$ is the sectional capacity of $\bar{L}$ as in (2.1). One has

$$\hat{\mu}^\pi(\bar{L}) \leqslant \int_{\mathbb{R}} x\,\nu_{\bar{L}}(dx).$$

The equality holds when L_K is ample. Observe that

$$\hat{\mu}_{\max}^\pi(\bar{L}) \geqslant \hat{\mu}_+^\pi(\bar{L}) \geqslant \hat{\mu}^\pi(\bar{L}). \tag{5.4}$$

Using Theorem 4.3 and the differentiability of geometric volume function in [7], we prove that the asymptotic positive slope $\hat{\mu}_+^\pi$ is differentiable and calculate its differential.

PROPOSITION 5.1. *Assume that $\bar{L}$ is a big Hermitian line bundle on X. For any Hermitian line bundle $\bar{M}$ the limit*

$$D_{\bar{M}}\hat{\mu}_+^\pi(\bar{L}) := \lim_{n\to+\infty}(\hat{\mu}_+^\pi(\bar{L}^{\otimes n}\otimes \bar{M}) - \hat{\mu}_+^\pi(\bar{L}^{\otimes n}))$$

exists in $\mathbb{R}$. Furthermore, one has

$$D_{\bar{M}}\hat{\mu}_+^\pi(\bar{L}) := \frac{\langle \hat{c}_1(\bar{L})^d\rangle\cdot\hat{c}_1(\bar{M})}{[K:\mathbb{Q}]\mathrm{vol}(L_K)} - \frac{d\langle c_1(L_K)^{d-1}\rangle\cdot c_1(M_K)}{\mathrm{vol}(L_K)}\hat{\mu}_+^\pi(\bar{L}),$$

where $\langle c_1(L_K)^{d-1}\rangle\cdot c_1(M_K)$ is the geometric positive intersection product [7, § 2].

Proof. This is a direct consequence of Theorem 4.3 and [7, Theorem A], where the latter asserts

$$\lim_{n\to+\infty}\frac{\mathrm{vol}(L_K^{\otimes n}\otimes M_K) - \mathrm{vol}(L_K^{\otimes n})}{n^{d-1}} = d\langle c_1(L_K)^{d-1}\rangle\cdot c_1(M_K). \qquad \square$$

We then deduce from Proposition 5.1 the expression of the distribution function of the measure $\nu_{\bar{L}}$.

PROPOSITION 5.2. *The distribution function $F_{\bar{L}}$ of $\nu_{\bar{L}}$ satisfies the equality*

$$F_{\bar{L}}(a) := \nu_{\bar{L}}(\,]-\infty,a]) = 1 - \frac{\langle\hat{c}_1(\bar{L}(-a))^d\rangle\cdot\hat{c}_1(\bar{\mathcal{O}}(1))}{[K:\mathbb{Q}]\mathrm{vol}(L_K)}, \quad a < \hat{\mu}_{\max}^\pi(\bar{L}).$$

Proof. One has

$$F_{\bar{L}}(a) = 1 + \frac{d}{da}\int_{\mathbb{R}}\max(x-a,0)\,\nu(dx) = 1 + \frac{d}{da}\hat{\mu}_+^\pi(\bar{L}(-a)).$$

By Proposition 5.1, one obtains

$$\frac{d}{da}\hat{\mu}_+^\pi(\bar{L}(-a)) = -\frac{\langle\hat{c}_1(\bar{L}(-a))^d\rangle\cdot\hat{c}_1(\bar{\mathcal{O}}(1))}{[K:\mathbb{Q}]\mathrm{vol}(L_K)}. \qquad \square$$

REMARK 12. (1) Since the support of $\nu_{\bar{L}}$ is bounded from above by $\hat{\mu}_{\max}(\bar{L})$, one has $F_{\bar{L}}(a) = 1$ for $a \geqslant \hat{\mu}_{\max}(\bar{L})$.

(2) As a consequence of Proposition 5.2, one obtains that the function

$$\frac{\langle \hat{c}_1(\bar{L}(-a))^d \rangle \cdot \hat{c}_1(\bar{\mathcal{O}}(1))}{[K:\mathbb{Q}]\mathrm{vol}(L_K)}$$

is decreasing with respect to a on $]-\infty, \hat{\mu}^{\pi}_{\max}(\bar{L})[$, which is also implied by Lemma 3.5. Furthermore, this function takes values in $]0,1]$, and converges to 1 when $a \to -\infty$.

(3) Let $a \in]-\infty, \hat{\mu}^{\pi}_{\max}(\bar{L})[$. The restriction of

$$\frac{1}{[K:\mathbb{Q}]\mathrm{vol}(L_K)} \langle \hat{c}_1(\bar{L}(-a))^d \rangle$$

on $C^0(X(\mathbb{C}))$ (considered as a subgroup of $\widehat{\mathrm{Pic}}(X)$ via the mapping $f \mapsto \mathcal{O}(f)$) is a positive linear functional and thus corresponds to a Radon measure on $X(\mathbb{C})$. Furthermore, by (1), its total mass is bounded from above by 1, and converges to 1 when $a \to -\infty$.

(4) By Remark 7, we observe from Proposition 5.2 that the only possible discontinuous point of the distribution function $F_{\bar{L}}(a)$ is $a = \hat{\mu}^{\pi}_{\max}(\bar{L})$.

5.2. *Applications in equidistribution*

In this section, we apply our differentiability result to an equidistribution problem in Arakelov geometry. We do not claim to have obtained new results but aim to give a more conceptual interpretation of the variational principle in the Arakelov geometry approach of the equidistribution problem developed in works such as [**1**, **2**, **8**, **9**, **24**–**26**, **30**]. Let $\pi : X \to \operatorname{Spec}\mathbb{Z}$ be a projective arithmetic variety. If $x \in X(\bar{\mathbb{Q}})$ is an algebraic point of X, then it defines a Borel probability measure

$$\eta_x := \frac{1}{[\mathbb{Q}(x):\mathbb{Q}]} \sum_{\sigma:\mathbb{Q}(x)\to\mathbb{C}} \delta_{\sigma(x)} \tag{5.5}$$

on the analytic space $X(\mathbb{C})$, where, for any $y \in X(\mathbb{C})$, δ_y denotes the Dirac measure concentrated on y. We consider a sequence $\bar{x} = (x_n)_{n\geqslant 1}$ of points in $X(\bar{\mathbb{Q}})$. In the equidistribution problem, we look for conditions under which the measure sequence $(\eta_{x_n})_{n\geqslant 1}$ converges weakly, or equivalently, the sequence of integrals $(\int_{X(\mathbb{C})} g\, d\eta_{x_n})_{n\geqslant 1}$ converges in $\mathbb{R}$ for any continuous function g on $X(\mathbb{C})$. Note that each continuous function g on $X(\mathbb{C})$ defines a Hermitian line bundle $\bar{\mathcal{O}}(g)$ on X as explained in § 2. Note that $g \mapsto \mathcal{O}(g)$ identifies $C^0(X(\mathbb{C}))$ with a subgroup of $\widehat{\mathrm{Pic}}(X)$. The integral with respect to the Galois orbit has the following interpretation:

$$\int_{X(\mathbb{C})} g\, d\eta_{x_n} = h_{\bar{\mathcal{O}}(g)}(x_n).$$

For $\bar{L} \in \widehat{\mathrm{Pic}}(X)$, we define

$$\varphi_{\bar{x}}(\bar{L}) := \liminf_{n\to+\infty} h_{\bar{L}}(x_n) \in \mathbb{R} \cup \{\pm\infty\}.$$

Since the Arakelov height $h_{\bar{L}}$ is additive with respect to $\bar{L}$, we obtain that $\varphi_{\bar{x}}(\cdot)$ is a superadditive function on $\widehat{\mathrm{Pic}}(X)$. Note that the sequence of measures $(\eta_{x_n})_{\geqslant 1}$ converges weakly if and only if the restriction of the function $\varphi_{\bar{x}}(\cdot)$ on $C^0(X(\mathbb{C}))$ is additive. In fact, if $(\eta_{x_n})_{\geqslant 1}$ converges weakly, then the restriction of the function $\varphi_{\bar{x}}(\cdot)$ on $C^0(X(\mathbb{C}))$ can be written as a limit of linear forms on $C^0(X(\mathbb{C}))$, and hence is itself a linear form. Conversely, if the restriction of $\varphi_{\bar{x}}(\cdot)$ on $C^0(X(\mathbb{C}))$ is additive, then one has $\varphi_{\bar{x}}(\bar{\mathcal{O}}(-g)) = -\varphi_{\bar{x}}(\bar{\mathcal{O}}(g))$ for any $g \in C^0(X(\mathbb{C}))$, which implies that $\varphi_{\bar{x}}(\bar{\mathcal{O}}(g))$ is actually the limit of $h_{\bar{\mathcal{O}}(g)}(x_n) = \int_{X(\mathbb{C})} g\, d\eta_{x_n}$.

For $\bar{L} \in \widehat{\mathrm{Pic}}(X)$, we define

$$C(X, \bar{L}) := \{\bar{L}^{\otimes n} \otimes \bar{\mathcal{O}}(f) \mid n \geqslant 1, f \in C^0(X(\mathbb{C}))\}.$$

It is a sub-semigroup of $\widehat{\mathrm{Pic}}(X)$ which is open with respect to $C^0(X(\mathbb{C}))$. Note that if $\varphi_{\bar{x}}(\bar{L}) \in \mathbb{R}$, then the function $\varphi_{\bar{x}}(\cdot)$ is finite on $C(X, \bar{L})$.

The following theorem shows that the problem of equidistribution can be interpreted as a differentiability property.

Theorem 5.3. *If $(\eta_{x_n})_{n \geqslant 1}$ converges weakly, then, for any $\bar{L} \in \widehat{\mathrm{Pic}}(X)$ such that $\varphi_{\bar{x}}(\bar{L}) \in \mathbb{R}$, the restriction of the function $\varphi_{\bar{x}}(\cdot)$ on $C(X, \bar{L})$ is differentiable at $\bar{L}$ along the directions in $C^0(X(\mathbb{C}))$. Conversely, if there exists a Hermitian line bundle $\bar{L} \in \widehat{\mathrm{Pic}}(X)$ such that $(h_{\bar{L}}(x_n))_{n \geqslant 1}$ converge in $\mathbb{R}$ and the function $\varphi_{\bar{x}}(\cdot)$ is differentiable at $\bar{L}$ along the directions in $C^0(X(\mathbb{C}))$, then the sequence of measures $(\eta_{x_n})_{n \geqslant 1}$ converges weakly.*

Proof. '$\Longrightarrow$' : For any integer $m \geqslant 1$ and any $f \in C^0(X(\mathbb{C}))$, one has

$$\varphi_{\underline{x}}(\bar{L}^{\otimes m} \otimes \bar{\mathcal{O}}(f)) := \liminf_{n \to \infty} (m h_{\bar{L}}(x_n) + h_{\bar{\mathcal{O}}(f)}(x_n)) = m\varphi_{\bar{x}}(\bar{L}) + \varphi_{\bar{x}}(\bar{\mathcal{O}}(f)), \tag{5.6}$$

where in the second equality we have used the assumption that $h_{\bar{\mathcal{O}}(f)}(x_n) = \int f \, d\eta_{x_n}$ converges in $\mathbb{R}$ when n tends to infinity. Hence, $D\varphi_{\underline{x}}(\bar{L}) = \varphi_{\underline{x}}$ is additive.

'$\Longleftarrow$' : The equality (5.6) still holds, but this time we use the convergence of $(h_{\bar{L}}(x_n))_{n \geqslant 1}$ to prove the second equality. Hence, $D\varphi_{\underline{x}}(\bar{L})$ equals the restriction of $\varphi_{\underline{x}}$ on $C^0(X(\mathbb{C}))$ and thus $\varphi_{\bar{x}}$ is additive on $C^0(X(\mathbb{C}))$. Therefore, the sequence $(\eta_{x_n})_{n \geqslant 1}$ converges weakly. □

Remark 13. From the proof of Theorem 5.3, we observe that if the sequence of measures $(\eta_{x_n})_{n \geqslant 1}$ converges weakly, then the limit measure is just the Radon measure on $X(\mathbb{C})$ corresponding to the positive linear functional

$$(f \in C^0(X(\mathbb{C}))) \longmapsto D\varphi_{\bar{x}}(\bar{L})(\bar{\mathcal{O}}(f)).$$

The variational principle can be interpreted as the following observation.

Proposition 5.4. *Let G be a commutative group, H be a subgroup of G, C be a sub-semigroup of G that is open with respect to H and $x \in C$. If f and g are two positively homogeneous functions on C such that:*
(1) *$\forall a, b \in C$, $f(a + b) \geqslant f(a) + f(b)$;*
(2) *$f \geqslant g$, $f(x) = g(x)$;*
(3) *g is differentiable at x along the directions in H;*
then the function f is also differentiable at x along the directions in H. Moreover, one has $Df(x) = Dg(x)$.

Proof. For any $w \in H$, there exists $n_0(w) \in \mathbb{N}_*$ such that $nx + w \in C$ for any $n \geqslant n_0(w)$. Moreover, the sequence $(f(nx + w) - f(nx))_{n \geqslant n_0(w)}$ is increasing. Therefore, the limit

$$D_w f(x) := \lim_{n \to \infty} f(nx + w) - f(nx)$$

is well defined in $\mathbb{R} \cup \{+\infty\}$. The functions f and g are positively homogeneous and $f(x) = g(x)$, hence $f(nx) = g(nx)$ for any $n \in \mathbb{N}_*$. Therefore, one has $Df(x) \geqslant Dg(x)$. If u and w are two elements in H, then

$$\begin{aligned}D_{u+w}f(x) &= \lim_{n\to\infty} f(2nx+u+w)-2nf(x)\\ &\geqslant \lim_{n\to\infty} f(nx+u)+f(nx+w)-2nf(x) = D_uf(x)+D_wf(x).\end{aligned}$$

Therefore, the function $Df(x)$ is super-additive. Moreover, $D_0f(x)=0$. By Proposition 4.1, we obtain that $Df(x)=Dg(x)$ is additive. □

COROLLARY 5.5. *Assume that the sequence $\bar{x}$ is generic (that is, any sub-sequence of $\bar{x}$ is dense in X). If there exists a big Hermitian line bundle $\bar{L}$ on X such that $h_{\bar{L}}(x_n)$ converges to $\hat{\mu}_+^\pi(\bar{L})$, then the sequence of measures $(\eta_{x_n})_{n\geqslant 1}$ converges weakly, and one has*

$$\lim_{n\to+\infty}\int f\,d\eta_{x_n} = \frac{\langle\hat{c}_1(\bar{L})^d\rangle\cdot\hat{c}_1(\bar{\mathcal{O}}(f))}{[K:\mathbb{Q}]\mathrm{vol}(L_K)}.$$

Proof. In the case where the sequence $\bar{x}$ is generic, the function $\varphi_{\bar{x}}(\cdot)$ is bounded from below by the essential minimum $\hat{\mu}_{\mathrm{ess}}(\cdot)$. Moreover, an application of the slope method shows that $\hat{\mu}_{\mathrm{ess}}(\cdot)$ is bounded from below by $\hat{\mu}_{\max}^\pi(\cdot)$ on the big cone. Hence, the corollary results from the inequality (5.4), Propositions 5.1, 5.4 and Theorem 5.3. □

REMARK 14. In Corollary 5.5, the hypothesis $\varphi_{\bar{x}}(\bar{L})=\hat{\mu}_+^\pi(\bar{L})$ implies that $\hat{\mu}_{\max}^\pi(\bar{L})=\hat{\mu}_+^\pi(\bar{L})$. In this case, the asymptotic measure $\nu_{\bar{L}}$ reduces to a Dirac measure. So one has

$$\varphi_{\bar{x}}(\bar{L}) = \int_{\mathbb{R}} x\,\nu_{\bar{L}}(dx),$$

and hence

$$\varphi_{\bar{x}}(\bar{L}) = \hat{\mu}^\pi(\bar{L}) := \frac{1}{[K:\mathbb{Q}]}\frac{S(\bar{L})}{(d+1)\mathrm{vol}(L_K)}.$$

Therefore, Corollary 5.5 is not more general than [**2**, Theorem 11.1]. We hope that a study on the differentiability domain of the function $\hat{\mu}_{\max}^\pi$ will provide new equidistribution criteria in Arakelov geometry.

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