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Harder–Narasimhan categories

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ABSTRACT

Semistability and the Harder–Narasimhan filtration are important notions in algebraic and arithmetic geometry. Although these notions are associated to mathematical objects of quite different natures, their definition and the proofs of their existence are quite similar. We propose in this article a generalization of Quillen's exact category and we discuss conditions on such categories under which one can define the notion of the Harder–Narasimhan filtrations and establish its functoriality.

1. Introduction

The notion of the *Harder–Narasimhan flag* of a vector bundle on a smooth projective curve defined over a field was firstly introduced by Harder and Narasimhan [1]. Let C be a smooth projective curve defined over a field and E be a non-zero vector bundle on C. Harder and Narasimhan have proved that there exists a flag $0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E$ of subbundles of E such that each subquotient E_i/E_{i-1} is a semistable vector bundle and that we have the inequalities of successive slopes

$$\mu_{\max}(E) := \mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1}) =: \mu_{\min}(E).$$

The avatar of the above constructions in Arakelov geometry was introduced by Stuhler [2] and Grayson [3]. Similar constructions exist also in the theory of filtered isocrystals. In [4], Faltings suggested the functoriality of the Harder–Narasimhan flags, but it is not evident how to state such a functoriality, because the length of the Harder–Narasimhan flag varies.

Previously, the categorical approach for studying semistability problems has been developed in various contexts such as [5–8].

The category of vector bundles is exact in the sense of Quillen [9]. However, it is not the case for the category of Hermitian vector bundles in Arakelov geometry. We shall propose a new notion – *geometric exact category* – which generalizes them simultaneously. A geometric exact category is a classical exact category equipped with geometric structures. After introducing a rank function and a degree function, the existence of the Harder–Narasimhan filtration results from a supplementary condition that the maximal destabilizing geometric subobject exists. The condition is automatically satisfied if the underlying exact category is an Abelian category. This formalism includes not only standard examples, but also examples like spectrum filtration of positive definite self-adjoint operator which is classical but less often interpreted in this way.

In order to state the functoriality, we need to take into account the successive slopes in the filtration. That leads to the notion of $\mathbb{R}$-indexed Harder–Narasimhan filtration. We expect that the subobject morphism, quotient morphism, and their compositions will be compatible with $\mathbb{R}$-indexed Harder–Narasimhan filtrations. However, we observe that the functoriality of the Harder–Narasimhan filtration does not follow from the existence of maximal destabilizing subobject. This is shown

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by a concrete example (see Section 4 Example 2, see also Remark 5.10). The functoriality is actually equivalent to the validity of a slope inequality, which compares the slopes of semistable objects.

The rest of the article is organized as follows. We recall the notions of $\mathbb{R}$-filtration and of Quillen's exact category in the second section. In the third section, we present the geometric exact categories. In the fourth section, we develop the formalism of the Harder–Narasimhan category, followed by several examples. In the fifth section, we discuss slope inequalities and the functoriality of the Harder–Narasimhan filtrations.

2. Notation and preliminaries

2.1. Filtrations in a category

We fix a category $\mathcal{C}$ which has an initial object. Let X be an object in $\mathcal{C}$. By *filtration* of X, we mean a family $\mathcal{F} = (X_t)_{t\in\mathbb{R}}$ of subobjects of X, which satisfies the following conditions:

(1) (decreasing property) if $s \leqslant t$ are two real numbers, then the canonical monomorphism $X_t \to X$ factorizes through X_s;
(2) (separation property) for sufficiently positive t, X_t is an initial object;
(3) (exhaustivity) for sufficiently negative t, $X_t = X$;
(4) (left locally constant property) for any $t \in \mathbb{R}$, there exists $\delta > 0$ such that, for any $s \in (t-\delta, t]$, the canonical morphism $X_t \to X_s$ is an isomorphism;
(5) (finite jump) the following *jump set* is finite:

$$J(\mathcal{F}) := \{s \in \mathbb{R} \mid \forall\, t > s,\ \text{the canonical map } X_t \to X_s \text{ is not an isomorphism}\}.$$

Assume that X and Y are two objects in $\mathcal{C}$, and $\mathcal{F} = (X_t)_{t\in\mathbb{R}}$ and $\mathcal{G} = (Y_t)_{t\in\mathbb{R}}$ are respectively filtrations of X and of Y. We call *morphism* from $\mathcal{F}$ to $\mathcal{G}$ any morphism f in $\mathcal{C}$ from X to Y such that, for any $t \in \mathbb{R}$, the composed map $X_t \xrightarrow{i_t} X \xrightarrow{f} Y$ factorizes through Y_t, where $i_t : X_t \to X$ denotes the canonical morphism. Alternatively, we say that the morphism f is *compatible* with the filtrations $\mathcal{F}$ and $\mathcal{G}$. All filtrations in $\mathcal{C}$ and all morphisms of filtrations form a category which we denote by $\mathbf{Fil}(\mathcal{C})$.

Suppose that all fiber products exist in $\mathcal{C}$. For any monomorphism $f : X \to Y$ in C and any filtration $\mathcal{G} = (Y_t)_{t\in\mathbb{R}}$ of Y, the family $f^*\mathcal{G} := (Y_t \times_Y X)_{t\in\mathbb{R}}$ is a filtration of X, called the *induced filtration*.

Let $f : X \to Y$ be an epimorphism in $\mathcal{C}$ and $\mathcal{F} = (X_t)_{t\in\mathbb{R}}$ be a filtration of X. For any $t \in \mathbb{R}$, denote by $i_t : X_t \to X$ the canonical monomorphism. Assume that, for any $t \in \mathbb{R}$, $fi_t : X_t \to Y$ has an image Y_t. Then $(Y_t)_{t\in\mathbb{R}}$ is a filtration of Y, denoted by $f_*\mathcal{F}$, called the *quotient filtration*.

2.2. Exact categories

The notion of exact category is defined by Quillen [9]. Let $\mathcal{C}$ be an essentially small additive category and let $\mathcal{E}$ be a class of diagrams of morphisms in $\mathcal{C}$ of the form

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0 .$$

If $0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0$ is a diagram in $\mathcal{E}$, we say that f is an *admissible monomorphism* and that g is an *admissible epimorphism*. We shall use the symbol "$\rightarrowtail$" to denote an admissible monomorphism, and "$\twoheadrightarrow$" for an admissible epimorphism.

If $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ and $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ are two diagrams of morphisms in $\mathcal{C}$, we call *morphism* from the first diagram to the second one any commutative diagram

$$(\Phi) : \quad \begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0. \end{array}$$

We say that (Φ) is an *isomorphism* if φ', φ and φ'' are all isomorphisms in $\mathcal{C}$.

Definition 2.1 (*Quillen*). We say that $(\mathcal{C}, \mathcal{E})$ is an *exact category* if the following axioms are verified:

(**Ex**1) For any diagram $0 \longrightarrow X' \xrightarrow{\varphi} X \xrightarrow{\psi} X'' \longrightarrow 0$ in $\mathcal{E}$, φ is a *kernel* of ψ and ψ is a *cokernel* of φ.
(**Ex**2) If X and Y are two objects in $\mathcal{C}$, then the following diagram is in $\mathcal{E}$:

$$0 \longrightarrow X \xrightarrow{(\mathrm{Id},0)} X \oplus Y \xrightarrow{\mathrm{pr}_2} Y \longrightarrow 0$$

(**Ex**3) Any diagram which is isomorphic to a diagram in $\mathcal{E}$ lies also in $\mathcal{E}$.
(**Ex**4) If $f : X \to Y$ and $g : Y \to Z$ are two admissible monomorphisms (resp. admissible epimorphisms), then so is gf.
(**Ex**5) For any admissible monomorphism $f : X' \rightarrowtail X$ and any morphism $u : X' \to Y$ in $\mathcal{C}$, the pushout of f and u exists. Furthermore, if the diagram

$$\begin{array}{ccc} X' & \overset{f}{\rightarrowtail} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ Y & \underset{g}{\longrightarrow} & Z \end{array}$$

is cocartesian, then g is an admissible monomorphism.
(**Ex**6) For any admissible epimorphism $f : X \twoheadrightarrow X''$ and any morphism $u : Y \to X''$ in $\mathcal{C}$, the fiber product of f and u exists. Furthermore, if the diagram

$$\begin{array}{ccc} Z & \overset{g}{\longrightarrow} & Y \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ X & \underset{f}{\twoheadrightarrow} & X'' \end{array}$$

is cartesian, then g is an admissible epimorphism.

Keller [10] has shown that the axioms above imply the following property, which was initially an axiom in Quillen's definition:

(**Ex**7) For any morphism $f : X \to Y$ in $\mathcal{C}$ having a kernel (resp. cokernel), if there exists an morphism $g : Z \to X$ (resp. $g : Y \to Z$) such that fg (resp. gf) is an admissible epimorphism (resp. admissible monomorphism), then also is f itself.

If $f : X \to Y$ is an admissible monomorphism, by (**Ex**1), the morphism f admits a cokernel, which is denoted by Y/X.

According to [9], if $\mathcal{C}$ is an Abelian category and if $\mathcal{E}$ is the class of all exact sequences in $\mathcal{C}$, then $(\mathcal{C}, \mathcal{E})$ is an exact category. Furthermore, any exact category can be naturally embedded (through the additive version of Yoneda's functor) into an Abelian category.

3. Geometric exact categories

Some natural categories, like the category of Hermitian spaces, are not exact categories. However, a Hermitian space can be considered as a vector space over $\mathbb{C}$ (which is an object in an Abelian category), equipped with a Hermitian inner product (which can be considered as a geometric structure). This observation leads to the following notion of geometric exact categories.

Definition 3.1. Let $(\mathcal{C}, \mathcal{E})$ be an exact category. We call *geometric structure* on $(\mathcal{C}, \mathcal{E})$ the data:

(1) a mapping A from *obj*$\mathcal{C}$ to the class of sets,
(2) for any admissible monomorphism $f : Y \to X$, a map $f^* : A(X) \to A(Y)$,
(3) for any admissible epimorphism $g : X \to Z$, a map $g_* : A(X) \to A(Z)$,

subject to the following axioms:

(**A**1) $A(0)$ is a one-point set,
(**A**2) if $X \overset{i}{\rightarrowtail} Y \overset{j}{\rightarrowtail} Z$ are admissible monomorphisms, then $(ji)^* = i^*j^*$,
(**A**3) if $X \overset{p}{\twoheadrightarrow} Y \overset{q}{\twoheadrightarrow} Z$ are admissible epimorphisms, then $(qp)_* = q_*p_*$,
(**A**4) for any object X of $\mathcal{C}$, $\mathrm{Id}_X^* = \mathrm{Id}_{X*} = \mathrm{Id}_{A(X)}$,
(**A**5) if $f : X \to Y$ is an isomorphism, then $f^*f_* = \mathrm{Id}_{A(X)}, f_*f^* = \mathrm{Id}_{A(Y)}$,
(**A**6) for any cartesian or[1] cocartesian square

$$\begin{array}{ccc} X & \overset{u}{\rightarrowtail} & Y \\ {\scriptstyle p}\big\downarrow\!\!\downarrow & & \big\downarrow\!\!\downarrow{\scriptstyle q} \\ Z & \underset{v}{\rightarrowtail} & W \end{array} \qquad (1)$$

1 Here we can prove that the square is actually cartesian **and** cocartesian.

in $\mathcal{C}$, where u and v (resp. p and q) are admissible monomorphisms (resp. admissible epimorphisms), we have $v^*q_* = p_*u^*$,

(**A**7) if $X \overset{u}{\twoheadrightarrow} Y \overset{v}{\rightarrowtail} Z$ is a diagram in $\mathcal{C}$ where u (resp. v) is an admissible epimorphism (resp. admissible monomorphism) and if $(h_X, h_Z) \in A(X) \times A(Z)$ satisfies $u_*(h_X) = v^*(h_Z)$, then there exists $h \in A(X \oplus Z)$ such that $(\mathrm{Id}, vu)^*(h) = h_X$ and that $\mathrm{pr}_{2*}(h) = h_Z$ (note that (Id, vu) is always an admissible monomorphism, since it is the composition of the isomorphism $(\mathrm{pr}_1, vu\mathrm{pr}_1 + \mathrm{pr}_2) : X \oplus Z \to X \oplus Z$ with the admissible monomorphism $(\mathrm{Id}, 0) : X \to X \oplus Z$).

The triplet $(\mathcal{C}, \mathcal{E}, A)$ is called a *geometric exact category*. For any object X of $\mathcal{C}$, we call any element h in $A(X)$ a *geometric structure* on X. The pair (X, h) is called a *geometric object* in $(\mathcal{C}, \mathcal{E}, A)$. If $p : X \to Z$ is an admissible epimorphism, $p_*(h)$ is called the *quotient geometric structure* on Z. If $i : Y \to X$ is an admissible monomorphism, i^*h is called the *induced geometric structure* on Y. $(Z, p_*(h))$ is called a *geometric quotient* of (X, h) and $(Y, i^*(h))$ is called a *geometric subobject* of (X, h).

Let $(\mathcal{C}, \mathcal{E})$ be an exact category. If for any object X of $\mathcal{C}$, we denote by $A(X)$ a one-point set, and we define induced and quotient geometric structures in the obvious way, then $(\mathcal{C}, \mathcal{E}, A)$ becomes a geometric exact category. The geometric structure A is called the *trivial geometric structure* on the exact category $(\mathcal{C}, \mathcal{E})$. Therefore, exact categories can be viewed as *trivial* geometric exact categories.

Let $(\mathcal{C}, \mathcal{E}, A)$ be a geometric exact category. If (X', h') and (X'', h'') are two geometric objects in $(\mathcal{C}, \mathcal{E}, A)$, we say that a morphism $f : X' \to X''$ in $\mathcal{C}$ is *compatible* with geometric structures if there exists a geometric object (X, h), an admissible monomorphism $u : X' \to X$ and an admissible epimorphism $v : X \to X''$ such that $h' = u^*(h)$ and that $h'' = v_*(h)$.

Remark 3.2. From the definition of morphisms compatible with geometric structures, we obtain the following results:

(1) If (X_1, h_1) and (X_2, h_2) are two geometric objects and if $f : X_1 \to X_2$ is an admissible monomorphism (resp. admissible epimorphism) such that $f^*h_2 = h_1$ (resp. $f_*h_1 = h_2$), then f is compatible with geometric structures.
(2) If (X_1, h_1) and (X_2, h_2) are two geometric objects and if $f : X_1 \to X_2$ is the zero morphism, then f is compatible with geometric structures.
(3) The composition of two morphisms compatible with geometric structure is also compatible with geometric structure. This is a consequence of axiom (**A**7).

Let $(\mathcal{C}, \mathcal{E}, A)$ be a geometric exact category. By argument (3) above, all geometric objects in $(\mathcal{C}, \mathcal{E}, A)$ and morphisms compatible with geometric structures form a category which we shall denote by $\mathcal{C}_A$.

We give below some examples of geometric exact categories.

3.1. Hermitian spaces

Let $\mathbf{Vec}_{\mathbb{C}}$ be the category of finite dimensional vector spaces over $\mathbb{C}$. It is an Abelian category. Let $\mathcal{E}$ be the class of all short exact sequences of finite dimensional vector spaces over $\mathbb{C}$. For any $X \in \mathrm{obj}(\mathbf{Vec}_{\mathbb{C}})$, denote by $A(X)$ the set of all Hermitian metrics on X. Let $h \in A(X)$. If $f : Y \to X$ is a vector subspace of X, then $f^*(h)$ denotes the induced metric on Y. If $\pi : X \to Z$ is a quotient space of X, then $\pi_*(h)$ denotes the quotient metric on Z. Note that for any $z \in Z$, one has $\|z\|_{\pi_*(h)} := \inf_{x \in X, \pi(x)=z} \|x\|_h$. We claim that $(\mathbf{Vec}_{\mathbb{C}}, \mathcal{E}, A)$ is a geometric exact category. The axioms (**A**1)–(**A**6) are easily verified. The verification of (**A**7) is as follows. Let $X \overset{u}{\twoheadrightarrow} Y \overset{v}{\rightarrowtail} Z$ be a diagram in $\mathbf{Vec}_{\mathbb{C}}$. Assume that X and Z are respectively equipped with Hermitian metrics $\|\cdot\|_X$ and $\|\cdot\|_Z$ such that the induced metric on Y from $\|\cdot\|_Z$ coincides with the quotient metric from $\|\cdot\|_X$. We equip $X \oplus Z$ with Hermitian metric $\|\cdot\|$ such that, for any $(x, z) \in X \oplus Z$,

$$\|(x, z)\|^2 = \|x - \varphi^\star(z)\|_X^2 + \|z\|_Z^2$$

where $\varphi = vu$, and $\varphi^\star$ denotes the adjoint of φ. With this metric, $\mathrm{pr}_2 : X \oplus Z \to Z$ is a projection of Hermitian spaces. Moreover, $u^\star$ is the identification of Y to $(Ker\ u)^\perp$, and $v^\star$ is the orthogonal projection of Z onto Y. Therefore, $\varphi^\star\varphi : X \to X$ is the orthogonal projection of X onto $(Ker\varphi)^\perp$. Hence, for any vector $x \in X$, we have

$$\|(x, \varphi(x))\|^2 = \|x - \varphi^\star\varphi(x)\|_X^2 + \|\varphi(x)\|_Z^2 = \|x - \varphi^\star\varphi(x)\|_X^2 + \|\varphi^\star\varphi(x)\|_X^2 = \|x\|_X^2.$$

The geometric objects in $(\mathbf{Vec}_{\mathbb{C}}, \mathcal{E}, A)$ are Hermitian spaces. From definition we see that if a linear mapping $\varphi : X \to Y$ of Hermitian spaces is compatible with geometric structure, then the norm of φ must be smaller than or equal to 1. The following proposition shows that the converse is also true.

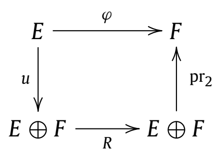
Proposition 3.3. *Let $\varphi : E \to F$ be a linear map of Hermitian spaces. If $\|\varphi\| \leqslant 1$, then there exists a Hermitian metric on $E \oplus F$ such that, in the decomposition $E \xrightarrow{(\mathrm{Id},\varphi)} E \oplus F \xrightarrow{\mathrm{pr}_2} F$ of φ, (Id, φ) is an inclusion of Hermitian spaces and pr_2 is a projection of Hermitian spaces.*

Proof. Since $\|\varphi\| \leqslant 1$, we have $\|\varphi^\star\| \leqslant 1$. Therefore, we obtain the inequalities $\|\varphi^\star\varphi\| \leqslant 1$ and $\|\varphi\varphi^\star\| \leqslant 1$. Hence $\mathrm{Id}_E - \varphi^\star\varphi$ and $\mathrm{Id}_F - \varphi\varphi^\star$ are Hermitian endomorphisms with non-negative eigenvalues. So there exist two Hermitian endomorphisms with non-negative eigenvalues P and Q of E and F respectively such that $P^2 = \mathrm{Id}_E - \varphi^\star\varphi$ and $Q^2 = \mathrm{Id}_F - \varphi\varphi^\star$.

If x is an eigenvector of $\varphi\varphi^\star$ associated to the eigenvalue λ, then $\varphi^\star x$ is an eigenvector of $\varphi^\star\varphi$ associated to the same eigenvalue. Therefore $\varphi^\star Qx = \sqrt{1-\lambda}\varphi^\star x = P\varphi^\star x$. As F is generated by eigenvectors of $\varphi\varphi^\star$, we have $\varphi^\star Q = P\varphi^\star$. For the same reason we have $Q\varphi = \varphi P$. Let $R = \begin{pmatrix} P & \varphi^\star \\ \varphi & -Q \end{pmatrix}$. As R is Hermitian, and verifies

$$R^2 = \begin{pmatrix} P^2 + \varphi^\star\varphi & P\varphi^\star - \varphi^\star Q \\ \varphi P - Q\varphi & \varphi\varphi^\star + Q^2 \end{pmatrix} = \mathrm{Id}_{E\oplus F},$$

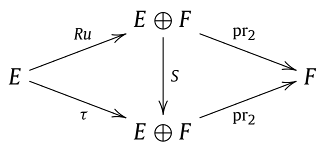
it is an isometry for the orthogonal sum metric on $E \oplus F$. Let $u : E \to E \oplus F$ be the mapping which sends x to $(x, 0)$. The diagram



is commutative. The endomorphism $\varphi^\star\varphi$ is auto-adjoint and positive semidefinite, and satisfies $\|\varphi^\star\varphi\| \leqslant 1$. Hence there exists an orthonormal basis $(x_i)_{1\leqslant i\leqslant n}$ of E such that $\varphi^\star\varphi x_i = \lambda_i x_i$ $(0 \leqslant \lambda_i \leqslant 1)$. Suppose that $0 \leqslant \lambda_j < 1$ for any $j \in \{1, \ldots, k\}$ and that $\lambda_j = 1$ for any $j \in \{k + 1, \ldots, n\}$. Let $B : E \to E$ be the $\mathbb{C}$-linear map such that $B(x_j) = \sqrt{1 - \lambda_j}x_j$ for $j \in \{1, \ldots, k\}$ and that $B(x_j) = x_j$ otherwise. Define $S = \begin{pmatrix} B & \varphi^\star \\ 0 & \mathrm{Id}_F \end{pmatrix} : E \oplus F \to E \oplus F$. Since $Ru = (P, \varphi)$ and since

$$(BP + \varphi^\star\varphi)(x_i) = \sqrt{1 - \lambda_i}Bx_i + \lambda_i x_i = \begin{cases} (1 - \lambda_i)x_i + \lambda_i x_i = x_i, & 1 \leqslant i \leqslant k, \\ 0Bx_i + x_i = x_i, & k < i \leqslant n, \end{cases}$$

the diagram



is commutative, where $\tau = (\mathrm{Id}_E, \varphi)$. We equip $E \oplus F$ with the Hermitian inner product $\langle\cdot,\cdot\rangle_0$ such that, for any $(\alpha, \beta) \in (E \oplus F)^2$, we have $\langle\alpha, \beta\rangle_0 = \langle S^{-1}\alpha, S^{-1}\beta\rangle$, where $\langle\cdot,\cdot\rangle$ is the orthogonal direct sum of Hermitian inner products on E and on F. Then, for any $x, y \in E$, one has

$$\langle\tau(x), \tau(y)\rangle_0 = \langle SRu(x), SRu(y)\rangle_0 = \langle Ru(x), Ru(y)\rangle = \langle u(x), u(y)\rangle = \langle x, y\rangle .$$

Finally, the kernel of pr_2 is stable by the action of S, so the projections of $\langle\cdot,\cdot\rangle_0$ and of $\langle\cdot,\cdot\rangle$ by pr_2 are the same. □

Remark 3.4. Similarly, Euclidean norms form a geometric structure on the category of all finite dimensional vector spaces over $\mathbb{R}$. The linear maps compatible with geometric structures are just those of norm $\leqslant 1$.

3.2. Ultranormed space

Let k be a field equipped with a non-Archimedean absolute value $|\cdot|$ under which k is complete. We denote by $\mathbf{Vec}_k$ the category of finite dimensional vector spaces over k, which is clearly an Abelian category. Let $\mathcal{E}$ be the class of short exact sequences in $\mathbf{Vec}_k$. For any finite dimensional vector space X over k, we denote by $A(X)$ the set of all ultranorms (that is, a norm $\|\cdot\|$ which verifies $\|x+y\| \leqslant \max(\|x\|, \|y\|)$) on X. Suppose that h is an ultranorm on X. If $f : Y \to Z$ is a subspace of E, $f^*(h)$ denotes the induced norm on Y; if $g : X \to Z$ is a quotient space of X, $g_*(h)$ denotes the quotient norm on F. Then $(\mathbf{Vec}_k, \mathcal{E}, A)$ is a geometric exact category. In particular, axiom (**A**7) is justified by the following proposition.

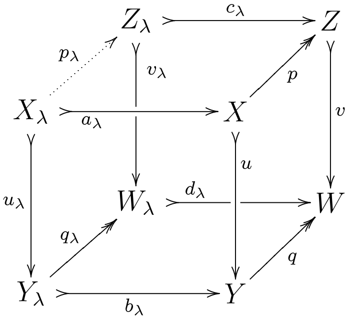
Proposition 3.5. *Let $\varphi : E \to F$ be a linear map of vector spaces over k. Suppose that E and F are equipped respectively with the ultranorms h_E and h_F such that $\|\varphi\| \leqslant 1$. If we equip $E \oplus F$ with the ultranorm h such that, for any $(x, y) \in E \oplus F$, $h(x, y) = \max(h_E(x), h_F(y))$, then in the decomposition $E \xrightarrow{(\mathrm{Id},\varphi)} E \oplus F \xrightarrow{\mathrm{pr}_2} F$ of φ, we have $(\mathrm{Id}, \varphi)^*(h) = h_E$ and $\mathrm{pr}_{2*}(h) = h_F$.*

Proof. In fact, for any element $x \in E$, $h(x, \varphi(x)) = \max(h_E(x), h_F(\varphi(x))) = h_E(x)$ since $h_F(\varphi(x)) \leqslant \|\varphi\|h_E(x) \leqslant h_E(x)$. Furthermore, by definition, one has $h_F = \mathrm{pr}_{2*}(h)$. Therefore the proposition is true. □

3.3. Filtrations in an Abelian category

Let $\mathcal{C}$ be an essentially small Abelian category and $\mathcal{E}$ be the class of all short exact sequences in $\mathcal{C}$. For any object X in $\mathcal{C}$, denote by $A(X)$ the set of isomorphism classes of $\mathbb{R}$-filtrations of X. The maps in (2)–(3) of Definition 3.1 are respectively chosen to be induced filtration map and quotient filtration map (see Section 2.1).

We claim that $(\mathcal{C}, \mathcal{E}, A)$ is a geometric exact category. In fact, axioms (**A**1)–(**A**5) are clearly satisfied. We now verify axiom (**A**6). Consider the diagram (1) in Definition 3.1, which is the right sagittal square of the following diagram (2). Suppose given an $\mathbb{R}$-filtration $(Y_\lambda)_{\lambda\in\mathbb{R}}$ of Y. For any $\lambda \in \mathbb{R}$, let $b_\lambda : Y_\lambda \to Y$ be the canonical morphism.

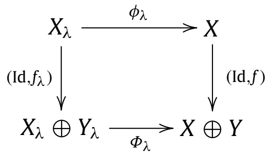


(2)

Let $d_\lambda : W_\lambda \to W$ be the image of qb_λ in W and $q_\lambda : Y_\lambda \to W_\lambda$ be the canonical epimorphism. Let $(Z_\lambda, c_\lambda, v_\lambda)$ be the fiber product of v and d_λ, and $(X_\lambda, a_\lambda, u_\lambda)$ be the fiber product of u and b_λ. Therefore, in diagram (2), the two coronal squares and the right sagittal square are cartesian, the inferior square is commutative. As $vpa_\lambda = qua_\lambda = qb_\lambda u_\lambda = d_\lambda q_\lambda u_\lambda$, there exists a unique morphism $p_\lambda : X_\lambda \to Z_\lambda$ such that $c_\lambda p_\lambda = pa_\lambda$ and $v_\lambda p_\lambda = q_\lambda u_\lambda$. It is then not hard to verify that the left sagittal square is cartesian, therefore p_λ is an epimorphism, so Z_λ is the image of pa_λ. Axiom (**A**6) is therefore verified. Finally, the verification of the axiom (**A**7) follows from the following proposition.

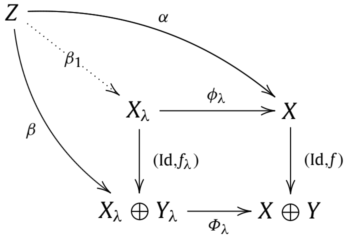
Proposition 3.6. *Let X and Y be two objects in $\mathcal{C}$ and let $\mathcal{F} = (X_\lambda)_{\lambda\in\mathbb{R}}$ (resp. $\mathcal{G} = (Y_\lambda)_{\lambda\in\mathbb{R}}$) be an $\mathbb{R}$-filtration of X (resp. Y). If $f : X \to Y$ is a morphism which is compatible with the filtrations $(\mathcal{F}, \mathcal{G})$, then the filtration $\mathcal{H} = (X_\lambda \oplus Y_\lambda)_{\lambda\in\mathbb{R}}$ of $X \oplus Y$ verifies $\Gamma_f^*\mathcal{H} = \mathcal{F}$ and $\mathrm{pr}_{2*}\mathcal{H} = \mathcal{G}$, where $\Gamma_f = (\mathrm{Id}, f) : X \to X \oplus Y$ is the graph of f and $\mathrm{pr}_2 : X \oplus Y \to Y$ is the projection onto the second factor.*

Proof. By definition, one has $\mathrm{pr}_{2*}\mathcal{H} = \mathcal{G}$. For any $\lambda \in \mathbb{R}$, consider the square



(3)

where $\phi_\lambda : X_\lambda \to X$ and $\psi_\lambda : Y_\lambda \to Y$ are canonical morphisms, $\Phi_\lambda = \phi_\lambda \oplus \psi_\lambda$, and $f_\lambda : X_\lambda \to Y_\lambda$ is the morphism through which the morphism $f\phi_\lambda$ factorizes. The square (3) is commutative. Suppose that $\alpha : Z \to X$ and $\beta = (\beta_1, \beta_2) : Z \to X_\lambda \oplus Y_\lambda$ are two morphisms such that $(\mathrm{Id}, f)\alpha = \Phi_\lambda \beta$.



(4)

Then we have $\alpha = \phi_\lambda\beta_1$ and $f\alpha = \psi_\lambda\beta_2$. So $\psi_\lambda\beta_2 = f\alpha = f\phi_\lambda\beta_1 = \psi_\lambda f_\lambda \beta_1$. As ψ_λ is a monomorphism, we obtain $f_\lambda\beta_1 = \beta_2$. So $\beta_1 : Z \to X_\lambda$ is the only morphism such that the diagram (4) commutes. Hence we get $\mathcal{F} = (\mathrm{Id}, f)^*\mathcal{H}$. □

4. Harder–Narasimhan categories

4.1. Degree function and rank function on a geometric exact category

Let $(\mathcal{C}, \mathcal{E}, A)$ be a geometric exact category. In the following, a geometrical object (X, h) in $(\mathcal{C}, \mathcal{E}, A)$ will be simply denoted by X if this notation does not lead to any ambiguity; and we use the expression h_X to denote the underlying geometric structure of X.

We say that a geometric object X is *non-zero* if its underlying object in $\mathcal{C}$ is non-zero. Since $\mathcal{C}$ is essentially small, the isomorphism classes of objects in $\mathcal{C}_A$ form a set.

We denote by $\mathcal{E}_A$ the class of diagrams of the form

$$0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \longrightarrow 0 \,, \tag{5}$$

where the underlying $\mathcal{C}$-diagram lies in $\mathcal{E}$, X' is a geometric subobject of X and X'' is a geometric quotient of X.

Let $K_0(\mathcal{C}, \mathcal{E}, A)$ be the free Abelian group generated by isomorphism classes in $\mathcal{C}_A$, modulo the subgroup generated by elements of the form $[X] - [X'] - [X'']$, where

$$0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{p} X'' \longrightarrow 0$$

is a diagram in $\mathcal{E}_A$. The group $K_0(\mathcal{C}, \mathcal{E}, A)$ is called the *Grothendieck group* of the geometric exact category $(\mathcal{C}, \mathcal{E}, A)$. We have a "*forgetful*" homomorphism from $K_0(\mathcal{C}, \mathcal{E}, A)$ to $K_0(\mathcal{C}, \mathcal{E})$, the Grothendieck group of the exact category $(\mathcal{C}, \mathcal{E})$, which sends $[X]$ to the class of the underlying $\mathcal{C}$-object of X.

In order to define the semistability of geometric objects and to establish the Harder–Narasimhan formalism, we need two auxiliary homomorphisms of groups. The first one, from $K_0(\mathcal{C}, \mathcal{E}, A)$ to $\mathbb{R}$, is called a *degree function* on $(\mathcal{C}, \mathcal{E}, A)$; and the second one, from $K_0(\mathcal{C}, \mathcal{E})$ to $\mathbb{Z}$, which takes strictly positive values on classes represented by non-zero objects in $\mathcal{C}$, is called a *rank function* on $(\mathcal{C}, \mathcal{E})$.

Now let $\deg : K_0(\mathcal{C}, \mathcal{E}, A) \to \mathbb{R}$ be a degree function on $(\mathcal{C}, \mathcal{E}, A)$ and $\mathrm{rk} : K_0(\mathcal{C}, \mathcal{E}) \to \mathbb{Z}$ be a rank function on $(\mathcal{C}, \mathcal{E})$. For any geometric object X in $(\mathcal{C}, \mathcal{E}, A)$, we denote by $\deg(X)$ the value $\deg([X])$, called the *degree* of X. Denote by $\mathrm{rk}(X)$ the function rk evaluated on the class of the underlying $\mathcal{C}$-object of X, called the *rank* of X. Note that $\mathrm{rk}(X)$ does not depend on the geometric structure of X.

If X is non-zero, the quotient $\mu(X) = \deg(X)/\mathrm{rk}(X)$ is called the *slope* of X. We say that a non-zero geometric object X is *semistable* if for any non-zero geometric subobject X' of X, we have $\mu(X') \leqslant \mu(X)$.

The following proposition provides some basic properties of degrees and slopes.

Proposition 4.1. *Let us keep the notation above.*

(1) *If* $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ *is a diagram in* $\mathcal{E}_A$*, then*

$$\deg(X) = \deg(X') + \deg(X'').$$

(2) *If X is a geometric object of rank 1, then it is semistable.*
(3) *Any non-zero geometric object X is semistable if and only if for any non-trivial geometric quotient X'' (i.e., X'' does not reduce to zero and is not canonically isomorphic to X), we have $\mu(X) \leqslant \mu(X'')$.*

Proof. Since deg is a homomorphism from $K_0(\mathcal{C}, \mathcal{E}, A)$ to $\mathbb{R}$, (1) is clear.

(2) Assume that X' is a non-zero geometric subobject of X. It fits into a diagram

$$0 \longrightarrow X' \xrightarrow{f} X \longrightarrow X'' \longrightarrow 0$$

in $\mathcal{C}_A$. Since X' is non-zero, $\mathrm{rk}(X') \geqslant 1$. Therefore $\mathrm{rk}(X'') = 0$ and hence $X'' = 0$. In other words, f is an isomorphism. So we have $\mu(X') = \mu(X)$.

(3) For any diagram $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in $\mathcal{E}_A$, X'' is non-trivial if and only if X' is non-trivial. If X' and X'' are both non-trivial, we have the following equality

$$\mu(X) = \frac{\mathrm{rk}(X')}{\mathrm{rk}(X)}\mu(X') + \frac{\mathrm{rk}(X'')}{\mathrm{rk}(X)}\mu(X'').$$

Therefore $\mu(X') \leqslant \mu(X) \Longleftrightarrow \mu(X'') \geqslant \mu(X)$. □

4.2. Harder–Narasimhan category and Harder–Narasimhan sequence

We are now able to introduce a condition ensuring the existence of the Harder–Narasimhan "flag".

Definition 4.2. Let $(\mathcal{C}, \mathcal{E}, A)$ be a geometric exact category, $\deg : K_0(\mathcal{C}, \mathcal{E}, A) \to \mathbb{R}$ be a degree function and $\mathrm{rk} : K_0(\mathcal{C}, \mathcal{E}) \to \mathbb{Z}$ be a rank function. We say that $(\mathcal{C}, \mathcal{E}, A, \deg, \mathrm{rk})$ is a *Harder–Narasimhan category* if the following axiom is verified:

(**HN**) For any non-zero geometric object X, there exists a geometric subobject X_{des} of X such that

$$\mu(X_{\mathrm{des}}) = \sup\{\mu(Y) \mid Y \text{ is a non-zero geometric subobject of } X\}.$$

Furthermore, any non-zero geometric subobject Z of X such that $\mu(Z) = \mu(X_{\mathrm{des}})$ is a geometric subobject of X_{des}.

Note that if X is a non-zero geometric object, then X_{des} is unique up to a unique isomorphism. Moreover, it is a semistable geometric object. If X is not semistable, we say that X_{des} *destabilizes* X.

Theorem 4.3. *Let* $(\mathcal{C}, \mathcal{E}, A, \deg, \mathrm{rk})$ *be a Harder–Narasimhan category. If* X *is a non-zero geometric object, then there exists a sequence of admissible monomorphisms*

$$0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X\ , \tag{6}$$

such that

(1) *for any integer* $j \in \{1, \ldots, n\}$*, the geometric quotient* X_j/X_{j-1} *is semistable;*
(2) *the inequalities* $\mu(X_1/X_0) > \mu(X_2/X_1) > \cdots > \mu(X_n/X_{n-1})$ *hold.*

Proof. We prove the existence by induction on the rank r of X. The case where X is semistable is trivial, and *a fortiori* the assertion is true for $r = 1$ (see Proposition 4.1 2). Now we consider the case where X is not semistable. Let $X_1 = X_{\mathrm{des}}$. It is a semistable geometric object, and $X' = X/X_1$ is non-zero. The rank of X' being strictly smaller than r, we can therefore apply the induction hypothesis on X'. We then obtain a sequence of admissible monomorphisms

$$0 = X_1' \xrightarrow{f_1'} X_2' \longrightarrow \cdots \longrightarrow X_{n-1}' \xrightarrow{f_{n-1}'} X_n' = X'$$

verifying the desired conditions.

Since the canonical morphism from X to X' is an admissible epimorphism, for any $i \in \{1, \ldots, n\}$, if we note $X_i = X \times_{X'} X_i'$, then by (**Ex**6), the projection $\pi_i : X_i \to X_i'$ is an admissible epimorphism. For any integer i, $1 \leqslant i < n$, we have a canonical morphism f_i from X_i to X_{i+1} and the square

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & X_{i+1} \\ {\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle \pi_{i+1}} \\ X_i' & \xrightarrow[f_i']{} & X_{i+1}' \end{array} \tag{7}$$

is cartesian. Since f_i' is an monomorphism, also is f_i (cf. [11] V. 7). Moreover, since the square (7) is cartesian, f_i is the kernel of the composed morphism $X_{i+1} \xrightarrow{\pi_{i+1}} X_{i+1}' \xrightarrow{p_i} X_{i+1}'/X_i'$, where p_i is the canonical morphism. Since π_{i+1} and p_i are admissible epimorphisms, also is $p_i\pi_{i+1}$ (by (**Ex**4)). Therefore f_i is an admissible monomorphism. Hence we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 = X_0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \longrightarrow \cdots \longrightarrow & X_{n-1} & \xrightarrow{f_{n-1}} & X_n = X \\ & & {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_{n-1}} & & \downarrow{\scriptstyle \pi_n} \\ & 0 = & X_1' & \xrightarrow[f_1']{} & X_2' & \longrightarrow \cdots \longrightarrow & X_{n-1}' & \xrightarrow[f_{n-1}']{} & X_n' = X' \end{array}$$

where the horizontal morphisms are admissible monomorphisms and the vertical morphisms are admissible epimorphisms. Furthermore, for any integer $i \in \{1, \ldots, n-1\}$, we have a natural isomorphism φ_i from X_{i+1}/X_i to X_{i+1}'/X_i'. We denote by g_i (resp. g_i') the canonical morphism from X_i (resp. X_i') to X (resp. X'). Let $h_i = g_i^*(h_X)$ (resp. $h_i' = g_i'^*(h_{X'})$) be the induced geometric structure on X_i (resp. X_i'). By (**A**6), one has $\pi_{i*}(h_i) = \pi_{i*}f_i^*(h_X) = f_i'^*\pi_*(h_X) = h_i'$. Therefore φ_{i*} sends the quotient geometric structure on X_{i+1}/X_i to that on X_{i+1}'/X_i'. Hence the geometric object X_{i+1}/X_i is semistable and we have the equality $\mu(X_{i+1}/X_i) = \mu(X_{i+1}'/X_i')$. Finally, since $X_1 = X_{\mathrm{des}}$, one has

$$\mu(X_2/X_1) = \frac{\mathrm{rk}(X_2)\mu(X_2) - \mathrm{rk}(X_1)\mu(X_1)}{\mathrm{rk}(X_2) - \mathrm{rk}(X_1)} < \mu(X_1).$$

Therefore the sequence $0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X$ satisfies the desired conditions. □

Definition 4.4. In the proof of Theorem 4.3, we have actually constructed by induction a sequence

$$0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X$$

of geometric subobjects of X such that $X_i/X_{i-1} = (X/X_{i-1})_{\mathrm{des}}$ for any $i \in \{1, \ldots, n-1\}$. This sequence satisfies the two conditions in Theorem 4.3. We call it the *Harder–Narasimhan sequence* of X. The real numbers $\mu(X_1)$ and $\mu(X/X_{n-1})$ are respectively called the *maximal slope* and the *minimal slope* of X, denoted by $\mu_{\max}(X)$ and $\mu_{\min}(X)$.

The following proposition shows that in the Abelian category case, condition (**HN**) is automatically satisfied.

Proposition 4.5. *Let* $(\mathcal{C}, \mathcal{E}, A)$ *be a geometric exact category,* deg *and* rk *are respectively a degree function and a rank function on it. If* $(\mathcal{C}, \mathcal{E})$ *is an Abelian category, then* $(\mathcal{C}, \mathcal{E}, A, \deg, \mathrm{rk})$ *is a Harder–Narasimhan category.*

Proof. We check condition (**HN**) by induction on $\mathrm{rk}(X)$. The condition is fulfilled when X is semistable, and in particular when X is of rank 1 (see Proposition 4.1 2). Assume that X is a geometric object of rank > 1 and which is not semistable. Let X' be a non-zero geometric subobject of X such that $\mu(X') > \mu(X)$ and $\mathrm{rk}(X')$ is maximal among the non-zero geometric subobjects Y of X verifying $\mu(Y) > \mu(X)$. By induction hypothesis there exists a geometric subobject X'_{des} verifying (**HN**) for X'. We shall show that $X_{\mathrm{des}} := X'_{\mathrm{des}}$ actually verifies (**HN**) for X. Let Y be a non-zero geometric subobject of X. If it is a geometric subobject of X', then one has $\mu(Y) \leqslant \mu(X')$. Otherwise the rank of $Y + X'$ will be strictly greater than $\mathrm{rk}(X')$ and by definition $\mu(Y + X') \leqslant \mu(X) < \mu(X')$. Furthermore, one has the following exact sequence

$$0 \longrightarrow Y \cap X' \longrightarrow Y \oplus X' \longrightarrow Y + X' \longrightarrow 0.$$

By the additivity of the degree function we obtain

$$\begin{aligned} \deg(Y) &= \deg(Y \cap X') + \deg(Y + X') - \deg(X') \\ &< \mu(X'_{\mathrm{des}})\mathrm{rk}(Y \cap X') + \mu(X')\big(\mathrm{rk}(Y + X') - \mathrm{rk}(X')\big) \\ &\leqslant \mu(X'_{\mathrm{des}})\big(\mathrm{rk}(Y \cap X') + \mathrm{rk}(Y + X') - \mathrm{rk}(X')\big) = \mu(X'_{\mathrm{des}})\mathrm{rk}(Y), \end{aligned} \tag{8}$$

where in the first inequality we have used the estimations $\mu(Y \cap X') \leqslant \mu(X'_{\mathrm{des}})$ and $\mu(Y + X') < \mu(X')$. Moreover, from the strict inequality (8), we obtain that, if Y is a non-zero geometric subobject of X such that $\mu(Y) = \mu(X'_{\mathrm{des}})$, then Y must be a geometric subobject of X', and therefore is a geometric subobject of X'_{des} by induction hypothesis. Thus condition (**HN**) holds for X. □

Example 1 (*Filtrations in an Abelian Category*). Let $\mathcal{C}$ be an Abelian category and $\mathcal{E}$ be the class of all short exact sequences in $\mathcal{C}$. Given a rank function $\mathrm{rk} : K_0(\mathcal{C}) \to \mathbb{Z}$. For any object X in $\mathcal{C}$, let $A(X)$ be the set of isomorphism classes of filtrations of X. We have shown in Section 3.3 that $(\mathcal{C}, \mathcal{E}, A)$ is a geometric exact category. Any geometric object can be considered, after choosing a representative in h, as an object X in $\mathcal{C}$ equipped with an $\mathbb{R}$-filtration $\mathcal{F} = (X_\lambda)_{\lambda\in\mathbb{R}}$. We define (by convention $\sup_\emptyset = 0$)

$$\deg(\mathcal{F}) = \sum_{\lambda\in\mathbb{R}} \lambda\Big(\mathrm{rk}(X_\lambda) - \sup_{s>\lambda} \mathrm{rk}(X_s)\Big) = -\int_{\mathbb{R}} \lambda \,\mathrm{d}\mathrm{rk}(X_\lambda). \tag{9}$$

The summation above is actually finite. We now show that the function deg defined above extends naturally to a homomorphism from $K_0(\mathcal{C}, \mathcal{E}, A)$ to $\mathbb{R}$. Let $0 \longrightarrow \mathcal{F}' \xrightarrow{u} \mathcal{F} \xrightarrow{p} \mathcal{F}'' \longrightarrow 0$ be a diagram $\mathcal{C}_A$, where $\mathcal{F}' = (X'_\lambda)_{\lambda\in I}$, $\mathcal{F} = (X_\lambda)_{\lambda\in I}$, $\mathcal{F}'' = (X''_\lambda)_{\lambda\in I}$ are respectively $\mathbb{R}$-filtrations of X', X and X''. Then, for any real number $\lambda \in I$, we have an exact sequence $0 \longrightarrow X'_\lambda \longrightarrow X_\lambda \longrightarrow X''_\lambda \longrightarrow 0$. Therefore, $\deg(\mathcal{F}) = \deg(\mathcal{F}') + \deg(\mathcal{F}'')$. Note that a filtration $\mathcal{F} = (X_\lambda)_{\lambda\in\mathbb{R}}$ is semistable if and only if its jump set $J(\mathcal{F})$ reduces to a one-point set.

Proposition 4.5 shows that $(\mathcal{C}, \mathcal{E}, A, \deg, \mathrm{rk})$ is a Harder–Narasimhan category. Suppose that $\mathcal{F} = (X_\lambda)_{\lambda\in\mathbb{R}}$ is a filtration of a non-zero object X in $\mathcal{C}$. Let $J(\mathcal{F}) = \{\lambda_1 > \lambda_2 > \cdots > \lambda_n\}$ be the jump set of $\mathcal{F}$. Then

$$0 \longrightarrow X_{\lambda_1} \longrightarrow X_{\lambda_2} \longrightarrow \cdots \longrightarrow X_{\lambda_n} = X$$

is the Harder–Narasimhan sequence of X. Furthermore, $\mu(X_{\lambda_1}) = \lambda_1$, and for any $i \in \{2, \ldots, n\}$, $\mu(X_{\lambda_i}/X_{\lambda_{i-1}}) = \lambda_i$.

Example 2 (*Vector Spaces with Two Norms*). Let $(\mathbf{Vec}_{\mathbb{C}}, \mathcal{E})$ be the Abelian category of finite dimensional vector spaces over $\mathbb{C}$. For any vector space $X \in \mathrm{obj}(\mathbf{Vec}_{\mathbb{C}})$, let $A(X)$ be the set of all pairs $(\|\cdot\|_1, \|\cdot\|_2)$ of Hermitian norms on X. By the results in Section 3.1, A is a geometric structure on $(\mathbf{Vec}_{\mathbb{C}}, \mathcal{E})$. For any X, let $\mathrm{rk}(X)$ be the rank of X. Then rk is a rank function. If $(X, \|\cdot\|_1, \|\cdot\|_2)$ is a geometric object, denote by $\deg(X, \|\cdot\|_1, \|\cdot\|_2)$ the logarithm of the ratio between the two metrics on $\det(X)$ induced by $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Namely, if $(e_i)_{i=1}^r$ is a basis of X, one has

$$\deg(X, \|\cdot\|_1, \|\cdot\|_2) = \log \frac{\det(\langle e_i, e_j\rangle_2)_{1\leqslant i,j\leqslant r}}{\det(\langle e_i, e_j\rangle_1)_{1\leqslant i,j\leqslant r}},$$

where $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are respectively the Hermitian inner products associated to $\|\cdot\|_1$ and to $\|\cdot\|_2$. Note that this definition does not depend on the choice of the basis $(e_i)_{i=1}^r$. If $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ is a short exact sequence of Hermitian spaces, then $\det(X)$ is canonically isometric to $\det(X') \otimes \det(X'')$. Therefore, the function deg is additive with respect to sequences in $\mathcal{E}_A$, and hence defines a degree function on $(\mathbf{Vec}_{\mathbb{C}}, \mathcal{E}, A)$. By Proposition 4.5, $(\mathbf{Vec}_{\mathbb{C}}, \mathcal{E}, A, \deg, \mathrm{rk})$ is a Harder–Narasimhan category.

Given a finite dimensional $\mathbb{C}$-vector space X equipped with a Hermitian norm $\|\cdot\|_1$ (and corresponding inner product $\langle\ ,\ \rangle_1$). The set of all Hermitian norms on X is in bijection with the set of all positive definite and self-adjoint (with respect to $\langle\ ,\ \rangle_1$) operators on X. Assume that $\|\cdot\|_2$ is a Hermitian norm on X which corresponds to the operator S (that is, $\forall x \in X$, $\|x\|_2^2 = \langle x, S(x)\rangle_1$). Let $\lambda_1 > \cdots > \lambda_n$ be eigenvalues of S and $E_1, \ldots, E_n$ be the corresponding eigenspaces. Then $(X, \|\cdot\|_1, \|\cdot\|_2)_{\mathrm{des}}$ is just E_1 equipped with induced metrics. The Harder–Narasimhan sequence

$$0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n = X$$

of $(X, \|\cdot\|_1, \|\cdot\|_2)$ satisfies $X_i = E_1 + \cdots + E_i$ ($i \in \{1, \ldots, n\}$), and the successive slopes are just $\log \lambda_i$. In particular, a geometric object $(X, \|\cdot\|_1, \|\cdot\|_2)$ is semistable if and only if the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are proportional.[2]

We show that the sequence (6) in Theorem 4.3 need not be unique for this example. Let $(X, \|\cdot\|_1, \|\cdot\|_2)$ be a geometric object such that $\mathrm{rk}(X) = 2$. Assume that $\|\cdot\|_2$ corresponds to a positive definite self-adjoint operator S (with respect to $\|\cdot\|_1$) whose eigenvalues are λ_1 and λ_2, where $\lambda_1 > \lambda_2$. Let v_1 and v_2 be corresponding eigenvectors such that $\|v_1\|_1 = \|v_2\|_1 = 1$. Let $\varepsilon \in [0, 1]$ and Y_ε be the subspace of X generated by $\sqrt{\varepsilon}v_1 + \sqrt{1-\varepsilon}v_2$, equipped with induced metrics. One has $\deg(Y_\varepsilon) = \frac{1}{2}\log(\varepsilon\lambda_1^2 + (1-\varepsilon)\lambda_2^2)$. Moreover, one has $\deg(X) = \log\lambda_1 + \log\lambda_2$. Therefore, the sequence $0 \longrightarrow Y_\varepsilon \longrightarrow X$ satisfies the two conditions in Theorem 4.3 as soon as $\varepsilon > \lambda_2/(\lambda_1 + \lambda_2)$ since in this case we have $\deg(Y_\varepsilon) > \deg(X/Y_\varepsilon)$.

Example 3 (*Filtered* (φ, N)*-Modules*). Let K be a field of characteristic 0, equipped with a discrete valuation v such that K is complete for the topology defined by v. Suppose that the residue field k of K is of characteristic $p > 0$. Let K_0 be the fraction field of Witt vector ring $W(k)$ and $\sigma : K_0 \to K_0$ be the absolute Frobenius endomorphism. We call (φ, N)*-module* any finite dimensional vector space D over K_0, equipped with

(1) a bijective σ-linear endomorphism $\varphi : D \to D$,
(2) a K_0-linear endomorphism $N : D \to D$ such that $N\varphi = p\varphi N$.

Let $(\mathcal{C}, \mathcal{E})$ be the category of all (φ, N)-modules. It is an Abelian category. There exists a natural rank function rk on the category $\mathcal{C}$ defined by the rank of the vector space over K_0. Consider the geometric structure A on $(\mathcal{C}, \mathcal{E})$ such that, for any (φ, N)-module D, $A(D)$ is the set of isomorphism classes of $\mathbb{Z}$-filtrations (i.e. an $\mathbb{R}$-filtration whose jump set is contained in $\mathbb{Z}$) of $D \otimes_{K_0} K$ in the category of vector spaces over K. Then $(\mathcal{C}, \mathcal{E}, A)$ becomes a geometric exact category. The objects in $\mathcal{C}_A$ are called *filtered* (φ, N)*-modules*.

To each filtered (φ, N)-module $(D, \mathcal{F} = (D_{K,\lambda})_{\lambda \in \mathbb{R}})$ we associate an integer

$$\deg(D, \mathcal{F}) = -v(\det\varphi) - \int_{\mathbb{R}} \lambda \,\mathrm{drk}_K(D_{K,\lambda}).$$

Then $(\mathcal{C}, \mathcal{E}, A, \deg, \mathrm{rk})$ becomes a Harder–Narasimhan category.

Note that semistable filtered (φ, N)-modules having slope 0 are nothing but admissible filtered (φ, N)-modules. In the classical literature, such filtered (φ, N)-modules were said to be weakly admissible. In fact, Colmez and Fontaine [13] have proved that all weakly admissible (φ, N)-modules are admissible, which had been a conjecture of Fontaine.

Example 4 (*Torsion Free Sheaves*). Let X be a geometrically normal projective variety of dimension $d \geqslant 1$ over a field K and L be an ample invertible $\mathcal{O}_X$-module. We denote by $\mathbf{TF}(X)$ the category of torsion free coherent sheaves on X. Notice that if $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ is an exact sequence of coherent $\mathcal{O}_X$-modules such that E' and E'' are torsion free, then also is E. Therefore, $\mathbf{TF}(X)$ is an exact subcategory of the Abelian category of all coherent $\mathcal{O}_X$-modules on X. Let $\mathcal{E}$ be the class of all exact sequences in $\mathbf{TF}(X)$ and let A be the trivial geometric structure on it. If E is a torsion free coherent $\mathcal{O}_X$-module, we denote by $\mathrm{rk}(E)$ its rank and by $\deg(E)$ the intersection number $c_1(L)^{d-1}c_1(E)$. The mapping deg (resp. rk) extends naturally to a homomorphism from $K_0(\mathbf{TF}(X))$ to $\mathbb{R}$ (resp. $\mathbb{Z}$). A classical result [14] (see also [15]) shows that $(\mathbf{TF}(X), \mathcal{E}, A, \deg, \mathrm{rk})$ is in fact a Harder–Narasimhan category.

Example 5 (*Hermitian Adelic Bundles*). Let K be a number field. Denote by Σ the set of all places of K. For any $v \in \Sigma$, denote by K_v the completion of K with respect to v and by $\mathbb{C}_v$ the completion of an algebraic closure of K_v. Let $|\cdot|_v$ be the canonical absolute value on $\mathbb{C}_v$, n_v be the degree of the residue field of v if v is finite, $n_v = 1$ if v is real and $n_v = 2$ if v is complex. The product formula asserts $\sum_v n_v \log|a|_v = 0$ for any $a \in K^\times$.

By *Hermitian adelic bundle* we mean a finite dimensional vector space E over K equipped with a family $(\|\cdot\|_v)_{v\in\Sigma}$, where $\|\cdot\|_v$ is a norm on $E \otimes_K \mathbb{C}_v$, invariant under the action of $\mathrm{Gal}(\mathbb{C}_v/K_v)$, such that

(1) $\|\cdot\|_v$ is a ultranorm if v is finite, a Hermitian norm if v is infinite,
(2) for all but finite number of places, the norms $\|\cdot\|_v$ have a common orthonormal basis.

Let $(\mathbf{Vec}_K, \mathcal{E})$ be the Abelian category of finite dimensional vector spaces over K. For each vector space X in $\mathbf{Vec}_K$, let $A(X)$ be the set of families $(\|\cdot\|_v)_{v\in\Sigma}$ such that $(X, (\|\cdot\|_v)_{v\in\Sigma})$ becomes a Hermitian adelic bundle. Then $(\mathbf{Vec}_K, \mathcal{E}, A)$ is a geometric exact category. We have an evident rank function on it, which is the rank of the vector space over K.

In the following, we consider a degree function, which is fundamental in Arakelov geometry. For any Hermitian adelic bundle $(X, (\|\cdot\|_v)_{v\in\Sigma})$, we define $\deg(X) = -\sum_v n_v \log\|s_1 \wedge \cdots \wedge s_r\|_v$, where $(s_1, \ldots, s_r)$ is a basis of X over K. By the product formula, this definition does not depend on the choice of the basis $(s_1, \ldots, s_r)$ (for more details, see [16–18]). Furthermore, it defines a degree function on $(\mathbf{Vec}_K, \mathcal{E}, A)$. Thus $(\mathbf{Vec}_K, \mathcal{E}, A, \deg, \mathrm{rk})$ is a Harder–Narasimhan category. This was a result of Stuhler [2] and Grayson [3] in the Hermitian vector bundle case, and of Gaudron [18] in the general case.

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2 Recently, André [12] proposed another formalism of slope filtrations in the protoabelian category framework. However, the definition of semistability in his setting requires the comparison of slopes for non-necessarily strict subobjects and hence his formalism does not contain this example.

5. Slope inequalities and functoriality

Throughout this section, we fix a Harder–Narasimhan category $(\mathcal{C}, \mathcal{E}, A, \deg, \mathrm{rk})$. We assume, in the case where A is non-trivial, that the exact category $(\mathcal{C}, \mathcal{E})$ verifies one of the following equivalent conditions:

(i) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms in $\mathcal{C}$ such that g and gf are admissible monomorphisms, then f is also an admissible monomorphism.

(ii) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms in $\mathcal{C}$ such that f and gf are admissible epimorphisms, then g is also an admissible epimorphism.

Note that these conditions are consequences of "weakly split idempotent" condition introduced in [19, A.5.1]. So they are satisfied for all examples in this article. Assuming these conditions, one has the following lemmas which are dual to each other.

Lemma 5.1. *Let X and Z be two geometric objects, $\pi : X \to Y$ be a geometric quotient of X, and $f : Y \to Z$ be a morphism in $\mathcal{C}$. If $f\pi$ is compatible with geometric structures, also is f.*

Proof. When A is trivial, all morphisms in $\mathcal{C}$ are compatible with geometric structures. So the result is trivial. In the following, we suppose that A is non-trivial and condition (ii) holds. By definition there exists a geometric object W and a decomposition $X \rightarrowtail^{i} W \twoheadrightarrow^{p} Z$ of $f\pi$ such that X is a geometric subobject of W and Z is a geometric quotient of W. Let T be the pushout of i and π (in $\mathcal{C}$) and let $j : Y \to T$ and $q : W \to T$ be canonical morphisms. By Axiom (**Ex**5), j is an admissible monomorphism. Let $\tau : U \rightarrowtail X$ be the kernel of π. Note that any exact category can be embedded as a full subcategory of an Abelian category which is closed under extensions, by diagram chasing we obtain that $q = \mathrm{Coker}(i\tau)$. Hence q is an admissible epimorphism. The morphisms $p : W \twoheadrightarrow Z$ and $f : Y \to Z$ induce a morphism $g : T \to Z$:

$$\begin{array}{ccccc} X & \overset{i}{\rightarrowtail} & W & & \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle q}\downarrow & \overset{p}{\searrow} & \\ Y & \underset{j}{\rightarrowtail} & T & \underset{g}{\longrightarrow} & Z \end{array}$$

which is an admissible epimorphism by condition (ii). If we equip T with the quotient geometric structure from W, we have $g_*(h_T) = p_*(h_W) = h_Z$ and $j^*(h_T) = \pi_*(i^*h_W) = \pi_*(h_X) = h_Y$. Therefore f is compatible with geometric structures. □

Lemma 5.2. *Let X and Z be two geometric objects, $u : Y \to Z$ be a geometric subobject of Z, and $f : X \to Y$ be a morphism in $\mathcal{C}$. If uf is compatible with geometric structures, also is f.*

5.1. Slope inequality condition

In the previous section, we have shown that to each non-zero geometric object X we can associate a sequence

$$0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n = X$$

such that $X_i/X_{i-1} = (X/X_{i-1})_{\mathrm{des}}$ for any $i \in \{1, \ldots, n-1\}$. All subquotients X_i/X_{i-1} are semistable. Furthermore, if we write $\mu_i = \mu(X_i/X_{i-1})$, then the inequalities $\mu_1 > \mu_2 > \cdots > \mu_n$ hold. These data define naturally an $\mathbb{R}$-filtration $\mathcal{F} = (\mathcal{F}_\lambda X)_{\lambda\in\mathbb{R}}$ of X as follows:

$$\mathcal{F}_\lambda X := \begin{cases} 0, & \lambda > \mu_1, \\ X_i, & \lambda \in (\mu_i, \mu_{i-1}],\ i \in \{1, \ldots, n-1\}, \\ X_n, & \lambda \leqslant \mu_n. \end{cases}$$

We call it the *Harder–Narasimhan filtration* of X. One has, for $M < \mu_n$,

$$\deg(X) = -\int_{\mathbb{R}} \lambda \,\mathrm{d}\mathrm{rk}(\mathcal{F}_\lambda X) = M\mathrm{rk}(X) + \int_M^{+\infty} \mathrm{rk}(\mathcal{F}_\lambda X)\,\mathrm{d}\lambda. \tag{10}$$

Note that we have actually defined a map from $obj(\mathcal{C}_A)$ to $obj(\mathbf{Fil}(\mathcal{C}))$. It is quite natural to ask if this construction is functorial, or in other words, if morphisms compatible with geometric structures are compatible with the Harder–Narasimhan filtrations.

Assume that X and Y are two semistable geometric objects such that $\mu(X) > \mu(Y)$. Note that the Harder–Narasimhan filtration of X (resp. Y) has only a jump point at $\mu(X)$ (resp. $\mu(Y)$). Hence the zero morphism is the only morphism from X to Y which is compatible with the Harder–Narasimhan filtrations. Therefore, the functoriality of the Harder–Narasimhan filtration requires the following condition:

(**SI**) If X_1 and X_2 are two semistable geometric objects such that $\mu(X_1) > \mu(X_2)$, there is no non-zero morphism from X_1 to X_2 which is compatible with geometric structures.

We call it the slope inequality condition. This condition is satisfied for all examples we have discussed in the previous section except the second one. We shall see that it is the source of many slope inequalities which are similar to the classical ones for vector bundles on curves. Let us begin by a result which claim that, under condition (**SI**), the Harder–Narasimhan sequence is the only sequence verifying the conditions in Theorem 4.3.

Proposition 5.3. *Assume that the Harder–Narasimhan category* $(\mathcal{C},\mathcal{E},A,\deg,\mathrm{rk})$ *satisfies condition (**SI**). Suppose that X is a non-zero geometric object and* $0=X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X$ *is a sequence which verifies conditions (1) and (2) of Theorem 4.3, then it is canonically isomorphic to the Harder–Narasimhan sequence of X.*

Proof. By induction we only need to prove that X_1 is isomorphic to X_{des}. Let i be the first index such that the canonical morphism $X_{\mathrm{des}} \to X$ factorizes through X_{i+1}. The composed morphism $X_{\mathrm{des}} \to X_{i+1} \to X_{i+1}/X_i$ is then non-zero and compatible with geometric structures (by Lemma 5.2). Since X_{des} and X_{i+1}/X_i are semistable, we have $\mu(X_{\mathrm{des}}) \leqslant \mu(X_{i+1}/X_i)$. This implies $i=0$ and $\mu(X_{\mathrm{des}})=\mu(X_1)$. By (**HN**), the morphism $X_1 \to X$ factorizes through X_{des}. So we have $X_{\mathrm{des}} \cong X_1$. □

Corollary 5.4. *Assume that* $(\mathcal{C},\mathcal{E},A,\deg,\mathrm{rk})$ *satisfies (**SI**). Let X be a non-zero geometric object.*
(1) *For any non-zero geometric subobject Y of X, one has* $\mu_{\max}(Y) \leqslant \mu_{\max}(X)$.
(2) *For any non-zero geometric quotient Z of X, one has* $\mu_{\min}(Z) \geqslant \mu_{\min}(X)$.
(3) *The inequalities* $\mu_{\min}(X) \leqslant \mu(X) \leqslant \mu_{\max}(X)$ *hold.*

Proof. Let $0=X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X$ be the Harder–Narasimhan sequence of X.

(1) After replacing Y by Y_{des} we may suppose that Y is semistable. Let i be the first index such that the canonical morphism $Y \to X$ factorizes through X_{i+1}. The composed morphism $Y \to X_{i+1} \to X_{i+1}/X_i$ is non-zero and compatible with geometric structures (by Lemma 5.2). Therefore $\mu(Y) \leqslant \mu(X_{i+1}/X_i) \leqslant \mu_{\max}(X)$.

(2) After replacing Z by a semistable quotient we may suppose that Z itself is semistable. Let $f : X \to Z$ be the canonical morphism. It is an admissible epimorphism. Let i be the smallest index such that the composed morphism $X_{i+1} \to X \xrightarrow{f} Z$ is non-zero. Since the composed morphism $X_i \to X \xrightarrow{f} Z$ is zero, we obtain a non-zero morphism from X_{i+1}/X_i to Z which is compatible with geometric structures (by Lemma 5.1). Therefore $\mu(Z) \geqslant \mu(X_{i+1}/X_i) \geqslant \mu_{\min}(X)$.

(3) We have $\deg(X)=\sum_{i=1}^{n}\deg(X_i/X_{i-1})$. Therefore

$$\mu(X)=\sum_{i=1}^{n}\frac{\mathrm{rk}(X_i/X_{i-1})}{\mathrm{rk}(X)}\mu(X_i/X_{i-1}) \in \big[\mu_{\min}(X),\mu_{\max}(X)\big]. \quad \square$$

Proposition 5.5. *Suppose that* $(\mathcal{C},\mathcal{E},A,\deg,\mathrm{rk})$ *satisfies condition (**SI**). If X and Y are two geometric objects and if* $f : X \to Y$ *is a non-zero morphism compatible with geometric structures, then* $\mu_{\min}(X) \leqslant \mu_{\max}(Y)$.

Proof. Let $0=X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{n-1} \longrightarrow X_n = X$ be the Harder–Narasimhan sequence of X. For any integer $i \in \{0,\ldots n\}$, let $\varphi_i : X_i \to X$ be the canonical monomorphism. Let $j \in \{1,\ldots,n\}$ be the first index such that $f\varphi_j$ is non-zero. Since $f\varphi_{j-1}=0$, the morphism $f\varphi_j$ factorizes through X_j/X_{j-1}, so we get a non-zero morphism g from X_j/X_{j-1} to Y. By Lemma 5.1, g is compatible with geometric structures. Let

$$0=Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{m-1} \longrightarrow Y_m$$

be the Harder–Narasimhan sequence of Y. Let $k \in \{1,\ldots n\}$ be the first index such that g factorizes through Y_k. If $\pi : Y_k \to Y_k/Y_{k-1}$ is the canonical morphism, then πg is non-zero since g does not factorize through Y_{k-1}. Furthermore, it is compatible with geometric structures. Therefore, we have

$$\mu_{\min}(X) \leqslant \mu(X_j/X_{j-1}) \leqslant \mu(Y_k/Y_{k-1}) \leqslant \mu_{\max}(Y). \quad \square$$

Corollary 5.6. *Keep the notation and the hypothesis of Proposition 5.5.*
(1) *If in addition f is monomorphic, then* $\mu_{\max}(X) \leqslant \mu_{\max}(Y)$.
(2) *If in addition f is epimorphic, then* $\mu_{\min}(X) \leqslant \mu_{\min}(Y)$.

Proof. Suppose that f is monomorphic. Let $i : X_{\mathrm{des}} \to X$ be the canonical morphism. Then the composed morphism $fi : X_{\mathrm{des}} \to Y$ is non-zero and compatible with geometric structures. Therefore $\mu_{\max}(X)=\mu_{\min}(X_{\mathrm{des}}) \leqslant \mu_{\max}(Y)$. The proof of the second assertion is similar. □

5.2. Functoriality

Theorem 5.7. *Assume that* $(\mathcal{C},\mathcal{E},A,\deg,\mathrm{rk})$ *satisfies condition (**SI**). Then any morphism in* $\mathcal{C}_A$ *is compatible with the Harder–Narasimhan filtrations.*

Proof. Let $f : X \to Y$ be a morphism which is compatible with geometric structures. Let $(\mathcal{F}_\lambda X)_{\lambda\in\mathbb{R}}$ and $(\mathcal{F}_\lambda Y)_{\lambda\in\mathbb{R}}$ be respectively the Harder–Narasimhan filtrations of X and Y. We shall prove that, for any $\lambda \in \mathbb{R}$, the morphism $f_\lambda := fi_\lambda : \mathcal{F}_\lambda X \to Y$ factorizes through $\mathcal{F}_\lambda Y$, where $i_\lambda : \mathcal{F}_\lambda X \to X$ denotes the canonical morphism. The case where $f_\lambda = 0$

is trivial. We suppose in the following that $f_\lambda \neq 0$. Let $0 = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n = Y$ be the Harder–Narasimhan sequence of Y. Let $i \in \{1, \dots, n\}$ be the smallest index such that f_λ factorizes through Y_i. Thus the composed morphism $\mathcal{F}_\lambda X \longrightarrow Y_i \longrightarrow Y_i/Y_{i-1}$ is non-zero, and compatible with geometric structures. Moreover, from the definition of $\mathcal{F}_\lambda X$ we obtain $\mu_{\min}(\mathcal{F}_\lambda X) \geqslant \lambda$. Therefore the slope inequality in Proposition 5.5 implies that $\lambda \leqslant \mu_{\max}(Y_i/Y_{i-1}) = \mu(Y_i/Y_{i-1})$. Thus Y_i is a geometric subobject of $\mathcal{F}_\lambda Y$ and hence f_λ factorizes through $\mathcal{F}_\lambda Y$. □

As an application, the following proposition compares the degree of two geometric objects provided an isomorphism compatible with geometric structures.

Corollary 5.8. *Under the conditions of the above theorem, if $f : X \to Y$ is an isomorphism compatible with geometric structures, then* $\deg(X) \leqslant \deg(Y)$.

Proof. Let $(X_\lambda)_{\lambda\in\mathbb{R}}$ and $(Y_\lambda)_{\lambda\in\mathbb{R}}$ be respectively the Harder–Narasimhan filtrations of X and of Y. Theorem 5.7 implies that f is compatible with filtrations. Hence $\mathrm{rk}(X_\lambda) \leqslant \mathrm{rk}(Y_\lambda)$ for any λ. Moreover, $\mathrm{rk}(X) = \mathrm{rk}(Y)$. By (10),

$$\deg(X) = M\mathrm{rk}(X) + \int_M^{+\infty} \mathrm{rk}(X_\lambda)\,\mathrm{d}\lambda \leqslant M\mathrm{rk}(Y) + \int_M^{+\infty} \mathrm{rk}(Y_\lambda)\,\mathrm{d}\lambda = \deg(Y),$$

where M is a sufficiently negative real number. □

Proposition 5.9. *Let $(\mathcal{C}, \mathcal{E}, A, \deg, \mathrm{rk})$ be a Harder–Narasimhan category. If $(\mathcal{C}, \mathcal{E})$ is an Abelian category, then condition (**SI**) is equivalent to the following condition:*

(**SI'**) *For any object X in $\mathcal{C}$ and for all geometric structures h_X and h'_X on X, if $\mathrm{Id}_X : (X, h_X) \to (X, h'_X)$ is compatible with geometric structures, then* $\deg(X, h_X) \leqslant \deg(X, h'_X)$.

Proof. "(**SI**) $\Longrightarrow$ (**SI'**)" is a consequence of Corollary 5.8. In the following, we prove the converse implication. Let X and Y be two semistable geometric objects. Suppose that there exists a non-zero morphism $f : X \to Y$ which is compatible with geometric structures. Let Z be the image of f in Y, $u : Z \to Y$ be the inclusion morphism and $\pi : X \to Z$ be the projection morphism. The fact that f is compatible with geometric structures implies that the identity morphism $\mathrm{Id}_Z : (Z, \pi_* h_X) \to (Z, u^* h_Y)$ is compatible with geometric structures (see Lemmas 5.1 and 5.2). Therefore, the semistability of X and Y, combined with condition (**SI'**), implies $\mu(X) \leqslant \mu(Z, \pi_* h_X) \leqslant \mu(Z, u^* h_Y) \leqslant \mu(Y)$. □

Remark 5.10. The functoriality of the Harder–Narasimhan filtration is not necessarily verified without condition (**SI**). Consider Example 2. In that setting, the Harder–Narasimhan filtration for a geometric object $(X, \|\cdot\|_1, \|\cdot\|_2)$ is just $X_\lambda = \{x \in X \mid \|x\|_2 \geqslant e^\lambda \|x\|_1\}$. Hence, with the notation introduced in the last paragraph of Example 2, the inclusion morphism $Y_\varepsilon \to X$ is not compatible with the Harder–Narasimhan filtrations if $\varepsilon > 0$, the quotient morphism $X \to X/Y_\varepsilon$ is not compatible with the Harder–Narasimhan filtrations if $\varepsilon < 1$.

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