Some asymptotics of TQFT via skein theory

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Abstract

For each oriented surface Σ of genus g we study a limit of quantum representations of the mapping class group arising in TQFT derived from the Kauffman bracket. We determine that these representations converge in the Fell topology to the representation of the mapping class group on $\mathcal{H}(\Sigma)$, the space of regular functions on the $SL(2,\mathbb{C})$-representation variety with its hermitian structure coming from the symplectic structure of the $SU(2)$-representation variety. As a corollary, we give a new proof of the asymptotic faithfulness of quantum representations.

1 Introduction

A topological quantum field theory in dimension 2+1 is an algebraic structure very close to topology: roughly speaking, it associates to each surface a finite dimensional vector space and to each cobordism a linear map between the vector spaces associated to the boundaries. Such theories have physical origins: they were introduced by Witten [Wit89] in the eighties from Chern-Simons actions and generated very rich mathematical developments. There are various rigorous constructions coming from geometric quantization, quantum groups and many other areas. Unfortunately, such constructions remain complicated and it is hard to make concrete computations.

In this paper, we preferred the approach of [BHMV95] which defines TQFTs in a purely combinatorial way: using skein theory and the Kauffman bracket, the authors defined a family of hermitian TQFTs $(V_p, \langle\cdot,\cdot\rangle_p)$ corresponding for $p = 2r$ to the $SU(2)$-theory at level $r-2$. Despite the simple and very beautiful structure of these combinatorial TQFTs, the connection with geometry is less clear than from other approaches. In this article, we show that some connections can be found in a simple and direct way. From the axioms, a TQFT generates for any closed surface Σ a family of representations of the extended mapping class group of Σ, a central extension of the mapping class group by $\mathbb{Z}$ coming from p_1-structures (see [MR95]). In some sense, these representations carry the main topological meaning of TQFTs. Hence we would like to link them with some geometrical representation. The basic idea for this comes from a general belief that when p goes to infinity, things become classical, by which we mean geometrical. This belief is based on the so-called semi-classical approximation. Hence we propose to study the limit of ρ_p, the quantum representations of Γ_g on $V_p(\Sigma)$.

For this purpose, let us describe two classical spaces on which the mapping class group acts.

Fix a closed oriented surface Σ of genus g. We call *multicurve* an isotopy class of 1-dimensional submanifold of Σ without component bounding a disc (the empty set is also considered as a multicurve). The mapping class group of Γ_g acts on the set of multicurves in a natural way. Call $C(\Sigma)$ the $\mathbb{C}$-vector space generated by multicurves: we obtain a representation of Γ_g on $C(\Sigma)$. This fundamental representation carries almost all information about the structure of Γ_g.

For instance, no non-trivial element of Γ_g acts trivially on multicurves, except for the elliptic and hyperelliptic involutions in genus 1 and 2.

Another very natural space on which the mapping class group acts is $\hom(\pi_1(\Sigma), G)/G$, the G-character variety of $\pi_1(\Sigma)$ for a fixed Lie group G. Let us denote it by $\mathcal{S}(\Sigma, G)$. Any element of the mapping class group of Σ may be represented as an automorphism of $\pi_1(\Sigma)$: its action on $\mathcal{S}(\Sigma, G)$ is then obtained by left composition on $\hom(\pi_1(\Sigma), G)$. In that way, we also obtain an action of the mapping class group on any space of functions defined on the G-character variety.

We are interested here in the cases $G = SU(2)$ and $G = SL(2, \mathbb{C})$. These spaces have a rich structure: we will use the natural symplectic structure ω on the smooth part of $\mathcal{S}(\Sigma, SU(2))$ (see [Gol84]) and the structure of an algebraic variety on $\mathcal{S}(\Sigma, SL(2, \mathbb{C}))$. We define $\mathcal{H}(\Sigma)$ as the ring of regular functions on $\mathcal{S}(\Sigma, SL(2, \mathbb{C}))$.

Using the natural inclusion of $\mathcal{S}(\Sigma, SU(2))$ in $\mathcal{S}(\Sigma, SL(2, \mathbb{C}))$, we can define a hermitian form on $\mathcal{H}(\Sigma)$ by the formula

$$\langle f, g \rangle = \int_{\mathcal{S}(\Sigma, SU(2))} f \overline{g} dV$$

Here, dV is the volume form on $\mathcal{S}(\Sigma, SU(2))$ induced by the symplectic form ω.

The following theorem can be interpreted in terms of Fell topology. For convenience, let us recall this notion briefly. Let G be a discrete group and $\rho_k : G \to U(V_k)$ a sequence of unitary representations of G into V_k. One says that this sequence converges to the representation $\rho : G \to U(V)$ in the Fell topology if for any unit vector $v \in V$ and any finite subset $S \subset G$ there is a sequence of unit vectors $v_k \in V_k$ such that for all $g \in S$, the sequence $\langle \rho_k(g) v_k, v_k \rangle$ converges to $\langle \rho(g) v, v \rangle$.

We obtain the following result:

Theorem. *Let Σ be a closed oriented surface of genus g. For all even integers p, there is a Γ_g-equivariant map $\varphi_p : \mathcal{H}(\Sigma) \to End(V_p(\Sigma))$ such that:*

$$\langle v, w \rangle = \lim_{p \to \infty} \left(\frac{2}{p} \right)^{d(g)} \langle \varphi_p(v), \varphi_p(w) \rangle_p \text{ for all } v, w \in \mathcal{H}(\Sigma).$$

Here, we have set $d(1) = 1$ and $d(g) = 3g - 3$ for $g > 1$. The hermitian form on $End(V_p(\Sigma))$ is defined by $\langle x, y \rangle_p = \mathrm{Tr}(xy^*)$. This implies in particular that the quantum representations $\rho_p \otimes \overline{\rho_p}$ converge in the Fell topology to $\rho : \Gamma_g \to U(\mathcal{H}(\Sigma))$, the natural representation coming from the action of Γ_g on $\mathcal{S}(\Sigma, SL(2, \mathbb{C}))$.

As a corollary, we obtain a new proof of the result of [And06] (see also [FWW02]) about asymptotic faithfulness of quantum representations.

Corollary. *Let Σ be a closed oriented surface of genus g. For any non-trivial h in Γ_g which is not the elliptic ($g = 1$) or hyperelliptic ($g = 2$) involution, there is some even p_0 such that $\rho_p(h) \neq \mathrm{Id}$ for all even $p \geq p_0$.*

Proof. One can associate to any curve γ on Σ a regular function f_γ on $\mathcal{S}(\Sigma, SL(2, \mathbb{C}))$ by the formula $f_\gamma(\rho) = -\operatorname{Tr} \rho(\gamma)$. For a disjoint union of curves, we associate the product of the functions associated to each component. In this way, we construct a map f from $C(\Sigma)$ to $\mathcal{H}(\Sigma)$. By a result of [Bul97] and [PS00], the map f is an isomorphism of vector spaces. Therefore, we can think of a regular function on $\mathcal{S}(\Sigma, SL(2, \mathbb{C}))$ as a linear combination of multicurves.

Recall that no element of Γ_g acts trivially on $C(\Sigma)$ except the identity and the elliptic and hyperelliptic involutions in genus 1 and 2. Hence, we can suppose that there is some v in $C(\Sigma) \simeq \mathcal{H}(\Sigma)$ such that $w = hv - v$ is non-zero. This implies that $\langle w, w\rangle$ is non-zero, because the form $\langle \cdot, \cdot \rangle$ is non-degenerate.

In fact, if $\langle w, w\rangle = 0$, then the regular function on $\mathcal{S}(\Sigma, SL(2,\mathbb{C}))$ associated to w satisfies

$$\int_{\mathcal{S}(\Sigma, SU(2))} |w|^2 dV = 0.$$

As w is continuous, it must vanish on $\mathcal{S}(\Sigma, SU(2))$. Moreover, as it is holomorphic on the space $\mathcal{S}(\Sigma, SL(2,\mathbb{C}))$ and 0 on $\mathcal{S}(\Sigma, SU(2))$, it vanishes identically. (See proof of Theorem 1.4.1 in [Gol04]).

Due to the equality $\langle w, w\rangle = \lim_{p\to\infty} (\frac{2}{p})^{d(g)} \langle \varphi_p w, \varphi_p w\rangle_p$, we can find even p_0 such that for all even $p \geq p_0$, $\varphi_p w \neq 0$. Hence $\varphi_p(hv) \neq \varphi_p(v)$ and $\rho_p(h)$ cannot be the identity. □

1.1 Plan of the proof of the theorem

The heart of the proof is the construction of the map φ_p, which is almost obvious, but is fundamental. As the space $\mathcal{H}(\Sigma)$ is isomorphic to $C(\Sigma)$, to define the map φ_p, it is sufficient to construct $\varphi_p(\gamma) \in V_p(\Sigma) \otimes V_p(\Sigma)^*$ for any multicurve γ.

For such a multicurve, we consider the cobordism $\Sigma \times [0,1]$ with the multicurve embedded as $\gamma \times \{\frac{1}{2}\}$. The TQFT naturally induces an element $Z_p(\Sigma \times [0,1], \gamma)$ in $V_p(\Sigma \amalg -\Sigma) = V_p(\Sigma) \otimes V_p(\Sigma)^*$. We call this element $\varphi_p(\gamma)$. We remark that it defines a self-adjoint element of $\mathrm{End}(V_p(\Sigma)$ as the cobordism $\Sigma \times [0,1]$ with the curve γ inside is isomorphic to the same cobordism with opposite orientation and exchanged boundaries. This gives our fundamental map φ_p, which is clearly equivariant because of the naturality of the construction.

To prove the theorem, one has to compute the limit of the expression $(\frac{2}{p})^{d(g)} \langle \varphi_p(\gamma), \varphi_p(\delta)\rangle_p$ for two multicurves γ and δ.

We do this in two steps. In the first step, we assume that δ is empty. Using combinatorial techniques from [BHMV95], we obtain for $\langle \varphi_p(\gamma), 1\rangle_p$ an explicit formula ressembling a Riemann sum. When we normalize it, it converges to an integral over a subspace of $\mathbb{R}^{d(g)}$: we denote its value by $\langle \gamma \rangle$. By linearity, we extend $\langle \cdot \rangle$ to a map from $C(\Sigma)$ to $\mathbb{C}$.

In the second step, we use the connection between the TQFT V_p and the Kauffman skein module at $A = -e^{\frac{i\pi}{p}}$. We find easily that $(\frac{2}{p})^{d(g)} \langle \varphi_p(\gamma), \varphi_p(\delta)\rangle_p$ converges to $\langle \gamma \cdot \delta \rangle$, where $\cdot$ is the multiplication induced on $C(\Sigma)$ by its identification with the Kauffman skein algebra of $\Sigma \times [0,1]$ at $A = -1$ (see [PS00]).

On the other hand, it is well-known that this multiplication on $C(\Sigma)$ is isomorphic to the natural multiplication on $\mathcal{H}(\Sigma)$, the space of regular functions on $\mathcal{S}(\Sigma, SL(2,\mathbb{C}))$ (see [Bul97, PS00]).

It remains to identify the linear form on $\mathcal{H}(\Sigma)$ defined by $f_\gamma \mapsto \langle \gamma \rangle$. Suppose that γ is a multicurve. We choose curves C_i on Σ which decompose the surface into pants such that all components of γ are parallel to some C_i. It is well known that the maps $f_i = f_{C_i}$ form a system of Poisson commuting functions on $\mathcal{S}(\Sigma, SU(2))$.

As shown in [JW94], the maps $(f_i) : \mathcal{S}(\Sigma, SU(2)) \to \mathbb{R}^{d(g)}$ are the moment map for an action of a torus of dimension $d(g)$ on a dense open subset of $\mathcal{S}(\Sigma, SU(2))$. The authors use

the Duistermaat-Heckman theorem to give an explicit formula for the volume form dV on $\mathcal{S}(\Sigma, SU(2))$. From their result, we deduce the following striking formula:

$$\langle \gamma \rangle = \int_{\mathcal{S}(\Sigma, SU(2))} f_\gamma dV.$$

The theorem follows from this formula.

1.2 Remarks and perspectives

The main motivation for this work came from the article [FK] about the asymptotics of quantum representations of the mapping class group of the torus. Our approach is different in the sense that we study the limit of $V_p \otimes V_p^*$ instead of simply V_p. We were also inspired by the ideas contained in the paper [Fre03] and [Mas03]. Our work is of course related to the article [And06] where similar ideas appear, and has also some intersection with [BFKB03].

There are many questions naturally linked to our results:

- How can we link our asymptotic result to the asymptotics considered in [FK]?
- Can we apply our result or some refinements to the problem of [AMU]? The Nielsen-Thurston classification of the elements of the mapping class group is directly related to their action on multicurves. As the quantum representations converge to this action, can we find some trace of this classification in quantum representations at finite level?
- In [BHMV95], one can choose any primitive $4r$-root of unity to construct a TQFT. We have chosen roots converging to -1. Is it possible to develop the same asymptotics for roots of unity converging to different complex numbers?
- Can we obtain a stronger convergence for the sequence involved in the theorem? For instance, what is the expansion of this sequence into powers of $\frac{1}{p}$?

We would like to thank Gregor Masbaum for his remarks, encouragements and his simplification of the proof of lemma 3.3. The results of this article were recovered and generalized to the $SU(n)$-case via the theory of Toeplitz operators in a purely geometric framework (see [And06]).

2 Review of TQFT

This part is a quick and formal review of the TQFT constructed in [BHMV95] which we give to fix notations and settings, and to recall results that will be used in this paper. We refer the interested reader to the beautiful original paper.

Fix an even integer $p = 2r$. The complex number $A = -e^{i\pi/2r}$ is a primitive $4r$-th root of unity. One can construct from it a 2+1 topological quantum field theory.

In the notations of [BHMV95], we set $\kappa = e^{-\frac{i\pi}{2r} - \frac{i\pi(2r+1)}{12}}$ and $\eta = \sqrt{\frac{2}{r}}\sin(\frac{\pi}{r})$. We define $C_r = \{0, 1, \cdots, r-2\}$ which will be called the set of *colors*.

A triple (a, b, c) of elements of C_r is called *r-admissible* if $a+b+c$ is even, the triangle inequality $|a-b| \leq c \leq a+b$ is satisfied and moreover we have $a+b+c < 2r-2$.

Define the quantum integer $[j]$ by the formula $[j] = \frac{\sin(\frac{\pi j}{r})}{\sin(\frac{\pi}{r})}$ and the quantum factorial by the formula $[j]! = \prod_{k=1}^{j}[k]$.

2.1 The cobordism category

A TQFT is a linear representation of a cobordism category. In our settings, the objects of our category are oriented surfaces with marked points and p_1-structures.

- A marking of a surface Σ is a finite family $(z_j, c_j)_{j\in J}$ where (z_j) is a family of distinct points in Σ with for all $j \in J$ a non zero tangent direction v_j at z_j on Σ. For all $j \in J$, c_j is a color in C_r.

- A p_1-structure is a somewhat complicated object, used to solve the so-called "framing anomaly". Consider the map $p_1 : BO \to K(\mathbb{Z}, 4)$ corresponding to the first Pontryagin class. Let X be its homotopy fiber, i.e. the set of couples $(x, \gamma) \in BO \times C([0, 1], K(\mathbb{Z}, 4))$ satisfying $\gamma(0) = *$, and $\gamma(1) = p_1(x)$. Let E be the universal stable bundle over BO, and E_X its pull-back over X. A p_1-structure on a manifold M is a fiber map from the stable tangent bundle of M to E_X.

In the notation of an object (Σ, z, c), we do not mention the directions v_j and the p_1-structure, although they are present.

We define now morphisms : let (Σ_1, z_1, c_1) and (Σ_2, z_2, c_2) be two objects as defined above. A morphism is

- An oriented 3-manifold M whose boundary is decomposed as $\partial M = -\Sigma_1 \amalg \Sigma_2$, where $-\Sigma$ means Σ with opposite orientation.

- A colored banded trivalent graph G embedded in M whose restriction to the boundary is compatible with the marked points.

- A p_1-structure on M extending the p_1-structure given on the boundary.

A banded trivalent graph G in M is a 1-3-valent graph contained in an oriented surface $SG \subset M$ such that

(i) G meets ∂M transversally on the set of 1-valent vertices of G noted ∂G.

(ii) The surface SG is a regular neighborhood of G in SG, and $SG \cap \partial M$ is a regular neighborhood of $G \cap \partial M$ in $SG \cap \partial M$.

A coloring of G is a map σ from the set of edges of G to C_r such that the colors of the edges meeting at each vertex are r-admissible. The restriction of a banded graph $G \subset M$ on ∂M gives marked points $(z_j)_{j\in J}$ with tangent directions $(v_j)_{j\in J}$, whereas the restriction of a coloring gives colors $(c_j)_{j\in J}$.

Two morphisms are called equivalent if the corresponding manifolds are isomorphic, the banded graphs are isotopic and the p_1-structures are homotopic relative to the boundary.

2.2 Main properties of TQFT

The theorem proved in [BHMV95] states that for each integer p, there is a functor (V_p, Z_p) from the precedent cobordism category to the category of finite dimensional $\mathbb{C}$-vector spaces.

This means that to every object (Σ, z, c) we can associate a vector space $V_p(\Sigma, z, c)$ and to any morphism (M, G) between two objects a linear map $Z_p(M, G)$ between the vector spaces corresponding to the objects. By convention, $V_p(\emptyset) = \mathbb{C}$, hence any closed manifold (M, G) acts as a scalar $\langle M, G\rangle_p$ which is a 3-manifold invariant. Moreover, there is natural hermitian form $\langle \cdot, \cdot \rangle_p$ on $V_p(\Sigma, z, c)$ such that for any two morphisms (M_1, G_1) and (M_2, G_2) from $\emptyset$ to (Σ, z, c), we have $\langle Z_p(M_1, G_1), Z_p(M_2, G_2)\rangle_p = \langle M_1 \cup (-M_2), G_1 \cup G_2\rangle_p$.

We give here some important results related to this construction:

Theorem 2.1 (BHMV)**.** *Let (Σ, z, c) be a surface with marked points and p_1-structure. Let H be a handlebody whose boundary is Σ and with a p_1-structure extending that of Σ. Let G be a 1-3-valent banded graph in H such that $\partial G = z$ and such that H is a tubular neighborhood of G. For each coloring σ of G compatible with the coloring of the boundary, we denote by u_σ the element induced by Z_p in $V_p(\Sigma, z, c)$.*

Then the elements u_σ form an orthogonal basis of $V_p(\Sigma, z, c)$, and if G does not contain any closed loop, we have

$$\langle u_\sigma, u_\sigma\rangle_p = \eta^{\#v-\#e}\frac{\prod_v \langle \sigma_v\rangle}{\prod_e \langle \sigma_e\rangle}$$

In this formula, v ranges over the set of vertices of G and e over the set of edges. Moreover, for any trivalent vertex v, σ_v is the triple of colors of the edges adjacent to this vertex and for any monovalent vertex v, σ_v is the color of the edge incoming to it.

We set $\langle j\rangle = (-1)^j[j+1]$ and $\langle a, b, c\rangle = (-1)^{\alpha+\beta+\gamma}\frac{[\alpha+\beta+\gamma+1]![\alpha]![\beta]![\gamma]!}{[a]![b]![c]!}$ where α, β and γ are defined by the equations $a = \beta + \gamma, b = \alpha + \gamma, c = \alpha + \beta$.

If G is reduced to a closed loop, then the formula is simply $\langle u_\sigma, u_\sigma\rangle_p = 1$.

Remark 2.2. *We check that for our choice of root of unity A and for a surface Σ without marked points, the hermitian pairing on $V_p(\Sigma)$ is positive definite.*

2.3 Kauffman Bracket and TQFT

We define $K(M)$ as the usual skein module of any oriented 3-manifold M. We refer to [PS00] for a complete account, but we will recall here what we need. Let A be some indeterminate. The $\mathbb{Z}[A, A^{-1}]$-module $K(M)$ is the free module generated by isotopy class of banded links in M including the empty link, $\emptyset$, quotiented by the submodule generated by the local relations of figure 1.

For any $u \in \mathbb{C} \setminus \{0\}$, we set $K(M, u) = K(M) \otimes_{\mathbb{Z}[A,A^{-1}]} \mathbb{C}$ where A acts on $\mathbb{C}$ by multiplication by u.

The following proposition is a consequence of the construction of the TQFT.

Proposition 2.3 (Proposition 1.9 in [BHMV95])**.** *Let M be an oriented connected 3-manifold with p_1-structure and boundary Σ (without boundary or marked points). Then there is a surjective map from $K(M, -e^{i\pi/p})$ to $V_p(\Sigma)$.*

This map is defined by sending the element $L \otimes 1$ to $Z_p(M, L)$ where L is considered as a banded link with color 1.

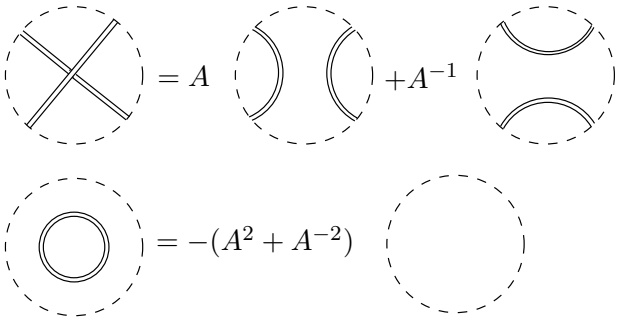

Figure 1: Kauffman relations

3 Convergence of TQFT

3.1 Settings

Let Σ be a closed oriented surface of genus g with p_1-structure. We denote by Γ_g the mapping class group of Σ. Fix $p = 2r$.

If h is an element of Γ_g, we can construct a cobordism C_h from Σ to itself as $\Sigma \times [0,1]$ where we identify the first boundary component with Σ using the identity and the second one using h. If h' is another element of Γ_g, the cobordisms $C_h \circ C_{h'}$ and $C_{hh'}$ are diffeomorphic. We should obtain a representation of Γ_g on $V_p(\Sigma)$ by considering the linear map $Z_p(C_h)$. The problem is that we have not chosen any p_1-structure on C_h, and we cannot make a canonical choice.

One way to get rid of this annoying fact is to consider the action of Γ_g on $V_p(\Sigma) \otimes V_p(\Sigma)^* = V_p(\Sigma \amalg -\Sigma)$. This action of h on this space is given by $Z_p(C_h \amalg -C_h)$ where we choose any p_1-structure on C_h and put the same one on $-C_h$. The action does not depend any more on the p_1-structure: in fact, if we change the p_1-structure in a cobordism M, the linear map $Z_p(M)$ is changed by a multiple of κ, a root of unity. When we take the dual, the root becomes its conjugate. Hence, the two "anomalies" cancel and we get a true representation of Γ_g.

We thus obtain a sequence of representations $(V_p(\Sigma) \otimes V_p(\Sigma)^*, Z_p \otimes Z_p^*)$ of Γ_g, and want to find their limit in some sense. The problem is that the spaces on which the mapping class group acts are a priori completely different. We need a way to compare them which is suggested by Proposition 2.3.

Given a multicurve γ in Σ, one can give it a banded structure by taking a neighborhood of it in Σ. We can consider the curve γ as a banded link in $\Sigma \times [0,1]$ by sending it to $\gamma \times \{1/2\}$. We use the same notation for the multicurve on Σ and its associated banded link in $\Sigma \times [0,1]$.

In [PS00], it is shown that the Kauffman skein module $K(\Sigma \times [0,1])$ is a free $\mathbb{Z}[A, A^{-1}]$-module with basis the isotopy classes of multicurves. It provides an isomorphism of vector spaces between $C(\Sigma)$ and $K(\Sigma \times [0,1], u)$ for any u in $\mathbb{C} \setminus \{0\}$. In particular, using Proposition 2.3, we get a surjective map

$$\varphi_p : C(\Sigma) \xrightarrow{\sim} K(\Sigma \times [0,1], -e^{i\pi/p}) \to V_p(\Sigma \amalg -\Sigma).$$

Theorem 3.1. *Let Σ be a closed oriented surface of genus g. There is a hermitian pairing $\langle \cdot, \cdot \rangle$ on $C(\Sigma)$ such that for all x and y in $C(\Sigma)$, the following holds, where $d(1) = 1$ and $d(g) = 3g-3$ for $g > 1$.*

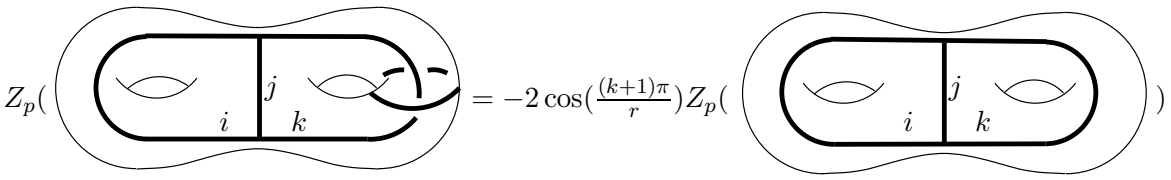

Figure 2: Action of a curve in TQFT

$$\langle x, y\rangle = \lim_{p\to\infty} \left(\frac{2}{p}\right)^{d(g)} \langle \varphi_p(x), \varphi_p(y)\rangle_p.$$

3.2 The trace function

Definition 3.2. *Let Σ be a closed oriented surface of genus g and γ be a multicurve on Σ. We set $\mathrm{Tr}_p(\gamma) = \langle \Sigma \times S^1, \gamma\rangle_p$. Here, γ is seen as a banded link with color 1 lying in the slice $\Sigma \times \{1/2\}$ of $\Sigma \times S^1$.*

Lemma 3.3. *Suppose that a surface Σ is presented as the boundary of a handlebody H which retracts on a trivalent banded graph G as in Theorem 2.1. We choose meridian disks D_e transverse to each edge of G and define $C_e = \partial D_e$: the curves C_e are disjoint on Σ. We choose a non-negative integer m_e for each edge of G.*

Then we define γ as the multicurve on Σ obtained by taking m_e parallel copies of C_e for each edge of G. We have

$$\mathrm{Tr}_p(\gamma) = \sum_{\sigma} \prod_{e} \left[-2\cos\left(\frac{(\sigma_e+1)\pi}{r}\right)\right]^{m_e}.$$

Here σ ranges over r-admissible colorings of G and e ranges over edges of G.

Proof. The proof is an easy consequence of the following fact from skein theory: a trivial curve colored with 1 and making a Hopf link with a curve colored with j may be removed and replaced by a factor $-A^{2j+2}-A^{-2j-2} = -2\cos(\frac{(j+1)\pi}{r})$. We refer for instance to Lemma 3.2 of [BHMV92].

We use the general trace formula of TQFT, see for instance formula 1.2 in [BHMV95]: let M be a cobordism from Σ to Σ and Γ be a colored banded graph in M. Let M_Σ be the closed manifold obtained from M by identifying the two copies of Σ. Then, $\langle M_\Sigma, \Gamma\rangle_p = \mathrm{Tr}\, Z_p(M, \Gamma)$.

Consider the basis $u_\sigma = Z_p(H, G_\sigma)$ of $V_p(\Sigma)$ involved in Theorem 2.1. For each curve C_e, the cobordism $(\Sigma \times [0,1], C_e)$ acts on u_σ by multiplication with $-2\cos(\frac{(\sigma_e+1)\pi}{r})$, as it is suggested in figure 2.

Then the cobordism $(\Sigma \times [0,1], \gamma)$ acts on u_σ by multiplication with $\prod_e \left(-2\cos(\frac{(\sigma_e+1)\pi}{r})\right)^{m_e}$. The formula for $\mathrm{Tr}_p(\gamma)$ comes now from the trace formula of TQFT.

□

3.3 Limit of the trace function

As before, fix a surface Σ presented as in Theorem 2.1 as the boundary of a handlebody H which retracts on a trivalent banded graph G.

The number of edges of G is $3g-3$ if $g>1$ or 1 if $g=1$. We denote this number by $d(g)$ and consider the subset U_g of $\mathbb{R}^{d(g)}$ consisting of all maps τ from the set of edges of G to $[0,1]$ such that for all triples of incoming edges (e,f,g) of some vertex, the following relations are satisfied:

$$\text{- } |\tau_f-\tau_g|\leq\tau_e\leq\tau_f+\tau_g$$

$$\text{- } \tau_e+\tau_f+\tau_g\leq 2$$

We use the formula of Lemma 3.3 to deduce the asymptotics of the trace function.

Lemma 3.4. *With the same hypothesis as in Lemma 3.3, let $F_\gamma : U_g\to\mathbb{R}$ be the map defined by $F_\gamma(\tau)=\prod_e(-2\cos(\tau_e\pi))^{m_e}$. Then the following formula holds:*

$$\lim_{r\to\infty}(\frac{2}{p})^{d(g)}\operatorname{Tr}_p(\gamma)=2^{g-d(g)}\int_{U_g}F_\gamma(\tau)d\tau.$$

Proof. The formula for $\operatorname{Tr}_p(\gamma)$ looks like a Riemann sum, hence the result should not be a surprise. To obtain the precise result, we have to decompose U_g into small pieces parametrized by r-admissible colorings σ.

Given a positive integer r and any coloring σ from the set of edges of G to C_r, we define the following set $A^r_\sigma=\prod_e[\frac{\sigma_e}{r},\frac{\sigma_e+1}{r})\subset\mathbb{R}^{d(g)}$. As σ runs over r-admissible colorings of G, these sets do not cover U_g because of the parity condition. We have to pack some sets A^r_σ together, which we do in the following way.

We denote by $C^1(G,\mathbb{Z}_2)$ the $\mathbb{Z}_2$-vector space of 1-cochains of G with $\mathbb{Z}_2$ coefficients. The subspace of 1-cycles will be denoted by $Z_1(G,\mathbb{Z}_2)$. Choose a subspace S of $C_1(G,\mathbb{Z}_2)$ such that $C_1(G,\mathbb{Z}_2)=S\oplus Z_1(G,\mathbb{Z}_2)$. The subspace S has dimension $d(g)-g$. For an admissible coloring σ of G, we define $B^r_\sigma=\bigcup_{\rho\in S}A^r_{\sigma+\rho}$. Here we have identified $\mathbb{Z}/2\mathbb{Z}$ with the set $\{0,1\}$. The sets B^r_σ are disjoint and almost cover U_g.

Let us prove that they are disjoint. Suppose we have $\sigma+\rho=\sigma'+\rho'$ with σ and σ' admissible and ρ,ρ' in S, then consider these maps modulo 2. If we apply the boundary map, the admissible colorings vanish by definition, and we have $\partial\rho=\partial\rho'$. But ∂ induces a bijection from S onto its image, hence we have $\rho=\rho'$, and it follows that $\sigma=\sigma'$. Hence the sets B^r_σ are actually disjoint. Moreover the measure of B^r_σ is $\frac{2^{d(g)-g}}{r^{d(g)}}$. It follows that $\sum_{\sigma,r-\text{admissible}}F_\gamma(\frac{\sigma_e+1}{r})\frac{2^{g-d(g)}}{r^{d(g)}}$ converges to $\int_{U_g}F_\gamma(\tau)d\tau$ and the result is proved. □

3.4 Proof of the Theorem 3.1

Let Σ be a closed oriented surface of genus g. We recall that $C(\Sigma)$ and $K(\Sigma\times[0,1],u)$ are isomorphic as vector spaces for any u in $\mathbb{C}\setminus\{0\}$. The stacking product induces on $K(\Sigma\times[0,1])$ a natural algebra structure which induces an algebra structure on $C(\Sigma)$ for each $u\in\mathbb{C}\setminus\{0\}$. We consider the algebra structure obtained for $u=-1$.

Fix γ and δ, two multicurves on Σ. We aim to compute the limit of the sequence $(\frac{2}{p})^{d(g)}\langle\varphi_p(\gamma),\varphi_p(\delta)\rangle_p$ as p goes to infinity. The right hand side is the quantum invariant of two thickened surfaces Σ with a multicurve inside, glued along their boundary. Instead of gluing the two boundaries simultaneously, we glue one and then the other. If we glue one boundary component, we obtain the stacking product of γ and δ. In the skein module for generic A, we have a decomposition

$\gamma \cdot \delta = \sum_i c_i \zeta_i$ for some multicurves ζ_i and some Laurent polynomials c_i in $\mathbb{Z}[A, A^{-1}]$. When evaluating this combination in $V_p(\Sigma\amalg - \Sigma)$, we have to specialize A to $-e^{\frac{i\pi}{p}}$. In formulas, we have $\varphi_p(\gamma\cdot\delta) = \sum_i c_i(-e^{\frac{i\pi}{p}})\varphi_p(\zeta_i)$. Then, we glue together the remaining boundary components and obtain $\langle\varphi_p(\gamma), \varphi_p(\delta)\rangle_p = \sum_i c_i(-e^{\frac{i\pi}{p}}) \operatorname{Tr}_p(\zeta_i)$.

The asymptotic formula becomes clear if we define the following linear form on $C(\Sigma)$:

Definition 3.5. *Let γ be a multicurve on Σ. Then there is a pants decomposition associated to γ such that all components of γ are parallel copies of the boundary circles. We define $\langle\gamma\rangle = 2^{g-d(g)} \int_{U_g} F_\gamma(\tau) d\tau$ where $F_\gamma(\tau) = \prod_e (-2\cos(\tau_e\pi))^{m_e}$. The expression of $\langle\gamma\rangle$ as a limit shows that this definition does not depend on the pants decomposition. We extend $\langle\cdot\rangle$ to a linear form on $C(\Sigma)$.*

Coming back to our computation, we obtain: $\lim\limits_{p\to\infty} (\frac{2}{p})^{d(g)} \langle\varphi_p(\gamma), \varphi_p(\delta)\rangle_p = \sum_i c_i(-1)\langle\zeta_i\rangle = \langle\gamma\delta\rangle$. Finally, we define an hermitian form on $C(\Sigma)$ by the formula $\langle x, y\rangle = \langle x\overline{y}\rangle$ where the product corresponds to the skein module product for $A = -1$, and the conjugation corresponds to conjugation of coefficients in $C(\Sigma)$. We have proven the following result:

For all $x, y \in C(\Sigma)$, we have $\langle x, y\rangle = \lim\limits_{p\to\infty} (\frac{2}{p})^{d(g)} \langle\varphi_p(x), \varphi_p(y)\rangle_p$.

4 Geometric interpretation

The heart of the following geometric interpretation is the theorem of [Bul97] and [PS00] stating that the algebra $K(\Sigma \times [0,1], -1)$ is isomorphic to $\mathcal{H}(\Sigma)$, the ring of regular functions on the $SL(2,\mathbb{C})$-character variety of Σ. Recall that the isomorphism is given by $f_\gamma(\rho) = -\operatorname{Tr}(\rho(\gamma))$ when γ is a connected curve on Σ and $\rho : \pi_1(\Sigma) \to SL(2,\mathbb{C})$ is a representation of $\pi_1(\Sigma)$.

This identifies $C(\Sigma)$ with its algebra structure. It remains to identify the linear form $\langle\cdot\rangle$ of definition 3.5.

Recall that the $SL(2,\mathbb{C})$-character variety contains the $SU(2)$-character variety, which carries a natural symplectic form ω defined in [AB83, Gol84]. Following [JW94], we define $\mathcal{S}_g$ to be the moduli space of irreducible representations of $\pi_1(\Sigma)$ on $SU(2)$ and $\overline{\mathcal{S}}_g$ the moduli space of all representations. Then it is known that $\mathcal{S}_g$ is a smooth $2d(g)$ manifold with symplectic form ω obtained by symplectic reduction from the form $\overline{\omega}(a,b) = \frac{1}{4\pi^2}\int_\Sigma \operatorname{Tr}(a\wedge b)$ for $a, b \in \Omega^1(\Sigma, su(2))$.

We denote the volume form on $\mathcal{S}_g$ by $dV = \frac{\omega^{d(g)}}{d(g)!}$.

Proposition 4.1. *For all multicurves γ on Σ, we have*

$$\langle\gamma\rangle = \int_{\mathcal{S}_g} f_\gamma dV$$

Proof. We give a proof of this proposition by adapting the results of [JW94].

Fix a pants decomposition of Σ associated to γ and denote the set of curves bounding the pants by C_e. We define the functions h_e on $\overline{\mathcal{S}}_g$ with values in $[0,1]$ by the formula $\operatorname{Tr}\rho(C_e) = 2\cos(\pi h_e(\rho))$.

Where the functions h_e are not equal to 0 or 1, they Poisson commute and their Hamiltonian flows define a torus action on $\overline{\mathcal{S}}_g$. In fact, we have the following theorem:

Theorem 4.2 ([JW94]). *Let U_g^{gen} be the interior of U_g in $\mathbb{R}^{d(g)}$ and $h = (h_1, \dots, h_{d(g)}) : \mathcal{S}_g \to \mathbb{R}^{d(g)}$ be the collection of the h_e functions.*

For x in $\overline{\mathcal{S}}_g$ such that $h(x) = y \in U_g^{gen}$, the torus action identifies $h^{-1}(y)$ with $U(1)^{d(g)}/\mathbb{Z}_2^{2g-2}$, where an element $(\varepsilon_v) \in \mathbb{Z}_2^{2g-2}$ acts on $U(1)^{d(g)}$ by the formula $e^{2i\pi x_e} \mapsto (-1)^{\varepsilon_v + \varepsilon_{v'}} e^{2i\pi x_e}$ where v and v' are the indices of the pants bounding C_e.

If we choose a Lagrangian submanifold L of $\overline{\mathcal{S}}_g$ transverse to the fibres of the torus action and which h maps diffeomorphically onto $V \subset U_g^{gen}$, then we can define canonical coordinates on $h^{-1}(V)$ by setting $x^e = 0$ on L and $y_e = h_e$.

The volume form is given on $h^{-1}(V)$ by $\prod dy_e \prod dx_e$.

We come back to the integral of the function associated to γ on the moduli space $\mathcal{S}_g$. Recall that γ was adapted to the pants decomposition. This means that γ is the union of parallel curves C_e with multiplicity m_e. The function f_γ is then defined by $f_\gamma(\rho) = \prod_e (-\operatorname{Tr} \rho(C_e))^{m_e} = \prod_e (-2\cos(\pi h_e(\rho)))^{m_e} = F_\gamma(h)$, where F_γ is the function of Lemma 3.4.

As this function only depends on the values of h, we can perform the integration on its fiber first. The fiber is isomorphic to $U(1)^{d(g)}/\mathbb{Z}_2^{2g-2}$. Hence $U(1)^{d(g)}$ is a Riemannian covering over the fiber and has volume equal to 1. To find the volume of the fiber, it is then sufficient to find the degree of this covering. Let G be the graph associated to the pants decomposition. The degree of the covering is equal to the dimension of the $\mathbb{Z}_2$-subspace of $C^1(G, \mathbb{Z}_2)$ generated by the family of vectors $u_v = e_a + e_b + e_c$ for each pant v bounding circles a, b and c. This subspace is the image of the coboundary map $d : C^0(G, \mathbb{Z}_2) \to C^1(G, \mathbb{Z}_2)$. Its dimension is then complementary to the dimension of $H^1(G, \mathbb{Z}_2)$ which is g. We find that the dimension is $d(g) - g$, hence the covering has degree $2^{d(g)-g}$ and the volume of the fiber is $2^{g-d(g)}$.

We finally obtain $\int_{\mathcal{S}_g} f_\gamma dV = 2^{g-d(g)} \int_{U_g} F_\gamma(\tau) d\tau = \langle \gamma \rangle$, which completes the proof.

□

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