Berezin-Toeplitz Operators, a semi-classical approach.

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Abstract: This article is devoted to the Toeplitz Operators [4] in the context of the geometric quantization [11], [15]. We propose an ansatz for their Schwartz kernel. From this, we deduce the main known properties of the principal symbol of these operators and obtain new results : we define their covariant and contravariant symbols, which are full symbol, and compute the product of these symbols in terms of the Kähler metric. This gives canonical star products on the Kählerian manifolds. This ansatz is also useful to introduce the notion of microsupport.

1. Introduction

Let M be a compact Kähler manifold with fundamental 2-form $\omega \in \Omega^2(M, \mathbb{R})$. Assume that there exists an Hermitian line bundle $L \to M$ with a covariant derivation ∇ of curvature $\frac{1}{i}\omega$. M and L are the data of geometric quantization introduced by Kostant [11] and Souriau [15] : the symplectic manifold (M, ω) is the classical phase space and the space $\mathcal{H}$ consisting of the holomorphic sections of L is the quantum space. The set of classical observables is the Poisson algebra $C^\infty(M)$. The quantum observables are the linear operators of $\mathcal{H}$.

To relate the classical and quantum observables, Berezin introduced in [2] the notions of covariant symbol and contravariant symbol. To describe this, introduce the space $L^2(M, L)$ which consists of the sections of $L \to M$ with finite L^2 norm, endowed with the scalar product

$$(s_1, s_2) = \int_M h(s_1, s_2)\ \mu_M$$

where h is the Hermitian metric and μ_M is the Liouville measure $\frac{1}{n!}\left|\omega^{\wedge n}\right|$. Since M is compact, $\mathcal{H}$ is finite dimensional subspace of $L^2(M, L)$. Denote by Π the orthogonal projector of $L^2(M,L)$ onto $\mathcal{H}$. From now on, we identify a quantum observable T with the bounded operator of $L^2(M,L)$ which vanishes on $\mathcal{H}^\perp$ and which restricts on $\mathcal{H}$ to T. So the quantum observables are the operators $T : L^2(M,L) \to L^2(M,L)$ such that $\Pi T \Pi = T$.

A contravariant symbol is a function $f \in C^\infty(M)$ to which we associate the operator $\Pi M_f \Pi$, where $M_f : L^2(M,L) \to L^2(M,L)$ is the multiplication by f.

$$f \xrightarrow{\text{contravariant}} \Pi M_f \Pi$$

On the covariant side, we start from a quantum observable T. We denote by $T(x_l,x_r)$ its Schwartz kernel. It is the section of $L \boxtimes L^{-1} \to M^2$ such that

$$(Ts)(x_l) = \int_M T(x_l,x_r).s(x_r)\ \mu_M$$

Since $L \otimes L^{-1} \simeq \mathbb{C}$, $T(x_l,x_r)$ restricts on the diagonal to a function. Assume that $\Pi(x,x)$ does not vanish. The covariant symbol of T is $f(x) = T(x,x)/\Pi(x,x)$.

$$\frac{T(x,x)}{\Pi(x,x)} \xleftarrow{\text{covariant}} T$$

It is natural to ask if one can find products on $C^\infty(M)$ corresponding to the product of the operators. Using the covariant symbol, Moreno and Ortega [13] defined star products on the projective space $\mathbb{CP}$ and the Poincaré disc. When M is a coadjoint orbit of a compact Lie group, similar results were obtained by Cahen, Rawnsley and Gutt [6]. More generally when M is a Kähler manifold, Bordemann, Meinrenken, Schlichenmaier [3] and Guillemin [9] deduced from the theory of Toeplitz operators of Boutet de Monvel and Guillemin [4] that the product of contravariant symbol is a star product.

These results involves the semi-classical limit defined in the following way. For every positive integer k, we replace in the previous constructions the line bundle L by L^k. We obtain a sequence of Hilbert spaces $\mathcal{H}_k$. The semi-classical limit is $k \to \infty$. Furthermore, we restrict our attention to a family of quantum observables called Toeplitz operators. By definition, a Toeplitz operator is a sequence (T_k) such that for every k,

$$T_k : L^2(M,L^k) \to L^2(M,L^k), \qquad T_k = \Pi_k M_{f(.,k)} \Pi_k + R_k \tag{1}$$

where

- $f(.,k)$ is a sequence of $C^\infty(M)$ which admits an asymptotic expansion of the form $\sum_{\ell=0}^{\infty} k^{-\ell} f_\ell$ for the C^∞ topology with f_0, f_1,.. C^∞ functions.
- (R_k) is a negligible operator, that is $\Pi_k R_k \Pi_k = R_k$ and its uniform norm $||R_k||$ is $O(k^{-\infty})$.

The interest to consider these operators is that the contravariant map of Berezin leads to a bijection between $C^\infty(M)[[\hbar]]$ and the set $\mathcal{T}$ of the Toeplitz operators modulo the negligible operators.

Theorem 1 ([3], [9]). *The product of two Toeplitz operators is a Toeplitz operator, so $\mathcal{T}$ is an algebra. The contravariant symbol map*

$$\sigma_{cont} : \mathcal{T} \to C^\infty(M)[[\hbar]]$$

which sends the operator (T_k) defined by (1) into $\sum_\ell \hbar^\ell f_\ell$ is well-defined. It is onto and its kernel is the set of negligible Toeplitz operators. Furthermore, if $\sigma_{cont}(T_k) = f_0 + O(\hbar)$ and $\sigma_{cont}(U_k) = g_0 + O(\hbar)$, then

$$\begin{aligned}
&\sigma_{cont}(T_k U_k) = f_0.g_0 + O(\hbar), \\
&\sigma_{cont}(T_k U_k - U_k T_k) = \frac{\hbar}{i}\{f_0, g_0\} + O(\hbar^2), \\
&||T_k|| = O(k^{-N}) \text{ iff } \sigma_{cont}(T_k) = O(\hbar^N), \\
&||T_k|| \sim \operatorname{Sup}|f^0| \text{ if } f_0 \neq 0.
\end{aligned}$$

The principal symbol of a Toeplitz operator (T_k) is the function f_0 such that $\sigma_{\mathrm{cont}}(T_k) = f_0 + O(\hbar)$. Observe that the map σ_{cont} is a full symbol in the sense that $\sigma_{\mathrm{cont}}(T_k) = 0$ if and only if $||T_k|| = O(k^{-\infty})$.

Our main result is an ansatz for the Schwartz kernel of a Toeplitz operator.

Theorem 2. *Let E be a section of $L \boxtimes L^{-1}$ such that $E(x,x) = 1$, $|E(x_l, x_r)| < 1$ if $x_l \neq x_r$ and*

$$\nabla_{\partial_{\bar{z}^i_l}} E(x_l, x_r) = \nabla_{\partial_{z^i_r}} E(x_l, x_r) = 0 + O(|x_r - x_l|^\infty)$$

for any complex coordinates system (z^i) of M. If (T_k) is a Toeplitz operator, its Schwartz kernel is of the form

$$T_k(x_l, x_r) = \Big(\frac{k}{2\pi}\Big)^n E^k(x_l, x_r) a(x_l, x_r, k) + R_k(x_l, x_r) \tag{2}$$

where $\big(a(.,k)\big)_k$ is a sequence of $C^\infty(M^2)$ which admits an asymptotic expansion $\sum_{\ell=0}^\infty k^{-\ell} a_\ell$ for the C^∞ topology whose coefficients satisfy

$$\partial_{\bar{z}^j_l}.a_\ell(x_l, x_r) = \partial_{z^j_r}.a_\ell(x_l, x_r) = 0 + O(|x_r - x_l|^\infty)$$

for any complex coordinates system (z^i) of M. (R_k) is negligible, that is R_k is uniformly $O(k^{-\infty})$ and the same holds for its successive covariant derivatives.

Conversely, if (T_k) is a sequence of operators whose Schwartz kernels are given by (2), then $||\Pi_k T_k \Pi_k - T_k|| = O(k^{-\infty})$ and $(\Pi_k T_k \Pi_k)$ is a Toeplitz operator.

For the projector (Π_k), this ansatz follows from a theorem of Boutet de Monvel and Sjöstrand about the Szegö kernel of a strictly pseudoconvex domain. This representation of the Schwartz kernel is actually similar to the representation of the Schwartz kernel of an $\hbar$-pseudodifferential operator as an oscillatory integral.

From this theorem, we can give a direct proof of theorem 1 and deduce many other properties of Toeplitz operators : if (T_k) is a Toeplitz operator, the sequence $(T_k(x,x))$ admits an asymptotic expansion

$$T_k(x,x) = \Big(\frac{k}{2\pi}\Big)^n \sum_\ell k^{-\ell} a_\ell(x,x) + O(k^{-\infty})$$

We set

$$\sigma(T_k) = \sum_\ell \hbar^\ell a_\ell(x,x) \quad \text{ and } \quad \sigma_{\text{cov}}(T_k) = \sigma(T_k).\sigma(\Pi_k)^{-1}.$$

So we obtain three full symbol maps

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "\sigma"'] \arrow[d, "\sigma_{\text{cont}}"] \arrow[dr, "\sigma_{\text{cov}}"] & \\ \mathcal{D} & \mathcal{D} \arrow[l, "\tilde{\psi}"] \arrow[r, "\psi"'] & \mathcal{D} \end{tikzcd} \qquad \text{with } \mathcal{D} = C^\infty(M)[[\hbar]]$$

that is σ, σ_{cov} and σ_{cont} are onto, their kernel is the set $O(k^{-\infty}) \cap \mathcal{T}$ of the negligible Toeplitz operators.

Since $\mathcal{T}$ is an algebra whose $O(k^{-\infty}) \cap \mathcal{T}$ is an ideal, we obtain three associative products $*$, $*_{\text{cov}}$ and $*_{\text{cont}}$ on $C^\infty(M)[[\hbar]]$. We prove that these are star products. Furthermore, the maps $\tilde{\psi}$ and ψ are equivalences of star product. We compute these products modulo $O(\hbar^2)$: if f and $g \in C^\infty(M)$, then

$$\begin{aligned} f * g =& f.g + \hbar\big(G^{ij}(\partial_{z^i} f)(\partial_{\bar{z}^j} g) - \tfrac{1}{2}\, r.f.g\big) + O(\hbar^2) \\ \Psi(f) =& f + \hbar G^{ij} \partial_{z^i}\partial_{\bar{z}^j} f + O(\hbar^2) \\ f *_{\text{cov}} g =& f.g + \hbar G^{ij}(\partial_{z^i} f)(\partial_{\bar{z}^j} g) + O(\hbar^2) \\ f *_{\text{cont}} g =& f.g - \hbar G^{ij}(\partial_{z^i} f)(\partial_{\bar{z}^j} g) + O(\hbar^2) \end{aligned}$$

where r is the scalar curvature of M, and the functions $G^{i,j}$ are defined by $G^{i,j}.G_{k,j} = \delta_{i,k}$ and $\omega = iG_{j,k} dz^j \wedge d\bar{z}^k$.

In fact, we can also compute the remainders $O(\hbar^2)$. We prove that the bidifferential operators B_ℓ associated to the star product $*$ are of the form

$$\begin{aligned} B_\ell(f,g) = \sum_{\alpha,\beta} \tilde{B}^\ell_{\alpha,\beta}\big([\det(G_{ij})]^{-1}, G_{\alpha',\beta'}\big)\, \partial^\beta_{\bar{z}} f.\partial^\alpha_z g \\ \text{if } f * g = \sum \hbar^\ell B_\ell(f,g), \quad \forall\ f,g \in C^\infty(M) \end{aligned} \tag{3}$$

where the functions $G_{\alpha,\beta}$ are the derivatives of the $G_{i,j}$ and $\tilde{B}^\ell_{\alpha,\beta}$ are polynomials in $[\det(G_{ij})]^{-1}$ and $G_{\alpha',\beta'}$. These polynomials are universal, that is they do neither depend on the choice of the complex coordinates system nor on the Kähler metric. Furthermore these formulas define a canonical star product on every Kähler manifold, that is on Kähler manifold which are neither necessarily compact and which nor have a prequantization bundle. We prove similar properties for the star products $*_{\text{cov}}$ et $*_{\text{cont}}$.

These results are connected with a theorem of Lu about the projector Π_k. The unit $\sigma(\Pi_k)$ of $(C^\infty(M)[[\hbar]], *)$ is not the formal series 1, but a formal series $1_* = \sum_l \hbar^l S_l$, with $S_0 = 1$ and $S_1 = \frac{r}{2}$, such that

$$\Pi_k(x,x) = \Big(\frac{k}{2\pi}\Big)^n \sum_\ell k^{-\ell} S_\ell(x) + O(k^{-\infty}) \tag{4}$$

The existence of this asymptotic expansion was proved by Zelditch [17], by using the result of Boutet de Monvel and Sjöstrand. In [12], Lu computed S_0, S_1, S_2, S_3 and S_4 and with his method we can also compute the other coefficients. Since 1_* is the unit of $(C^\infty(M)[[\hbar]], *)$, we can also compute it from the formulas for the star product $*$.

In a next article, we will explain how we can generalize the ansatz for the kernel of Toeplitz operators to define Lagrangian sections similar to the Lagrangian distributions of the theory of $\hbar$-pseudodifferential operators. We will deduce from this the Bohr-Sommerfeld conditions for a Toeplitz operator. To prepare this, we introduce the notion of microsupport. This is fairly easy, because the quantum states are defined on the phase space.

The paper is organized as follows. The second section is devoted to introduce our notations. In the third one, we consider an algebra $\mathcal{F}$, which contains as a subalgebra the set of the Toeplitz operators. We prove that (Π_k) belongs to this algebra, introduce the full symbol of its operators and compute the product of the symbols. In the following section, we derive from this the properties of the Toeplitz operator. Finally we define the notion of microsupport and consider the functional calculus of Toeplitz operators.

This article is a part of our PHD-thesis [7]. It is self-contained, expect that we use the essential result of Boutet de Monvel and Sjöstrand on the Szegö kernel and we apply the stationary phase lemma of Hörmander.

2. Notations

2.1. Geometric objects. First if $L \to M$ is a Hermitian fiber bundle, we denote by $h(u,v)$ the scalar product of $u, v \in L_x$ and by $|u| = h(u,u)^{\frac{1}{2}}$ the norm of u. When L is endowed with a connection, we denote by $\nabla : C^\infty(M,L) \to \Omega^1(M,L)$ the covariant derivation. We use the same notation for the induced Hermitian structure and covariant derivation on $L^k \to M$ and $L^k \boxtimes L^{-k} \to M \times M$.

If $D : C^\infty(M, L^k) \to C^\infty(M, L^k)$ is a differential operator, we define the differential operators D_l and D_r by

$$D_l = D \otimes \mathrm{Id} : C^\infty(M \times M, L^k \boxtimes L^{-k}) \to C^\infty(M \times M, L^k \boxtimes L^{-k}),$$
$$D_r = \mathrm{Id} \otimes D : C^\infty(M \times M, L^k \boxtimes L^{-k}) \to C^\infty(M \times M, L^k \boxtimes L^{-k}).$$

2.2. Negligible terms. First, if $(f(,.k))_k$ is a sequence of $C^\infty(X)$, we say that $(f(.,k))$ is *negligible* if for every integers ℓ, N, for every vector fields $X_1, ..., X_\ell$ and for every compact $K \subset X$, there exists C such that

$$\big|(X_1 ... X_\ell . f)(x,k)\big| \leqslant Ck^{-N}, \quad \forall x \in K.$$

Consider now a line bundle $L \to X$ endowed with a Hermitian structure. Let $(s_k)_k$ be a sequence such that $s_k \in C^\infty(M, L^k)$ for all k. Introduce a covariant derivation $\nabla : C^\infty(M,L) \to \Omega^1(M,L)$. We say that (s_k) is *negligible* if for every integers ℓ, N, for every vector fields $X_1, ..., X_\ell$ and for every compact $K \subset X$, there exists C such that

$$\big|\nabla_{X_1} ... \nabla_{X_\ell} s_k(x)\big| \leqslant Ck^{-N}, \quad \forall x \in K.$$

It is easy to see that this definition depends on the choice of h, but does not depend of ∇. Locally, if $t: U \to L$ is a unitary gauge (i.e. $|t(x)| = 1,\ \forall x \in U$) and $s_k = f(.,k)t^k$ on U, the fact that (s_k) is negligible means that the sequence $(f(.,k))$ is negligible.

Let (T_k) be a sequence such that for every k, T_k is an operator $C^\infty(X, L^k) \to C^\infty(X, L^k)$ with a smooth Schwartz kernel. Using a density $\mu \in C^\infty(X, |\Omega|(X))$, the kernel $T_k(x_l, x_r)$ can be viewed as a section of $L^k \boxtimes L^{-k}$. We say that (T_k) is a *smoothing operator* if $(T_k(x_l, x_r))$ is a negligible sequence. This definition does not depend on the choice of μ.

We will denote by $O(k^{-\infty})$ a negligible sequence of functions or the set of the negligible sequences of functions. We use the same notation for sequences of sections or for smoothing operators.

2.3. Symbols. A symbol of order N is a sequence of functions $(f(.,k))$ in $C^\infty(X)$ which admits an asymptotic expansion

$$f(.,k) = \sum_{\ell=N}^{\infty} k^{-\ell} f_\ell + O(k^{-\infty})$$

in the C^∞ topology. We denote by $S^N(X)$ the set of the symbols of order N defined on X. We associate to $(f(.,k)) \in S^0(X)$ the formal symbol $\sum_\ell \hbar^\ell f_\ell$. This defines a map

$$S^0(X) \to C^\infty(X)[[\hbar]]$$

By Borel lemma, this map is onto, its kernel is $O(k^{-\infty})$.

2.4. Taylor expansions. We say that a function $f \in C^\infty(X)$ vanishes to order k along a submanifold $Y \subset X$, if for every differential operator D of order $k-1$,

$$D.f\big|_Y = 0.$$

We say that a function $f \in C^\infty(X)$ vanishes to order ∞ along Y, if it vanishes to order k along Y for every k. We denote by $\mathcal{I}^k(Y)$ the ideal of $C^\infty(M)$ which is the set of the functions which vanish to order k along Y. The Taylor series of $f \in C^\infty(M)$ along Y is the class of f in $C^\infty(M)/\mathcal{I}^\infty(Y)$.

Lemma 1. *Let X be a submanifold of an open set Ω of $\mathbb{R}^k$. Let $d \in C^\infty(\Omega, \mathbb{R}^+)$ be a non negative function which vanishes along X to order 2, does not vanishes outside of X and whose kernel of its Hessian is T_xX for all x in X. Let $\big(a(.,k)\big)$ be a sequence of $C^\infty(\Omega)$ which has an asymptotic expansion $\sum_{i=0}^\infty a_i(x)k^{-i}$ in the C^0 topology. Let N be a non negative integer. The following assertions are equivalent.*

i. $\forall$ *compact K of Ω, $\exists\, C$ such that* $\left| e^{-kd(x)} a(x,k) \right| \leqslant C\, k^{-\frac{N}{2}}$ *on K.*

ii. $a_i \in \mathcal{I}^{N-2i}(X), \quad \forall i$ *such that* $N \geqslant 2i$.

Proof. Let ℓ be some integer larger than $\frac{N}{2}$. We have $\left| \; a(x,k) - \sum_{i=0}^{\ell} a_i(x) k^{-i} \right| \leqslant C_K \; k^{-\frac{N}{2}-1}$ on every compact K of Ω. Consequently, $i.$ is equivalent to

$$\left| \; e^{-kd(x)} \sum_{i=0}^{\ell} a_i(x) k^{-i} \right| \leqslant C_K \; \tau^{-\frac{N}{2}}. \tag{5}$$

Moreover, assertion $ii.$ is equivalent to $\left| a_i(x) \right| \leqslant C\delta(x)^{\frac{N}{2}-i}$ on every compact K of Ω. The function $y \rightarrow y^m e^{-y}$ is bounded on $\mathbb{R}^+$. It follows that

$$\left| \; e^{-k\delta(x)} a_i(x) k^{-i} \right| \leqslant C a_i(x) d(x)^{-\frac{N}{2}+i} \; k^{-\frac{N}{2}}.$$

This prove that $ii.$ implies $i.$. Conversely, we introduce the set $D = \{x \in \Omega \; / \; d(x)^{-1} \in \mathbb{N}\}$. Consider an integer j between 1 and $\ell + 1$ and we use the inequality (5) where $x \in D$ and $k = j/d(x)$. We obtain that the function

$$b_j(x) = j^{\ell} a_0(x) d(x)^{-\frac{N}{2}} + j^{\ell-1} a_1(x) d(x)^{-\frac{N}{2}+1} + ... + j^0 a_{\ell}(x) d(x)^{-\frac{N}{2}+\ell}$$

is bounded on $K \cap D$ if K is a compact of Ω. The functions $b_j(x)$ are obtained from the functions $a_j(x) d(x)^{-\frac{N}{2}+j}$ by a linear equations system of Vandermonde type. By solving this system, we obtain that $a_j(x) d(x)^{-\frac{N}{2}+j}$ is bounded on $K \cap D$ if K is a compact of Ω. Using the Taylor expansions of a_j and d along X, $i.$ follows. $\square$

3. The algebra $\mathcal{F}$

This section is devoted to an algebra of operators defined in the following way.

Definition 1. *$\mathcal{F}$ is the space of operators $\left(Q_k : C^{\infty}(M, L^k) \rightarrow C^{\infty}(M, L^k) \right)_{k \geqslant 0}$, whose kernel is of the form*

$$Q_k(x_l, x_r) = \left(\frac{k}{2\pi} \right)^n E^k(x_l, x_r) \; a(x_l, x_r, k) + O(k^{-\infty}) \tag{6}$$

where

- *E satisfies the same assumptions as in theorem 2*
- *$(a(.,k))$ is a symbol in $S^0(M^2)$.*

Our basic interest in this algebra is that it contains as a subalgebra the set of Toeplitz operators. In the next section we will derive many properties of the Toeplitz operators from those of operators of $\mathcal{F}$. The first subsection is devoted to the section E defined in theorem 2. We prove its existence and give its main properties. In the following two subsections, we prove that (Π_k) is an operator of $\mathcal{F}$. In the last subsections we define the full symbol of an operator of $\mathcal{F}$, prove that $\mathcal{F}$ is an algebra and compute the product of symbols.

3.1. The section E. In the following, g denote the Riemannian metric of M defined by $g(X,Y) = \omega(X, JY)$.

Proposition 1. *There exists a section E of $L \boxtimes L^{-1}$ such that*

$$\begin{aligned} &E\big|_{\operatorname{diag}(M)} = 1 \\ &\nabla_{\bar{Z}_l} E \equiv \nabla_{Z_r} E \equiv 0 \mod \mathcal{I}^\infty(\operatorname{diag}(M)) \end{aligned} \tag{7}$$

for all holomorphic vector field Z of M. This section is unique modulo a section which vanishes, with all its derivatives, along $\operatorname{diag}(M)$. *The function*

$$\delta = -2 \ln |E|$$

of $C^\infty(M \times M)$ vanishes, with its first derivatives, along $\operatorname{diag}(M)$. *If $x \in M$, the Hessian at (x,x) of δ is the quadratic form, whose kernel is* $\operatorname{diag}(T_x M)$ *and restriction on $T_x M \times (0) \subset T_x M \times T_x M$ is $\frac{1}{2} g$. Furthermore,*

$$\nabla E \equiv -E \otimes (\partial_l + \bar{\partial}_r)\delta \mod \mathcal{I}^\infty(\operatorname{diag}(M)). \tag{8}$$

On a neighborhood of $\operatorname{diag}(M)$, we have $\delta(x_l, x_r) < 0$ if $x_l \neq x_r$. By modifying E outside this neighborhood, we may assume that $\delta(x_l, x_r) < 0$ if $x_l \neq x_r$ for all $(x_l, x_r) \in M \times M$.

Remark 1. Let $t : U \to L$ be a holomorphic gauge. Let $\rho \in C^\infty(U)$ be such that $|t| = e^{-\rho}$ and introduce the unitary gauge $s = e^\rho t$. Then we will prove that

$$E = e^{i\psi}\ s \otimes s^{-1} \text{ with } \psi(x_l, x_r) = i(\rho(x_l) + \rho(x_r)) + \tilde{\psi}(x_l, x_r) \tag{9}$$

where $\tilde{\psi}$ is such that

$$\tilde{\psi}(x,x) = -2i\rho(x) \text{ and } \partial_{\bar{z}_l^i} \tilde{\psi} \equiv \partial_{z_r^i} \tilde{\psi} \equiv 0 \mod \mathcal{I}^\infty(\operatorname{diag}(U)). \tag{10}$$

This local expression will be useful, especially to apply the stationary phase lemma for the composition of operators. □

Proof. We introduce the same local data as in the previous remark and look for a section E verifying (9). Then equations (7) are equivalent to (10). There is a unique function $\tilde{\psi}$ satisfying (10) modulo $\mathcal{I}^\infty(\operatorname{diag}(U))$. Using the local uniqueness, we can construct with a partition of unity the global section E required. We have

$$\delta(x_l, x_r) = 2\rho(x_l) + 2\rho(x_r) - i\tilde{\psi}(x_l, x_r) - i\tilde{\psi}(x_r, x_l) \mod \mathcal{I}^\infty(\operatorname{diag}(M)).$$

From $\partial_{z_l^j} . \tilde{\psi} \equiv (\partial_{z_l^j} + \partial_{z_r^j})\tilde{\psi}$ modulo $\mathcal{I}^\infty(\operatorname{diag}(U))$ it follows that $\partial_{z_l^j} \delta(x,x)$ vanishes. Similarly we show that the other derivatives of δ vanish along $\operatorname{diag}(M)$. To compute the Hessian of δ, observe that

$$\partial_{z_l^j} \partial_{z_r^k} \delta\big|_{(x,x)} = \partial_{\bar{z}_l^j} \partial_{\bar{z}_r^k} \delta\big|_{(x,x)} = 0 \qquad \partial_{z_l^j} \partial_{\bar{z}_l^k} \delta\big|_{(x,x)} = \partial_{z_r^j} \partial_{\bar{z}_r^k} \delta\big|_{(x,x)} = G_{j,k} \tag{11}$$

with $G_{j,k} = \partial_{z^j} \partial_{\bar{z}^k} (\rho + \bar{\rho})$. Let X and Y be two vectors in $T_x M$.

$$\begin{aligned} (X, 0) &= (Z, Z) + (\bar{Z}, -Z) \text{ with } Z = \tfrac{1}{2}(X - iJX) \in T_x^{1,0} M \\ (Y, 0) &= (\bar{W}, \bar{W}) + (W, -\bar{W}) \text{ with } W = \tfrac{1}{2}(Y - iJY) \in T_x^{1,0} M \end{aligned}$$

Since the kernel of $\operatorname{Hess}\delta\big|_{(x,x)}$ contains certainly $\operatorname{diag} T_xM$, we have

$$\begin{aligned}\operatorname{Hess}\delta\big|_{(x,x)}\big((X,0),(Y,0)\big) &= \operatorname{Hess}\delta\big|_{(x,x)}\big((\bar{Z},-Z),(W,-\bar{W})\big)\\ &= \tfrac{1}{2i}\omega(W,\bar{Z}) + \tfrac{1}{2i}\omega(Z,\bar{W})\end{aligned}$$

using (11) and since $\omega = i\partial\bar{\partial}(\rho+\bar{\rho}) = i\sum G_{j,k}dz^j\wedge d\bar{z}^k$, we have

$$\operatorname{Hess}\delta\big|_{(x,x)}\big((X,0),(Y,0)\big) = \tfrac{1}{2}g(X,Y).$$

By derivating $h(E,E) = \exp(-\delta)$ we obtain (8). $\square$

3.2. The Szegö projector. We recall first the result of Boutet de Monvel and Sjöstrand that we will apply. Let Y be a complex manifold of dimension $k+1$. Let D be a domain of Y with compact C^∞ boundary. Let $E\to\partial D$ be the complex subbundle of $T(\partial D)\otimes\mathbb{C}$, which consists of the holomorphic vectors of Y tangent to ∂D. The complex dimension of E is k. Let $r: Y\to\mathbb{R}$ be a defining function for the boundary of D, i.e. $D=\{r\leqslant 0\}$ and $r'(y)\neq 0$ if $y\in\partial D$. Assume that D is strictly pseudoconvexe, i.e. the sesquilinear form of $E\big|_y$

$$(X,Y)\to\langle\partial\bar{\partial}r, X\wedge\bar{Y}\rangle \qquad X,Y\in E\big|_y$$

is positive definite at every point $y\in\partial D$. Then the restriction of $-i\partial r$ at the boundary ∂D is a contact form.

Let $\mu\in C^\infty(\partial D,|\Omega|(\partial D))$ be a volume form. Hence $L^2(\partial D)$ is endowed with a Hilbertian structure. $\mathcal{H}$ is the set of the functions of $L^2(\partial D)$ satisfying induced Cauchy-Riemann equations:

$$\mathcal{H} = \Big\{f\in L^2(\partial D)\ /\ \bar{Z}.f = 0,\ \forall Z\in C^\infty(\partial D, E)\Big\}$$

The Szegö projector $\Pi: L^2(\partial D)\to L^2(\partial D)$ is the orthogonal projection onto $\mathcal{H}$.

Let $\phi\in C^\infty(Y\times Y)$ be a function such that

$$\phi(y,y) = \frac{1}{i}r(y) \text{ and } \bar{Z}_l\phi\equiv Z_r\phi\equiv 0 \mod \mathcal{I}^\infty(\operatorname{diag}(Y)) \tag{12}$$

for all holomorphic vector field Z. Define $\varphi\in C^\infty(\partial D\times\partial D)$ by $\varphi(u_l,u_r) = \phi(u_l,u_r)$. $d\varphi$ doesn't vanish on $\operatorname{diag}(\partial D)$. $d\mathrm{Im}\,\varphi$ vanishes identically on $\operatorname{diag}(\partial D)$ and the Hessian of $\mathrm{Im}\,\varphi$ at (u,u) is negative with kernel $\operatorname{diag}(T_u\partial D)$. So by modifying φ outside a neighborhood of $\operatorname{diag}(\partial D)$, we may assume that $\mathrm{Im}\,\varphi(u_l,u_r)<0$ if $u_l\neq u_r$. The map

$$\mathbb{R}^+\times\partial D\times\partial D\to\mathbb{C},\quad (\tau,u_l,u_r)\to\tau\varphi(u_l,u_r)$$

is a non-degenerate phase function of positive type (cf. [10]) and parametrizes a positive canonical ideal $\mathcal{C}$. Let $\mathcal{F}^0(\mathcal{C})$ be the set of the Fourier integral operators of order 0 associated with $\mathcal{C}$. It consists of the operators $T : C^\infty(\partial D) \to C^\infty(\partial D)$ whose Schwartz kernel is the sum of an oscillatory integral and a C^∞ function :

$$T(u_l, u_r) = \int_{\mathbb{R}^+} e^{i\tau\varphi(u_l,u_r)} s(u_l, u_r, \tau)|d\tau| + f(u_l, u_r) \tag{13}$$

where s is a classical symbol of $S^n_{1,0}(\partial D \times \partial D \times \mathbb{R}^+)$ which admits an asymptotic expansion

$$s(u_l, u_r, \tau) \sim \sum_{\ell=0}^{\infty} \tau^{n-\ell} s_\ell(u_l, u_r).$$

These operators are continuous $L^2(P) \to L^2(P)$.

Theorem 3 (Boutet de Monvel, Sjöstrand [5]). *Π is an elliptic Fourier integral operator of order 0 associated with the canonical ideal $\mathcal{C}$.*

To apply this result, we introduce the principal bundle $\pi : P \to M$ with structural group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ such that $L^k \simeq P \times_{s_k} \mathbb{C}$, where $s_k : \mathbb{T} \to Gl(\mathbb{C})$ is the representation defined by $s_k(\theta).v = e^{-ik\theta}v$. Consider the embedding i of the principal bundle P into $L^* \simeq P \times_{s_{-1}} \mathbb{C}$ defined by

$$i(u) = [u, 1] \in P \times_t \mathbb{C}, \quad \forall\, u \in P$$

The covariant derivation ∇ induces a connection 1-form $\alpha \in \Omega^1(P)$. Let $\mathrm{Hor}^{1,0} \to P$ denote the subbundle of $TP \otimes \mathbb{C}$, which consists of the horizontal lifts of holomorphic vectors. Let $H : L^* \longrightarrow \mathbb{R}$ denote the function sending $u \in L^*$ into $|u|^2$. The following result is well-known.

Proposition 2. *$D = \{H \leqslant 1\}$ is a strictly pseudoconvex domains of L^* with boundary $i(P)$. The fiber bundle of holomorphic vectors of L^* tangent to $i(P)$ is $i_* \mathrm{Hor}^{1,0}$. Moreover $i^*\partial \ln H = i\alpha$.*

$\mu_P = \frac{1}{2\pi n!}\left|\alpha \wedge (d\alpha)^{\wedge n}\right|$ is a volume form. So we obtain a scalar product on $L^2(P)$, a Szegö projector Π and a subspace $\mathcal{H} = \mathrm{Im}\ \Pi \subset L^2(P)$.

Since $L^k \simeq P \times_{s_k} \mathbb{C}$, we have an identification between sections of L^k and functions $f \in C^\infty(P)$ such that $R^*_\theta f = e^{ik\theta} f$. If $s : M \to L^k$ is associated to $f \in C^\infty(P)$, then $\nabla_X s$ is associated to $X^{\mathrm{hor}}.f$, where X^{hor} denotes the horizontal lift of the vector field X. So s is holomorphic if and only if f satisfies induced Cauchy-Riemann equations. Furthermore, this identification is compatible with scalar products, that is $(s_1, s_2) = (f_1, f_2)$ if s_1 and s_2 are respectively associated to f_1 and f_2. By Fourier decomposition, we obtain

$$L^2(P) \simeq \bigoplus_{k=-\infty}^{k=\infty} L^2(M, L^k) \qquad \text{and} \qquad \mathcal{H} \simeq \bigoplus_{k=0}^{k=\infty} \mathcal{H}_k.$$

Using the first sum, we associate to a bounded family $(T_k)_{k\in\mathbb{Z}}$ of bounded operators of $L^2(M, L^k)$ a bounded operator T of $L^2(P)$ which commutes with the action of $\mathbb{T}$, and conversely. The sequence $(T_k)_{k\geqslant 0}$ is called the sequence of positive Fourier coefficients of T. In particular, the sequence of positive Fourier coefficients of the Szegö projector Π is the sequence (Π_k). In the next section we will prove the following theorem.

Theorem 4. *The operators of $\mathcal{F}$ are the sequences of positive Fourier coefficients of the Fourier integral operators of $\mathcal{F}^0(\mathcal{C})$ which commute with the action of $\mathbb{T}$. Furthermore if (T_k) is the sequence of positive Fourier coefficients of $T \in \mathcal{F}^0(\mathcal{C})$, then T is smoothing if and only if (T_k) is smoothing.*

Applying the theorem of Boutet de Monvel and Sjöstrand, we obtain the

Corollary 1. *The projector (Π_k) belongs to $\mathcal{F}$.*

3.3. Proof of theorem 4. First, we prove that the section E of $L \boxtimes L^{-1}$ determines a non degenerate phase function of positive type which parametrizes the canonical ideal $\mathcal{C}$. $P \times P \to M \times M$ is a $\mathbb{T}^2$-principal bundle and

$$L \boxtimes L^{-1} \simeq (P \times P) \times_s \mathbb{C},$$

where $s : \mathbb{T}^2 \to Gl(\mathbb{C})$ is the representation defined by $s(\theta_l, \theta_r).v = e^{i(\theta_l - \theta_r)}.v$. In this way the section E is associated to a function $\tilde{E} \in C^\infty(P \times P)$ such that

$$R^*_{(\theta_l,\theta_r)} \tilde{E} = e^{i(\theta_l - \theta_r)} \tilde{E}.$$

$E(x,x) = 1$ implies $\tilde{E}(u,u) = 1$. Let V be the neighborhood of $\operatorname{diag} P$ given by $V = \{|\tilde{E} - 1| \leqslant \frac{1}{2}\}$. Define the function $\varphi = \frac{1}{i} \ln \tilde{E}$ on V.

Lemma 2. *The function $\tau\varphi(u_l, u_r)$ defined on $R^+ \times V$ is a non degenerate phase function of positive type which parametrizes the canonical ideal $\mathcal{C}$.*

Proof. Introduce the same notation as in remark 1. We identify $U \times \mathbb{C}$ with L^{-1} over U, by sending (x, z) into $zs^{-1}(x)$. In the same way, the bundle P over U can be identified with $U \times \mathbb{T}$ in such a way that $i(x, \theta) = e^{i\theta} s^{-1}(x)$. Introduce a complex coordinates system (z^j) over U. Recall that $e^\rho s^{-1}$ is a holomorphic gauge and let w denote the linear holomorphic coordinate of L^{-1} such that $w\big(e^\rho s^{-1}\big) = 1$. Then $H = w\bar{w}e^{\rho + \bar{\rho}}$ and the embedding i is given by

$$U \times \mathbb{T} \longrightarrow U \times \mathbb{C} \qquad (z, \theta) \longrightarrow (z, w = e^{i\theta - \rho}).$$

The function φ is given by $\varphi = \theta_l - \theta_r + i(\rho(x_l) + \rho(x_r)) + \tilde{\psi}(x_l, x_r)$. It extends to a function ϕ defined on a neighborhood of $\operatorname{diag}(L^{-1}) \subset L^{-1} \times L^{-1}$ by

$$\phi = -i \ln(w_l \bar{w}_r) + \tilde{\psi}(x_l, x_r).$$

From equations (10) which determine $\tilde{\psi}$, it follows that ϕ satisfies equations (12), where the function r is $\ln H$. $\square$

To prove theorem 4, we will apply the stationary phase lemma to obtain expression (6) from expression (13). By the previous lemma, the oscillatory term $e^{i\tau\varphi(u_l,u_r)}$ becomes $E^k(x_l, x_r)$. The amplitude $s(u_l, u_r, \tau)$ gives the symbol $a(x_l, x_r, k)$.

In connection with negligible terms, observe that if $T : C^\infty(P) \to C^\infty(P)$ is a smoothing operator (i.e. its kernel is C^∞) which commutes with the action of $\mathbb{T}$, then the family $(T_k)_{k \in \mathbb{Z}}$ of its Fourier coefficients is smoothing (i.e. the kernels

$T_k(x_l, x_r)$ are C^∞, $|T_k(x_l, x_r)| = O(|k|^{-\infty})$ as $k \to \pm\infty$ and the same holds for their successive covariant derivatives), and conversely.

Proof of theorem 4. Consider an operator $T \in \mathcal{F}^0(\mathcal{C})$ which commutes with the action of $\mathbb{T}$. Its kernel is of the form

$$T(u_l, u_r) = \int_{\mathbb{R}^+} e^{i\tau\varphi(u_l,u_r)} s(u_l, u_r, \tau)|d\tau| + f(u_l, u_r)$$

where φ is defined as in lemma 2 and the classical symbol $s \in S^n_{1,0}$ has support in $V \times \mathbb{R}^+$ and asymptotic expansion

$$s(u_l, u_r, \tau) \sim \sum_{\ell=0}^{\infty} \tau^{n-\ell} s_\ell(u_l, u_r). \tag{14}$$

We compute the Fourier coefficients of $T(u_l, u_r)$. We may assume $f = 0$ since its Fourier coefficients are negligible. Since T commutes with the action of $\mathbb{T}$, its kernel is $\mathbb{T}$-invariant, i.e. $T(R_\theta.u_l, R_\theta.u_r) = T(u_l, u_r)$. So by averaging, we may assume that s and the coefficients s_ℓ of its asymptotic expansion are $\mathbb{T}$-invariant. Let Q denote the quotient of $P \times P$ by the diagonal action and $p : P \times P \to Q$ the associated projection. The push-forward of $T(u_l, u_r)$ by $p : P \times P \to Q$ is

$$p_*T(q) = \int_{\mathbb{R}^+} e^{i\tau\tilde{\varphi}(q)} \tilde{s}(q, \tau)|d\tau|$$

where $\tilde{s}$ and $\tilde{\varphi}$ are such that $p^*\tilde{s} = s$ and $p^*\tilde{\varphi} = \varphi$. Furthermore $\tilde{s} \sim \sum_{\ell=0}^{\infty} \tau^{n-\ell}\tilde{s}_\ell$ where the functions $\tilde{s}_\ell$ are defined by $p^*\tilde{s}_\ell = s_\ell$. Q is a $\mathbb{T}$-principal bundle with base $M \times M$. The action of $\theta \in \mathbb{T}$ is given by

$$R_\theta.p(u_l, u_r) = p(R_\theta.u_l, u_r).$$

We have to compute the positive Fourier coefficients of p_*T for this action. We may assume that $P \simeq U \times \mathbb{T} \ni (x, \theta)$ and $Q \simeq U \times U \times \mathbb{T} \ni (x_l, x_r, \gamma)$ with $p^*\gamma = \theta_l - \theta_r$. Using the same notation as in the proof of lemma 2, we have $\tilde{\varphi}(x_l, x_r, \gamma) = \gamma + \psi(x_l, x_r)$. The Fourier coefficients of p_*T are

$$I_k(x_l, x_r, \gamma) = e^{ik\gamma} \int_{\mathbb{T}\times\mathbb{R}^+} e^{-ik\theta} e^{i\tau(\theta+\Psi(x_l,x_r))} \tilde{s}(x_l, x_r, \theta, \tau)|d\theta||d\tau|.$$

The support of $\tilde{s}$ is included in $p(V)\times\mathbb{R}^+ \subset U\times U\times(-\alpha, \alpha)\times\mathbb{R}^+$ with $0 < \alpha < \pi$. We replace τ by $k\tau$.

$$I_k(x_l, x_r, \gamma) = e^{ik\gamma} \int_{\mathbb{T}\times\mathbb{R}^+} e^{-i|k|\phi(\theta,\tau,x_l,x_r)} \tilde{s}(x_l, x_r, \theta, k\tau)k|d\theta||d\tau|$$

$$\text{with } \phi(\theta, \tau, x_l, x_r) = \begin{cases} \theta + \tau\theta + \tau\Psi(x_l, x_r) \text{ if } k > 0 \\ -\theta - \tau\theta - \tau\Psi(x_l, x_r) \text{ if } k < 0 \end{cases}$$

To estimate this as $|k|$ tends to ∞, we follow the method of stationary phase ([10], section 7.7). First observe that if $k < 0$, the phase ϕ does not have critical point, so $|I_k(x_l, x_r, \gamma)|$ is uniformly $O(|k|^{-\infty})$ as $k \to -\infty$ and the same holds for its successive derivatives. Consequently, T is a smoothing operator iff the sequence (T_k) of its positive Fourier coefficients is smoothing.

Assume now that $k > 0$. The first step is to restrict the integral at a small neighborhood of the critical locus of the phase ϕ by integrating by parts.

$$(\partial_\tau \phi = \partial_\theta \phi = 0) \text{ iff } (x_l = x_r,\ \theta = 0 \text{ et } \tau = 1)$$

Recall that the imaginary part of $\psi(x_l, x_r)$ is positive if $x_l \neq x_r$. We obtain

$$I_k(x_l, x_r, \gamma) = e^{ik\gamma} \int_D e^{-ik\phi(\theta, x_l, x_r, \tau)} \tilde{s}(x_l, x_r, \theta, k\tau) k |d\theta||d\tau| + e^{ik\gamma} g_k(x_l, x_r)$$

where $D = (-\epsilon, \epsilon) \times (1-\epsilon, 1+\epsilon)$ and the semi-norms $C^0(K)$ of g_k are $O(k^{-\infty})$ if K is compact. We now apply theorem 7.7.12 of [10]. Observe that $\phi = \psi + (\partial_\tau \phi)(\partial_\theta \phi)$. Hence

$$I_k(x_l, x_r, \gamma) = \Big(\frac{k}{2\pi}\Big)^n e^{ik(\gamma + \psi(x_l, x_r))} a(x_l, x_r, k) + e^{ik\gamma} g'_k(x_l, x_r)$$

where $\big(a(.,k)\big)$ admits an asymptotic expansion $\sum_{\ell=0}^{\ell=\infty} k^{-\ell} a_\ell$ in the C^∞ topology and the semi-norms $C^0(K)$ of g'_k are $O(k^{-\infty})$. Actually it follows from theorem 7.7.12 of [10] that I_k has an asymptotic expansion in the C^0 topology, but the coefficients a_l are C^∞ and by Borel process there exists a sequence $\big(a(.,k)\big)$ as above. Using the identification between the functions I_k and the sections of $L^k \boxtimes L^{-k}$, we express the kernels of the positive Fourier coefficients of T in the form

$$T_k(x_l, x_r) = \Big(\frac{k}{2\pi}\Big)^n E^k \ a(x_l, x_r, k) + R_k(x_l, x_r) \tag{15}$$

where $|R_k(x_l, x_r)|$ is uniformly $O(k^{-\infty})$. We have to improve this, that is to show that $\big|\nabla_{X_1} ... \nabla_{X_\ell} R_k\big| \leqslant C_{K,N} k^{-N}$ on every compact K and for all N. The sections $\nabla_{X_1} ... \nabla_{X_\ell} F_k$ are the positive Fourier coefficients of $X_1^{\mathrm{hor}} ... X_\ell^{\mathrm{hor}} T$. We can estimate them in the same way as the Fourier coefficients of T. Consequently their norm is $O(k^N)$ on every compact with N sufficiently large. The derivatives of $E^k a(.,k)$ satisfy the same estimate, so the same holds for $\nabla_{X_1} ... \nabla_{X_\ell} R_k$. By applying lemma 3.2 of [14], we obtain that (R_k) is smoothing.

Conversely, we have to show that for every sequence (a_k) of $C^\infty(M \times M)$ which admits an asymptotic expansion $\sum k^{-\ell} a^\ell$ in the C^∞ topology, there exists a symbol $s \in S^n_{1,0}(P \times P \times \mathbb{R}^+)$ which admits an asymptotic expansion $\sum \tau^\ell s_\ell$ such that (15) is satisfied. Assume that s_0 is locally $\mathbb{T}^2$-invariant on a neighborhood of $\operatorname{diag} P$. We can easily compute (cf theorem 7.7.2 [10]) the first coefficient a_0 of the asymptotic expansion. It is such that $\tilde{p}^* a_0 = s_0$ on a neighborhood of $\operatorname{diag}(P)$, where $\tilde{p}$ is the projection of $P \times P$ onto $M \times M$. So we can choose the convenient s_0, and by successive iterations the other coefficients s_l. Finally we obtain s by Borel process. $\square$

3.4. Symbol of the operators of $\mathcal{F}$. Let us define the full symbol of an operator of $\mathcal{F}$. Let $\mathcal{J}$ denote the space $C^\infty(M^2)/\mathcal{I}^\infty(\operatorname{diag} M)$ which consists of the Taylor expansions along $\operatorname{diag}(M)$ of the functions in $C^\infty(M^2)$.

Definition 2. *The symbol $S(T_k)$ of an operator $(T_k) \in \mathcal{F}$ is the formal series*

$$S(T_k) = \sum \hbar^\ell [a_\ell] \in \mathcal{J}[[\hbar]]$$

where the kernel of (T_k) is given by (6) and the symbol $(a(.,k)) \in S^0(M^2)$ has the asymptotic expansion $\sum_{\ell=0}^{\infty} k^{-\ell} a_\ell(x_l, x_r)$.

From lemma 1, we deduce that $S(T_k)$ is well-defined, i.e. it does not depend on the choice of the section E nor on the choice of the symbol $(a(.,k))$. Furthermore, Borel process and lemma 1 imply that

Lemma 3. *The map $S : \mathcal{F} \to \mathcal{J}[[\hbar]]$ is onto and its kernel is the set of smoothing operators.*

Since M is compact and the kernel $T_k(x_l, x_r)$ is C^∞, T_k is a bounded operator of $L^2(M, L^k)$ for every k. Furthermore the sequence (T_k^*) of adjoints belongs to $\mathcal{F}$ and

$$S(T_k^*)(x_l, x_r) = \overline{S(T_k)}(x_r, x_l).$$

For every k, T_k is trace class and we have the asymptotic expansion

$$\operatorname{Tr} T_k = \Big(\frac{k}{2\pi}\Big)^n \sum_\ell k^{-\ell} \int_M f_\ell(x,x)\ \mu_M(x) + O(k^{-\infty}) \text{ if } S(T_k) = \sum_\ell \hbar^\ell f_\ell.$$

3.5. Symbolic calculus. We discuss now the composition of the operators of $\mathcal{F}$. We will prove that the product of two operators of $\mathcal{F}$ belongs to $\mathcal{F}$. The set of smoothing operators is an ideal of $\mathcal{F}$, so that S induced a product in $\mathcal{J}[[\hbar]]$. We will also compute this product.

Let us introduce some notations. Let (z^i) be a complex coordinates system defined on an open set U of M. Using these coordinates the Taylor expansion along the diagonal of a function $f \in C^\infty(U^2)$ can be seen as a formal series of $C^\infty(U)[[\bar{Z}_l, Z_r]]$.

Lemma 4. *The map $\mathcal{D}_2 : C^\infty(U^2) \to C^\infty(U)[[\bar{Z}_l, Z_r]]$ defined by*

$$[\mathcal{D}_2 f](x, \bar{Z}_l, Z_r) = \sum_{\alpha,\beta} f_{\alpha,\beta}(x) \bar{Z}_l^\alpha Z_r^\beta \text{ where } f_{\alpha,\beta}(x) = \tfrac{1}{\alpha!\beta!}\, \partial^\alpha_{\bar{z}_l} \partial^\beta_{z_r} f(x_l, x_r)\big|_{x = x_l = x_r}$$

induces an algebra isomorphism from $C^\infty(U^2)/\mathcal{I}^\infty(\operatorname{diag} U)$ onto $C^\infty(U)[[\bar{Z}_l, Z_r]]$.

We need also to consider Taylor expansions of functions in $C^\infty(U^3)$ along the set $\operatorname{trig}(U) = \{(x,x,x) \ / \ x \in U\}$. We use the indices l, m, r for the first, second and third factors of U^3.

Lemma 5. *The map $\mathcal{D}_3 : C^\infty(U^3) \to C^\infty(U)[[\bar{Z}_l, Z_m, \bar{Z}_m, Z_r]]$ defined by*

$$[\mathcal{D}_3 f](x, \bar{Z}_l, Z_m, \bar{Z}_m, Z_r) = \sum_{\alpha,\gamma,\delta,\beta} f_{\alpha,\gamma,\delta,\beta}(x) \bar{Z}_l^\alpha Z_m^\gamma \bar{Z}_m^\delta Z_r^\beta$$

$$\text{where } f_{\alpha,\gamma,\delta,\beta}(x) = \tfrac{1}{\alpha!\gamma!\delta!\beta!}\, \partial^\alpha_{\bar{z}_l} \partial^\gamma_{z_m} \partial^\delta_{\bar{z}_m} \partial^\beta_{z_r} f(x_l, x_m, x_r)\big|_{x = x_l = x_m = x_r}$$

induces an algebra isomorphism from $C^\infty(U^3)/\mathcal{I}^\infty(\operatorname{trig} U)$ onto the algebra of formal series $C^\infty(U)[[\bar{Z}_l, Z_m, \bar{Z}_m, Z_r]]$

Let us define the functions G_{ij}, G^{ij} and $G_{\alpha,\beta}$ associated to the Kähler metric. The functions G_{ij} are given by

$$\omega = i \sum_{i,j} G_{ij} dz^i \wedge d\bar{z}^j.$$

The functions G^{ij} are such that $(G^{ji})_{i,j}$ is the inverse of $(G_{ij})_{i,j}$. To define the functions $G_{\alpha,\beta}$, observe that $d\omega = 0$ implies $\partial_{z^k} G_{ij} = \partial_{z^i} G_{kj}$ and $\partial_{\bar{z}^k} G_{ij} = \partial_{\bar{z}^j} G_{ik}$. Consequently

$$\partial_{z^{i_1}} \partial_{z^{i_2}} ... \partial_{\bar{z}^{j_1}} \partial_{\bar{z}^{j_2}} ... G_{i_0 j_0}$$

is symmetric with respect to $i_0, i_1, i_2, ...$ and $j_0, j_1, j_2, ...$ Let $G_{\alpha,\beta}$ denote this function where α (resp. β) is the multiindice such that $\alpha(l)$ (resp. $\beta(l)$) is the number of indices i_k (resp. j_k) equal to l.

Theorem 5. *If (P_k) and (Q_k) are operators in $\mathcal{F}$, then the same holds for $(P_k \circ Q_k)$. The product*

$$A : \mathcal{J}[[\hbar]] \times \mathcal{J}[[\hbar]] \rightarrow \mathcal{J}[[\hbar]], \qquad A\big(S(P_k), S(Q_k)\big) = S(P_k \circ Q_k)$$

is associative and $\mathbb{C}[[\hbar]]$-bilinear. The operators A_ℓ defined by

$$A_\ell : \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}, \qquad A(F,G) = \sum \hbar^\ell A_\ell(F,G) \quad \forall\, F, G \in \mathcal{J}$$

are given with the previous notations by

$$A_\ell(F,G) = [\det(G_{ij})]^{-1} \sum_{k=\ell}^{3l} \frac{(-1)^{\ell-k}}{k!(k-\ell)!} \left[\Delta^k (R^{k-\ell} H.D)\right]_{Z_m = \bar{Z}_m = 0}$$

where R, D and H are the formal series of $C^\infty(U)[[\bar{Z}_l, Z_m, \bar{Z}_m, Z_r]]$ defined by

$$R = \sum_{\substack{|\alpha|>0, |\beta|>0, \\ |\alpha|+|\beta| \geqslant 3}} \frac{G_{\alpha,\beta}(x)}{\alpha!\beta!} Z_m^\alpha \bar{Z}_m^\beta, \qquad D = \sum_{\alpha,\beta} \frac{\partial_z^\alpha \partial_{\bar{z}}^\beta [\det(G_{ij})](x)}{\alpha!\beta!} Z_m^\alpha \bar{Z}_m^\beta$$

$$H = [\mathcal{D}_3(f(x_l, x_m).g(x_m, x_r))](x, \bar{Z}_l, Z_m, \bar{Z}_m, Z_r)$$

and Δ is the operator $\sum G^{ij}(x) \partial_{Z_m^i} \partial_{\bar{Z}_m^j}$ which acts on $C^\infty(U)[[\bar{Z}_l, Z_m, \bar{Z}_m, Z_r]]$.

Remark 2. If $\sum \hbar^\ell f_\ell$ is the full symbol of (T_k), its principal symbol is f_0. The formula for the composition of principal symbols is

$$\mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}, \qquad F(x, \bar{Z}_l, Z_r), G(x, \bar{Z}_l, Z_r) \rightarrow F(x, \bar{Z}_l, 0).G(x, 0, Z_r)$$

Proof. We compute the kernel $T_k(x_l, x_r)$ of the product of two operators in $\mathcal{F}$ whose symbols F and G belong to $\mathcal{J}$. We will estimate this kernel by applying the stationary phase lemma.

$$T_k(x_l, x_r) = \Big(\frac{k}{2\pi}\Big)^{2n} \int_M E^k(x_l, x_m, x_r) F(x_l, x_m) G(x_m, x_r) \mu_M(x_m)$$

where $E(x_l, x_m, x_r) \in L^*_{x_l} \otimes L_{x_r}$ is the contraction of $E(x_l, x_m) \in L^*_{x_l} \otimes L_{x_m}$ with $E(x_m, x_r) \in L^*_{x_m} \otimes L_{x_r}$. $|E(x_l, x_m, x_r)| < 1$ if $(x_l, x_m, x_r) \notin \mathrm{trig}(M)$. So the sequence $T_k(.,k)$ is negligible on every open set which does not meet the diagonal $\{x_l = x_r\}$, and to estimate T_k on the neighborhood of (x,x) modulo a negligible sequence it suffices to integrate on a small neighborhood of (x,x,x). So we may assume that M is an open set U and use the notations introduced in remark 1. Then

$$E^k(x_l, x_m, x_r) = e^{ik\phi(x_l,x_m,x_r)}\ s(x_l) \otimes s^{-1}(x_r)$$

where $\phi(x_l, x_m, x_r) = \psi(x_l, x_m) + \psi(x_m, x_r)$.

Lemma 6. *The formal series* $[\mathcal{D}_3\phi(x_l, x_m, x_r) - \mathcal{D}_3\psi(x_l, x_r)](x, \bar{Z}_l, Z_m, \bar{Z}_m, Z_r)$ *is equal to*

$$\sum_{|\alpha|>0,|\beta|>0} i\frac{G_{\alpha,\beta}(x)}{\alpha!\beta!} Z_m^\alpha \bar{Z}_m^\beta = iR(x, Z_m, \bar{Z}_m) + i\sum_{j,k} G_{j,k} Z_m^j \bar{Z}_m^k \tag{16}$$

where R is the formal series defined in theorem 5.

Proof. Set $G(x) = \rho(x) + \bar{\rho}(x)$. Since $\partial_{z^i}\partial_{\bar{z}^j} = G_{i,j}$, we have $G_{\alpha,\beta} = \partial_z^\alpha \partial_{\bar{z}}^\beta G$ if $|\alpha|, |\beta| > 0$. Define the functions $G_{0,\beta} = \partial_{\bar{z}}^\beta G$ and $G_{\alpha,0} = \partial_z^\alpha G$. From remark 1,

$$\phi(x_l, x_m, x_r) - \psi(x_l, x_r) = i\big(G(x_m) - \tilde{\psi}(x_l, x_m) - \tilde{\psi}(x_m, x_r) + \tilde{\psi}(x_l, x_r)\big)$$

To compute the successive derivatives of $\tilde{\psi}(x_l, x_r)$ with respect to z_l^k or $\bar{z}_r^k$, observe that $\big(\partial_{z_l^k}\tilde{\psi}\big)(x,x) = \partial_{z^k} G(x)$ and

$$\partial_{\bar{z}_l^j}\partial_{z_l^k}\tilde{\psi}(x_l, x_r) \equiv \partial_{z_r^j}\partial_{z_l^k}\tilde{\psi}(x_l, x_r) \equiv 0 \mod \mathcal{I}^\infty(\{x_l = x_r\}).$$

By iterating this and doing the same with $\bar{z}_r^k$, we obtain

$$\big(\partial_{z_l}^\alpha \tilde{\psi}\big)(x,x) = G_{\alpha,0}, \qquad \big(\partial_{\bar{z}_r}^\beta \tilde{\psi}\big)(x,x) = G_{0,\beta}.$$

It follows that

$$[\mathcal{D}_3\tilde{\psi}(x_l, x_m)] = \sum \frac{G_{0,\beta}(x)}{\beta!}\bar{Z}_m^\beta, \qquad [\mathcal{D}_3\tilde{\psi}(x_m, x_r)] = \sum \frac{G_{\alpha,0}(x)}{\alpha!} Z_m^\alpha.$$

Moreover, we have

$$[\mathcal{D}_3\tilde{\psi}(x_l, x_r)] = G(x), \qquad [\mathcal{D}_3 G(x_m)] = \sum \frac{G_{\alpha,\beta}(x)}{\alpha!\beta!} Z_m^\alpha \bar{Z}_m^\beta.$$

By adding up these series, we obtain the result. $\square$

To apply the stationary phase lemma, we show that $d^2_{x_m}\phi$ is non-degenerate at (x,x,x). By lemma 6, the matrix of $d^2_{x_m}\phi$ is written

$$\begin{pmatrix} 0 & -iG_{jk}(x) \\ -iG_{kj}(x) & 0 \end{pmatrix} \tag{17}$$

in terms of the basis $(\partial_{z^i_m}, \partial_{\bar{z}^i_m})$, and the result follows. Let us determine the ideal $\mathcal{J}$ of $C^\infty(U^3)$ generated by $\partial_{z^j_m}\phi, \partial_{\bar{z}^j_m}\phi$.

Lemma 7. *A function $f \in C^\infty(U^3)$ belongs to $\mathcal{J}$ if and only if $[\mathcal{D}_3 f]$ belongs to the ideal generated by $Z^j_m, \bar{Z}^j_m$.*

Proof. By lemma 6, $[\mathcal{D}_3 \partial_{z^j_m} \phi]$ and $[\mathcal{D}_3 \partial_{\bar{z}^j_m} \phi]$ belong to the ideal generated by $Z^j_m, \bar{Z}^j_m$, so the same holds for every function of $\mathcal{J}$. Conversely, consider the ideal $\mathcal{J}'$ generated by the functions $u^j = z^j_m - z^j_l$ and $v^j = \bar{z}^j_m - \bar{z}^j_r$. The function $u^j \bar{u}^j + v^j \bar{v^j}$ vanishes on $\operatorname{trig} U$ to order 2, its Hessian is non-degenerate in the directions transversal to $\operatorname{trig} U$. So $\mathcal{I}^\infty(\operatorname{trig} U) \subset \mathcal{J}'$. Moreover $[\mathcal{D}_3 u^j] = Z^j_m$ and $[\mathcal{D}_3 v^j] = \bar{Z}^j_m$. We obtain that

$$f \in \mathcal{J}' \Leftrightarrow [\mathcal{D}_3 f] \in \langle Z^j_m, \bar{Z}^j_m \rangle.$$

From lemma 6, we see that the functions $\partial_{z^j_m} \phi, \partial_{\bar{z}^j_m} \phi$ belong to $\mathcal{J}'$, that is they are linear combinations of the functions u^j and v^j with C^∞ coefficients. This gives a linear system which is inversible on a neighborhood of $\operatorname{trig} U$ since the coefficients along $\operatorname{trig} U$ are those of the matrix (17). We obtain that $\mathcal{J}' \subset \mathcal{J}$ and the result follows. $\square$

If $f \in C^\infty(U^3)$, let $f^r \in C^\infty(U^2)$ denote a function such that $f(x_l, x_m, x_r) \equiv f^r(x_l, x_r)$ modulo $\mathcal{I}$. By lemma 7, such a function exists, is unique modulo $\mathcal{I}^\infty(\operatorname{diag} U)$ and

$$[\mathcal{D}_2 f^r](x, \bar{Z}_l, Z_r) = [\mathcal{D}_3 f](x, \bar{Z}_l, 0, 0, Z_r).$$

Lemma 6 implies that $\phi^r = \psi$. The final result follows then from theorem 7.7.12 of [10] by using that $\mu_M = \det(G_{ij}) |dz^1 ... dz^n . d\bar{z}^1 ... d\bar{z}^n|$, (17) and (16). $\square$

4. Toeplitz Operators

In this chapter we prove theorem 2 and give the properties of the full symbol σ, the covariant symbol and the contravariant one.

The fist task is to compute the symbol of the projector (Π_k). To do this we consider the set $\tilde{\mathcal{T}}$ of the operators $(T_k) \in \mathcal{F}$ such that

$$\forall Z \in C^\infty(M, T^{1,0}M), \ \nabla_{\bar{Z}} \circ T_k \equiv T_k \circ \nabla_Z \equiv 0 \mod O(k^{-\infty})$$

where $O(k^{-\infty})$ is the set of smoothing operators. $\tilde{\mathcal{T}}$ is a subalgebra of $\mathcal{F}$ and (Π_k) is an operator of $\tilde{\mathcal{T}}$. As we shall see, $\tilde{\mathcal{T}} = \mathcal{T} + O(k^{-\infty})$, that is every operator of $\tilde{\mathcal{T}}$ is the sum of a Toeplitz operator and a smoothing operator, and conversely.

Lemma 8. *Let (T_k) be an operator of $\mathcal{F}$ with symbol $S(T_k) = \sum \hbar^l [f_l]$. Then (T_k) belongs to $\tilde{\mathcal{T}}$ if and only if*

$$\bar{Z}_l . f_\ell(x_l, x_r) \equiv Z_r . f_\ell(x_l, x_r) \equiv 0 \mod \mathcal{I}^\infty(\operatorname{diag} M) \tag{18}$$

for every integer ℓ and holomorphic vector field $Z \in C^\infty(M, T^{1,0}M)$

Proof. If (T_k) is an operator in $\mathcal{F}$ with symbol $\sum \hbar^\ell [f_\ell]$ and Z is a holomorphic vector field, then $(\nabla_{\bar{Z}} \circ T_k)$ and $(T_k \circ \nabla_Z)$ are operators in $\mathcal{F}$ with symbol $\sum \hbar^\ell [\bar{Z}_l . f_\ell(x_l, x_r)]$ and $\sum \hbar^\ell [\bar{Z}_r . f_\ell(x_l, x_r)]$. $\square$

Let us define the full symbol map $\tilde{\sigma} : \tilde{\mathcal{T}} \rightarrow C^{\infty}(M)[[\hbar]]$ by

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{S} & \mathcal{J}[[\hbar]] \\ \uparrow & & \Big\downarrow r \\ \tilde{\mathcal{T}} & \xrightarrow{\tilde{\sigma}} & C^{\infty}(M)[[\hbar]] \end{array}$$

where r is the restriction
$$r\big(\textstyle\sum \hbar^{\ell} f_{\ell}(x_l, x_r)\big) = \sum \hbar^{\ell} f_{\ell}(x, x)$$

From the properties of S and lemma 8, it follows that the map $\tilde{\sigma}$ is onto and its kernel is the set $O(k^{-\infty})$ which consists of the smoothing operators. Since $O(k^{-\infty})$ is an ideal of $\tilde{\mathcal{T}}$, we obtain an associative product $C^{\infty}(M)[[\hbar]] \times C^{\infty}(M)[[\hbar]] \rightarrow C^{\infty}(M)[[\hbar]]$, $(f, g) \rightarrow f * g$.

Lemma 9. *The product $*$ is $\mathbb{C}[[\hbar]]$-bilinear. The operators*

$$B_{\ell} : C^{\infty}(M) \times C^{\infty}(M) \rightarrow C^{\infty}(M)$$

defined by $f * g = \sum \hbar^{\ell} B_{\ell}(f, g)$ for every $f, g \in C^{\infty}(M)$ are bidifferential. Locally, they are given by

$$B_{\ell}(f, g) = [\det(G_{ij})]^{-1} \sum_{k=\ell}^{3\ell} \frac{(-1)^{\ell-k}}{k!(k-\ell)!} \left[\Delta^{k}(R^{k-\ell} H.D)\right]_{Z_m = \bar{Z}_m = 0} \tag{19}$$

where Δ, R and D are defined as in theorem 5 and

$$H = \Big(\sum_{\beta} \tfrac{1}{\beta!} \partial_{\bar{z}}^{\beta} g(x) \bar{Z}_m^{\beta}\Big) . \Big(\sum_{\alpha} \tfrac{1}{\alpha!} \partial_{z}^{\alpha} f(x) Z_m^{\alpha}\Big).$$

Proof. This follows from theorem 5. It suffices to compute $[\mathcal{D}_3 F(x_l, x_m)]$, where $F(x, x) = f(x)$ and $\bar{Z}_l.F(x_l, x_r)$ and $Z_r.F(x_l, x_r)$ vanish to order ∞ along $\{x_l = x_r\}$ if Z is a holomorphic vector field. This computation can be done as in the proof of lemma 6. We compute in the same way $[\mathcal{D}_3 G(x_m, x_r)]$. $\square$

We obtain that $B_0(f, g) = f.g$. Consequently $*$ has a unit and it is determined as being the unique formal series $1_* = \sum_{\ell} \hbar^{\ell} S_{\ell}$ such that $1_* \neq 0$ and $1_* * 1_* = 1_*$. The symbol $\tilde{\sigma}(\Pi_k)$ satisfies $\tilde{\sigma}(\Pi_k) * \tilde{\sigma}(\Pi_k) = \tilde{\sigma}(\Pi_k)$. Furthermore $\tilde{\sigma}(\Pi_k) \neq 0$, because (Π_k) is not a smoothing operator. So $\tilde{\sigma}(\Pi_k) = 1_*$. To compute it, we can use that $1_* * 1 = 1$, which gives

$$S_0 = 1, \qquad S_{\ell} = -\sum_{i=0}^{i=\ell-1} B_{\ell-i}(S_i, 1). \tag{20}$$

Let (T_k) be an operator of $\mathcal{F}$. Using that (Π_k) is the unit of $\tilde{\mathcal{T}}/\mathcal{R}$, we obtain that $(T_k) \in \tilde{\mathcal{T}}$ if and only if $\Pi_k T_k \Pi_k \equiv T_k \mod O(k^{-\infty})$.

We now come to the Toeplitz operators.

Proposition 3. *The following assertions are equivalent and define the set $\mathcal{T}$ of the Toeplitz operators (T_k):*

i. $(T_k) \in \mathcal{F}$ *et* $\Pi_k T_k \Pi_k = T_k$

ii. $\exists \sum \hbar^{\ell} f_{\ell} \in C^{\infty}(M)[[\hbar]]$ *such that* $T_k \equiv \Pi_k M_{f(.,k)} \Pi_k + O(k^{-\infty})$
where $f(., k) = \sum k^{-\ell} f_{\ell} + O(k^{-\infty})$ *and* $\Pi_k T_k \Pi_k = T_k$

$\mathcal{T}$ is a $*$-algebra, that is if (R_k) and (S_k) belong to $\mathcal{T}$, then the same holds for $(R_k \circ S_k)$ and (R_k^*).

Define the full symbol map $\sigma : \mathcal{T} \to C^\infty(M)[[\hbar]]$ by

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{S} & \mathcal{J}[[\hbar]] \\ \uparrow & & \downarrow r \\ \mathcal{T} & \xrightarrow{\sigma} & C^\infty(M)[[\hbar]] \end{array}$$

or equivalently σ is the restriction

$$\sigma : \mathcal{T} \to \tilde{\mathcal{T}} \xrightarrow{\tilde{\sigma}} C^\infty(M)[[\hbar]]$$

Then σ is onto and its kernel is $\mathcal{T} \cap O(k^{-\infty})$. Since $\mathcal{T} \cap O(k^{-\infty})$ is an ideal of $\mathcal{T}$, we obtain an associative product $C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$. It is the same as the product $*$ described in lemma 9.

The map $\sigma_{cont} : \mathcal{T} \to C^\infty(M)[[\hbar]]$ such that $\sigma_{cont}(T_k) = \sum \hbar^\ell f_\ell$ if

$$T_k \equiv \Pi_k M_{f(.,k)} \Pi_k + O(k^{-\infty}) \text{ and } f(.,k) = \sum k^{-\ell} f_\ell + O(k^{-\infty})$$

is well-defined. It is onto and its kernel is $\mathcal{T} \cap O(k^{-\infty})$.

The map $\tilde{\Psi} : C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$, which sends $\sigma_{cont}(T_k)$ into $\sigma(T_k)$ if $(T_k) \in \mathcal{T}$, is $\mathbb{C}[[\hbar]]$-linear. The operators $\tilde{\Psi}_\ell$ such that $\tilde{\Psi}(f) = \sum \hbar^\ell \tilde{\Psi}_\ell(f)$ for every $f \in C^\infty(M)$, are differential of order 2ℓ. Furthermore $\tilde{\Psi}_0(f) = f$.

Remark 3. Recall that $\tilde{\mathcal{T}}$ is the set of the operators $(T_k) \in \mathcal{F}$ which satisfy $\Pi_k T_k \Pi_k \equiv T_k \mod O(k^{-\infty})$. Using the definition $i.$ of a Toeplitz operator, we obtain that $\tilde{\mathcal{T}} = \mathcal{T} + O(k^{-\infty})$. Now theorem 2 of the introduction follows from lemma 8.

Proof. First define a Toeplitz operator by assertion i. The properties of σ follow from those of $\tilde{\sigma}$ and the fact that $\tilde{\mathcal{T}} = \mathcal{T} + O(k^{-\infty})$. To prove that $ii. \Rightarrow i.$, observe that $M_{f(.,k)} \Pi_k \in \mathcal{F}$ and so $\Pi_k M_{f(.,k)} \Pi_k \in \mathcal{F}$. To prove the converse, we compute $\sigma(\Pi_k M_{f(.,k)} \Pi_k)$. We have

$$S(M_{f(.,k)} \Pi_k)(x_l, x_r, \hbar) = \sum \hbar^\ell f_\ell(x_l).S(\Pi_k)(x_l, x_r, \hbar)$$

and by applying theorem 5, we obtain that

$$\sigma(\Pi_k M_{f(.,k)} \Pi_k) = \sum \hbar^{\ell+m} \tilde{\Psi}'_\ell(f_m)$$

where the operators $\tilde{\Psi}'_\ell$ are differential of order 2ℓ and $\tilde{\Psi}'_0$ is the identity. This defines a map $\tilde{\Psi}' = \sum \hbar^\ell \psi'_\ell$, which is bijective. We obtain that $i. \Rightarrow ii.$. Now by definition $\sigma_{\mathrm{cont}} = \tilde{\Psi}'^{-1} \circ \sigma$ and the properties of σ_{cont} follow from those of σ. Finally, observe that $\tilde{\Psi} = \tilde{\Psi}'$ and this completes the proof. □

In the last proposition, we defined the symbol σ and the contravariant symbol. The third full symbol is the covariant symbol.

Definition 3. *The covariant symbol map $\sigma_{\mathrm{cov}} : \mathcal{T} \to C^\infty(M)[[\hbar]]$ is the map*

$$(T_k) \to \sigma_{\mathrm{cov}}(T_k) = \sigma(T_k) \big(\sum \hbar^\ell S_\ell \big)^{-1}$$

We denote by Ψ the map $C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$, which sends $\sigma_{\mathrm{cont}}(T_k)$ into $\sigma_{\mathrm{cov}}(T_k)$. It satisfies the same properties as $\tilde{\Psi}$. So we have the following commutative diagram

$$\begin{tikzcd} & \mathcal{T}/O(k^{-\infty}) \arrow[dl, "\sigma"'] \arrow[d, "\sigma_{\mathrm{cont}}"] \arrow[dr, "\sigma_{\mathrm{cov}}"] & \\ \mathcal{D} & \mathcal{D} \arrow[l, "\tilde{\psi}"] \arrow[r, "\psi"'] & \mathcal{D} \end{tikzcd} \qquad \text{with } \mathcal{D} = C^\infty(M)[[\hbar]]$$

where each map is a bijection. Using the symbol maps σ_{cov} and σ_{cont}, we define the products $*_{\mathrm{cov}}$ and $*_{\mathrm{cont}}$ of $C^\infty(M)[[\hbar]]$. These are associative product with unit 1.

We describe the symbolic calculus modulo $O(h^2)$. To do this, we introduce the bivector $G^{-1} \in C^\infty(M, T^{1,0}M \otimes T^{0,1}M)$ and the Laplacian $\Delta : C^\infty(M) \to C^\infty(M)$ defined locally by

$$G^{-1} = \sum_{i,j} G^{ij} \partial_{z^i} \otimes \partial_{\bar{z}^j}, \qquad \Delta = G^{ij} \partial_{z^i} \partial_{\bar{z}^j}$$

where (z^i) are complex coordinates. We denote by $r \in C^\infty(M)$ the scalar curvature of (M, g).

Proposition 4. *If f and $g \in C^\infty(M)$, we have*

$$\begin{aligned} f * g =& f.g + \hbar\big(\langle dg \otimes df, G^{-1}\rangle - \tfrac{1}{2}\, r.f.g\big) + O(\hbar^2), \\ \Psi(f) =& f + \hbar\Delta f + O(\hbar^2), \\ f *_{\mathrm{cov}} g =& f.g + \hbar\langle dg \otimes df, G^{-1}\rangle + O(\hbar^2), \\ f *_{\mathrm{cont}} g =& f.g - \hbar\langle df \otimes dg, G^{-1}\rangle + O(\hbar^2). \end{aligned}$$

Consequently, we have $f * g = f.g + O(\hbar)$, $f * g - g * f = \frac{\hbar}{i}\{f, g\} + O(\hbar^2)$ and the same holds for $*_{\mathrm{cov}}$ and $*_{\mathrm{cont}}$. Furthermore, if (T_k) is a Toeplitz operator then $\sigma(T_k) = \sigma_{\mathrm{cov}}(T_k) + O(\hbar) = \sigma_{\mathrm{cont}}(T_k) + O(\hbar)$.

We say that the function $f \in C^\infty(M)$ such that $\sigma(T_k) = f + O(\hbar)$ is the *principal symbol* of (T_k).

Proof. Let x be a point of M and (z^i) a complex coordinates system such that $z^i(x) = 0$ and

$$G_{i,j}(x) = \delta_{i,j} \qquad G_{i,\alpha}(x) = G_{\alpha,i}(x) = 0 \text{ if } |\alpha| = 2.$$

We have to show that the operator B_1 defined in lemma 9 is given by

$$B_1(f,g)\big|_x = \sum_i (\partial_{\bar{z}^i} f)(\partial_{z^i} g) + \frac{1}{2}\sum_{i,j} G_{ij,ij}.f.g\big|_x \tag{21}$$

with $G_{ij,kl} = G_{\alpha,\beta}$ where $\alpha(s) = \delta_{si} + \delta_{sj}$ and $\beta(s) = \delta_{sk} + \delta_{sl}$. The formula (19) gives for $B_1(f,g)$

$$\Big[\Delta(F.G.D) - \frac{1}{2}\Delta^2(R.F.G.D) + \frac{1}{12}\Delta^3(R^2.F.G.D)\Big]\big|_{Z_m = \bar{Z}_m = 0}$$

Since R vanishes to order 4 at x, the third term of the sum vanishes at x. We have

$$D \equiv 1 + \sum_{i,j,k} G_{ij,ik}(x) Z_m^j \bar{Z}_m^k$$

modulo some terms of order larger than 3. So, at x, $\Delta(F.G.D)\big|_{Z_m=\bar{Z}_m=0}$ is equal to

$$\begin{aligned}&\Big[\Delta\Big(\sum_{i,j}(\partial_{\bar{z}^i} f)(\partial_{z^j} g) Z_m^j \bar{Z}_m^i + f.g \sum_{i,j,k} G_{ij,ik}(x) Z_m^j \bar{Z}_m^k\Big)\Big]\big|_{Z_m=\bar{Z}_m=0}\\ =&\sum_i (\partial_{\bar{z}^i} f)(\partial_{z^i} g) + f.g \sum_{i,j} G_{ij,ij}.\end{aligned}$$

On the other hand

$$R\big|_x \equiv \frac{1}{4} \sum_{i,j,k,\ell} G_{ij,k\ell}(x) Z_m^i Z_m^j \bar{Z}_m^k \bar{Z}_m^\ell$$

modulo some terms of order larger than 5. So, at x

$$\begin{aligned}\Delta^2(R.F.G.D)\big|_{Z_m=\bar{Z}_m=0} =& \Big[\Delta^2\Big(f.g. \sum_{i,j,k,\ell} G_{ij,k\ell}(x) Z_m^i Z_m^j \bar{Z}_m^k \bar{Z}_m^\ell\Big)\Big]\big|_{Z_m=\bar{Z}_m=0}\\ =& f.g \sum_{i,j} G_{ij,ij}.\end{aligned}$$

By adding up, we obtain (21). In the same way, we compute

$$\tilde{\Psi}(f)\big|_x = f + \hbar \sum_i (\partial_{\bar{z}^i} \partial_{z^i} f + \tfrac{1}{2} \sum_{i,j} G_{ij,ij}.f.g\big|_x) + O(\hbar^2)$$

And we obtain the formulas of the proposition. $\square$

By applying equation (20), we compute $\sigma(\Pi_k)$ modulo $O(\hbar^2)$

Corollary 2. $\sigma(\Pi_k) = 1 + \frac{\hbar}{2} r + O(\hbar^2)$.

From this we obtain the first and second terms of Riemann-Roch-Hirzebruch formula

$$\dim \mathcal{H}_k = \Big(\frac{k}{2\pi}\Big)^n \int_M (1 + \tfrac{1}{2k}\, r) \tfrac{\omega^n}{n!} + O(k^{n-2}).$$

Applying Lemma 9, proposition 3 and proposition 4, we obtain :

Proposition 5. *The products* $*$, $*_{\mathrm{cov}}$ *and* $*_{\mathrm{cont}}$ *are equivalent star products.*

Let $V_\ell, N_\ell : C^\infty(M^2) \to C^\infty(M)$ denote the bidifferential operators such that

$$f *_{\mathrm{cov}} g = \sum \hbar^\ell V_\ell(f,g), \qquad f *_{\mathrm{cont}} g = \sum \hbar^\ell N_\ell(f,g)$$

and Ψ_ℓ, Ψ_ℓ^{-1} the differential operators such that

$$\Psi(f) = \sum \hbar^\ell \Psi_\ell(f), \qquad \Psi^{-1}(f) = \sum \hbar^\ell \Psi_\ell^{-1}(f)$$

where Ψ^{-1} is the inverse of Ψ. The operators V_ℓ may be easily computed using the following equation

$$V_\ell(f,g) = \sum_{\ell_1+\ell_2+\ell_3+\ell_4=\ell} S^{-1}_{\ell_1} B_{\ell_2}(S_{\ell_3} f, S_{\ell_4} g) \tag{22}$$

where $\sum_\ell \hbar^\ell S_\ell^{-1}$ is the inverse of $\sum \hbar^\ell S_\ell$ for the usual product. Then we can deduce N_ℓ, Ψ_ℓ and Ψ_ℓ^{-1} from the following proposition.

Proposition 6. *Let U be an open set of M endowed with a system of complex coordinates. Denote by $(a_{\alpha,\beta})$ the family of $C^\infty(U)$ such that $\Psi_\ell = a_{\alpha,\beta}\partial_z^\alpha \partial_{\bar z}^\beta$, then*

$$V_\ell(f,g) = \sum_{\alpha,\beta} a_{\alpha,\beta}(\partial_z^\alpha g)(\partial_{\bar z}^\beta f). \tag{23}$$

In a similar way, if we denote by $b_{\alpha,\beta}$ the functions of $C^\infty(U)$ such that $\Psi_\ell^{-1} = \sum_{\alpha,\beta} b_{\alpha,\beta}\partial_z^\alpha \partial_{\bar z}^\beta$, then $N_\ell(f,g) = \sum_{\alpha,\beta} b_{\alpha,\beta}(\partial_z^\alpha f)(\partial_{\bar z}^\beta g)$.

Using this, we may deduce that the functions $a_{\alpha,\beta}$, $b_{\alpha,\beta}$ are given by universal polynomials in $[\det(G_{i,j})]^{-1}$ and $G_{\alpha',\beta'}$.

Proof. From (22), we deduce that the operators V_ℓ act by antiholomorphic derivations on the first factor and holomorphic derivations on the second factor. Observe that $\Psi(f) = f$ and $\Psi(\bar f) = \bar f$ over an open set V, if f is holomorphic on V. Indeed, since $S(M_f \Pi_k)(x_l, x_r) = f(x_l) S(\Pi_k)(x_l, x_r)$ satisfies equations (18) over V, we have $A(S(\Pi_k), S(M_f\Pi_k)) = S(\Pi_k M_f \Pi_k)$ over V which leads to $\Psi(f) = f$. $\Psi(\bar f) = \bar f$ can be proved in the same way. Let us prove that the operators N_ℓ act by holomorphic derivations on the first factor and antiholomorphic derivations on the second factor. It suffices to prove that $f *_{\mathrm{cont}} g = f.g$ and $\bar g *_{\mathrm{cont}} \bar f = \bar g *_{\mathrm{cont}} \bar f$ on V if g is holomorphic on V. The second equation follows from the first one by considering adjoints. $f *_{\mathrm{cont}} g = f.g$ is a consequence of

$$\begin{aligned} S(\Pi_k M_f \Pi_k M_g \Pi_k)\big|_V =& A(S(\Pi_k M_f), S(\Pi_k M_g \Pi_k))\big|_V \\ =& A(S(\Pi_k M_f), S(M_g \Pi_k))\big|_V \end{aligned}$$

Finally, if f and g are holomorphic over V, then

$$\Psi(\bar f.g)\big|_V = \Psi(\bar f *_{\mathrm{cont}} g)\big|_V = \Psi(\bar f) *_{\mathrm{cov}} \Psi(g)\big|_V = \bar f *_{\mathrm{cov}} g\big|_V$$

that is $\Psi_\ell(\bar f g) = V_\ell(\bar f, g)$ over V, which proves (23). In the same way, we obtain that $N_\ell(f, \bar g) = \Psi_\ell^{-1}(f.\bar g)$ on V. $\square$

Let us explain how we can define the symbolic calculus on a Kähler manifold which is not necessarily compact or which does not admit a prequantization bundle.

Observe that the star products $*$, $*_{\mathrm{cov}}$, $*_{\mathrm{cont}}$ and the equivalence maps do not depend on the choice of the prequantization bundle L. Indeed, this is clear for $*$ because the formula (19) depends only on the Kähler metric. So the unit 1_* of $(C^\infty(M)[[\hbar]], *)$ does not depend on L and consequently the same holds for the covariant star product. Finally, we compute the contravariant symbol by the formulas given in proposition 6, which do not depend on L.

Now assume that M is a Kähler manifold endowed with a prequantization bundle which is not necessarily compact. We can define the algebra $\mathcal{F}$ in the following way : it consists of the operators which satisfy the assumptions of definition 1 and whose kernel is properly supported. Then we define as previously the subalgebra $\tilde{\mathcal{T}}$ and introduce the symbol σ, the covariant symbol and the contravariant symbol. Their products still define the star products $*$, $*_{\mathrm{cov}}$, $*_{\mathrm{cont}}$ and all the formulas are the same as in the compact case.

Finally, if M does not admit a prequantization bundle, we can not construct an algebra of operators. Using complex coordinate systems, we can still define the products $*$, $*_{\mathrm{cov}}$, $*_{\mathrm{cont}}$ and the equivalence maps. We have to prove that these definitions do not depend on the choice of coordinates and that the products obtained are associative. But it suffices to prove this locally and for every $x \in M$ there exists a neighborhood U of x endowed with a prequantization bundle $L \to U$. So we can apply the previous observations on the non compact case.

5. Microsupport, characterization by the coherent states and functional calculus of Toeplitz operators

We begin with the microsupport. First we introduce the coherent states. Let $P \subset L$ be the set which consists of the vectors $u \in L$ such that $|u| = 1$. Denote by $\pi : P \to M$ the canonical projection. For every k, the map which sends $s \in \mathcal{H}_k$ into $h\big(s(\pi(u)), u^k\big)$ is continuous. Let $u \in P$. By Riesz lemma, there exists a unique vector e_k^u of $\mathcal{H}_k$ such that

$$(s, e_k^u) = h\big(s(\pi(u)), u^k\big), \ \forall\, s \in \mathcal{H}_k$$

that is, $s(\pi(u)) = (s, e_k^u)u^k$, for all $s \in \mathcal{H}_k$. e_k^u is the *coherent state* at u. If T_k is an operator $C^\infty(M, L^k) \to C^\infty(M, L^k)$ such that $\Pi_k T_k \Pi_k = T_k$, we have

$$T_k e_k^u(x) = T_k(x, \pi(u)).u^k \tag{24}$$

$$(T_k e_k^u, e_k^v) = v^{-k}.T_k\big(\pi(v), \pi(u)\big).u^k \tag{25}$$

where the points are contractions. These properties can be proved by writing them in terms of an orthogonal base of $\mathcal{H}_k$. By choosing $T_k = \Pi_k$, we deduce that

$$e_k^u(x) = \Pi_k(x, \pi(u)).u^k, \qquad (e_k^u, e_k^v) = O(k^{-\infty}) \text{ if } \pi(u) \neq \pi(v)$$

$$(e_k^u, e_k^u) = \Big(\frac{k}{2\pi}\Big)^n \sum_l k^{-l} S_l(\pi(u)) + O(k^{-\infty}).$$

Proposition 7. *Let (u_k) be a sequence of $\mathcal{H}_k$. The following assertions are equivalent :*

i. $\exists\, N,\ ||u_k|| = O(k^N)$

ii. $\exists\, N,\ \mathrm{Sup}_M\, |u_k| = O(k^N)$

iii. $\forall\, l \geqslant 0$, $\forall$ *vector fields* $X_1, ..., X_l$ *of* M, $\exists\, N,\ \mathrm{Sup}_M\, |\nabla_{X_1} ... \nabla_{X_l} u_k| = O(k^N)$

When they are satisfied, we say that (u_k) is admissible.

Proof. Obviously, $iii \Rightarrow ii \Rightarrow i$. To prove that $i \Rightarrow iii$, we introduce the vectors $(e_k^{Q,u})$ which generalize the coherent states. Let $Q : C^\infty(M, L^k) \to C^\infty(M, L^k)$ be a differential operator of the form $\nabla_{X_1} \circ ... \circ \nabla_{X_l}$, where X_1,...,X_l are l vector fields. Since $\mathcal{H}_k$ is a finite dimensional subspace of $C^\infty(M, L^k)$, the map which sends $v \in \mathcal{H}_k$ into $h(Q.v\big|_u, u^k)$ is continuous. By Riesz lemma, there exists $e_k^{u,Q} \in \mathcal{H}_k$ such that $(v, e^k_{u,Q}) = h(Q.v\big|_u, u^k)$ for all $v \in \mathcal{H}_k$. We have

$$\begin{aligned} T_k e_k^{Q,u}(x) &= \Big(Q_r . T_k \big|_{(x, \pi(u))} \Big) . u^k \\ (T_k e_k^{Q,u}, e_k^{R,v}) &= v^{-l} . \Big(\bar{R}_l \otimes Q_r . T_k \big|_{\pi(v), \pi(u))} \Big) . u^k \end{aligned} \tag{26}$$

Hence the norm of $(e_k^{Q,u})_k$ is $O(k^N)$ uniformly with respect to $u \in P$ for some N (which depends on the order of Q). The result follows by applying Cauchy-Schwarz lemma. $\square$

Proposition 8. *Let (u_k) be an admissible sequence of $\mathcal{H}_k$ and $x \in M$. The following assertions are equivalent.*

i. $\exists$ *a neighborhood V of x such that* $\displaystyle\int_V |u_k|^2 \, \mu_M = O(k^{-\infty})$

ii. $\exists$ *a neighborhood V of x such that* $\mathrm{Sup}_V \, |u_k| = O(k^{-\infty})$

iii. $\exists$ *a neighborhood V of x,* $\forall \, l \geqslant 0$, $\forall$ *vector fields $X_1, ..., X_l$ of M,* $\exists \, N$ *such that* $\mathrm{Sup}_V \, |\nabla_{X_1} ... \nabla_{X_l} u_k| = O(k^{-\infty})$

When they are satisfied, we say that (u_k) is negligible at x.

Proof. Obviously, $iii \Rightarrow ii \Rightarrow i$. Let us prove that $i \Rightarrow iii$. Choose a neighborhood W of x such that $\overline{W} \subset V$ and a section $f : W \to P$. Let us write for all $y \in W$,

$$|\nabla_{X_1} ... \nabla_{X_l} u_k|(y) = \Big| \int_M h\big(u_k, e_k^{f(y),Q}\big) \mu_M \Big|$$

$\big|h\big(u_k, e_k^{f(y),Q}\big)(x)\big|$ is smaller than $|u_k(x)|.\big|e_k^{f(y),Q}(x)\big|$. The first term of this product is $O(k^N)$ since u_k is admissible and the second one is uniformly $O(k^{-\infty})$ when $(x, y) \in V^c \times W$. Hence, the integral on the complementary set V^c of V is $O(k^{-\infty})$. By applying Cauchy-Schwarz lemma and assumption i, we can estimate the integral on V. $\square$

Remark 4. If (s_k) is a sequence of $\mathcal{H}_k$, we prove in the same way that

$$(s_k) \text{ is negligible iff } ||s_k|| = O(k^{-\infty})$$

Remark 5. Let (T_k) be a sequence such that for every k, T_k is an operator $C^\infty(M, L^k) \to C^\infty(M, L^k)$ and $\Pi_k T_k \Pi_k = T_k$. Note that we can apply the previous propositions to the sequence $(T_k(x_l, x_r))$ of kernels. Indeed, $T_k(x_l, x_r)$ is a holomorphic section of $L^k \boxtimes L^{-k} \to M \times \overline{M}$, $M \times \overline{M}$ is a Kählerian manifold whose fundamental 2-form is $\omega_l - \omega_r$ and the curvature of $L^k \boxtimes L^{-k}$ is $\frac{k}{i}(\omega_l - \omega_r)$. We can also apply the previous remark and deduce that

$$(T_k) \text{ is a smoothing operator iff } ||T_k|| = O(k^{-\infty}).$$

Definition 4. *The microsupport of an admissible sequence* (u_k) *of* $\mathcal{H}_k$ *is the complementary set of*

$$\{x \in M \;/\; (u_k) \text{ is negligible at } x\}$$

Let (T_k) *be a sequence such that for every* k, T_k *is an operator* $C^\infty(M, L^k) \to C^\infty(M, L^k)$ *and* $\Pi_k T_k \Pi_k = T_k$. *We say that* (T_k) *is admissible if the kernel sequence* $(T_k(x_l, x_r))$ *is admissible. In this case, the microsupport of* (T_k) *is the microsupport of the sequence* $(T_k(x_l, x_r))$.

The microsupport is a closed set. We denote it by $\mathrm{MS}(u_k)$ or $\mathrm{MS}(T_k)$. We have

$$\begin{aligned}
\mathrm{MS}(T_k.s_k) &\subset \mathrm{MS}(T_k).\,\mathrm{MS}(s_k) \\
&= \big\{x \;/\; \exists\, y \in M,\; y \in \mathrm{MS}(s_k) \text{ and } (x,y) \in \mathrm{MS}(T_k)\big\} \\
\mathrm{MS}(T_k \circ T'_k) &\subset \mathrm{MS}(T_k) \circ \mathrm{MS}(T'_k) \\
&= \big\{(x,z) \;/\; \exists\, y \in M,\; (x,y) \in \mathrm{MS}(T_k) \text{ et } (y,z) \in \mathrm{MS}(T'_k)\big\}
\end{aligned}$$

The microsupport of a Toeplitz operator (T_k) with symbol $\sum_k \hbar^k f_k$ is a subset of $\mathrm{diag}(M)$. By identifying $\mathrm{diag}(M)$ with M, we have

$$\mathrm{MS}(T_k) = \overline{\cup_k \operatorname{Supp} f_k}.$$

We say that (T_k) is *elliptic* at x if $f_0(x) \neq 0$, or equivalently if there exists a Toeplitz operator (S_k) such that $(T_k S_k - \Pi_k)$ and $(S_k T_k - \Pi_k)$ are negligible at (x,x).

Proposition 9. *Let* (s_k) *be an admissible sequence of* $\mathcal{H}_k$. *A point* x *of* M *does not belong to the microsupport of* (s_k) *if and only if there exists a Toeplitz operator* (T_k) *elliptic at* x *such that* $(T_k.s_k)$ *is negligible at* x.

Proof. If $s_k(y)$ is $O(k^{-\infty})$ on a neighborhood V of x, we introduce a Toeplitz operator (T_k) elliptic at x and whose microsupport is a subset of V. This implies that $(T_k s_k)$ is negligible. Conversely, assume that $T_k.s_k(y)$ is $O(k^{-\infty})$ on a neighborhood of x and (T_k) is elliptic at x. By multiplying (T_k) by a Toeplitz operator (S_k) such that $(S_k T_k - \Pi_k)$ is negligible at (x,x), we may assume that $(T_k - \Pi_k)$ is negligible at (x,x). If f is a section of P defined on a neighborhood of x, we have

$$(T_k s_k, e_k^{f(y)}) = (s_k, T_k^* e_k^{f(y)}).$$

So it suffices to prove that when y belongs to some neighborhood of x

$$T_k^* e_k^{f(y)} = e_k^{f(y)} + r_k^y$$

where $||r_k||$ is $O(k^{-\infty})$ uniformly with respect to y. This follows from (24) which implies that

$$T_k e_k^u(x) = \frac{T_k(x, \pi(u))}{\Pi_k(x, \pi(u))} e_k^u(x) \tag{27}$$

and the fact that $(T_k - \Pi_k)$ is negligible at (x,x). $\square$

The computation modulo $O(k^{-1})$ of the L^2-norm of a Toeplitz operator was done in [3]. We recall the proof which is an easy consequence of the previous results. Then we give a characterization by the coherent states of the Toeplitz operators. We end with the functional calculus.

Proposition 10. *Let (T_k) be a Toeplitz operator whose symbol is $\sum_{l \geqslant N} \hbar^l f_l$ with $f_N \neq 0$. We have*

$$||T_k|| \sim k^{-N} \operatorname{Sup} |f_N|.$$

Remark 6. We have the same estimation with the covariant symbol and the contravariant one since $\sigma(T_k) = O(\hbar^N)$ implies $\sigma_{\mathrm{cov}}(T_k) = \sigma(T_k) + O(\hbar^{N+1})$ and $\sigma_{\mathrm{cont}}(T_k) = \sigma(T_k) + O(\hbar^{N+1})$.

Proof. By using contravariant symbols, we can prove that

$$T_k = k^{-N} \Pi_k M_{f_N} \Pi_k + k^{-N-1} \Pi_k M_{g(.,k)} \Pi_k + R_k$$

where (R_k) is a smoothing operator and $(g(.,k))$ is a sequence of $C^\infty(M)$ whose norm is uniformly $O(k^{-N-1})$. It follows that there exists C such that $||T_k|| \leqslant k^{-N} \operatorname{Sup} |f_N| + C k^{-N-1}$. Let u be in P such that $\operatorname{Sup} |f_N| = |f_N(\pi(u))|$. Using that $||T_k e^k_u||^2 = (T_k^* T_k e^u_k, e^u_k)$ and (25), we obtain

$$\frac{||T_k e^k_u||^2}{||e^k_u||^2} = k^{-2N} |f_N(\pi(u))|^2 + O(k^{-2N-1}).$$

We deduce from this that $||T_k|| \geqslant k^{-N} \operatorname{Sup} |f_N| + C' k^{-N-1}$. $\square$

Proposition 11. *Let (T_k) be a sequence such that for every k, T_k is an operator of $C^\infty(M, L^k)$ and $\Pi_k T_k \Pi_k = T_k$. Then (T_k) is a Toeplitz operator if and only if there exists a symbol $(f(.,k))$ of $S^0(M \times M)$ such that*

$$\big(T_k e^u_k\big)(x) = f(x, \pi(u), k) e^u_k(x) + r^u_k(x) \tag{28}$$

where (r^u_k) is a uniformly negligible sequence with respect to u. In this case, the covariant symbol of (T_k) is $\sum_l \hbar^l f_l(x,x)$ where $f(.,k) = \sum_l k^{-l} f_l + O(k^{-\infty})$.

This result can be compared with the characterisation of the $\hbar$-pseudodifferential operators from their action on the oscillatory functions $e^{\frac{i}{\hbar} x.\xi}$ (cf. [8]).

Proof. If (T_k) is Toeplitz operator, we can prove (28) by using (27) and the expression of the kernels of (T_k) and (Π_k). Conversely, if s is a section of $\mathcal{H}_k$, we have

$$s = \int_P (s, e^u_k)\; e^u_k \; \mu_P(u).$$

Consequently,

$$T_k s = \int_P (s, e^u_k)\; T_k e^u_k \; \mu_P(u).$$

Using (28), we obtain that

$$T_k(x_l, x_r) = f(x_l, x_r, k) \Pi_k(x_l, x_r) + u^{-k}.r^u_k(x) \text{ with } \pi(u) = x_r$$

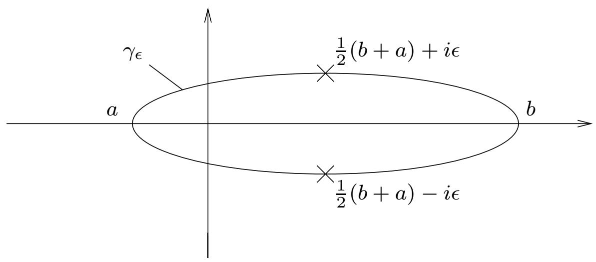
Hence, $(T_k) \in \mathcal{F}$ and by assumption, $\Pi_k T_k \Pi_k = T_k$, that is (T_k) a Toeplitz operator. $\square$

Proposition 12. *Let (T_k) be a selfadjoint Toeplitz operator with symbol $\sum_l \hbar^l f_l$ and g be a function of $C^\infty(\mathbb{R}, \mathbb{C})$. Then $(g(T_k))$ is a Toeplitz operator with principal symbol $g(f_0)$.*

Proof. By the previous proposition, the spectrum of T_k is a subset of

$$[-\operatorname{Sup}|f_0| - 1, \operatorname{Sup}|f_0| + 1]$$

if k is sufficiently large. By modifying g outside this interval, we may assume that g has compact support. So g extends to a function G of $C^\infty(\mathbb{C})$ with compact support and such that $\partial_{\bar{z}} G$ vanishes to order ∞ along $\mathbb{R} \subset \mathbb{C}$. Let $a, b \in \mathbb{R}$ be such that Supp $g \subset (a, b)$. Introduce the loops γ_ϵ :



Since $\partial_{\bar{z}} G$ vanishes to order ∞ along $\mathbb{R}$,

$$g(T_k) = \frac{1}{2i\pi} \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} G(z)(z - T_k)^{-1} dz.$$

By applying Stokes theorem

$$g(T_k) = \frac{1}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} G(z)(z - T_k)^{-1} |dz d\bar{z}|$$

where the integral is well-defined since $||(z - T_k)^{-1}|| = O(|y|^{-1})$ (y is the imaginary part of z). For $y \neq 0$, we denote by $\sum_l \hbar^\ell h_\ell(z, x)$ the inverse of $z - \sum_l \hbar^\ell f_\ell$ in $(C^\infty(M)[[\hbar]], *)$. Using that the bidifferential operators B_ℓ associated to $*$ are of degree ℓ in each argument, we obtain that

$$h_\ell(z, x) = P_\ell(z, x)(z - f_0)^{-(\ell+1)} \tag{29}$$

where the functions P_ℓ are polynomial in z with coefficients in $C^\infty(M)$. So the functions $y^{-1} h_\ell(z, x) \partial_{\bar{z}} G(z)$ are C^∞. By applying Borel process, we construct a symbol $H(z, x_l, x_r, k)$ in $S^0(\mathbb{C} \times M \times M)$ with asymptotic expansion $\sum_\ell k^{-\ell} H_\ell(z, x_l, x_r)$ such that

$$H_\ell(z, x, x) = y^{-1} h_\ell(z, x) \partial_{\bar{z}} G(z)$$
$$\partial_{\bar{z}_l^i} H_\ell = O(|x_l - x_r|^\infty) \text{ and } \partial_{z_r^i} H_\ell = O(|x_l - x_r|^\infty)$$

where the estimations are uniform with respect to z. Introduce the operators L^z_k of kernel $(\frac{k}{2\pi})^n E^k H(z, x_l, x_r, k)$. We have

$$L^z_k.(z - T_k) = y^{-1}\partial_{\bar{z}} G \Pi_k + S^z_k$$

where $||S^z_k|| = O(k^{-\infty})$ uniformly with respect to z. We deduce from this that

$$\partial_{\bar{z}} G (z - T_k)^{-1} = y L^z_k - y S^z_k (z - T_k)^{-1}$$

and then that $g(T_k)$ is a Toeplitz operator with symbol

$$\sum_l \frac{\hbar^l}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} G(z) h_l(z, x) |dz d\bar{z}|. \tag{30}$$

□

The full calculus of the symbol $\sum_\ell \hbar^\ell G_\ell$ of $g(T_k)$ can be done by the following way : write the Taylor series $\sum_p \frac{1}{p!} g^{(p)}\big(f_0(x)\big)(y - f_0(x))^p$ of g at $f_0(x)$. Then

$$\sum_\ell \hbar^\ell G_\ell\big|_x = \sum_p \frac{1}{p!} g^{(p)}\big(f_0(x)\big)\big(\sum \hbar^\ell f_\ell(y) - f_0(x) 1_*(y)\big)^{*p}\big|_{y=x}$$

where 1_* is the unit of $(C^\infty(M)[[\hbar]], *)$ and

$$\big(\sum \hbar^\ell f_\ell\big)^{*0} = 1_*, \quad \big(\sum \hbar^\ell f_\ell(y)\big)^{*1} = \sum \hbar^\ell f_\ell, \quad \big(\sum \hbar^\ell f_\ell(y)\big)^{*2} = \sum \hbar^\ell f_\ell * \sum \hbar^\ell f_\ell,$$

and so on. Indeed, the right hand side is well-defined. Then assume that g vanishes to order $q+1$ at $f_0(x)$, we deduce from (29) and (30) that $G_0(x) = ... = G_q(x) = 0$. So we may replace g with its Taylor series.

In particular, if $\sigma(T_k) = f_0 + \hbar f_1 + O(\hbar^2)$, then $\sigma(g(T_k))$ is equal to

$$g(f_0) + \hbar\big(g(f_0)\tfrac{r}{2} + g'(f_0)(f_1 - \tfrac{r}{2} f_0) + g''(f_0) G^{i,j} (\partial_{z^i} f_0)(\partial_{\bar{z}^i} f_0)\big) + O(\hbar^2)$$

The same formulas apply for the covariant symbol and contravariant symbol. Hence if $\sigma_{\mathrm{cov}}(T_k) = f_0 + \hbar f_1 + O(\hbar^2)$, then

$$\sigma_{\mathrm{cov}}(g(T_k)) = g(f_0) + \hbar\big(g'(f_0) f_1 + g''(f_0) G^{i,j} (\partial_{z^i} f_0)(\partial_{\bar{z}^i} f_0)\big) + O(\hbar^2)$$

and if $\sigma_{\mathrm{cont}}(T_k) = f_0 + \hbar f_1 + O(\hbar^2)$, then

$$\sigma_{\mathrm{cont}}(g(T_k)) = g(f_0) + \hbar\big(g'(f_0) f_1 - g''(f_0) G^{i,j} (\partial_{z^i} f_0)(\partial_{\bar{z}^i} f_0)\big) + O(\hbar^2).$$

References

1. V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, *Comm. Pure Appl. Math*, Vol **14**, pp 187-214, 1961.
2. F.A. Berezin, General concept of quantization, *Commun.math. Phys.*, Vol **40**, pp. 153-174, 1975.
3. M. Bordemann, E. Meinrenken, M. Schlichenmaier, Toeplitz Quantization of Kähler Manifolds and $gl(N)$, $N \to \infty$ Limits, *Commun. Math. Phys.*, Vol. **165**, pp.281-296, 1994.
4. L. Boutet de Monvel, V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of math. stud. 99, Princeton University, 1981.
5. L. Boutet de Monvel, J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegö, *Astérisque*, Vol. **34,35**, pp. 123-164, 1976.
6. M. Cahen, S. Gutt, J. Rawnsley, Quantization of Kähler manifolds I *J. Geom. Phys.*, Vol **7**, pp 45-62, 1990; II *Trans. Amer. Math. Soc.*, Vol **337**, pp 73-98, 1993; III *Lett. Math. Phys*, Vol **30**, pp 291-305, 1994.
7. L.Charles, Semi-classical aspects of Geometric quantization, PHD-Thesis, University Paris-Dauphine, 2000.
8. Y. Colin de Verdière, *Méthodes semi-classiques et théorie spectrale*, en préparation.
9. V. Guillemin, Star products on compact pre-quantizable symplectic manifolds *Lett. Math. Phys*, Vol **35**, pp 85-89, 1995
10. L. Hörmander, *The analysis of linear partial differential operators I,IV*, Springer Verlag, 1983.
11. B. Kostant, Quantization and unitary representation, *Lect. Notes Math.*, vol **170**, pp 87-208, 1970.
12. Z. Lu, On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch, *Amer. J. Math*, Vol **122**, pp 235-273, 2000.
13. C. Moreno, P. Ortega-Navarro, Deformations of the algebra of functions on Hermitian symmetric spaces resulting from quantization, *Ann. Inst. H. Poincaré Sect. A*, Vol **38**, pp 215-241, 1983; $*$-products on $D^1(\mathbb{C})$, S^2 and related spectral analysis, *Lett. Math. Phys.*, Vol **7**, 1983.
14. M.A. Shubin, *Pseudodifferential operator and spectral theory*, Springer Verlag, 1987.
15. J.M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris 1970.
16. G.Tian, On a set of polarized Kähler metrics on algebraic manifolds *J. Diff. Geom.* Vol **32**, pp 99-130, 1990.
17. S. Zelditch, Szegö Kernels and a theorem of Tian, *Int. Math. Research Notices* , Vol **5**, 1998.