A note on Chern-Simons bundles and the Mapping Class Group

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Abstract

Chern-Simons bundles are prequantum bundles for moduli spaces of flat principal bundles over a compact surface. We recall their construction by gauge theory in the case where the surface has a boundary. We discuss their equivariance with respect to the mapping class group of the surface.

1 Introduction

Let Σ be a compact connected oriented surface with n boundary components. Consider a compact simple simply-connected Lie group G. Choose a conjugacy class $\mathcal{C}_i$, $i = 1, \ldots, n$ for each boundary component. The moduli space $\mathcal{M}(\Sigma, G, (\mathcal{C}_i))$ is by definition the space of gauge isomorphism classes of flat G-principal bundles whose holonomy around the i-th boundary component belongs to $\mathcal{C}_i$.

The subset $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$ of $\mathcal{M}(\Sigma, G, (\mathcal{C}_i))$ consisting of classes of irreducible bundles, is a smooth symplectic manifold, cf. [AB83], [Gol84]. When the conjugacy classes $\mathcal{C}_i$ satisfy some integrality condition, $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$ has a canonical prequantum bundle, that is a Hermitian line bundle endowed with a connection whose curvature is the symplectic form, cf. [RSW89], [Fre95], [DW97]. This bundle is sometimes called the Chern-Simons bundle, because in the case Σ is closed and X is a three dimensional oriented manifold with boundary Σ, the Chern-Simons invariant of a flat G-principal bundle $F \to X$ is an element of the fiber of the Chern-Simons bundle at $F|_{\Sigma}$.

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The pure modular group of Σ acts on the moduli space $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$. It is not a priori clear whether this action is compatible with the geometric structures of the moduli space. For instance, does the pure modular group act by symplectomorphisms ? Can we lift this action to the Chern-Simons bundle ? These are the questions we adress in this note.

Our motivation is to understand the semi-classical limit of the quantum representation of the mapping class group. In [Cha10], the results of this note will be used in the proof of the Witten asymptotic conjecture for mapping tori.

First let us state the integrality condition for the existence of the Chern-Simons bundle. Consider a maximal torus of G with Lie algebra $\mathfrak{t}$ and denote by Λ the integral lattice $\ker(\exp|_{\mathfrak{t}})$ and $P^\vee$ the dual lattice to the root lattice. Choose $\lambda_i \in \mathfrak{t}$ such that $\mathcal{C}_i$ is the class of $\exp(\lambda_i)$, for $i = 1, \ldots, r$. Let $k \in \mathbb{Z}^*$. Then if the λ_i's satisfy

$$\forall i, \ \forall \xi \in \Lambda, \ \langle \lambda_i, \xi \rangle \in k^{-1}\mathbb{Z}, \quad \text{and}$$
$$\forall \xi \in 2i\pi P^\vee, \ \langle \lambda_1 + \ldots + \lambda_r, \xi \rangle \in k^{-1}\mathbb{Z}.$$

the moduli space $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$ has a canonical prequantum bundle L^k_{CS} with curvature k times the symplectic form

Recall that the pure modular group, denoted by $\mathrm{PMod}(\Sigma)$, is the group of orientation preserving diffeomorphisms of Σ fixing each boundary component as a set, up to isotopy. $\mathrm{PMod}(\Sigma)$ has a natural extension

$$0 \to \mathbb{Z}^{\pi_0(\partial\Sigma)} \to \mathrm{Mod}(\Sigma) \to \mathrm{PMod}(\Sigma) \to 0$$

Here $\mathrm{Mod}(\Sigma)$ is the group of preserving orientation diffeomorphisms of Σ relative to the boundary up to isotopy and $\mathbb{Z}^{\pi_0(\Sigma)}$ is sent isomorphically to the subgroup of $\mathrm{Mod}(\Sigma)$ generated by the Dehn twists around the boundary components of Σ.

Theorem 1.1. *The action of* $\mathrm{PMod}(\Sigma)$ *on* $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$ *preserves the symplectic form. When* $(\mathcal{C}_i)$ *satisfies the previous integrality condition, there is a natural action by prequantum bundle automorphisms of* $\mathrm{Mod}(\Sigma)$ *on the Chern-Simons bundle* L^k_{CS}*, which lifts the action of* $\mathrm{PMod}(\Sigma)$ *on the base. The Dehn twist around the* i*-th boundary component acts by multiplication by* $\exp(ik\pi|\lambda_i|^2)$.

Here $|\cdot|$ is the norm for the basic inner product. So the action of the modular group on L^k_{CS} factors through an action of the pure modular group only for particular values of k and λ_i.

Let $\bar{\Sigma}$ be the closed surface obtained from Σ by filling each boundary component by a disk. When each conjugacy class $\mathcal{C}_i$ consists of one element c_i of the center of G, the moduli space $\mathcal{M}^s(\Sigma, G, (\{c_i\}))$ is very close to a moduli space of bundles on $\bar{\Sigma}$. Indeed $\mathcal{M}^s(\Sigma, G, (\{c_i\}))$ and $\mathcal{M}^s(\bar{\Sigma}, G)$ have the same dimension. Furthermore the tangent space at F to $\mathcal{M}^s(\Sigma, G, (\{c_i\}))$ is isomorphic with $H^1(\bar{\Sigma}, \mathrm{Ad}\, F)$, where $\mathrm{Ad}\, F \to \bar{\Sigma}$ is the flat vector bundle whose restriction to Σ is the vector bundle associated to F throught the adjoint representation. For some choices of c_i, the moduli space has the advantage of being closed: $\mathcal{M}^s(\Sigma, G, (\{c_i\})) = \mathcal{M}(\Sigma, G, (\{c_i\}))$, that is any G-principal flat bundles on Σ satisfying the holonomy condition on the boundary is irreducible. To the contrary, $\mathcal{M}^s(\bar{\Sigma}, G)$ is never closed because the trivial bundle is not irreducible. For these reasons, in some paper, $\mathcal{M}^s(\Sigma, G, (\{c_i\}))$ is used as a smooth replacement of $\mathcal{M}(\bar{\Sigma}, G)$.

We would like to know whether the action of $\mathrm{PMod}(\Sigma)$ in these cases factors through an action of $\mathrm{Mod}(\bar{\Sigma})$. We will prove it is not true in some very simple case. Let us assume that the boundary of Σ has only one connected component. Recall the Birman exact sequence

$$0 \to \pi_1(\bar{\Sigma}) \to \mathrm{PMod}(\Sigma) \to \mathrm{Mod}(\bar{\Sigma}) \to 0$$

We aim to compute the action of $\pi_1(\bar{\Sigma})$ on $\mathcal{M}(\Sigma, G, \{c\})$ where c is an element of the center $Z(G)$.

There is a natural action of $H^1(\bar{\Sigma}, Z(G))$ on $\mathcal{M}(\Sigma, G, \{c\})$ considered by Goldman in [Gol84]: identifying the moduli space with conjugacy classes of morphisms $\pi_1(\Sigma) \to G$, the action is the pointwise multiplication.

Theorem 1.2. *The action of $\pi_1(\bar{\Sigma})$ on $\mathcal{M}(\Sigma, G, \{c\})$ factors through the action of $H^1(\bar{\Sigma}, Z(G))$ by the morphism $\Phi : \pi_1(\bar{\Sigma}) \to H^1(\bar{\Sigma}, Z(G))$ defined by:*

$$\Phi(\gamma)(\alpha) = c^{i(\gamma,\alpha)}, \qquad \forall \alpha \in \pi_1(\bar{\Sigma}).$$

where i is the algebraic intersection number of $\pi_1(\bar{\Sigma})$.

When c is a generator of the center, the morphism Φ is onto. So the action of $\mathrm{PMod}(\Sigma)$ factors through an action of $\mathrm{Mod}(\bar{\Sigma})$ if and only if the action of $H^1(\bar{\Sigma}, Z(G))$ is trivial. We will prove that for $G = SU(2)$, $c = -\mathrm{id}$ and Σ of genus $\geqslant 2$, the action of $H^1(\bar{\Sigma}, Z(G))$ is not trivial.

In the first part of the paper, we will recall a construction of the Chern-Simons bundle. Following Meinrenken and Woodward [MW98], we will introduce a prequantum bundle on the moduli space of flat bundles of Σ. When the boundary of Σ is not empty, a central extension of the gauge group of the boundary acts on the prequantum bundle. Doing a symplectic reduction at a level corresponding to an integral coadjoint orbit of this central extension, we obtain the Chern-Simons bundle.

There exists alternative construction of the Chern-Simons bundle, for instance [Kre08] based on the holonomy description of the moduli space. This method has the advantage of avoiding the infinite dimensional geometry, but it uses a particular presentation of the fondamental group of the surface, and so it does not seem to be convenient for describing the mapping class group action.

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2 Symplectic Preliminaries

In the first section we recall some general well-known facts on Hamiltonian spaces, their prequantizations, their automorphisms and their reductions.

Then we discuss an extension of this theory where the group action admits a momentum which is not necessarily equivariant. The basic example is the group of affine symplectomorphisms acting on a symplectic vector space. In these cases, the action may not lift to the prequantum bundle, but the action of a convenient central extension by the structure group of the prequantum bundle does. Then we can perform symplectic reduction in this setting.

2.1 Prequantum bundle and symplectic reduction

2.1.1 Prequantum bundle

Let (M, ω) be a symplectic manifold. Denote by $\mathbb{T}$ the group $\mathbb{R}/\mathbb{Z}$. A prequantum bundle of (M, ω) is a principal bundle P over M with structure group $\mathbb{T}$ endowed with a connection $\beta \in \Omega^1(P, \mathbb{R})$ whose curvature is ω. Our sign convention for the curvature is $d\beta + \pi^*\omega = 0$, with π the projection from P to M.

The Lie algebra $\operatorname{aut}(P, \beta)$ of infinitesimal automorphisms of the prequantum bundle (P, β) consists of the vector fields Y of P which are $\mathbb{T}$-invariant and satisfy $\mathcal{L}_Y \beta = 0$. It is well-known that $\operatorname{aut}(P, \beta)$ is isomorphic to the Poisson Lie algebra of (M, ω). Indeed, a vector field Y of P belongs to aut(P,β) if and only if

$$Y = X_f^{\mathrm{hor}} + (\pi^* f)\partial_\theta \tag{1}$$

where $f \in \mathcal{C}^\infty(M)$. Here we denote by $X_f \in \mathcal{C}^\infty(M, TM)$ the Hamiltonian vector field of f,

$$\omega(X_f, \cdot) = df,$$

by $X_f^{\mathrm{hor}} \in \mathcal{C}^\infty(P, TP)$ the horizontal lift of X_f and by $\partial_\theta \in \mathcal{C}^\infty(P, TP)$ the infinitesimal generator of the $\mathbb{T}$-action. So Y is determined by the function f and conversely $\pi^* f = \beta(Y)$. Furthermore the Lie bracket of vector fields corresponds to minus the Poisson bracket of functions.

Let ρ be the character

$$\rho : \mathbb{T} \to U(1), \qquad \rho(t) = e^{-2i\pi t}.$$

For any prequantum bundle (P,β), the associated bundle $L := P \times_\rho \mathbb{C}$ is a Hermitian line bundle with a covariant derivation ∇ of curvature $-2\pi i\omega$. In particular if $P = M \times \mathbb{T}$ with $\beta = -\alpha + dt$, then L is isomorphic to the trivial complex line bundle with base M and covariant derivative $\nabla = d + \frac{2\pi}{i}\alpha$. Conversely one may start from (L,∇) and define (P,β) as the subbundle of L which consists of the vectors with norm 1.

2.1.2 Hamiltonian spaces and symplectic reduction

A Hamiltonian space (M,ω,G,μ) is a symplectic manifold (M,ω) with an action of a Lie group G and an equivariant momentum $\mu : \mathrm{Lie}(G) \to M$. So

$$d\mu^Y = \omega(Y^{\sharp_M}, \cdot), \qquad \forall\, Y \in \mathrm{Lie}(G)$$

where $Y^{\sharp M}$ is the infinitesimal action of $Y \in \mathrm{Lie}(G)$. Let H be a closed normal subgroup of G whose action on

$$\{x \in M / \langle \mu(x), Y\rangle = 0,\ \forall\, Y \in \mathrm{Lie}(H)\}$$

is free and proper. Then the symplectic quotient $M/\!\!/H$ is a smooth manifold and we obtain a Hamiltonian space $(M/\!\!/H, G/H, \mu^{G/H})$.

Let (P,β) be a prequantum bundle of a symplectic manifold M. Assume that a Lie group G acts on P by prequantum bundle automorphisms, that is by diffeomorphisms commuting with the $\mathbb{T}$-action and preserving β. By (1), the action of G on the base M has a natural momentum μ defined by the condition that for any $Y \in \mathrm{Lie}(G)$,

$$\pi^* \mu^Y = \beta(Y^{\sharp_P}) \tag{2}$$

with π the projection from P to M. This momentum is equivariant [1].

Assume that H is a closed normal subgroup of G satisfying the same assumptions as above. Then P descends to a prequantum bundle of $M/\!\!/H$ on which G/H acts by prequantum bundle automorphisms.

2.1.3 Lifting the action

We have seen that when a group acts on a prequantum bundle (P, β), the infinitesimal action defines a momentum for the action on the base. We have the following converse.

Theorem 2.1. *Let (P, β) be a prequantum bundle with base M and curvature ω. Assume G is a connected and simply connected Lie group acting on (M, ω) with an equivariant momentum μ. Then there exists a unique lift to $\mathrm{Aut}(P, \beta)$ of the action on M such that the corresponding momentum is μ.*

Proof. By Section 2.1.1, the map sending $Y \in \mathrm{Lie}(G)$ into the vector field of P

$$Y^{\sharp_P} := (Y^{\sharp_M})^{\mathrm{hor}} + (\pi^* \mu^Y)\partial_\theta$$

is a Lie algebra anti-morphism. Furthermore, $Y^{\sharp_P}$ is complete for any $Y \in \mathrm{lie}(G)$. Indeed, its flow at time t is given by

$$\varphi_t(y) = \theta_t(\pi(y)).\mathcal{T}_t(y) \tag{3}$$

where $\mathcal{T}_t$ is the $\mathbb{T}$-equivariant diffeomorphism of P obtained by lifting the flow of $Y^{\sharp M}$ by parallel transport, and $\theta_t(x) = t\mu^Y(x)$ modulo $\mathbb{Z}$.

By Palais' theorem (Theorem 10.5.1 of [HN12]), G being simply connected, there exists an action of G on P with infinitesimal action $Y \to Y^{\sharp_P}$. $\square$

2.1.4 Automorphisms

A Hamiltonian space automorphism is a pair (φ_M, φ_G) consisting of a symplectomorphism of M and an automorphism of G such that

- $\varphi_M(g.x) = \varphi_G(g).\varphi_M(x)$ for all $g \in G$ and $x \in M$
- $\langle \mu(\varphi_M(x)), T_e\varphi_G(\xi)\rangle = \langle \mu(x), \xi\rangle$ for all $x \in M$ and $\xi \in \mathrm{Lie}(G)$.

[1] Indeed the action of any $g \in G$ preserves β and sends $Y^{\sharp_P}$ into $(Ad_g Y)^{\sharp_P}$ so that

$$\langle \mu(x), Y\rangle = \langle \beta|_x, Y^{\sharp_P}(x)\rangle = \langle \beta|_{gx}, (\mathrm{Ad}_g Y)^{\sharp_P}(gx)\rangle = \langle \mu(gx), \mathrm{Ad}_g Y\rangle.$$

If furthermore $\varphi_G(H) \subset H$, then (φ_M, φ_G) induces an automorphism of the reduced Hamiltonian space $(M/\!\!/H, G/H)$.

An automorphism of (P, β, G) is a pair (φ_P, φ_G) consisting of a prequantum bundle automorphism of P and an automorphism of G such that

$$\varphi_P(g.z) = \varphi_G(g).\varphi_P(z) \quad \text{for all } g \in G \text{ and } z \in P.$$

Such an automorphism induces an automorphism of the associated Hamiltonian space. If $\varphi_G(H) \subset H$, then the pair (φ_P, φ_G) induces an automorphism of $(P/\!\!/H, G/H)$.

2.2 Non-equivariant momentum

2.2.1 Central extension and prequantum bundle

Consider a Lie group G with a central extension

$$1 \to \mathbb{T} \to \hat{G} \to G \to 1.$$

Since the extension is central, the adjoint action of any element in the image of $\mathbb{T} \to \hat{G}$ is trivial. Hence the adjoint and the coadjoint action of $\hat{G}$ factor through actions of G.

Denote by j the injection from $\mathbb{R} = \text{Lie}(\mathbb{T})$ into $\text{Lie}(\hat{G})$ and by

$$p : \text{Lie}(\hat{G}) \to \mathbb{R}$$

the adjoint map. Since the extension is central, for any $g \in \hat{G}$, $\text{Ad}_g\, j = j$ and consequently $\text{Ad}_g^*\, p = p$. So the coadjoint action of $\hat{G}$ preserves the level sets of p.

Let (P, β) be a prequantum bundle of a symplectic manifold M. Assume that $\hat{G}$ acts on P by prequantum bundle automorphisms in such a way that the kernel of the projection $\hat{G} \to G$ acts as the structure group of P. Under this asumption, the action of $\hat{G}$ on M factors through an action of G. Observe also that the momentum μ of the $\hat{G}$-action defined by Equation (2) takes its value in $p^{-1}(1)$.

Remark. Introduce a section of the projection $\text{Lie}(\hat{G}) \to \text{Lie}(G)$ so that the Lie algebra of $\hat{G}$ identifies with $\text{Lie}(G) \oplus \mathbb{R}$ and its dual with $\text{Lie}(G)^* \oplus \mathbb{R}$. Then

$$\mu = (\mu_M, 1) \in \text{Lie}(G)^* \oplus \mathbb{R}.$$

The map μ_M is a momentum of the G-action on M not necessarily equivariant. Actually, μ_M is infinitesimally equivariant if and only if the section of $\text{Lie}(\hat{G}) \to \text{Lie}(G)$ we consider is a Lie algebra morphism.

2.2.2 Lifting the action

The current setting with central extension is convenient to lift the actions admitting a momentum non equivariant. Indeed we have the following

Theorem 2.2. *Let (P, β) be a prequantum bundle with a connected base M. Assume that G is a connected Lie group acting on M by Hamiltonian symplectomorphisms. Then the group $\hat{G}$, consisting of the pairs $(\varphi, g) \in \operatorname{Aut}(P, \beta) \times G$ such that φ lifts the action of g, is a central extension of G by the structure group $\mathbb{T}$ of P.*

Proof. The only difficulty is to show the existence of a pair (φ, g) for any g. Let $(g_t, t \in [0,1])$ be a path of G from 1 to g. We are going to lift the action of g_t on M to the prequantum bundle. Let $\xi_t := \dot{g}_t g_t^{-1} \in \operatorname{Lie}(G)$. Choose a momentum $\mu : M \to \operatorname{Lie}(G)$ of the action. Then the flow φ_t at time t of the non-autonomous vector field $(\xi_t^{\sharp_M})^{\text{hor}} + \pi^* \mu^{\xi_t} \partial_\theta$ is well-defined and has the form (3) with

$$\theta_t(x) = \int_0^t \mu^{\xi_s}(g_s.x)\, ds.$$

Furthermore $\varphi_t \in \operatorname{Aut}(P, \beta)$ and lifts g_t. □

It is not difficult to see that $\hat{G}$ has a unique structure of Lie group such that the morphisms $\mathbb{T} \to \hat{G}$ and $\hat{G} \to G$ are Lie group morphisms. So we are exactly in the situation considered in Section 2.2.1

We do not need Theorem 2.2 for our application to gauge theory. Actually we will only consider the case of a (infinite dimensionnal) symplectic vector space acted on by affine symplectomorphisms.

2.2.3 Symplectic reduction

Consider as in Section 2.2.1 a Lie group G with a central extension $\hat{G}$ by $\mathbb{T}$. Recall that the coadjoint action preserves the level sets of the projection $p : \operatorname{Lie}(\hat{G})^* \to \mathbb{R}$. Let $\mathcal{O}$ be a coadjoint orbit of $\hat{G}$ on the level set $p^{-1}(1)$ endowed with the Kostant-Souriau form $\omega_{\mathcal{O}}$. Then $(\mathcal{O}, \omega_{\mathcal{O}}, \hat{G}, j_{\mathcal{O}})$ is a Hamiltonian space, where the momentum is the injection $j_{\mathcal{O}}$ from $\mathcal{O}$ into $\operatorname{Lie}(\hat{G})^*$. Denote by $\mathcal{O}^-$ the coadjoint orbit $\mathcal{O}$ with the same action but with opposite symplectic form and momentum.

Let $(M, \omega, \hat{G}, \mu)$ be a Hamiltonian space whose momentum takes its values in $p^{-1}(1)$. Consider the symplectic product $M \times \mathcal{O}^-$, with the diagonal action of $\hat{G}$. This action factors through an action of G. In addition, the momentum of this diagonal action takes its value in $p^{-1}(0) = \ker p \simeq \operatorname{Lie}(G)^*$.

So we obtain a Hamiltonian G-action on $M \times \mathcal{O}^-$ with an equivariant momentum

$$\nu : M \times \mathcal{O}^- \to \mathrm{Lie}(G)^*.$$

Under the assumption that G acts freely and properly on the null level-set of ν, we may perform the symplectic quotient

$$(M \times \mathcal{O}^-)/\!\!/G := \nu^{-1}(0)/G.$$

It has the alternative presentation $\mu^{-1}(\mathcal{O})/G$ or $\mu^{-1}(\lambda)/G_\lambda$ where $\lambda \in \mathcal{O}$.

2.2.4 Reduction of prequantum bundle

Consider a prequantum bundle P of the coadjoint orbit $(\mathcal{O}, \omega_{\mathcal{O}}, \hat{G}, j_{\mathcal{O}})$. Let $\hat{G}_\lambda$ be the isotropy subgroup in $\hat{G}$ of $\lambda \in \mathcal{O}$. $\hat{G}_\lambda$ preserves the fiber P_λ and acts on it through a character $\hat{G}_\lambda \to \mathbb{T}$ with derivative

$$\hat{\mathfrak{g}}_\lambda \to \mathbb{R}, \qquad \xi \to \lambda(\xi).$$

This follows from Equations (1) and (2). By Kostant Theorem [Kos70], this defines a one to one correspondence between prequantizations of $(\mathcal{O}, \omega_{\mathcal{O}}, \hat{G}, j_{\mathcal{O}})$ and the characters of $\hat{G}_\lambda$ whose derivative is (11).

Given a prequantization of the coadjoint orbit and the same data as in Section 2.2.3, we obtain a prequantum bundle over $M \times \mathcal{O}^-$ with an action of G. To describe this it is more convenient to use complex line bundles instead of $\mathbb{T}$-principal bundle: the prequantum bundle of $\mathcal{O}^-$ is the dual of the prequantum bundle $L_{\mathcal{O}}$ of $\mathcal{O}$, the prequantum bundle of $M \times \mathcal{O}^-$ is the external tensor product $L \boxtimes L_{\mathcal{O}}^{-1}$. Since the momentum of the diagonal action takes its values in $p^{-1}(0)$, the action on $L \boxtimes L_{\mathcal{O}}^{-1}$ factors through an action of G. So we are in the usual situation without central extension. Hence, under the assumption that G acts freely and properly on $\nu^{-1}(0)$, the restriction of $L \boxtimes L_{\mathcal{O}}^{-1}$ to $\nu^{-1}(0)$ descends to a prequantum bundle over the symplectic quotient $(M \times \mathcal{O}^-)/\!\!/G$.

Alternatively, we obtain this bundle by restricting L to $\mu^{-1}(\lambda)$, tensoring it by $L_{\mathcal{O}}^{-1}|_\lambda$, and dividing out by G_λ. Observe that $L|_{\mu^{-1}(\lambda)} \otimes L_{\mathcal{O}}^{-1}|_\lambda$ is isomorphic to $L|_{\mu^{-1}(\lambda)}$ as a bundle with connection, but not as a G_λ-bundle. Actually the action of $\hat{G}_\lambda$ on $L_{\mathcal{O}}|_\lambda$ is given by a character whose derivative is the restriction of λ, so it doesn't even descend to an action of G_λ. Nevertheless if G_λ is connected, the action on $L|_{\mu^{-1}(\lambda)} \otimes L_{\mathcal{O}}^{-1}|_\lambda$ is uniquely determined by the connection since its momentum vanishes. In this way we can recover the prequantum bundle on the quotient without knowing $L_{\mathcal{O}}$, but only its existence.

We shall also consider powers of prequantum bundles. Let k be any nonvanishing integer. Assume that $(\mathcal{O}, k\omega_{\mathcal{O}})$ is endowed with a $\hat{G}$-equivariant prequantum bundle with associated momentum $kj_{\mathcal{O}}$. Then we may apply the same construction by replacing L with L^k and we obtain a prequantum bundle over the symplectic quotient $(M \times \mathcal{O}^-)/\!\!/G$ whose curvature is k times the symplectic form.

2.3 Affine symplectomorphisms

Let (E, ω) be a symplectic vector space and let G be the group $E \rtimes \mathrm{Symp}(E)$ of affine symplectomorphisms. Assume that P is the trivial $\mathbb{T}$-principal bundle over E with connection $-\alpha + dt$ where $\alpha_x(\dot{x}) = \frac{1}{2}\omega(x, \dot{x})$.

By Theorem 2.2, G has a central extension $\hat{G}$ acting on P. It is explicitely given by $\hat{G} = G \times \mathbb{T}$, the action being

$$(u, A, s).(x, t) = \big(u + Ax, t + s + \tfrac{1}{2}\omega(u, Ax)\big), \qquad (u, A, s) \in \hat{G},\ (x, t) \in P$$

The product of $\hat{G}$ is $(g, t).(h, s) = (gh, t + s + C(g, h))$ where the cocycle is

$$C((u, A), (v, B)) = \tfrac{1}{2}\omega(u, Av).$$

The adjoint action is $\mathrm{Ad}_{(g,t)}(X, s) = (\mathrm{Ad}_g X, s + D_g(X))$ with D_g given by

$$D_{(u,A)}(\dot{v}, \dot{B}) = \omega(u, A\dot{v}) - \tfrac{1}{2}\omega(u, (\mathrm{Ad}_A \dot{B})u)$$

The Lie bracket is $[(X, t), (Y, s)] = ([X, Y], c(X, Y))$ with the cocycle

$$c((\dot{u}, \dot{A}), (\dot{v}, \dot{B})) = \omega(\dot{u}, \dot{v}).$$

Finally the momentum of $(\dot{u}, \dot{A}, \dot{t})$ is $\mu^{(\dot{u}, \dot{A})} + \dot{t}$ where

$$\mu^{(\dot{u}, \dot{A})}(x) = \omega(\dot{u}, x) + \tfrac{1}{2}\omega(\dot{A}x, x).$$

μ is a momentum of the G-action which is not equivariant. Indeed

$$\{\mu^X, \mu^Y\} = \mu^{[X,Y]} + \omega(\dot{u}, \dot{v})$$

where X and Y are respectively $(\dot{u}, \dot{A})$ and $(\dot{v}, \dot{B})$.

3 Chern-Simons bundle

In the following we apply the construction of the previous section to a space of a connections acted on by a gauge group. We will treat these infinite dimensional manifold in a formal way. Nevertheless this can be made rigorous by introducing the appropriate Sobolev completions as in [AB83], [Don92], [MW98] and [Woo06].

The construction of the Chern-Simons bundle we present is mainly inspired from [MW98]. We start with the symplectic vector space of connections and the gauge group action. We do a first symplectic quotient and obtain a Hamiltonian space with a prequantum bundle acted on by a central extension of the gauge group of the boundary. After a second quotient we get our moduli space and the Chern-Simons bundle.

Lie group notations

Let $\mathfrak{g}$ be a compact simple Lie algebra, and G the corresponding compact connected and simply-connected Lie group. Choose a maximal torus T of G and denote by $\mathfrak{t}$ its Lie algebra. The integral lattice Λ of $\mathfrak{t}$ is defined as the kernel of the restriction of the exponential map to $\mathfrak{t}$. Since G is simply-connected, the coroot lattice and $(2i\pi)^{-1}\Lambda$ are equal. The basic inner product $\langle \cdot, \cdot \rangle$ is the unique invariant inner product on $\mathfrak{g}$ such that for each long root α, its coroot satisfies $\langle 2i\pi\alpha^\vee, 2i\pi\alpha^\vee \rangle = 2$. Let $P^\vee$ be the dual lattice of the root lattice. Recall that $\Lambda \subset 2i\pi P^\vee$ and that the exponential map induces an isomorphism from $2i\pi P^\vee/\Lambda$ to the center of G.

We denote by $\theta, \bar{\theta} \in \Omega^1(G, \mathfrak{g})$ the left and right Maurer-Cartan forms. Define the Cartan three-form

$$\chi = \frac{1}{12}\langle [\theta, \theta], \theta \rangle = \frac{1}{12}\langle\langle [\bar{\theta}, \bar{\theta}], \bar{\theta} \rangle$$

The cohomology class of χ is integral and is a generator of $H^3(G, \mathbb{Z})$.

3.1 The space of connections

Let Σ be an oriented compact surface with boundary possibly empty. Consider the vector space $\Omega^1(\Sigma, \mathfrak{g})$ endowed with the symplectic product

$$\omega(a, b) = \int_\Sigma \langle a, b \rangle \tag{4}$$

Viewing the elements of $\Omega^1(\Sigma, \mathfrak{g})$ as connections on the trivial G-principal bundle with base Σ, the gauge group $G(\Sigma) = \mathcal{C}^\infty(\Sigma, G)$ acts on $\Omega^1(\Sigma, \mathfrak{g})$.

The action is explicitly given by

$$a^g = \operatorname{Ad}_g .a - g^*\bar{\theta}, \qquad a \in \Omega^1(\Sigma, G),\ g \in G(\Sigma)$$

Observe that each g acts as an affine symplectomorphism. So we can apply the construction of section 2.3. Let P be the trivial principal bundle over $\Omega^1(\Sigma, \mathfrak{g})$ with structure group $\mathbb{T}$ and connection form $-C + dt$ where

$$C_a(\dot{a}) = \frac{1}{2}\omega(a, \dot{a}). \tag{5}$$

The group of automorphisms of P preserving the connection and lifting the action of the gauge group is $G(\Sigma) \times \mathbb{T}$ with product

$$(g,t).(h,s) = \Big(gh, t + s + \frac{1}{2}\int_\Sigma \langle g^*\theta, h^*\overline{\theta}\rangle\Big). \tag{6}$$

The action on P is

$$(g,t).(a,s) = \Big(a^g, s + t - \frac{1}{2}\int_\Sigma \langle g^*\theta, a\rangle\Big) \tag{7}$$

Let $\mathfrak{g}(\Sigma) = \mathcal{C}^\infty(\Sigma, \mathfrak{g})$ be the Lie algebra of the gauge group. The adjoint action and the Lie bracket of $G(\Sigma) \times \mathbb{T}$ are given by

$$\operatorname{Ad}_{(g,t)}(\xi, s) = \Big(\operatorname{Ad}_g \xi, s - \int_{\partial\Sigma} \langle \xi, g^*\theta\rangle\Big) \tag{8}$$

$$\big[(\xi,t),(\eta,s)\big] = \Big([\xi,\eta], \int_{\partial\Sigma} \langle \xi, d\eta\rangle\Big) \tag{9}$$

And the momentum of $(\xi, 0) \in \mathfrak{g}(\Sigma) \oplus \mathbb{R}$ is

$$\mu^\xi(a) = \int_\Sigma \langle F(a), \xi\rangle - \int_{\partial\Sigma} \langle a, \xi\rangle \tag{10}$$

where $F(a) = da + \frac{1}{2}[a,a]$ is the curvature of a. The moment is not equivariant because of the boundary term. The previous formulas (6), (7), (8), (9) and (10) follow from the corresponding ones in section 2.3.

Let $G_\partial(\Sigma)$ be the normal subgroup of $G(\Sigma)$ consisting of the gauge transforms equal to the identity on the boundary of Σ. Its Lie algebra $\mathfrak{g}_\partial(\Sigma)$ consists of the $\xi \in \mathfrak{g}(\Sigma)$ vanishing along $\partial\Sigma$. It is a subalgebra of the Lie algebra $\mathfrak{g}(\Sigma) \oplus \mathbb{R}$ with the bracket given by (9). Passing from Lie algebra to Lie group, we obtain the following.

Proposition 3.1. *There exists a unique group morphism $s : G_\partial(\Sigma) \to G(\Sigma) \times \mathbb{T}$ with differential the injection from $\mathfrak{g}_\partial(\Sigma)$ into $\mathfrak{g}(\Sigma) \oplus \mathbb{R}$. For any $g \in G_\partial(\Sigma)$, we have $s(g) = (g, t(g))$ with*

$$t(g) := -\int_{\Sigma\times[0,1]} \tilde{g}^* \chi$$

and $\tilde{g}$ a map from $\Sigma \times [0,1]$ to G restricting to g on $\Sigma \times \{1\}$ and to the identity on $\Sigma \times \{0\}$.

Proof. Since $\pi_0(G) = \pi_1(G) = \pi_2(G) = 0$, $G_\partial(\Sigma)$ is connected, which proves the uniqueness of s. This also shows the existence of the extension $\tilde{g}$ for any $g \in G_\partial(\Sigma)$. $t(g)$ does not depend on the choice of $\tilde{g}$ because the cohomology class of χ is integral. The fact that s is a group morphism can be directly proved using Equation (6). Since the derivative of t vanishes at the identity, the differential of s is the injection from $\mathfrak{g}_\partial(\Sigma)$ into $\mathfrak{g}(\Sigma) \oplus \mathbb{R}$. □

Assume temporarily that Σ has no boundary and consider the quotient

$$\mathcal{M}^s(\Sigma, G) := \{a \in \Omega^1(\Sigma, \mathfrak{g}) / F(a) = 0 \text{ and } a \text{ irreducible}\}/G(\Sigma)$$

Here a is irreducible means that the isotropy group of a consists of the gauge transform with value in the center of G. $\mathcal{M}^s(\Sigma, G)$ is a smooth finite dimensional manifold. A proof is provided in [MW98], page 426. By the previous consideration, $\mathcal{M}^s(\Sigma, G)$ is the smooth part of the symplectic quotient $\Omega^1(\Sigma, \mathfrak{g}) /\!\!/ G(\Sigma)$. It inherits by reduction a prequantum bundle, as explained in Section 2.1.2.

3.2 Moduli space of flat bundles

Assume from now on that the boundary of Σ is not empty. The quotient

$$\mathcal{M}_\partial = \{a \in \Omega^1(\Sigma, \mathfrak{g}) / F(a) = 0\}/G_\partial(\Sigma)$$

is a smooth infinite dimensional Banach manifold, cf. [Woo06], page 317-318. It is the symplectic quotient $\Omega^1(\Sigma, \mathfrak{g}) /\!\!/ G_\partial(\Sigma)$ and has a residual action of $G(\partial\Sigma) \simeq G(\Sigma)/G_\partial(\Sigma)$. Applying again the considerations of Section 2.1.2, $\mathcal{M}_\partial$ inherits a prequantum bundle P_∂. Furthermore the group

$$\widehat{G(\partial\Sigma)} := (G(\Sigma) \times \mathbb{T})/s(G_\partial(\Sigma))$$

acts on P_∂ by prequantum bundle automorphisms lifting the action of $G(\partial\Sigma)$ on $\mathcal{M}_\partial$. Observe that we are exactly in the situation considered in Section 2.2.1 with the central extension:

$$0 \to \mathbb{T} \to \widehat{G(\partial\Sigma)} \to G(\partial\Sigma) \to 0.$$

The Lie algebra of $\widehat{G(\partial\Sigma)}$ identifies with $\mathfrak{g}(\partial\Sigma) \oplus \mathbb{R}$ in such a way that the adjoint action and the Lie bracket are still given by formulas (8) and (9). Let us identify the dual of $\mathfrak{g}(\partial\Sigma)$ with $\Omega^1(\partial\Sigma, \mathfrak{g})$ through the pairing

$$(\xi, a) \to \int_{\partial\Sigma} \langle \xi, a \rangle.$$

Then the coadjoint action factors through the following action of $G(\partial\Sigma)$

$$g.(b, \lambda) = (\operatorname{Ad}_g b + \lambda g^* \bar{\theta}, \lambda)$$

Furthermore the momentum is $\mathcal{M}_\partial \ni [a] \to (-a|_{\partial\Sigma}, 1) \in \Omega^1(\partial\Sigma, \mathfrak{g}) \oplus \mathbb{R}$.

The map from $\Omega^1(\partial\Sigma, \mathfrak{g})$ to $\Omega^1(\partial\Sigma, \mathfrak{g}) \oplus \mathbb{R}$ sending a into $(-a, 1)$ intertwins the gauge group action on the space of connection of $\partial\Sigma$ with the coadjoint action. For any connection a of $\partial\Sigma$, denote by $\operatorname{hol}_{\partial_i}(a)$ the holonomy of the i-th boundary component of Σ. Then the map

$$\Omega^1(\partial\Sigma, \mathfrak{g}) \oplus \mathbb{R} \to G^r, \qquad (a, 1) \to (\operatorname{hol}_{\partial_i}(-a))$$

induces a one-to-one correspondance betwwen the coadjoint orbits of $\widehat{G(\partial\Sigma)}$ at level 1 and the conjugacy classes of G^r. Let $\mathcal{C}_1, \dots, \mathcal{C}_r$ be conjugacy classes of G and denote by $\mathcal{O}$ the corresponding coadjoint orbit. Doing the symplectic reduction at level $\mathcal{O}$ as explained in Section 2.2.3, we obtain a smooth symplectic finite dimensional manifold $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$. It has several presentations

$$\begin{aligned}
\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i)) &= \big\{ [a] \in \mathcal{M}_\partial / a \text{ irreducible}, \ (-a|_{\partial\Sigma}, 1) \in \mathcal{O} \big\} / G(\partial\Sigma) \\
&\simeq \big\{ a \in \Omega^1(\Sigma, \mathfrak{g}) / a \text{ irreducible}, \ F(a) = 0, \ \operatorname{hol}_{\partial_i}(a) \in \mathcal{C}_i, \ \forall i \big\} / G(\Sigma) \\
&\simeq \big\{ a \in \Omega^1(\Sigma, \mathfrak{g}) / a \text{ irreducible}, \ F(a) = 0, \ a|_{(\partial\Sigma)} = b \big\} / G(\Sigma)_b
\end{aligned}$$

Here b is any form in $\Omega^1(\partial\Sigma, \mathfrak{g})$ whose holonomy along the i-th boundary component of Σ is in $\mathcal{C}_i$. $G(\Sigma)_b$ is the subgroup of $G(\Sigma)$ consisting of the elements whose restriction to the boundary fix b.

3.3 Integral coadjoint orbits

The next question is to know when the orbit $\mathcal{O}$ of $\widehat{G(\partial\Sigma)}$ corresponding to the conjugacy classes $\mathcal{C}_i$ has a prequantization in the sense of Section 2.2.4.

Theorem 3.2. *Let k be a non-vanishing integer. Then the coadjoint orbit $(\mathcal{O}, k\omega_{\mathcal{O}})$ is prequantizable if and only if there exists $\lambda_1, \dots, \lambda_r \in k^{-1}\Lambda^*$ such that $\mathcal{C}_i = G.\exp(\lambda_i)$ for each i. When it exists, the prequantization is unique up to isomorphism.*

Because of the Lie bracket (9), we recognize that $\widehat{G(\partial\Sigma)}$ is isomorphic to the universal central extension of the loop group of G^r. The prequantizable coadjoint orbits of this group are well-known [PS86].

Proof. We prove the condition is necessary. Choose a component Γ of $\partial\Sigma$. Consider the map from $\mathfrak{g}$ to $\mathfrak{g}(\partial\Sigma) \oplus \mathbb{R}$ sending ξ into the element equal to ξ on Γ and vanishing on the other components. By Equation (9), this map is a Lie algebra morphism. G being simply connected, we can integrate it and we obtain a group morphism Φ from G to $\widehat{G(\partial\Sigma)}$. Denote by p the projection from $\widehat{G(\partial\Sigma)}$ onto $G(\partial\Sigma)$. For any $\xi \in \mathfrak{g}$, $p(\Phi(\exp(\xi)))$ is equal to $\exp(\xi)$ on Γ and to the identity on the other components.

Assume now that $(\mathcal{O}, k\omega_{\mathcal{O}})$ is endowed with a $\widehat{G(\partial\Sigma)}$-prequantum bundle P with momentum $kj_{\mathcal{O}}$, where $j_{\mathcal{O}}$ is the injection from $\mathcal{O}$ into the Lie algebra. Recall from Section 2.2.4 that for any $(a, 1) \in \mathcal{O}$, the isotropy group of $(a, 1)$ acts on the fiber $P_{(a,1)}$ through a morphism $\widehat{G(\partial\Sigma)}_{(a,1)} \to \mathbb{T}$ whose derivative is $k\chi_a$ where

$$\chi_a(\xi, t) = \int_{\partial\Sigma} \langle \xi, a\rangle + t. \tag{11}$$

We may choose a so that $a = \lambda d\theta$ on Γ with $\lambda \in \mathfrak{t}$ and θ a coordinate. Then for any $\xi \in \mathfrak{t}$, $p(\Phi(\exp(\xi)))$ fixes a so that $\Phi(\exp(\xi))$ fixes $(a, 1)$. By (11), $\Phi(\exp(\xi))$ acts on $P_{(a,1)}$ by mutiplication by $k\langle\xi, \lambda\rangle$. When $\xi \in \Lambda$, $\Phi(\exp \xi) = \mathrm{id}$ and consequently $k\langle\xi, \lambda\rangle \in \mathbb{Z}$.

To show that the condition is sufficient recquires more work. We have to prove the existence of a character of $\widehat{G(\partial\Sigma)}_{(a,1)}$ with derivative $k\chi_a$. A proof is given in [BL99], section 4.2. The uniqueness of the prequantization follows from the fact that $\widehat{G(\partial\Sigma)}_{(a,1)}$ is connected. □

3.4 Chern-Simons bundle

Assume that $(\mathcal{O}, k\omega_{\mathcal{O}})$ is prequantizable. Following section 2.2.4, we obtain a prequantization of the $G(\partial\Sigma)$-Hamiltonian space $\mathcal{M}_\partial \times \mathcal{O}^-$. We can not directly perform the reduction because the center of G which acts trivially on $\mathcal{M}_\partial \times \mathcal{O}^-$ may act non trivially on the prequantum bundle. Recall that the restriction of the exponential map to $2i\pi P^\vee$ induces an isomorphism from $2i\pi P^\vee/\Lambda$ onto the center of G.

Lemma 3.3. *For any $\xi \in 2i\pi P^\vee$, $\exp(\xi)$ acts on the prequantum bundle of $\mathcal{M}_\partial \times \mathcal{O}^-$ by multiplication by $-k\sum_i\langle\xi, \lambda_i\rangle \in \mathbb{T}$.*

Proof. By formula (7), $\exp(\xi)$ being in the center of G, for any $t \in \mathbb{R}$, $\exp(\xi, t)$ acts on the k-th power of the prequantum bundle of $\mathcal{M}_\partial$ by multiplication by kt. On the other hand, since $\mathcal{O}$ is connected and $\exp(\xi)$ acts trivially on it, $\exp(\xi, t) \in \widehat{G(\partial\Sigma)}$ acts on the prequantum bundle of $\mathcal{O}$ by multiplication by a constant in $\mathbb{T}$. To compute this constant, introduce $a \in \Omega^1(\partial\Sigma, \mathfrak{g})$ such that $a = \lambda_i d\theta_i$ on the ith boundary component with θ_i a coordinate. So $(a, 1) \in \mathcal{O}$. By equation (11), $\exp(\xi, t)$ acts on the fiber at $(a, 1)$ by multiplication by $k(\sum_i \langle \xi, \lambda_i \rangle + t) \in \mathbb{T}$. ☐

We are now ready to apply the reduction described in Section 2.2.4. We obtain the following theorem.

Theorem 3.4. *Let $k \in \mathbb{N}^*$ and $\lambda_1, \ldots, \lambda_r \in \mathfrak{t}$. Assume that*

$$k\lambda_i \in \Lambda^*, \forall i = 1, \ldots, r \qquad \text{and} \qquad k \sum_i \langle \xi, \lambda_i \rangle \in \mathbb{Z}, \ \forall \xi \in 2i\pi P^\vee \tag{12}$$

Then the moduli space $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$, where $\mathcal{C}_i = G.\exp(\lambda_i)$ for $i = 1, \ldots, r$, inherits by reduction a prequantum bundle L^k_{CS} whose curvature is k times the symplectic form.

We call the bundle L^k_{CS} the Chern-Simons bundle at level k. As it is explained in section 2.2.4, the bundle has several presentations. In particular we will use later the following one. Choose $b \in \Omega^1(\partial\Sigma, \mathfrak{g})$ whose holonomy for of the i-th component is in $\mathcal{C}_i$. Recall that $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$ is the quotient of

$$\mathcal{A}^s_b := \big\{ a \in \Omega^1(\Sigma, \mathfrak{g}) / \ a \text{ irreducible}, F(a) = 0, a|_{\partial\Sigma} = b \big\}, \tag{13}$$

by $G(\Sigma)_b$. Consider as in section 3.1 the trivial prequantum bundle over $\Omega^1(\Sigma, \mathfrak{g})$ with connection $-C + dt$. Restrict this bundle to $\mathcal{A}^s_b$ and lift the action of $G(\Sigma)_b$ by parallel transport, we know this is possible by the previous considerations. Then dividing out by this action, we get the desired Chern-Simons bundle.

Remark. When G is simply-laced, $2i\pi P^\vee = \Lambda^*$. So in this case, the second assumption in (12) is equivalent to $k(\lambda_1 + \ldots + \lambda_r) \in \Lambda$.

Remark. For $r = 1$, the second assumption in (12) implies the first one.

Remark. Assume that $G = SU(2)$. Let us parametrize the conjugacy classes by $[0, 1]$, where $t \in [0, 1]$ corresponds to $\exp(t\rho).G$ with $\rho = \mathrm{diag}(i\pi, -i\pi)$. Then $\mathcal{M}^s(G, \Sigma, (\mathcal{C}_i))$ is closed, that is $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i)) = \mathcal{M}(\Sigma, G, (\mathcal{C}_i))$, if and only if

$$t_1 \pm \ldots \pm t_r \notin \mathbb{Z}$$

for any choices of signs. The moduli space inherits by reduction a prequantum bundle at level k if each t_i belongs to $k^{-1}\mathbb{Z}$ and $k\sum t_i$ is even.

Remark. Assume that $G = SU(n)$ and $r = 1$. Then the Chern-Simons bundle is defined at level k iff the eigenvalues of each element of the conjugacy class $\mathcal{C}$ are roots of unity of order k.

4 Modular group action

We start with the action of the preserving orientation diffeomorphisms of Σ on the space of connections and its prequantum bundle. Using the explicit formulas given in section 3.1, it is easy to see that we obtain an action on $\mathcal{M}_\partial$ and its prequantum bundle by prequantum bundle automorphisms. We then proved that this induces an action of the modular group of Σ. Finally we explain why the action of the modular group on the Chern-Simons bundle does not factor through an action of the pure modular group and why the action of the pure modular group on the moduli space does not factor through an action of the modular group of the filled surface.

4.1 Diffeomorphism group and $\mathcal{M}_\partial$

Introduce the group $\mathrm{Diff}^+(\Sigma)$ of diffeomorphisms of Σ preserving the orientation.

Proposition 4.1. $\mathrm{Diff}^+(\Sigma)$ *acts on the prequantum bundle* P_∂ *of* $\mathcal{M}_\partial$ *by prequantum bundle automorphisms. It also acts on* $\widehat{G(\partial\Sigma)}$ *by group automorphisms. These actions are compatible.*

Proof. $\mathrm{Diff}^+(\Sigma)$ acts by pull-back on $\Omega^1(\Sigma, \mathfrak{g})$. Because the orientation is preserved, each diffeomorphism preserves the symplectic form (4) and the trivial lift to the prequantum bundle $\Omega^1(\Sigma, \mathfrak{g}) \times \mathbb{T}$ preserves the connection one-form (5).

The group $\mathrm{Diff}^+(\Sigma)$ acts also by pull-back on the gauge group $G(\Sigma)$. Lifting trivially this action to the extension $G(\Sigma) \times \mathbb{T}$, we obtain an action by group automorphisms. This follows easily from Equation (6) giving the product in $G(\Sigma) \times \mathbb{T}$. Furthermore this action is compatible with the action on the prequantum bundle as follows from Equation (7).

Each diffeomorphism of Σ preserves the boundary, so its action on $G(\Sigma)$ preserves the subgroup $G_\partial(\Sigma)$. Furthermore the morphism $s : G_\partial(\Sigma) \to G(\Sigma) \times \mathbb{T}$ defined in Proposition 3.1 is $\mathrm{Diff}^+(\Sigma)$ equivariant. Indeed the Wess-Zumino-Witten term satisfies $t(\varphi^* g) = t(g)$ for any diffeomorphism φ.

So we get an action of $\mathrm{Diff}^+(\Sigma)$ on the central extension $\widehat{G(\Sigma)}$ by group automorphisms.

Then as a consequence of the general principles of Section 2.1.4, we obtain an action of $\mathrm{Diff}^+(\Sigma)$ on the prequantum bundle P_∂ of $\mathcal{M}_\partial$ by prequantum bundle automorphisms, and an action on the central extension $\widehat{G(\partial\Sigma)}$ by group automorphisms. These actions are compatible. □

4.2 Modular group and $\mathcal{M}_\partial$

Consider the group $\mathrm{Diff}^+(\Sigma, \partial)$ of diffeomorphism preserving the orientation and fixing the boundary points by points. Let $\mathrm{Diff}^+_0(\Sigma, \partial)$ be its subgroup consisting of diffeomorphism isotopic to the identity. Define the modular group of Σ by

$$\mathrm{Mod}(\Sigma) = \mathrm{Diff}^+(\Sigma, \partial) / \mathrm{Diff}_0(\Sigma, \partial).$$

Proposition 4.2. *Consider the actions of* $\mathrm{Diff}^+(\Sigma)$ *introduced in Proposition 4.1. Then the action of the subgroup* $\mathrm{Diff}^+(\Sigma, \partial)$ *on the prequantum bundle* P_∂ *factors through an action of* $\mathrm{Mod}(\Sigma)$*. The action of* $\mathrm{Diff}^+(\Sigma, \partial)$ *on* $\widehat{G(\partial\Sigma)}$ *is trivial.*

The proof starts by the following Lemma.

Lemma 4.3. *For all* $a \in \Omega^1(\Sigma, \mathfrak{g})$ *and isotopy* $(\varphi_t,\ t \in [0, 1])$ *in* $\mathrm{Diff}^+(\Sigma)$*, we have*

$$\varphi_t^* a = a^{g_t}$$

where g_t *is the gauge transform such that* $g_t^{-1}(x)$ *is the holonomy of* α *along the path* $s \in [0, 1] \to \varphi_{st}(x) \in \Sigma$.

If (φ_t) is a isotopy in $\mathrm{Diff}^+(\Sigma, \partial)$, the gauge transform g_1 is trivial on the boundary. So the action of $\mathrm{Diff}_0(\Sigma, \partial)$ on $\mathcal{M}_\partial$ is trivial.

Proof. It suffices to show that all paths γ of Σ have the same holonomy for $\varphi_t^* a$ and a^{g_t}. On one hand, we have

$$\mathrm{hol}_\gamma(\varphi_t^* a) = \mathrm{hol}_{\varphi_t \circ \gamma}(a).$$

On the other hand, if x and y are the endpoints of γ,

$$\begin{aligned} \mathrm{hol}_\gamma(a^{g_t}) =& g_t(x) \,\mathrm{hol}_\gamma(a) g_t(y)^{-1} \\ =& \,\mathrm{hol}_{\gamma_1}(\alpha).\, \mathrm{hol}_\gamma(a).\, \mathrm{hol}_{\gamma_2}(a) \\ =& \,\mathrm{hol}_{\varphi_t \circ \gamma}(a) \end{aligned}$$

where $\gamma_1(s) = \varphi_{t(1-s)}(x)$ and $\gamma_2(s) = \varphi_{st}(x)$. At the last line, we used that $\varphi_t \circ \gamma$ and the concatenation of γ_1, γ and γ_2 are isotopic with fixed endpoints. □

Lemma 4.4. *For any $a \in \Omega^1(\Sigma, \mathfrak{g})$ with vanishing curvature and any isotopy $(\varphi_t,\ t \in [0,1])$ in $\mathrm{Diff}^+(\Sigma)$, the parallel transport along the path $t \in [0,1] \to a_t := \varphi_t^* a$ in the prequantum bundle of $\Omega^1(\Sigma, \mathfrak{g})$ sends (a, s) to $(\varphi_1^* a, s')$ with*

$$s' = s + \frac{1}{2} \int_0^1 ds \int_{\partial\Sigma} \langle a_s, \iota_{X_s} a_s \rangle$$

where $X_s(\varphi_s(x)) = \frac{d}{ds}(\varphi_s(x))$.

If (φ_t) is an isotopy in $\mathrm{Diff}^+(\Sigma, \partial)$, the infinitesimal generator X_t vanishes on the boundary, so the lift by parallel transport is trivial. Hence the action of $\mathrm{Diff}_0(\Sigma, \partial)$ on the prequantum bundle of $\mathcal{M}_\partial$ is trivial. This proves the first part of Proposition 4.2.

To prove that $\mathrm{Diff}^+(\Sigma, \partial)$ acts trivially on $\widehat{G(\partial\Sigma)}$, consider first the action on the Lie algebra of the central extension and then use that the central extension is connected.

Proof of lemma 4.4. Recall that the connection of the prequantum bundle of $\Omega^1(\Sigma, \mathfrak{g})$ is given in equation (5). So

$$s' = s - \int_0^1 ds \ \frac{1}{2} \int_\Sigma \langle a_s, \dot{a}_s \rangle$$

One has $\dot{a}_t = \mathcal{L}_{X_t} a_t$. Hence to conclude, it is sufficient to show that

$$\langle b, \mathcal{L}_Y b \rangle = -d \langle b, \iota_Y b \rangle,$$

for any flat connection $b \in \Omega^1(\Sigma, \mathfrak{g})$ and vector field Y of Σ. Since we have

$$\langle b, \mathcal{L}_Y b \rangle = \langle b, \iota_Y db + d\iota_Y b \rangle = \langle b, \iota_Y db \rangle + \langle db, \iota_Y b \rangle - d \langle b, \iota_Y b \rangle.$$

it is sufficient to prove that $\langle b, \iota_Y db \rangle = \langle db, \iota_Y b \rangle = 0$. Since Σ is a surface, $\langle b, db \rangle = 0$, so

$$\langle \iota_Y b, db \rangle = \langle b, \iota_Y db \rangle \tag{14}$$

We have

$$\langle b, \iota_Y [b, b] \rangle = \langle b, 2[\iota_Y b, b] \rangle = -2 \langle [b, b], \iota_Y b \rangle$$

Since b is flat, this gives

$$\langle b, \iota_Y db \rangle = -2 \langle db, \iota_Y b \rangle \tag{15}$$

Comparing equations (14) and (15), we obtain $\langle \iota_Y b, db \rangle = \langle b, \iota_Y db \rangle = 0$. □

4.3 Modular group and Chern-Simons bundle

Consider conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$ of G and let

$$\mathcal{O} := \{(a,1) \in \Omega^1(\partial\Sigma, \mathfrak{g}) \times \mathbb{R} / \ \mathrm{hol}(a|_{(\partial\Sigma)_i}) \in \mathcal{C}_i, \ i = 1, \ldots, r\}$$

be the corresponding coadjoint orbit of $\widehat{G(\partial\Sigma)}$. Recall that the moduli space is the symplectic reduction of the Hamiltonian space $(\mathcal{M}_\partial \times \mathcal{O}^-, G(\partial\Sigma))$. Consider the trivial action of the modular group $\mathrm{Mod}(\Sigma)$ on the orbit. It follows from the previous section that the modular group acts by automorphisms of Hamiltonian space on $(\mathcal{M}_\partial \times \mathcal{O}^-, G(\partial\Sigma))$. So applying the general principle of section 2.1.4, we get an action by symplectomorphisms of the mapping class group on $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$.

Assume that the conjugacy classes satisfy (12), so that the orbit has a prequantum bundle and we can define a Chern-Simons bundle by symplectic reduction. Applying again the general principle of section 2.1.4, we obtain an action of the modular group on the Chern-Simons bundle. So we have proved the following proposition.

Proposition 4.5. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be conjugacy classes of G satisfying (12). Then the action of $\mathrm{Mod}(\Sigma)$ on P_∂ given in Proposition 4.2 descends to an action on the Chern-Simons bundle $L^k_{\mathrm{CS}} \to \mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$.*

4.4 Pure modular group action

Recall that the pure mapping class group $\mathrm{PMod}(\Sigma)$ is the group of preserving orientation diffeomorphisms of Σ that fix each boundary component as a set, modulo isotopies in $\mathrm{Diff}^+(\Sigma)$. One has a short exact sequence

$$1 \to \mathbb{Z}^r \to \mathrm{Mod}(\Sigma) \to \mathrm{PMod}(\Sigma) \to 1$$

where the image of the obvious generators of $\mathbb{Z}^r$ are Dehn twists on the boundary components.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be conjugacy classes of G. The action of $\mathrm{Mod}(\Sigma)$ on the space $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$ factors through an action of the pure mapping class group. This follows from lemma 4.3. Alternatively, it is a well known fact that the pull-backs of the same bundle by homotopic maps are isomorphic. So if we think of $\mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$ as a moduli space of principal bundles with a discrete structure group, we obtain another proof that the mapping class group action descend to the pure mapping class group.

Let us make now the integrality assumption (12) and consider the action of the modular group on the Chern-Simons bundle $L^k_{\mathrm{CS}} \to \mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$.

Let us parametrize a neighborhood of the j-th boundary component by $]0,1] \times S^1 \ni (\rho,\theta)$. Let $f \in \mathcal{C}^\infty(]0,1])$ be a non-decreasing function equal to 0 (resp. 1) on a neighborhood of 0 (resp. 1). Consider the twist T of Σ given by $T(\rho,\theta) = (\rho, \theta + f(\rho))$ on $]0,1] \times S^1$ and equal to the identity outside $]0,1] \times S^1$.

Proposition 4.6. *The twist T acts on the Chern-Simons bundle $L^k_{\mathrm{CS}} \to \mathcal{M}^s(\Sigma, G, (\mathcal{C}_i))$ by multiplication by $\frac{k}{2}|\lambda_j|^2 \in \mathbb{T}$.*

Proof. We will use the presentation of the Chern-Simons bundle given after Equation (13). Let $b \in \Omega^1(\partial\Sigma, \mathfrak{g})$ be equal to $\lambda_j d\theta$ on the j-th boundary component. Let us compute the action of T on the fiber of $[a]$, where $a \in \mathcal{A}^s_b$, cf. definition (13).

Consider the family Φ_t, $t \in [0,1]$ of diffeomorphisms of Σ defined on $]0,1] \times S^1$ by $\Phi_t(\rho,\theta) = (\rho, \theta + f(t\rho))$ and equal to the identity everywhere else. Observe that $\Phi_t^* a$ belongs to $\mathcal{A}^s_b$ for all t. Furthermore $\Phi_t^* a = a^{g_t}$ with g_t a curve in $G(\Sigma)_b$ by lemma 4.3. The parallel transport along the curve $(\Phi_t^* a)$ in the prequantum bundle of $\Omega^1(\Sigma, \mathfrak{g})$ is the translation by $-\frac{1}{2}|\lambda_j|^2$ by lemma 4.4. So we have the following equality in the Chern-Simons bundle of $\mathcal{M}(\Sigma, G, (\mathcal{C}_i))$

$$[a, s] = \left[T^* a, -\frac{k}{2}\ |\lambda_j|^2 + s\right]$$

By definition, the Dehn twist T sends $[a,s]$ to $[T^*a, s]$. □

4.5 Birman exact sequence and central conjugacy classes

Let $\bar{\Sigma}$ be the closed surface obtained by filling the holes of Σ. There is a natural epimorphism $\mathrm{PMod}(\Sigma) \to \mathrm{Mod}(\bar{\Sigma})$. One can ask whether the action of $\mathrm{PMod}(\Sigma)$ factors through an action of the mapping class group of $\bar{\Sigma}$. We will prove it is not true even in a very simple case.

Assume that the boundary of Σ is connected. Let c be an element of the center $Z(G)$ of G. The moduli space $\mathcal{M}(\Sigma, G, \{c\})$ has the following presentation

$$\mathcal{M}(\Sigma, G, \{c\}) \simeq \mathrm{Mor}(\pi_1(\Sigma), G, c)/G$$

where $\mathrm{Mor}(\pi_1(\Sigma), G, c)$ consists of the morphisms from the fundamental group of Σ to G sending any loop homotopic to the boundary to c. The group

$$H^1(\bar{\Sigma}, Z(G)) = \mathrm{Mor}(\pi_1(\bar{\Sigma}), Z(G))$$

acts on $\mathrm{Mor}(\pi_1(\bar{\Sigma}), G, c)$ by pointwise multiplication: $(\varphi.h)(\gamma) = \varphi(\gamma)h(\gamma)$. This induces an action of $H^1(\bar{\Sigma}, Z(G))$ on $\mathcal{M}(\bar{\Sigma}, G, \{c\})$.

Recall the Birman exact sequence:

$$0 \to \pi_1(\bar{\Sigma}) \to \mathrm{PMod}(\Sigma) \to \mathrm{Mod}(\bar{\Sigma}) \to 0$$

Introduce the morphism $\Phi : \pi_1(\bar{\Sigma}) \to H^1(\bar{\Sigma}, Z(G))$ defined by:

$$\Phi(\gamma)(\alpha) = c^{i(\gamma,\alpha)}, \qquad \forall \alpha \in \pi_1(\bar{\Sigma}).$$

where i is the algebraic intersection number of $\pi_1(\bar{\Sigma})$.

Proposition 4.7. *The action of $\pi_1(\bar{\Sigma})$ on $\mathcal{M}(\Sigma, G, \{c\})$ factors through the action of $H^1(\bar{\Sigma}, Z(G))$ by the morphism Φ.*

Proof. Let us prove that for any $\gamma \in \pi_1(\bar{\Sigma})$, the actions of γ and $\Phi(\gamma)$ on $\mathcal{M}(\Sigma, G, \{c\})$ coincide. We may assume that γ is represented by a simple curve of Σ since these element generate $\pi_1(\bar{\Sigma})$. Choose a neighborhood of this curve diffeomorphic to an annulus $S^1 \times [-1, 1]$, the marked point of Σ being sent to $(0, 0)$.

The morphism $\pi_1(\bar{\Sigma}) \to \mathrm{PMod}(\Sigma)$ in the Birman exact sequence maps γ to the class of ST, where T (resp. S) is a diffeomorphism of Σ equal to the identity outside of $S^1 \times [-1, 0]$ (resp. $S^1 \times [0, 1]$), and whose restriction to $S^1 \times [-1, 0]$ (resp. $S^1 \times [0, 1]$) is a Dehn Twist. The two twists have opposite orientations so that $ST = \mathrm{id}$ in $\mathrm{Mod}(\bar{\Sigma})$.

Denote by u, C_- and C_+ the paths of $S^1 \times [-1, 1]$ defined by $u(t) = (-1, t)$ with $t \in [-1, 1]$ and $C_\pm(t) = (\exp(\mp 2i\pi t), \pm 1)$ with $t \in [0, 1]$. The image of u by ST is isotopic with fixed endpoints to the concatenation $C_+ u C_- \simeq u u^{-1} C_+ u C_-$, and $u^{-1} C_+ u C_-$ is a loop winding around the marked point. Hence for any flat connection α whose holonomy around the marked point is c, one has

$$\mathrm{hol}_{ST(u)}(\alpha) = c\,\mathrm{hol}_u(\alpha).$$

So ST modify the holonomy of a path β intersecting transversally α by a factor c (resp. c^{-1}) at each positive (resp. negative) intersection. $\square$

Assume from now on that $G = \mathrm{SU}(2)$ and $c = -\mathrm{id}$. For any closed curve γ of Σ, introduce the function $f_\gamma : \mathcal{M}(\Sigma, \mathrm{SU}(2), -\mathrm{id}) \to \mathbb{R}$ defined by

$$f_\gamma([\alpha]) = \arccos\Big(\frac{1}{2}\,\mathrm{tr}(\mathrm{hol}_\rho(\gamma))\Big)$$

where $\mathrm{hol}_\rho(\gamma)$ is the holonomy of ρ along the curve γ.

Lemma 4.8. *Consider any simple non-separating closed curve γ of Σ. Then the set of $\rho \in \mathcal{M}(\Sigma, \mathrm{SU}(2), -\mathrm{id})$ such that $f_\gamma(\rho) \neq 1/2$ is nonempty.*

Proof. Choose a maximal family of non-intersecting simple curves $\gamma_1, \ldots, \gamma_N$ so that $\gamma_1 = \gamma$ and $\gamma_N = \partial\Sigma$, and the associated graph is the one given in figure 1. It follows from Jeffrey-Weitsman [JW92] that for any $t \in [1/4, 3/4]$, there exists $\rho \in \mathcal{M}(\Sigma, \mathrm{SU}(2), -\mathrm{id})$ such that

$$f_\gamma(\rho) = t, \qquad f_{\gamma_2}(\rho) = f_{\gamma_3}(\rho) = \ldots = f_{\gamma_{N-1}}(\rho) = 1/2, \qquad f_{\gamma_N}(\rho) = 1.$$

Indeed the triples $(t, t, 1/2)$, $(1/2, 1/2, 1/2)$ and $(1/2, 1/2, 1)$ satisfy the

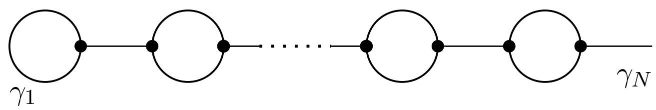

Figure 1: trivalent graph

quantum Clebsh-Gordon conditions

$$a + b + c \leqslant 2, \quad a \leqslant b + c, \quad b \leqslant c + a, \quad c \leqslant a + b.$$

$\square$

Consider $\varphi \in H^1(\bar{\Sigma}, \{\pm\,\mathrm{id}\})$, $\gamma \in \pi_1(\bar{\Sigma})$ and $\rho \in \mathcal{M}(\Sigma, \mathrm{SU}(2), -\mathrm{id})$. Then it is easy to see that

$$f_\gamma(\varphi.\rho) = \begin{cases} f_\gamma(\rho) \text{ if } \varphi(\rho) = \mathrm{id} \\ 1 - f_\gamma(\rho) \text{ if } \varphi(\rho) = -\mathrm{id} \end{cases}$$

Choose any non-seperating closed curve γ of Σ and $\varphi \in H^1(\bar{\Sigma}, \{\pm\,\mathrm{id}\})$ such that $\varphi(\gamma) = -\mathrm{id}$. Then Lemma 4.8 implies that the action of φ is not trivial.

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