A Lefschetz fixed point formula for symplectomorphisms

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May 19, 2010

Abstract

Consider a compact Kähler manifold endowed with a prequantum bundle. Following the geometric quantization scheme, the associated quantum spaces are the spaces of holomorphic sections of the tensor powers of the prequantum bundle. In this paper we construct an asymptotic representation of the prequantum bundle automorphism group in these quantum spaces. We estimate the characters of these representations under some transversality assumption. The formula obtained generalizes in some sense the Lefschetz fixed point formula for the automorphisms of the prequantum bundle preserving its holomorphic structure. Our results will be applied in two forthcoming papers to the quantum representation of the mapping class group.

Consider a compact Kähler manifold M endowed with a Hermitian holomorphic bundle $L \to M$ whose curvature is the fundamental two-form. In the point of view of geometric quantization, M is the classical phase space and the space $H^0(M, L)$ of holomorphic sections of L is the quantum space.

The group of holomorphic automorphisms of L acts naturally on the quantum space. Furthermore, if the higher cohomology groups ($H^q(M, L)$, $q \geqslant 1$) of the sheaf of holomorphic sections of L are all trivial, the holomorphic Lefschetz fixed point formula expresses the characters of this representation in terms of characteristic classes of M and L.

With the physical interpretation in mind, it is natural to consider the prequantum bundle automorphisms instead of the holomorphic automorphisms. These are the automorphisms of L preserving the Chern connection and the metric but not necessarily the holomorphic structure. Whereas the

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group of holomorphic automorphism is finite dimensional, the group of the prequantum bundle automorphisms is infinite-dimensional. Its Lie algebra identifies with the Poisson algebra of M. Furthermore each Hamiltonian symplectomorphism of M lifts to a prequantum bundle automorphism, and if M is simply connected, each symplectomorphism isotopic to the identity is Hamiltonian. The goal of this paper is to define an asymptotic representation of these automorphisms such that a suitable version of the Lefschetz fixed point formula holds. Here the terms asymptotic refers to the semi-classical limit, obtained by replacing the prequantum bundle L by its k-th tensor power with large value of k.

It is convenient to work with the metaplectic correction. Let $\delta \to M$ be a half-form bundle, that is a square root of the canonical bundle of M. Such a bundle exists if and only if the second Stiefel-Whitney class of M vanishes. We will define the notion of half-form bundle automorphisms. For the introduction, it is sufficient to know that any symplectomorphism of M isotopic to the identity lifts to a half-form bundle automorphism. Furthermore on any component of M, this lifts is unique up to a sign.

Consider two automorphisms Φ_L and Ψ of the prequantum bundle and the half-form bundle respectively which lifts the same symplectomorphism Φ of M. Then we will define a class $U(\Phi_L, \Psi)$ which consists of sequences

$$T_k : H^0(M, L^k \otimes \delta) \to H^0(M, L^k \otimes \delta), \qquad k = 1, 2, \ldots$$

of unitary maps whose Schwartz kernel has a precise asymptotic. Without going into the complete definition, let us describe the main characteristics.

- The Schwartz kernel concentrate on the graph of Φ^{-1} in the sense that
$$T_k(y, x) = O(k^{-N}), \quad \forall N$$
for any y and $x \in M$ such that $y \neq \Phi(x)$.

- The asymptotic on the graph is given in terms of Φ_L and Ψ by
$$T_k(\Phi(x), x) = \left(\frac{k}{2\pi} \right)^n \Phi_L(x)^k \otimes (\Psi(x) + O(k^{-1})).$$

The precise definition is given in sections 4.2 and 5.2. The main properties of these operators are the following.

- $U(\Phi_L, \Psi)$ is not empty. For any sequences (T_k) and (T_k') in $U(\Phi_L, \Psi)$ we have that
$$T_k' T_k^{-1} = \mathrm{id} + O(k^{-1})$$
in uniform norm.

- We have an asymptotic representation in the sense that

$$U(\Phi_L, \Psi)U(\Phi'_L, \Psi') = U(\Phi_L \circ \Phi'_L, \Psi \circ_{\mathcal{D}} \Psi').$$

- When the graph of Φ intersects transversally the diagonal, we can estimate the trace of any (T_k) in $U(\Phi_L, \Psi)$:

$$\mathrm{Tr}(T_k) = \sum_{x = \Phi(x)} \frac{i^{m_x} u_x^k}{|\det(\mathrm{id} - T_x\Phi)|^{1/2}} + O(k^{-1}) \tag{1}$$

  where for any fixed point x of Φ, $u_x \in \mathbb{C}$ is the trace of the endomorphism $\Phi_L(x)$ and $m_x \in \mathbb{Z}/4\mathbb{Z}$ depends only on $T_x\Phi$ and $\Psi(x)$.

Precise statements are given in theorems 5.2.1 and 5.3.1. These results are consequences of more general statements where the half-form line bundle is replaced by any auxiliary line bundle, cf. theorem 4.2.2 and 4.3.1. However the quantization with metaplectic correction presented here has the following particular features. First, we have an asymptotic representation of a finite cover of the prequantum bundle automorphism group, whereas with a general auxiliary line bundle we have to consider a $U(1)$-extension. Second, the formula giving the asymptotic of the trace is much more complicated for a general auxiliary line bundle. It involves the complex structure of M in a essential way, whereas it depends only on the symplectic data in the half-form case.

The index m_x appearing in the estimate of the trace (1) is one of the main point of this paper. It is similar to a Maslov or a Conley-Zehnder index. Our definition requires a choice of a complex polarization, whereas the already known definitions of indices for symplectomorphisms involved a choice of a real polarization. Besides the definition given in section 2.2, we propose a simple useful characterization in section 2.3. We also compute the index of the elements of the metalinear and unitary groups.

The results of the paper relies on the articles [4] and [5]. In [4] we proposed an elementary definition of a Fourier integral operator in the context of geometric quantization of Kähler manifolds. Previously, a definition was given by Zelditch [10] using the general theory of Toeplitz operators of Boutet de Monvel and Guillemin [2]. The interest of half-form bundle for these Fourier integral operators was understood in [5] where the spaces $U(\Phi_L, \Psi)$ were introduced. The estimate of the trace with the definition of the index is new. Of course, it is very similar to the known formula for the usual Fourier integral operators. It was one of our goal to obtain the closest formula to the usual case.

In a sequel of this paper we will apply our result to the quantum representations of the mapping class group defined in topological quantum field theory. Estimating the character of these representations with our formula, we will obtain the leading order behavior of the quantum invariant of some 3-dimensional manifolds in the large level limit. These asymptotics were initially obtained by Witten in [9] by doing the perturbative theory of some Feynman path integral and have been rigorously proved only in few cases. The article [7] will be devoted to $\mathrm{Sl}(2, \mathbb{Z})$ and [6] to the mapping class group of surfaces with genus greater than 2.

The layout of this paper is as follows. Sections 1, 2 and 3 are devoted to linear algebra preliminaries. In section 1, we define the half-form morphisms and the symplectic linear category with half-forms. The automorphism group of an object in this category is a concrete realization or the metaplectic group. It is the subject of section 2, where the index of some of its element is defined. In section 3, we generalize the previous considerations, which is necessary for the application to the quantum representation of the mapping class group. In section 4, we consider the quantization of Kähler manifold with an auxiliary line bundle, introduce the operator quantizing the symplectomorphisms and estimate their trace. In section 5, we treat the particular case where the auxiliary bundle is a half-form bundle. In appendix A, we introduce a functor from the category of symplectic vector spaces with polarization and half-form to the category of Hilbert spaces. Applying this functor to the automorphism group of a single object, we recover the well-known metaplectic representation. With this elementary construction in mind, one can view the quantization of symplectomorphisms studied in this article as a generalization of the metaplectic representation.

1 Half-forms

1.1 Complex structures and canonical lines

Let S be a symplectic real vector space. A positive polarization E of S is a Lagrangian subspace of $S \otimes \mathbb{C}$ such that

$$\tfrac{1}{i}\omega(x, \overline{x}) > 0$$

for any non-vanishing $x \in E$. Any positive polarization has a canonical Hermitian scalar product given by

$$(x, y) \rightarrow \tfrac{1}{i}\omega(x, \overline{y}).$$

The set of positive polarizations of S is a contractible topological space.

For any positive polarizations, we consider the canonical line $\wedge^{\text{top}} E^*$ with its associated Hermitian product. For any positive polarizations E_a and E_b, $S \otimes \mathbb{C}$ is the direct sum of E_a and $\overline{E}_b$. Let $\pi_{E_b, E_a} : E_b \to E_a$ be the restriction of the projection onto E_a with kernel $\overline{E}_b$. This map is invertible. Define

$$\Psi_{E_a, E_b} = \pi^*_{E_b, E_a} : \wedge^{\text{top}} E_a^* \to \wedge^{\text{top}} E_b^*$$

Given three positive polarizations, let $\zeta(E_a, E_b, E_c)$ be the complex number such that

$$\Psi_{E_a, E_c} = \zeta(E_a, E_b, E_c) \Psi_{E_b, E_c} \circ \Psi_{E_a, E_b} \tag{2}$$

Define $\zeta^{1/2}(E_a, E_b, E_c)$ to be the square root depending continuously on E_a, E_b and E_c and taking the value 1 when $E_a = E_b = E_c$.

1.2 Half-form lines

For any positive polarization E, a half-form line of E is a complex line δ together with an isomorphism $\varphi : \delta^{\otimes 2} \to \wedge^{\text{top}} E^*$. We endow δ with the scalar product making φ a unitary map.

Let us consider the category $\mathcal{D}(S)$ with objects the triples (E, δ, φ) consisting of a positive polarization together with a half-form line. The morphisms from $(E_a, \delta_a, \varphi_a)$ to $(E_b, \delta_b, \varphi_b)$ are the linear maps $\Psi : \delta_a \to \delta_b$ satisfying

$$\varphi_b \circ \Psi^{\otimes 2} = \Psi_{E_a, E_b} \circ \varphi_a.$$

The composition of a morphism $\Psi : (E_a, \delta_a, \varphi_a) \to (E_b, \delta_b, \varphi_b)$ with a morphism $\Psi' : (E_b, \delta_b, \varphi_b) \to (E_c, \delta_c, \varphi_c)$ is defined as

$$\Psi' \circ_{\mathcal{D}} \Psi := \zeta^{1/2}(E_a, E_b, E_c) \Psi' \circ \Psi$$

where the product in the right-hand side is the usual composition of maps.

Proposition 1.2.1. *The product $\circ_{\mathcal{D}}$ is well-defined and associative. For any object (E, j, φ), the identity of δ is a unit of (E, δ, φ). So $\mathcal{D}(S)$ is a category. Furthermore each morphism is invertible, its inverse being its adjoint.*

Proof. Because of equation (2), $\Psi' \circ_{\mathcal{D}} \Psi$ is a morphism. We deduce from the associativity

$$\Psi_{E_c, E_d} \circ (\Psi_{E_b, E_c} \circ \Psi_{E_a, E_b}) = (\Psi_{E_c, E_d} \circ \Psi_{E_b, E_c}) \circ \Psi_{E_a, E_b}$$

the cocycle identity

$$\zeta(E_b, E_c, E_d)\zeta(E_a, E_b, E_d) = \zeta(E_a, E_c, E_d)\zeta(E_a, E_b, E_c).$$

The square root satisfying the same identity, the product $\circ_{\mathcal{D}}$ is associative. Since $\Psi_{E,E}$ is the identity map, the identity of δ is a half-form morphism. Furthermore,

$$\Psi_{E_b,E_b} \circ \Psi_{E_a,E_b} = \Psi_{E_a,E_b}, \qquad \Psi_{E_a,E_b} \circ \Psi_{E_a,E_a} = \Psi_{E_a,E_b}$$

imply that

$$\zeta(E_a, E_b, E_b) = \zeta(E_a, E_a, E_b) = 1.$$

So the square root of ζ satisfies the same equation, consequently the identity of δ is a unit of (E, δ, φ).

Since $\overline{E}_a$ and E_b are Lagrangian spaces, one has

$$\omega(x - \pi_{E_a,E_b}x, \overline{\pi_{E_b,E_a}y}) = 0, \qquad \omega(\pi_{E_a,E_b}x, \overline{y} - \overline{\pi_{E_b,E_a}y}) = 0$$

for any $x \in E_a$ and $y \in E_b$. Consequently

$$\omega(x, \overline{\pi_{E_b,E_a}y}) = \omega(\pi_{E_a,E_b}x, \overline{y})$$

which proves that π_{E_b,E_a} is the adjoint of π_{E_a,E_b}. So Ψ_{E_b,E_a} is the adjoint of Ψ_{E_a,E_b}. Hence the adjoint $\Psi_{b,a}$ of a half-form morphism $\Psi_{a,b}$ is a half-form morphism. Furthermore, since the only automorphisms of a half-form bundle are id and $-$ id, one has

$$\Psi_{b,a} \circ_{\mathcal{D}} \Psi_{a,b} = \pm\,\mathrm{id}, \qquad \Psi_{a,b} \circ_{\mathcal{D}} \Psi_{b,a} = \pm\,\mathrm{id}\,.$$

Deforming from $E_a = E_b$ to remove the sign ambiguity, we obtain that $\Psi_{b,a}$ is the inverse of $\Psi_{a,b}$. $\qquad\square$

1.3 Symplectic linear category with half-forms

We introduce now a category $\mathcal{D}$ with objects the quadruples (S, E, δ, φ) consisting of a symplectic vector space with a positive polarization and a half-form line. Let us define the morphisms.

Consider two symplectic vector spaces S_a and S_b with positive polarizations E_a and E_b respectively. A symplectic linear isomorphism g from S_a to S_b sends isomorphically E_a to the positive polarization gE_a of S_b. So composing the pull-back by g^{-1} with the morphism Ψ_{gE_a,E_b} from $\wedge^{\mathrm{top}}(gE_a)^*$ to $\wedge^{\mathrm{top}}E_b^*$, we obtain an isomorphism

$$\Psi_{g,E_a,E_b} := \Psi_{gE_a,E_b} \circ \left(g^{-1}\right)^* : \wedge^{\mathrm{top}}E_a^* \to \wedge^{\mathrm{top}}E_b^*.$$

The morphisms from $a = (S_a, E_a, \delta_a, \varphi_a)$ to $b = (S_b, E_b, \delta_b, \varphi_b)$ are defined as the pairs (g, Ψ) consisting of a symplectic linear isomorphism from S_a to S_b and a linear morphism $\delta_a \to \delta_b$ such that

$$\varphi_b \circ \Psi^{\otimes 2} = \Psi_{g, E_a, E_b} \circ \varphi_a.$$

The composition of a morphism $(g, \Psi) : a \to b$ with a morphism $(g', \Psi') : b \to c$ is defined as

$$(g', \Psi') \circ_{\mathcal{D}} (g, \Psi) := \big(g'g, \zeta^{1/2}(g'gE_a, g'E_b, E_c)\Psi' \circ \Psi\big)$$

where the product in the right-hand side is the usual composition of maps.

Proposition 1.3.1. $\mathcal{D}$ *is a category where each morphism is invertible.*

Proof. This follows from proposition 1.2.1. Indeed, given a morphism $(g, \Psi) : a \to b$, one may identify S_a and S_b with S through symplectomorphisms in such a way that g becomes the identity. Then Ψ is a morphism from $(E_a, \delta_a, \varphi_a)$ to $(E_b, \delta_b, \varphi_b)$ in the category $\mathcal{D}(S)$. With these identifications, the composition of morphisms in $\mathcal{D}$ corresponds to the composition in $\mathcal{D}(S)$. $\square$

In appendix A, we will define a functor from the category $\mathcal{D}$ to the category of Hilbert spaces.

2 Metaplectic group

2.1 The automorphism group

Consider a fixed symplectic vector space S together with a positive polarization E. Denote by $\mathrm{Sp}(S)$ the group of linear symplectomorphism. For any $g \in \mathrm{Sp}(S)$, observe that the endomorphism $\Psi_{g,E,E}$ of $\wedge^{\mathrm{top}} E^*$ is the multiplication by $\det(g^{-1}\pi_{E,gE} : E \to E)$.

Let (δ, φ) be a half-form line of (S, E). Then identifying the automorphisms of δ with complex numbers, the automorphisms of (S, E, δ, φ) are the pairs (g, z) consisting of a linear symplectomorphism $g \in \mathrm{Sp}(S)$ with a complex number z such that

$$z^2 = \det(g^{-1}\pi_{E,gE} : E \to E).$$

The composition of two automorphisms is given by

$$(g', z') \circ_{\mathcal{D}} (g, z) = \big(g'g, \zeta^{1/2}(g'gE, g'E, E)z'z\big).$$

Since the previous formulas don't depend on (δ, φ), we denote by $\mathrm{Mp}(S, E)$ the group of automorphisms of (S, E, δ, φ). The projection

$$\mathrm{Mp}(S, E) \to \mathrm{Sp}(S)$$

is two to one. Recall that $\mathrm{Sp}(S)$ is connected with fundamental group $\mathbb{Z}$. The metaplectic group is defined as the 2-cover of the symplectic group.

Proposition 2.1.1. *The group $\mathrm{Mp}(S, E)$ is connected, so it is isomorphic to the metaplectic group.*

Proof. Choose a line D of E. Then for any $\theta \in \mathbb{R}$, consider the symplecto-morphism g_θ which preserves D and E and such that for any $x \in E$,

$$g_\theta x = \begin{cases} \exp(i\theta)x & \text{if } x \in D \\ x & \text{if } x \perp D \end{cases}$$

Then $\theta \to (g_\theta, \exp(-i\theta/2))$ is a path of automorphisms connecting $(\mathrm{id}, 1)$ with $(\mathrm{id}, -1)$. $\qquad\square$

2.2 Definition of the index

Consider the complex structure j of S such that $\ker(j - i\,\mathrm{id}) = E$. Then $(X, Y) \to \omega(X, jY)$ is a scalar product of S. Let $\mathrm{Sym}(S, j)$ be the space of linear endomorphisms of S symmetric with respect to this scalar product. For any $A \in \mathrm{Sym}(S, j)$, define the square root

$$\det{}^{1/2}\left(\tfrac{1}{2}\,\mathrm{id} + iA\right)$$

in such a way that it depends continuously on A and is positive for $A = 0$. It is well-defined because $\mathrm{Sym}(S, j)$ is contractible, as a vector space. Observe also that any symmetric endomorphism A is diagonalisable with a real spectrum so that $\tfrac{1}{2}\,\mathrm{id} + iA$ is invertible.

Let $\mathrm{Sp}_*(S)$ be the set of symplectic linear isomorphism g of S such that $\mathrm{id} - g$ is invertible. For any such g, it is easily checked that

$$A(g) := \tfrac{1}{2}(\mathrm{id} + g) \circ (\mathrm{id} - g)^{-1} \circ j$$

belongs to $\mathrm{Sym}(S, j)$. Denote by $\mathrm{Mp}_*(S, E)$ the subset of $\mathrm{Mp}(S, E)$ consisting of the pairs (g, z) such that $g \in \mathrm{Sp}_*(S)$.

Proposition 2.2.1. *For any automorphism $(g, z) \in \mathrm{Mp}_*(S, E)$, one has*

$$z \det{}^{1/2}\left(\tfrac{1}{2}\,\mathrm{id} + iA(g)\right) = \frac{i^{m(g,z)}}{|\det(\mathrm{id} - g)|^{1/2}}$$

where $m(g, z) \in \mathbb{Z}/4\mathbb{Z}$.

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Proof. Since the inverse of $(\mathrm{id} - g)^{-1} j$ is $-j(\mathrm{id} - g)$, one has

$$\begin{aligned}
\tfrac{1}{2}\,\mathrm{id} + iA(g) &= \tfrac{1}{2}\big(-j(\mathrm{id} - g) + i(\mathrm{id} + g)\big)(\mathrm{id} - g)^{-1} j \\
&= -j(\overline{\pi} - \pi g)(\mathrm{id} - g)^{-1} j
\end{aligned}$$

where $\pi = \tfrac{1}{2}(\mathrm{id} - ij)$ is the projector of $S \otimes \mathbb{C}$ with image E and kernel $\overline{E}$. From this one deduces that

$$\det\big(\tfrac{1}{2}\,\mathrm{id} + iA(g)\big) = \frac{\det(\overline{\pi} - \pi g)}{\det(\mathrm{id} - g)} = (-1)^n \frac{\det(\pi g : E \to E)}{\det(\mathrm{id} - g)}$$

The inverse of $\pi_{E,gE}$ is the restriction of π from gE to E. So

$$z^2 = \det(g^{-1}\pi_{E,gE} : E \to E) = \det^{-1}(\pi g : E \to E). \tag{3}$$

The two previous equations imply that

$$z^2 \det\big(\tfrac{1}{2}\,\mathrm{id} + iA(g)\big) = \frac{(-1)^n}{\det(\mathrm{id} - g)}, \tag{4}$$

which concludes the proof. $\qquad\square$

We call $m(g, z)$ the index of (g, z). This defines a locally constant map m from $\mathrm{Mp}_*(S, E)$ to $\mathbb{Z}/4\mathbb{Z}$ which distinguishes the components of $\mathrm{Mp}_*(S, E)$, as proves the following proposition.

Proposition 2.2.2. $\mathrm{Mp}_*(S, E)$ *has four components:* $m^{-1}(0)$, $m^{-1}(1)$, $m^{-1}(2)$ *and* $m^{-1}(3)$.

Observe also that $m(-\,\mathrm{id}, e^{i\frac{\pi}{2}p}) = p$ for $p = n \bmod 2$. When S is two-dimensional one can can easily characterize the index of (g, z) in term of the trace of g and the argument of z, cf. equation (5).

Proof. It is proved in [8] lemma 1.7, that $\mathrm{Sp}_*(S)$ has two components, distinguished by the sign of $\det(g - \mathrm{id})$. So $\mathrm{Mp}_*(S, E)$ has at most 4 components. To conclude it suffices to prove that m is onto. This follows from equation (4), because the right hand side can be positive and negative. Actually, it is also proved in [8] lemma 1.7 that any loop in $\mathrm{Sp}_*(S)$ is contractible in $\mathrm{Sp}(S)$, which gives another proof that $\mathrm{Mp}_*(S, E)$ has 4 components. $\quad\square$
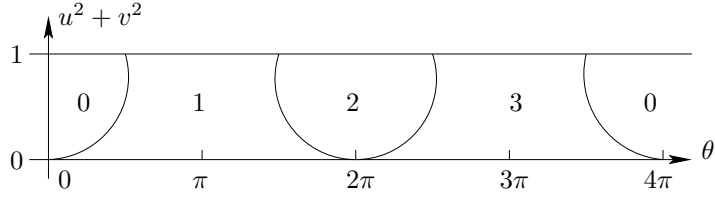
Figure 1: the index

2.3 A characterization of the index

Assume S is two-dimensional. By choosing a basis (e, f) of S such that the symplectic product of e with f is equal to 1, we identify S with $\mathbb{R}^2$ and $\mathrm{Sp}(S)$ with $\mathrm{Sl}(2, \mathbb{R})$. For any $g \in \mathrm{Sp}(S)$,

$$\det(\mathrm{id} - g) = 2 - \mathrm{tr}(g).$$

So the set of g with $\det(\mathrm{id} - g) < 0$ consists of the hyperbolic elements with negative trace, whereas the set of g with $\det(\mathrm{id} - g) > 0$ consists of the elliptic elements together with the hyperbolic ones with a positive trace.

Recall that $\mathrm{Sl}(2, \mathbb{R})$ is diffeomorphic to the product of the circle and the unit disc. Such a diffeomorphism is the map sending $\theta \in S^1$ and $(u, v) \in D$ into

$$g(\theta, u, v) := \left(1 - u^2 - v^2\right)^{-1/2} \begin{pmatrix} \cos(\theta) + u & -\sin(\theta) + v \\ \sin(\theta) + v & \cos(\theta) - u \end{pmatrix}$$

where $D = \{(u, v) / \ u^2 + v^2 < 1\}$.

Let E be the positive polarization generated by $e - if$ and let $\mathrm{Mp}(S, E)$ be the associated metaplectic group. Let us parametrize $\mathrm{Mp}(S, E)$ by $S^1 \times D$

$$(\theta, u, v) \to (g(-2\theta, u, v), e^{i\theta}(1 - u^2 - v^2)^{-\frac{1}{4}})$$

Then $\mathrm{Mp}_*(S, E)$ is the image of $\left\{\cos(2\theta) \neq \left(1 - u^2 - v^2\right)^{1/2}\right\}$. We find easily the index of any element of $\mathrm{Sp}_*(S)$ by computing explicitly the index of one element in each component. cf. figure 1.

Observe that when the argument of z runs over an interval $\left[k\frac{\pi}{2}, (k+1)\frac{\pi}{2}\right[$ with $k \in \mathbb{Z}$, the index $m(g, z)$ takes two distinct values depending on the sign of $\mathrm{tr}(g) - 2$. Checking the various cases, one obtains the following formula

$$m(g, z) = k + \tfrac{1}{2}(1 - (-1)^{k+\epsilon}) \tag{5}$$

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where k and ϵ are determined by

$$\arg(z) \in \left[k\tfrac{\pi}{2}, (k+1)\tfrac{\pi}{2} \right[, \qquad \epsilon = \begin{cases} 0 \text{ if } \operatorname{tr}(g) > 2 \\ 1 \text{ otherwise.} \end{cases}$$

The formula does not depend on the parametrization. Unfortunately such a simple description doesn't generalize in higher dimension. Nevertheless we can characterize the index in any dimension by considering product.

Let S_1 and S_2 be two symplectic vector spaces with positive polarizations E_1 and E_2. Then $E = E_1 \times E_2$ is a positive polarization of $S = S_1 \times S_2$. We have a morphism from $\operatorname{Mp}(S_1, E_1) \times \operatorname{Mp}(S_2, E_2)$ to $\operatorname{Mp}(S, E)$ sending $((g_1, z_1), (g_2, z_2))$ into $(g_1 \times g_2, z_1 z_2)$. Furthermore,

$$m(g_1 \times g_2, z_1 z_2) = m(g_1, z_1) + m(g_2, z_2). \tag{6}$$

So the image of $\operatorname{Mp}_*(S_1, E_1) \times \operatorname{Mp}_*(S_2, E_2)$ meets each connected component of $\operatorname{Mp}_*(S, E)$. This gives the following characterization.

Proposition 2.3.1. *The collection $(m_{S,E} : \operatorname{Mp}(S, E) \to \mathbb{Z}/4\mathbb{Z})$, where (S, E) runs over the symplectic vector space endowed with a positive polarization, is the unique collection of continuous map satisfying (5) for any two dimensional space and (6) for any product.*

2.4 Unitary group of E

The subgroup of $\operatorname{Sp}(S)$ consisting of the elements commuting with j is isomorphic with the unitary group of E. The isomorphism is the map ι sending $h \in U(E)$ to $g \in \operatorname{Sp}(S)$ whose complexification acts as

$$g(x) = \begin{cases} h(x) \text{ if } x \in E \\ \overline{h}(x) \text{ if } x \in \overline{E} \end{cases}$$

Denote by $U_*(E)$ the subset of $U(E)$ consisting of the h such that $h - id$ is invertible. Obviously $h \in U_*(E)$ iff $\iota(h) \in \operatorname{Sp}_*(S)$. Next lemma will be used to compare our trace estimates with the holomorphic Lefschetz fixed point formula.

Lemma 2.4.1. *For any $h \in U_*(E)$, we have*

$$\det^{1/2}\left(\tfrac{1}{2} \operatorname{id} + iA(\iota(h))\right) = \frac{1}{\det(\operatorname{id} - h^{-1})}.$$

Proof. Let (e_i) be an orthonormal basis of E diagonalising h. Then the matrix of $A(g)$ in the base $(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n)$ is given by

$$\begin{pmatrix} D & 0 \\ 0 & \overline{D} \end{pmatrix}$$

with D the diagonal matrix with entries $d_i = \frac{i}{2}(1 + u_i)(1 - u_i)^{-1}$. Here the u_i's are the eigenvalues of h, $h(e_i) = u_i e_i$. A straightforward computation using that u_i is a complex number with modulus 1 gives

$$(\tfrac{1}{2} + id_i)(\tfrac{1}{2} + i\overline{d_i}) = \frac{1}{(1 - \overline{u}_i)^2}$$

so that

$$\det\left(\tfrac{1}{2}\operatorname{id} + iA(j(h))\right) = \frac{1}{\det^2(\operatorname{id} - h^{-1})}$$

This proves the result up to a plus or minus sign. $U_*(E)$ being connected, it suffices now to check the result for one element h. It is obvious for $h = -\operatorname{id}$, because $A(\iota(h)) = 0$. $\qquad\square$

Let $\tilde{U}(E)$ be the subgroup of $U(E) \times \mathbb{C}$

$$\tilde{U}(E) = \{(h, z)/\ z^2 = \det h\}$$

It is isomorphic to the twofold cover of $U(E)$. By equation (3), ι lifts to the embedding from $\tilde{U}(E)$ to $\operatorname{Mp}(S, E)$ sending (h, z) to $(\iota(h), z^{-1})$. This map is a group morphism. Finally observe that $\tilde{U}_*(E)$ has 2 components, one containing $(-\operatorname{id}, \epsilon)$ and the other $(-\operatorname{id}, -\epsilon)$ with $\epsilon^2 = (-1)^n$. So the index m takes two distinct values on $\tilde{U}_*(E)$.

2.5 Metalinear group of a Lagrangian subspace

Let Λ be a Lagrangian subspace of S. Let $\operatorname{Sp}(S, \Lambda, j\Lambda)$ be the subgroup of $\operatorname{Sp}(S)$ consisting of the elements preserving Λ and $j\Lambda$. Let $\operatorname{Ml}(\Lambda)$ be the metalinear group of Λ, i.e. the subgroup of $\operatorname{Gl}(\Lambda) \times \mathbb{C}^*$ consisting of the pairs (h, z) such that $z^2 = \det h$.

Proposition 2.5.1. *For any $(h, z) \in \operatorname{Ml}(\Lambda)$, there is a unique pair $(g, z') \in \operatorname{Sp}(S, \Lambda, j\Lambda) \times \mathbb{C}^*$ such that*

- *h is the restriction of g to Λ.*

- *$z'/z \in \mathbb{R}_+$.*

The map $j_\Lambda : \mathrm{Ml}(\Lambda) \to \mathrm{Mp}(S, E)$ sending (h, z) into (g, z') is an injective group morphisms, with image the set of (g, z') such that $g \in \mathrm{Sp}(S, \Lambda, j\Lambda)$.

Proof. Let (e_i) be a basis of Λ, orthonormal for the scalar product $\omega(x, jy)$. Then $g \in \mathrm{Sp}(S, \Lambda, j\Lambda)$ if and only if its matrix in the basis $e_1, \ldots, e_n, je_1,$ $\ldots, je_n$ is of the form

$$\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \tag{7}$$

with $BC^t = \mathrm{id}$. So g is determined by its restriction h to Λ which is an arbitrary element of $\mathrm{Gl}(\Lambda)$.

Denote by π the projection onto E with kernel $\overline{E}$. Then the matrix of $\pi g : E \to E$ in the basis $(e_i - ije_i)_i$ is $\frac{1}{2}(B + C)$. Furthermore $\det(B + C)$ and $\det B$ have the same sign. So $\det(\pi g : E \to E)$ is real and have the same sign of $\det(h)$. This proves the existence and unicity of z'.

To prove that j_Λ is a morphism, we have to show that $\zeta^{1/2}(g'gE, g'E, E)$ is a positive number for any $g, g' \in \mathrm{Sp}(S, \Lambda, j\Lambda)$. By the first part of the proof, $\zeta(g'gE, g'E, E)$ is real. So it suffices to prove it is positive for one pair (h, h') in each component of $\mathrm{Gl}(\Lambda) \times \mathrm{Gl}(\Lambda)$. If any two of three polarizations E, F and G are equal, then

$$\zeta^{1/2}(G, F, E) = 1.$$

One deduces that $\zeta^{1/2}(g'gE, g'E, E) = 1$ for $(h, h') = (\mathrm{id}, \mathrm{id})$, (id, k), (k, id) and (k, k) where k is any involution of Λ with negative determinant. $\square$

Proposition 2.5.2. *For any $(h, z) \in \mathrm{Ml}(\Lambda)$ such that $h - \mathrm{id}$ is invertible, the index m of $j_\Lambda(h, z)$ is determined by $z = i^m |z|$.*

Proof. One has to prove that $\det^{1/2}\!\left(\frac{1}{2}\mathrm{id} + iA(g)\right)$ is positive for any $g \in \mathrm{Sp}(S, \Lambda, j\Lambda)$. If the matrix of g is (7), the one of $A(g)$ is

$$-\tfrac{1}{2} \begin{pmatrix} 0 & D \\ D^t & 0 \end{pmatrix} \qquad \text{with } D = (1 + B)(1 - B)^{-1}$$

So

$$\det\!\left(\tfrac{1}{2}\mathrm{id} + iA(g)\right) = \det\!\left(\tfrac{1}{2}(\mathrm{id} + D^t D)\right).$$

Deforming $A(g)$ to 0 through a radial homothety we obtain that the square root of $\det\!\left(\frac{1}{2}\mathrm{id} + iA(g)\right)$ is positive. $\square$

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3 Generalized half-form lines

Let p be a positive integer. A generalized half-form line of a symplectic vector space S equipped with a positive polarization E is a complex line δ together with an isomorphism

$$\varphi : \delta^{\otimes 2p} \to \left(\wedge^{\mathrm{top}} E^*\right)^{\otimes p}$$

We have a category whose objects are the quadruples (S, E, δ, φ). The morphisms from $(S_a, E_a, \delta_a, \varphi_a)$ to $(S_b, E_b, \delta_b, \varphi_b)$ are the pairs consisting of a linear symplectomorphism $g : S_a \to S_b$ together with a morphism $\Psi : \delta_a \to \delta_b$ such that

$$\varphi_b \circ \Psi^{\otimes 2p} = \Psi_{g,E_a,E_b}^{\otimes p} \circ \varphi_a$$

where the map Ψ_{g,E_a,E_b} is defined as in section 1.3. The composition of a morphism $(g, \Psi) : a \to b$ with a morphism $(g', \Psi') : b \to c$ is defined as

$$(g', \Psi') \circ_{\mathcal{D}} (g, \Psi) := \left(g'g, \zeta^{1/2}(g'gE_a, g'E_b, E_c)\Psi' \circ \Psi\right)$$

where the product in the right-hand side is the usual composition of maps.

The automorphism group $\mathrm{Mp}_p(S, E)$ of (S, E, δ, φ) consists of the pair (g, z) where g is a linear symplectomorphism of S and z a complex number such that

$$z^{2p} = \det^p(g^{-1}\pi_{E,gE} : E \to E).$$

Let U_{2p} be the group of $2p$-th roots of unity. The map sending $((g, z), u)$ to (g, zu) is a surjective morphism

$$\mathrm{Mp}(S, E) \times U_{2p} \to \mathrm{Mp}_p(S, E)$$

with kernel $\left\{((\mathrm{id}, 1), 1), ((\mathrm{id}, -1), -1)\right\}$. For any element $(g, z) \in \mathrm{Mp}_p(S, E)$ such that 1 is not an eigenvalue of g, we define its index in the following way. Set $p' = p$ if p is even and $p' = 2p$ if p is odd. Then $m_p(g, z)$ is the unique element of $\mathbb{Z}$ mod $2p'\mathbb{Z}$ such that

$$z \det^{1/2}\left(\tfrac{1}{2}\,\mathrm{id} + iA(g)\right) = \frac{e^{i\frac{\pi}{p'}m_p(g,z)}}{|\det(\mathrm{id} - g)|^{1/2}}$$

The existence of $m_p(g, z)$ follows from proposition 2.2.1.

4 Quantization of Kähler manifolds

4.1 Hilbert space

Consider a compact Kähler manifold M with a prequantization bundle L, that is L is a holomorphic Hermitian line bundle such that the curvature of its Chern connection is $\frac{1}{i}\omega$ where ω is the fundamental two-form of M. Let $K \to M$ be a holomorphic Hermitian line bundle. Define the sequence of vector spaces

$$\mathcal{H}_k := \big\{\text{holomorphic sections of } L^k \otimes K\big\}, \qquad k = 1, 2, \ldots$$

Since M is compact, $\mathcal{H}_k$ is a finite dimensional vector space. It has a natural scalar product defined by means of the Hermitian structure of $L^k \otimes K$ and the Liouville measure of M.

4.2 Fourier integral operators

Consider a symplectomorphism $\Phi : M \to M$ together with an automorphism Φ_L of the bundle L lifting Φ and preserving the connection and the metric. To these data is associated a space $\mathcal{F}(\Phi_L)$ of Fourier integral operators, that we define now.

Consider a family of operators $(S_k : \mathcal{H}_k \to \mathcal{H}_k, \ k = 1, 2, \ldots)$. The scalar product of $\mathcal{H}_k$ gives us an isomorphism

$$\operatorname{Hom}(\mathcal{H}_k, \mathcal{H}_k) \simeq \mathcal{H}_k \otimes \overline{\mathcal{H}}_k.$$

The latter space can be regarded as the space of holomorphic sections of

$$(L^k \otimes K) \boxtimes (\overline{L}^k \otimes \overline{K}) \to M^2,$$

where M^2 is endowed with the complex structure $(j, -j)$. The section $S_k(x, y)$ associated in this way to S_k is its Schwartz kernel.

By definition (S_k) is a Fourier integral operator of $\mathcal{F}(\Phi_L)$ if

$$S_k(x, y) = \left(\frac{k}{2\pi}\right)^n F^k(x, y) g(x, y, k) + O(k^{-\infty}) \tag{8}$$

where

- F is a section of $L \boxtimes \bar{L} \to M^2$ such that $\|F(x, y)\| < 1$ if $x \neq \Phi(y)$,

$$F(\Phi(x), x) = \Phi_L(u) \otimes \bar{u}, \quad \forall u \in L_x \text{ such that } \|u\| = 1,$$

  and $\bar{\partial} F \equiv 0$ modulo a section vanishing to any order along the graph of Φ^{-1}.

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- $g(., k)$ is a sequence of sections of $K \boxtimes \bar{K} \to M^2$ which admits an asymptotic expansion in the $\mathcal{C}^\infty$ topology of the form

$$g(., k) = g_0 + k^{-1}g_1 + k^{-2}g_2 + ...$$

whose coefficients satisfy $\bar{\partial}g_i \equiv 0$ modulo a section vanishing to any order along the graph of Φ^{-1}.

Let us define the principal symbol of (S_k) to be the map $x \to g_0(\Phi(x), x)$. Using the Hermitian structure of K, we regard it as a section of the bundle $\text{Hom}(K, \Phi_* K) \to M$. The principal symbol map

$$\sigma : \mathcal{F}(\Phi_L) \to \mathcal{C}^\infty(M, \text{Hom}(K, \Phi_* K))$$

satisfies the expected property.

Theorem 4.2.1. *The following sequence is exact*

$$0 \to \mathcal{F}(\Phi_L) \cap O(k^{-1}) \to \mathcal{F}(\Phi_L) \xrightarrow{\sigma} \mathcal{C}^\infty(M, \text{Hom}(K, \Phi_* K)) \to 0,$$

where the $O(k^{-1})$ is for the uniform norm of operators.

Consider two symplectomorphisms Φ and Φ'. Define the product of two sections Ψ, Ψ' of $\text{Hom}(K, \Phi_* K)$ and $\text{Hom}(K, \Phi'_* K)$ respectively as the section of $\text{Hom}(K, (\Phi' \circ \Phi)_* K)$ given at x by

$$\Psi' \circ_{\mathcal{D}} \Psi(x) = \zeta^{1/2}(g'_{\Phi(x)}g_x E_x, g'_{\Phi(x)}E_{\Phi(x)}, E_{(\Phi' \circ \Phi)(x)})\Psi'(\Phi(x)) \circ \Psi(x) \quad (9)$$

where $E_y = T_y^{1,0}M$, $g_y = T_y\Phi$ and $g'_y = T_y\Phi'$ for $y = x$, $\Phi(x)$ or $\Phi'(\Phi(x))$. Here the product on the right hand side is the usual composition of homomorphism.

Theorem 4.2.2. *Let Φ_L and Φ'_L be two automorphisms of the prequantum bundle L lifting Φ and Φ' respectively. If $T \in \mathcal{F}(\Phi_L)$ and $S \in \mathcal{F}(\Phi'_L)$, then $S \circ T$ is a Fourier integral operator of $\mathcal{F}(\Phi'_L \circ \Phi_L)$. Its symbol is given by*

$$\sigma(S \circ T) = \sigma(S) \circ_{\mathcal{D}} \sigma(T).$$

Theorems 4.2.1 and 4.2.2 are consequences of theorems 3.1 and 3.2 of [5].

4.3   Trace estimate

Consider a symplectomorphism Φ of M together with a lift Φ_L to the pre-quantum bundle. Assume that the graph of Φ intersects transversally the diagonal.

Theorem 4.3.1. *For any $(S_k) \in \mathcal{F}(\Phi_L)$ with symbol Ψ, we have*

$$\mathrm{Tr}(S_k) = \sum_{x=\Phi(x)} z_x \det{}^{1/2}\left(\tfrac{1}{2}\,\mathrm{id} + iA_x\right) u_x^k$$

where for any fixed point x of M,

- z_x *and* u_x *are the traces of* $\Psi(x) : K_x \to K_x$ *and* $\Phi_L(x) : L_x \to L_x$ *respectively.*

- $A_x = \tfrac{1}{2}(\mathrm{id} + T_x\Phi) \circ (\mathrm{id} - T_x\Phi)^{-1} \circ j_x$ *and the square root of the determinant is determined as in section 2.2.*

Proof. By assumption the Schwartz kernel of S has the form (8). One has

$$\mathrm{Tr}(S_k) = \left(\frac{k}{2\pi}\right)^n \int_M F^k(x,x) f(x,x,k) \mu_M(x) + O(k^{-\infty})$$

Since $|F(x,y)| < 1$ outside $\Gamma = \{(\Phi(x), x)/\ x \in M\}$, one can restrict the integral to a neighborhood of the fixed point set of Φ. Let us write on a neighborhood of Γ

$$\nabla^{L \boxtimes \overline{L}} F = \beta \otimes F$$

In the proposition 2.2 of [4], the first derivatives of β along Γ are computed. Denote by E_x the space $T_x^{1,0} M$.

Lemma 4.3.2. *The form β vanishes along Γ. For any vector fields X and Y of M^2, one has at any point of Γ*

$$\mathcal{L}_X \langle \beta, Y \rangle = \tfrac{1}{i} \omega_{M \times M^-}(q(X), Y)$$

where $\omega_{M \times M^-}$ is the symplectic form of the product of (M, ω) with $(M, -\omega)$. And for any $x \in M$, $q_{(\Phi(x),x)}$ is the projection of $T_{\Phi(x)} M \times T_x M$ onto $\overline{E}_{\Phi(x)} \times E_x$ with kernel the tangent space of Γ at $(\Phi(x), x)$.

Let us write on a neighborhood of a fixed point x_0 of Φ

$$F(x,x) = \exp(-\varphi(x))$$

where φ is a complex valued function. By lemma 4.3.2, the first derivatives of φ vanishes at x_0.

Lemma 4.3.3. *The Hessian of φ at x_0 is given by*

$$\mathrm{Hess}\,\varphi(X, X')(x_0) = \omega\big(\big(\tfrac{1}{2}\,\mathrm{id} + iA_{x_0}\big)^{-1}X, jX'\big)$$

for any tangent vectors X, X' of M at x_0.

Proof. By lemma 4.3.2, the Hessian is given by

$$\mathrm{Hess}\,\varphi(X, X')(x_0) = i\omega_{M \times M^-}(q(X, X), (X', X'))$$

Let π be the projector of $T_x M \otimes \mathbb{C}$ with image E_x and kernel $\overline{E}_x$. Then

$$q(X, X) = (\overline{\pi}(V), \pi(V))$$

for a unique $V \in T_x M \otimes \mathbb{C}$. Using that $\pi = \tfrac{1}{2}(\mathrm{id} - ij_x)$, we obtain that

$$\begin{aligned}
\mathrm{Hess}\,\varphi(X, X')(x_0) &= i\omega(\overline{\pi}(V), X') - i\omega(\pi(V), X') \\
&= \omega(V, j_x X')
\end{aligned}$$

To compute V one has to solve the following system

$$\begin{cases} X = T_x\Phi(Y) + \tfrac{1}{2}(\mathrm{id} + ij_x)V \\ X = Y + \tfrac{1}{2}(\mathrm{id} - ij_x)V \end{cases}$$

Adding and subtracting both equations, we obtain

$$\begin{cases} 2X = (\mathrm{id} + T_x\Phi)(Y) + V \\ 0 = (\mathrm{id} - T_x\Phi)(Y) - ij_x V \end{cases}$$

With the second equation, we compute Y in terms of V. Inserting the result in the first equation, we obtain

$$V = \big(\tfrac{1}{2}\,\mathrm{id} + iA_{x_0}\big)^{-1}X$$

with proves the lemma. $\qquad\square$

Now the theorem follows from stationary phase lemma by using that $\exp(-\varphi(x_0)) = u_{x_0}$ and $g(x_0, x_0, k) = z_{x_0} + O(k^{-1})$. Observe furthermore that the Liouville measure is the Riemannian volume of the metric $\omega(X, j_x Y)$. So for an orthogonal basis $X_1, \ldots, X_{2n}$, we have

$$\mu_M(X_1 \wedge \ldots \wedge X_{2n}) = 1$$

and

$$\det\big(\mathrm{Hess}\,\varphi(X_i, X_j)(x_0)\big)_{i,j} = \det^{-1}\big(\tfrac{1}{2}\,\mathrm{id} + iA_{x_0}\big)$$

which leads to the result. $\qquad\square$

4.4 The holomorphic Lefschetz fixed point formula

As in the previous section, consider a symplectomorphism Φ of M together with a lift Φ_L to the prequantum bundle. Assume that Φ is a holomorphic map. Since the holomorphic structure of L is characterized by the connection, Φ_L is a holomorphic bundle homomorphism. Consider a holomorphic bundle homomorphism Ψ of K lifting Φ. Then we have a map

$$(\Phi_L^k \otimes \Psi)_* : \mathcal{H}_k \to \mathcal{H}_k$$

defined as the inverse of the pull-back by $\Phi_L^k \otimes \Psi$. More generally, $\Phi_L^k \otimes \Psi$ acts on the q-th cohomology group of the sheaf of holomorphic sections of $L^k \otimes K$. One defines the holomorphic Lefschetz number

$$L(\Phi_L^k \otimes \Psi) := \sum_{q=0}^{n} (-1)^q \operatorname{Tr}\big((\Phi_L^k \otimes \Psi)_*|_{H^q(M, L^k \otimes K)}\big)$$

Then assuming that the graph of Φ intersects transversally the diagonal, the holomorphic Lefschetz fixed point theorem, cf. [1] theorem 4.12, says

$$L(\Phi_L^k \otimes \Psi) = \sum_{x=\Phi(x)} \frac{z_x u_x^k}{\det\big(\mathrm{id} - h_x^{-1}\big)}$$

where the complex numbers z_x and u_x are defined as in theorem 4.3.1 and h_x is the holomorphic tangent map of Φ at x, that is the restriction of $T_x\Phi \otimes \mathbb{C}$ to $E_x = T_x^{1,0}M$.

When k is sufficiently large, Kodaira's vanishing theorem implies that $H^q(M, L^k \otimes K) = 0$ for every positive q. If the latter is the case, the holomorphic Lefschetz number is the trace of the action of $\Phi_L^k \otimes \Psi$ on $\mathcal{H}_k$.

Furthermore, the family of operators $((\Phi_L^k \otimes \Psi)_*|_{\mathcal{H}_k}, k = 1, 2, \ldots)$ is a Fourier integral operator of $\mathcal{F}(\Phi_L)$ with symbol Ψ. This is an easy consequence of the fact that the sequence $(\mathrm{id}_{\mathcal{H}_k}, k = 1, 2, \ldots)$ belongs to $\mathcal{F}(\mathrm{id}_L)$ and has the symbol id_K.

So theorem 4.3.1 gives the asymptotic behaviour of the Lefschetz numbers. That the result agrees with holomorphic Lefschetz fixed point theorem is a consequence of lemma 2.4.1.

5 Quantization with half-form bundle

5.1 Hilbert spaces

Let p be a positive integer. Let (δ, φ) be a generalized half-form bundle of M, i.e. δ is a complex line bundle over M and φ is an isomorphism form $\delta^{\otimes 2p}$

to $\left(\wedge^{n,0}T^*M\right)^{\otimes p}$. So at each point $x \in M$, we have a positive polarization

$$T_x^{1,0}M = \ker(j_x - i\,\mathrm{id})$$

of T_xM and a generalized half-form line (δ_x, φ_x) of this polarization.

The half-form bundle δ has a natural metric and holomorphic structure such that φ is an isomorphism of Hermitian holomorphic bundle. We apply the previous constructions with $K = \delta$, which defines the Hilbert space $\mathcal{H}_k$.

5.2   Unitary maps

Let Ψ be an automorphism of the bundle δ lifting a symplectomorphism Φ of M. One says that Ψ is a half-form bundle automorphism if for any point x of M,

$$\Psi(x) : \delta_x \to \delta_{\Phi(x)}$$

is a morphism of half-form lines, cf. section 1.3 for $p = 1$ and section 3 for any p. Observe that for any two half-form bundle automorphisms Ψ and Ψ', the product $\Psi' \circ_{\mathcal{D}} \Psi$ defined in (9) is a half-form bundle morphism.

Theorem 5.2.1. *For any automorphisms Φ_L, Ψ of the prequantum bundle L and the half-form bundle δ respectively which lift the same symplectomorphism of M, let $U(\Phi_L, \Psi)$ be the set of unitary Fourier integral operators of $\mathcal{F}(\Phi_L)$ with symbol Ψ. Then*

- *$U(\Phi_L, \Psi)$ is not empty.*

- *$T_k \in U(\Phi_L, \Psi)$ and $T'_k \in U(\Phi'_L, \Psi') \Rightarrow (T'_kT_k) \in U(\Phi'_L \circ \Phi_L, \Psi' \circ_{\mathcal{D}} \Psi)$*

- *$U(\mathrm{id}_L, \mathrm{id}_\delta)$ consists of the sequences $\exp(ik^{-1}T_k)$ where (T_k) runs over the self-adjoint operators of $\mathcal{F}(\mathrm{id}_L)$.*

Proof. We only give an outline since the ideas of the proof are standard. To show that $U(\Phi_L, \Psi)$ is not empty, consider a Fourier integral operator (T_k) of $\mathcal{F}(\Phi_L)$ with symbol Ψ. Then its adjoint is a Fourier integral operator of $\mathcal{F}(\Phi_L^{-1})$ with symbol Ψ^*. By proposition 1.2.1, Ψ^* is the inverse of Ψ. So by theorem 4.2.2, $(T_k^*T_k)$ is a Fourier integral operator of $\mathcal{F}(\mathrm{id}_L)$ with symbol the identity. $\mathcal{F}(\mathrm{id}_L)$ is the algebra of Toeplitz operators. By ellipticity, $T_k^*T_k$ is an invertible self-adjoint operator when k is sufficiently large. By changing the first values of T_k, $T_k^*T_k$ is invertible for any k. Then using the functional calculus of Toeplitz operators (cf. proposition 12 of [3]), one proves that $(T_k^*T_k)^{-1/2}$ is a Toeplitz operator with principal symbol equal to 1. This implies that $\left(T_k(T_k^*T_k)^{-1/2}\right)$ belongs to $U(\Phi_L, \Psi)$.

The second part of the theorem follows directly from theorem 4.2.2. To show the last part, one constructs the operator T_k by successive approximations using the functional calculus of Toeplitz operators. $\qquad\square$

5.3 Trace estimates

Consider two automorphisms Φ_L, Ψ of the prequantum bundle L and the half-form bundle δ respectively which lift the same symplectomorphism Φ of M.

Theorem 5.3.1. *Assume that the graph of Φ intersects transversally the diagonal of M^2. Then for any $(T_k) \in U(\Phi_L, \Psi)$, one has*

$$\text{Tr}(T_k) = \sum_{x=\Phi(x)} \frac{e^{i\frac{\pi}{p'}m_p(g_x, z_x)}u_x^k}{|\det(\text{id}-g_x)|^{1/2}} + O(k^{-1})$$

where for any fixed point x of Φ,

- *g_x is the linear tangent map to Φ at x and $z_x \in \mathbb{C}$ is the trace of the endomorphism $\Phi_{\delta,x} : \delta_x \to \delta_x$*

- *$p' = p$ (resp. $2p$) if p is even (resp. odd) and $m_p(g_x, z_x) \in \mathbb{Z}$ mod $2p'\mathbb{Z}$ is the index defined in section 3.*

- *$u_x \in \mathbb{C}$ is the trace of the endomorphism $\Phi_{L,x} : L_x \to L_x$*

This is an immediate consequence of theorem 4.3.1 and the definition of the index.

A   Linear Quantization

In this appendix we define a functor from the category of symplectic space with polarization and half-form to the category of Hilbert space.

A.1   Hilbert space

Let S be a symplectic vector space. Consider the trivial bundle L_S with base S, fiber $\mathbb{C}$ and endowed with the connection $d + \frac{1}{i}\alpha$ where $\alpha \in \Omega^1(S)$ is given by

$$\alpha|_x(y) = \tfrac{1}{2}\omega(x, y).$$

Let E be a positive polarization and (δ, φ) be a half-form line. Abusing notation, we denote by δ the trivial bundle with base S, fiber δ and endowed with the trivial connection.

Consider the space $\mathcal{H}(S, E, \delta, \varphi)$ which consists of the holomorphic sections Ψ of $L_S \otimes \delta$ with respect to the polarization E such that

$$\int_S |\Psi(x)|^2 \mu(x) < \infty$$

where μ the Liouville measure of S. Here $|\Psi(x)|$ denote the punctual norm in $L_S \otimes \delta$. That a section Ψ is holomorphic with respect to E means that its covariant derivative with respect to any vector of $\overline{E}$ vanishes.

$\mathcal{H}(S, E, \delta, \varphi)$ is an abstract presentation of the Bargmann space. It is a Hilbert space with the scalar product $\int_S (\Psi, \Psi')(x) \, \mu(x)$.

A.2   Unitary map

Consider symplectic vector spaces (S_a, E_a) and (S_b, E_b) with positive polarizations. Let g be a linear symplectomorphism from S_a to S_b.

Lemma A.2.1. *There exists a unique quadratic function $\Phi : S_b \times S_a \to \mathbb{C}$ vanishing on the graph of g and such that $\exp(\Phi)$ is a holomorphic section of $L_{S_b} \boxtimes \overline{L}_{S_a}$ with respect to $E_b \times \overline{E}_a$.*

Consider now a half-form line (δ_i, φ_i) of E_i for $i = a, b$. Then for any morphism (g, Ψ) from $a = (S_a, E_a, \delta_a, \varphi_a)$ to $b = (S_b, E_b, \delta_b, \varphi_b)$ we define a map from $\mathcal{H}(a)$ to $\mathcal{H}(b)$ by

$$\big(U(g, \Psi)f\big)(x) = (2\pi)^{-n} \int_{S_a} \exp(\Phi(x, y)) \Psi(f(y)) \mu_b(y)$$

where μ_b is the Liouville measure of S_b.

Theorem A.2.2. *For any morphism (g, Ψ), the operator $U(g, \Psi)$ is unitary. Furthermore the map sending (S, E, δ, φ) to $\mathcal{H}(S, E, \delta, \varphi)$ and (g, Ψ) to $U(g, \Psi)$ is a functor from the category $\mathcal{D}$ to the category of Hilbert spaces.*

The elementary but long proof of this result will be provided somewhere else. Applying the functor to the automorphism group of a symplectic space with polarization and half-form, we obtain the well-known metaplectic representation.

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