Asymptotic properties of the quantum representations of the mapping class group

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Abstract

We establish two results on the large level limit of projective quantum representations of surface mapping class groups obtained by quantizing moduli spaces of flat $SU(n)$-bundle with Hitchin's connection. First we prove that these projective representations lift to asymptotic representations. Second we show that under an infinitesimal rigidity assumption the characters of these representations have an asymptotic expansion. The leading order term of this expansion agrees with the formula derived heuristically by E. Witten in *Quantum field theory and the Jones polynomial*.

1 Introduction

In 1988, Witten introduced new invariants of three-dimensional manifolds [33], by quantizing Chern-Simons field theory. Furthermore he showed that these invariants have in the semi-classical limit, an asymptotic behavior governed by the Chern-Simons invariants and the Reidemeister torsions. The quantum invariants were later defined rigorously by Reshetikhin and Turaev [30]. The asymptotic behavior predicted by Witten is now referred to as the Witten asymptotic conjecture. The conjecture has been settled for many Seifert manifolds [22], [31], [25], [18], [19], [17], [1]. Nevertheless, since Thurston's work, it is believed that hyperbolic manifolds are the most prevalent type of three-dimensional manifold, so that Seifert manifolds are relatively rare.

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Recently, J. Marché and the author proved the conjecture for an infinite family of hyperbolic manifold, obtained by surgery on the figure eight knot [13], [14]. In this paper, we adress the case of the mapping tori of surface diffeomorphisms, whose quantum invariants will be defined by the Hitchin's connection [20].

More precisely, the monodromy of the Hitchin's connection provides a family of projective representation of the mapping class group of any surface with genus $\geqslant 2$, called quantum representations. It is a general property of topological quantum field theories that the trace of the quantum representation of a surface diffeomorphism ϕ is equal to the quantum invariant of the mapping torus of ϕ. So the objective is to understand the asymptotic behaviour of the character of the quantum representations. But this does not really make sense because the trace is not defined on the projective linear group.

We will first prove that the projective representations coming from Hitchin's connection can be naturally lifted to asymptotic representations. Then we will show that the characters of these asymptotic representations have an asymptotic expansion whose leading order terms agrees with the Witten's formula. For this second result we will need to assume that a specific moduli space of flat bundle on the mapping torus is a 0-dimensional manifold. This rigidity assumption is natural in the statement of Witten's conjecture. Consequently for any pseudo-Anosov diffeomorphism satisfying this rigidity condition, we obtain a hyperbolic manifold satisfying the Witten asymptotic conjecture.

Asymptotic representations

Let Σ be a compact connected oriented surface with genus $\geqslant 2$ and whose boundary C is a circle. Let n, d be two coprime integers with $n \geqslant 2$. Consider the moduli space $\mathcal{M}$ of flat $\mathrm{SU}(n)$-principal bundles over Σ such that the holonomy of C is equal to $\exp(2i\pi d/n)\,\mathrm{id}$.

$\mathcal{M}$ is a smooth compact manifold. It has a natural symplectic form $\omega_{\mathcal{M}}$ [5] and is the base of a Hermitian line bundle $L \rightarrow \mathcal{M}$, equipped with a connection of curvature $\frac{n}{i}\omega_{\mathcal{M}}$ [28]. Any class σ in the Teichmuller space $\mathcal{T}$ of Σ induces a complex structures on $\mathcal{M}$ and L making them respectively a Kähler manifold $\mathcal{M}_\sigma$ and a holomorphic line bundle L_σ. Introduce for any positive integer k the vector space

$$\mathcal{H}_{k,\sigma} = H^0(\mathcal{M}_\sigma, \mathcal{O}(L_\sigma^k))$$

of holomorphic sections of the k-th tensor power of L_σ. The vector spaces

$(\mathcal{H}_{k,\sigma}, \sigma \in \mathcal{T})$ are the fibers of a smooth vector bundle $\mathcal{H}_k$ with base $\mathcal{T}$. In [20], Hitchin proved that $\mathcal{H}_k$ admits a natural projectively flat connection.

The mapping class group $\mathrm{Mod} = \pi_0(\mathrm{Diff}^+(\Sigma, C))$ acts naturally on $\mathcal{H}_k$ preserving the Hitchin connection. Consequently, we obtain a projective representation of Mod on any fiber of $\mathcal{H}_k$. In the following, we work with a fixed $\sigma \in \mathcal{T}$ and denote by

$$\rho_k : \mathrm{Mod} \to \mathrm{PGl}(\mathcal{H}_{k,\sigma}), \qquad k \in \mathbb{N}^* \tag{1}$$

the corresponding group morphism.

Our first result says that we may lift this projective representation to an asymptotic representation. The precise statement involves the metaplectic correction. According to [29], $\mathcal{M}$ admits a metaplectic structure Met. Since $\mathcal{M}$ is simply connected, Met is unique up to isomorphism, and each symplectomorphism of $\mathcal{M}$ lifts to an automorphism of Met, unique up to a plus or minus sign. The mapping class group acting on $\mathcal{M}$ by symplectomorphisms, we obtain a central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{Mod}} \to \mathrm{Mod} \to 1.$$

Theorem 1.1. *There exists a sequence of maps* $\tilde{\rho}_k : \widetilde{\mathrm{Mod}} \to \mathrm{GL}(\mathcal{H}_{k,\sigma})$, $k \in \mathbb{N}^*$ *such that the following diagrams commute*

$$\begin{array}{ccc} \widetilde{\mathrm{Mod}} & \xrightarrow{\tilde{\rho}_k} & \mathrm{GL}(\mathcal{H}_{k,\sigma}) \\ \downarrow & & \downarrow \\ \mathrm{Mod} & \xrightarrow{\rho_k} & \mathrm{PGL}(\mathcal{H}_{k,\sigma}) \end{array}$$

and for any $h, h' \in \widetilde{\mathrm{Mod}}$,

$$\tilde{\rho}_k(h)\tilde{\rho}_k(h') = c_k(h,h')\tilde{\rho}_k(hh')$$

with $c_k(h,h')$ *a sequence of* $\mathbb{C}$ *such that* $c_k(h,h') = 1 + \mathcal{O}(k^{-1})$.

To compare with, the projective representation on $L^2(\mathbb{R}^n)$ of the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ provided by the Stone-Von Neumann theorem, may be lifted to a genuine representation of the metaplectic group.

Witten asymptotic conjecture

Our second result is an estimation of the trace of $\tilde{\rho}_k(h)$ for any $h \in \widetilde{\mathrm{Mod}}$ satisfying some rigidity assumption. Let us introduce the various terms involved in the result. Let $\phi \in \mathrm{Diff}^+(\Sigma, C)$ be a representative of the image of h in Mod. Denote by Σ_ϕ the mapping torus

$$\Sigma_\phi := (\Sigma \times \mathbb{R})/(x, t+1) \sim (\phi(x), t).$$

Embed C into Σ_ϕ by $x \to [x, 0]$. Introduce the moduli space $\mathcal{M}'$ consisting of isomorphism classes of flat $SU(n)$-principal bundle with base Σ_ϕ and whose holonomy along C is $\exp(2i\pi d/n)\,\mathrm{Id}$.

Let $\bar{\Sigma} \supset \Sigma$ be the closed surface obtained by gluing a disk D along C. Extend ϕ to $\bar{\Sigma}$ by setting $\phi(x) = x$, for any $x \in D$. The mapping torus $\bar{\Sigma}_\phi$ is a closed manifold containing Σ_ϕ. For any $P \in \mathcal{M}'$, the bundle associated to P via the adjoint representation is a flat real vector bundle over Σ_ϕ, which is the restriction of a flat real vector bundle $\mathrm{Ad}\, P \to \bar{\Sigma}_\phi$, unique up to isomorphism.

Our rigidity assumption is that for any $P \in \mathcal{M}'$, the group $H^1(\bar{\Sigma}_\phi, \mathrm{Ad}\, P)$ is trivial. Since this cohomology group is isomorphic to the Zariski tangent space of $\mathcal{M}'$ at P, this assumption says that $\mathcal{M}'$ is a smooth 0-dimensional manifold. It is believed that this assumption is generically satisfied for closed 3-dimensional manifolds. Nevertheless, we do not know any result supporting this, cf. however [?] for the case of manifolds obtained from the 3-sphere by surgery on a knot.

Our result involves two quantities naturally associated to any element P of $\mathcal{M}'$: the Reidemeister torsion of $\mathrm{Ad}\, P$ and the Chern-Simons invariant of P. For the former, note that the cohomology group $H^k(\bar{\Sigma}_\phi, \mathrm{Ad}\, P)$ is trivial for $k = 0$ or 2. So if it is also the case for $k = 1$, then the Reidemeister torsion of $\mathrm{Ad}\, P$ is a real number.

Introducing the Chern-Simons invariant of P requires some care because Σ_ϕ is not closed. For any $a \in \Omega^1(\Sigma_\phi, \mathfrak{su}(n))$, let

$$\mathrm{CS}(a) = \frac{1}{4\pi} \int_{\Sigma_\phi} \mathrm{tr}(da \wedge a + \frac{2}{3} a^3).$$

Contrarily to the case of a closed manifold, it is false that the class modulo 2π of $\mathrm{CS}(a)$ only depends on the gauge class of a. Nevertheless let us fix a diffeomorphism between C and $\mathbb{R}/\mathbb{Z}$, and identify accordingly the boundary T of Σ_ϕ with $(\mathbb{R}/\mathbb{Z})^2 \ni (x, y)$. Then for any $P \in \mathcal{M}'$, there exists a flat connection $a \in \Omega^1(\Sigma_\phi, \mathfrak{su}(n))$ inducing the flat structure of P and whose restriction to the boundary T has the form $pdx + qdy$ with $p, q \in \mathfrak{su}(n)$. Then one shows that the class modulo $2\pi\mathbb{Z}$ of $n\phi(P)$, where

$$\phi(P) := \mathrm{CS}(a) + \frac{1}{4\pi}\,\mathrm{tr}(pq), \tag{2}$$

only depends of P.

Theorem 1.2. *Let $\phi \in \mathrm{Diff}^+(\Sigma, C)$ and $h \in \widetilde{\mathrm{Mod}}$ lifting the mapping class of ϕ. Assume that for any $P \in \mathcal{M}'$, the cohomology group $H^1(\Sigma_\phi, \mathrm{Ad}\, P)$ is trivial. Then*

$$\mathrm{tr}\big(\tilde{\rho}_k(h)\big) = \frac{1}{n} \sum_{P \in \mathcal{M}'} i^{m(h,P)}\ e^{in(k+1)\phi(P)}\ |\tau(\mathrm{Ad}\, P)|^{1/2} + \mathcal{O}(k^{-1})$$

where for any $P \in \mathcal{M}'$, $\phi(P)$ is defined as in equation (2), $\tau(\mathrm{Ad}\, P)$ is the Reidemeister torsion of $\mathrm{Ad}\, P$ and $m(h, P) \in \mathbb{Z}/4\mathbb{Z}$.

Let us state a local version of this result, which only requires that the rigidity assumption is satisfied at some point. The endomorphism $\tilde{\rho}_k(h)$ of $\mathcal{H}_{k,\sigma}$ has a Schwartz kernel, whose restriction to the diagonal identifies with a function on $\mathcal{M}$ that we will denote by $x \to \tilde{\rho}_k(h)(x)$. We have

$$\mathrm{tr}\big(\tilde{\rho}_k(h)\big) = \int_{\mathcal{M}} \tilde{\rho}_k(h)(x)\ \mu(x)$$

where μ is the Liouville measure of $\mathcal{M}$. Instead of this integral, we will estimate the following one

$$I_k(f) = \int_{\mathcal{M}} f(x)\tilde{\rho}_k(h)(x)\ \mu(x), \qquad f \in \mathcal{C}^\infty(\mathcal{M}), \tag{3}$$

with f conveniently chosen.

Let j be the injection of Σ into Σ_ϕ given by $j(x) = [x, 0]$. Introduce the map $p : \mathcal{M}' \to \mathcal{M}$ sending P into j^*P. Each fiber of p contains n points, and the image of p is the set of fixed points of the action of ϕ on $\mathcal{M}$.

Theorem 1.3. *Let $\phi \in \mathrm{Diff}^+(\Sigma, C)$ and $h \in \widetilde{\mathrm{Mod}}$ lifting the mapping class of ϕ. Let $f \in \mathcal{C}^\infty(\mathcal{M})$ and define $I_k(f)$ by Equation (3).*

- *If the support of f is disjoint from $p(\mathcal{M}')$, then $I_k(f) = \mathcal{O}(k^{-N})$, for any N.*
- *If for any $P \in p^{-1}(\mathrm{Supp}(f))$, $H^1(\bar{\Sigma}_\phi, \mathrm{Ad}\, P) = 0$, then*

  $$I_k(f) = \frac{1}{n} \sum_{P \in p^{-1}(\mathrm{Supp}(f))} i^{m(h,P)}\ e^{in(k+1)\phi(P)}\ |\tau(\mathrm{Ad}\, P)|^{1/2} f(P) + \mathcal{O}(k^{-1})$$

  where $m(h, P)$, $\phi(P)$ and $\tau(\mathrm{Ad}\, P)$ are defined as in Theorem 1.2.

Remark 1.1. Actually we will prove that the sequence of functions $x \to \tilde{\rho}_k(h)(x)$ admits a complete asymptotic expansion with an explicit leading order term. The estimation in the previous theorems follows from application of stationnary phase lemma. We have worked with the usual assumption that the phase has non-degenerate critical points. When this assumption is not satisfied, we can still give an estimate using a more refined version of stationnary phase lemma.

Remark 1.2. In three-dimensional topological quantum field theory (cf. for instance Chapter 4 of [6]), for any diffeomorphism ϕ of a closed surface Σ, the quantum invariant $Z_k(\Sigma_\phi)$ of the mapping torus is the trace of the endomorphism $Z_k(\phi) : V_k(\Sigma) \to V_k(\Sigma)$. Actually, this equality holds up to some multiplicative constant due to the anomaly.

A similar result is satisfied for a surface with a non-empty colored boundary. More precisely, if $\partial\Sigma$ is connected, we define a vector space $V_k(\Sigma, c)$ for any color c in the set $\mathcal{C}_k$ of admissible colors at level k for the group $SU(n)$. Furthermore for any diffeomorphism ϕ of Σ relative to the boundary, we have an endomorphism $Z_k(\phi, c)$ of $V_k(\Sigma, c)$. On the other hand, the boundary of Σ_ϕ being the torus T, $Z_k(\Sigma_\phi) \in V_k(T)$. The vector space $V_k(T)$ admits a canonical basis $(\Psi_c, c \in \mathcal{C}_k)$ called the Verlinde basis. The relation between these invariants is the following

$$\mathrm{Tr}(Z_k(\phi, c)) \equiv \langle Z_k(\Sigma_\phi), \Psi_c \rangle$$

up to a multiplicative power of $\exp(2i\pi c/24)$ where c is the central charge,

$$c = (n^2 - 1)\frac{k}{k+n}.$$

The set of color $\mathcal{C}_k$ may be viewed as a set of conjugacy classes of $SU(n)$. Furthermore $c_d = \{\exp(2i\pi d/n)\,\mathrm{id}\} \in \mathcal{C}_k$ for any k.

It is likely that there exists an isomorphism between $V_k(\Sigma, c_d)$ and $\mathcal{H}_{k,\sigma}$ such that $Z_k(\phi)$ is sent to $\tilde{\rho}_k(h)$. Indeed, it has been proved in [24], [7] that the Hitchin connection is isomorphic to the connection of conformal block theory. Furthermore the topological quantum field theory defined from conformal block theory is equivalent to the Reshetikhin-Turaev construction, [6] and [2]. Observe also that the extension $\widetilde{\mathrm{Mod}}$ should be related to the asymptotics of the anomaly.

Proofs and comments

For the proof of the three theorems, we have to gather and adapt various results in the litterature related to Hitchin connections [20] [4], to the semi-classical limit of Kähler quantization [10] [15] and to the geometry of the moduli spaces of flat bundles [16] [26] [12]. More precisely, for Theorem 1.1, we consider the connection introduced by Andersen, Gammelgaard and Lauridsen [4]. We show that this connection is the same as the one introduced by Hitchin [20] up to a projective correction, and we prove that it is asymptotically flat. This last fact is based on a miraculous cancellation that was already observed in our previous work [10]. The proof of Theorems 1.2, 1.3 is based on microlocal analysis technics for the quantization of Kähler manifolds. First, we deduce from [10] that the parallel transport for the Hitchin connection is similar to a Fourier integral operator. We then estimate the trace of these operators by applying a semi-classical Lefschetz formula [15]. Finally we interpret the various quantities appearing in the asymptotic expansion as invariants of mapping tori.

The proof of the Witten asymptotic conjecture for the manifolds obtained by surgery on the figure eight knot is also based on microlocal technics. Interestingly the tools from microlocal analysis are needed for the hyperbolic manifolds, whereas the case of Seifert manifolds can be handled with methods more elementary. For instance it is instructive to compare with the previous work of Andersen [1] for the mapping tori of finite order diffeomorphisms. In this case, on one hand, the mapping torus is Seifert. On the other hand, the surface has a complex structure preserved by the diffeomorphism. Consequently the action on the quantum space is defined without using the Hitchin connection and the trace can be estimated with the standard holomorphic Lefschetz formula. To the contrary, consider a pseudo-Anosov diffeomorphism ϕ of a surface Σ. On one hand, by Thurston's theorem, its mapping torus is hyperbolic. On the other hand, by the Nielsen-Thurston classification, such a diffeomorphism does not fix any point of the Teichmüller space. So the operator $Z_k(\phi) : V_k(\Sigma) \to V_k(\Sigma)$ is a quantization of a symplectomorphism of the moduli space which does not preserve a priori a complex structure. In microlocal analysis, the operators quantizing symplectomorphisms which do not preserve any polarization, are an essential tool, introduced by Hormander in [21] under the name of Fourier integral operators.

This work suggests further developpments:

- extend this result to Hitchin's connection for moduli spaces with parabolic datas.
- understand better the relation between combinatorial topological quantum field theory and the Hitchin's connection
- compute $H^1(X, \mathrm{Ad}\, F)$ for flat G-principal bundle F on three-dimensional

manifold X.

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2 Hitchin's connection

2.1 Holomorphic differential operators

Let M be a complex manifold and $F \to M$ be a holomorphic line bundle. Consider the algebra of differential operators acting on $\Gamma(M, F)$. It is the direct sum of the subalgebra of holomorphic differential operators and the left-ideal $\mathcal{I}$ generated by the anti-holomorphic derivations. More explicitly, introduce a local holomorphic trivialization of F and a system (z^i) of holomorphic coordinates of M. Then each differential operator is of the form

$$\sum_{\alpha \in \mathbb{Z}^n} a_\alpha \partial_{z^1}^{\alpha(1)} ... \partial_{z^n}^{\alpha(n)} + \sum_{\alpha, \beta \in \mathbb{Z}^n, \beta \neq 0} a_{\alpha,\beta} \partial_{z^1}^{\alpha(1)} ... \partial_{z^n}^{\alpha(n)} \partial_{\bar{z}^1}^{\beta(1)} ... \partial_{\bar{z}^n}^{\beta(n)},$$

where the coefficients a_α and $a_{\alpha,\beta}$ are smooth functions. The first summand is a holomorphic differential operator and the second one belongs to the ideal $\mathcal{I}$.

We denote by $D_k^{\mathrm{hol}}(F)$ the bundle whose sections are the holomorphic differential operators of order k acting on $\Gamma(M, F)$. $D_k^{\mathrm{hol}}(F)$ has a natural holomorphic structure, such that its holomorphic sections are the holomorphic differential operators with holomorphic coefficients. Observe that for any holomorphic differential operator P and smooth section Z of $T^{1,0}M$, one has

$$D_{\bar{Z}} P = [D_{\bar{Z}}, P] \mod \mathcal{I} \tag{4}$$

where $D_{\bar{Z}}$ denote the derivative of sections of $D_k^{\mathrm{hol}}(F)$ (resp. F) on the left hand side (resp. right hand side).

2.2 Variations of complex structures

Let U and M be two manifolds. Consider a smooth family $(j_u)_{u \in U}$ of complex structures of M. Denote by M_u the complex manifold $(\{u\} \times M, j_u)$. Let E be the complex vector bundle over $U \times M$ with fibers

$$E_{u,x} = T_x^{1,0} M_u.$$

We call E the relative holomorphic tangent bundle. We shall often consider the decomposition of the tangent space of $U \times M$ given by

$$T_{u,x}(U \times M) \otimes \mathbb{C} = (T_u U \otimes \mathbb{C}) \oplus E_{u,x} \oplus \bar{E}_{u,x}$$

Let X be a vector field of U. Since $j_u^2 = -\operatorname{id}$, the derivative of (j_u) with respect to X has the form

$$X.j = \mu(X) + \bar{\mu}(X)$$

where μ is a section of $\operatorname{Hom}(\bar{E}, E)$. Let Z be a smooth section of E. Consider Z and X as vector fields of $U \times M$. Observe that the Lie bracket $[X, \bar{Z}]$ is tangent to $E \oplus \bar{E}$. Furthermore,

$$[X, \bar{Z}] = \tfrac{i}{2}\mu(X)(\bar{Z}) \mod \bar{E}. \tag{5}$$

which follows easily from the fact that the projection from $E \oplus \bar{E}$ onto $\bar{E}$ is $\frac{1}{2}(\operatorname{id} + ij)$.

2.3 Connection

Consider as previously two manifolds U and M with a smooth family $(j_u)_{u \in U}$ of complex structures. Let $(F_u \to M_u)_{u \in U}$ be a smooth family of holomorphic line bundles. Assume that M is compact and that the dimension of the space $H^0(M_u, F_u)$ of holomorphic sections does not depend on u. Then by elliptic regularity, there exists a smooth vector bundle $\mathcal{H}$ with base U and fibers $H^0(M_u, F_u)$, such that its smooth sections are the smooth families of holomorphic sections (cf [8], chapter 9.2).

Let F and $D_2^{\mathrm{hol}}(F)$ be the bundles over $U \times M$ which restrict over any slice M_u to F_u and $D_2^{\mathrm{hol}}(F_u)$ respectively. If s is a section of F, we denote by $\bar{\partial} s$ the section of $F \otimes \bar{E}^*$ which restricts over M_u to $\bar{\partial}_u s_u$. So a smooth section of $\mathcal{H}$ is by definition a smooth section of F satisfying $\bar{\partial} s = 0$. We use the same notation with $D_2^{\mathrm{hol}}(F)$ and more generally with any family of holomorphic bundles.

Introduce a connection ∇ on F, such that its restriction to any M_u is compatible with the holomorphic structure of F_u. Introduce a section P of $D_2^{\mathrm{hol}}(F) \otimes p^*(T^*U)$ where p is the projection from $U \times M$ to U. We would like to define a connection on $\mathcal{H} \to U$ whose covariant derivative in the direction of $X \in \Gamma(U, TU)$ is given by

$$\nabla_X + P(X). \tag{6}$$

The following lemma provides a sufficient condition for the connection to be well-defined. We denote by R^∇ the curvature of ∇.

Proposition 2.1. *Assume that for any section Z of E, we have*

$$[\bar{\partial} P(X)](\bar{Z}) = \frac{i}{2} \nabla_{\mu(X)(\bar{Z})} + R^{\nabla}(X, \bar{Z})$$

Then if s is a section of F whose restriction to each M_u is holomorphic, the same holds for $(\nabla_X + P(X))s$.

Proof. We show that the assumption implies that for any smooth section s of F and any point u of U

$$\big([\nabla_{\bar{Z}}, \nabla_X + P(X)] s \big)_u = Q_u s_u$$

with Q_u a differential operator of F_u in the ideal $\mathcal{I}_u$ generated by the anti-holomorphic derivations. By (5), we have that

$$[\nabla_{\bar{Z}}, \nabla_X] = -\tfrac{i}{2} \nabla_{\mu(X)(\bar{Z})} - R^{\nabla}(X, \bar{Z}) \mod \mathcal{I}$$

By (4), we have that

$$[\nabla_{\bar{Z}}, P(X)] = [\bar{\partial} P(X)](\bar{Z}) \mod \mathcal{I}.$$

The conclusion follows. □

2.4 Unicity

Let us discuss the unicity of a connection of the form (6) and satisfying the assumption of proposition 2.1. Assume that M is connected and that for any u, M_u has no holomorphic vector field. Suppose that (∇, P) satisfies the hypothesis of proposition 2.1 for any vector field X. Let (∇', P') be another pair satisfying the same assumption. Assume that for any $u \in U$ and $X \in T_u U$, $P(X)_u$ and $P'(X)_u$ are second-order differential operators with the same principal symbol. Then there exists a form $\alpha \in \Omega^1(U)$ such that

$$\nabla'_X + P'(X) = \nabla_X + P(X) + \alpha(X) \,\mathrm{id} \tag{7}$$

for any vector field X of U.

Indeed, ∇ and ∇' differ by a one-form $\beta \in \Omega^1(U \times M, \operatorname{End} F)$ which vanishes in the directions tangent to $\bar{E}$. Let $\tilde{\beta}$ be the section of $\operatorname{End} F \otimes p^* T^* U$ such that $\beta(X) = \tilde{\beta}(X)$, for any vector field X of U. Then it is easily checked that the pair $(\nabla' - \beta, P' + \tilde{\beta})$ satisfies the assumption of proposition 2.1. Since

$$\nabla'_X + P'(X) = \nabla'_X - \beta(X) + P'(X) + \tilde{\beta}(X) = \nabla_X + P'(X) + \tilde{\beta}(X)$$

we may assume that $\nabla = \nabla'$. Next, the hypothesis of proposition 2.1 implies that

$$\bar{\partial}(P(X) - P'(X)) = 0 \tag{8}$$

Since $P(X)_u$ and $P'(X)_u$ have the same principal symbol, $P(X)_u - P'(X)_u$ is a first order holomorphic differential operator. Since $H^0(M_u, E_u) = 0$, equation (8) implies that $P(X)_u - P'(X)_u$ is a zero-order holomorphic differential operator. In other words, $P(X)_u - P'(X)_u$ is the multiplication by a function. By equation (8), this function is holomorphic, hence constant because M is compact and connected. This proves that $P' = P + \alpha$ with $\alpha \in \Omega^1(U)$. The desired equation (7) follows.

2.5 A preliminary computation

To apply proposition 2.1, we need to compute the $\bar{\partial}$ of some second order differential operator. The parameter space doesn't enter in the calculation, so we assume in this subsection that $U = \{pt\}$. Suppose that the complex manifold M has a Kähler metric, and that the holomorphic line bundle F has a Hermitian metric. Hence F and $T^{1,0}M$ have canonical connections compatible with the metric and the holomorphic structure (Chern connection). Let G be a section of the second symmetric tensor power $S^2(T^{1,0}M)$. Define the holomorphic differential operator acting on the sections of F

$$\Delta^G s = \mathrm{Tr}^{\mathrm{End}(T^{1,0}M)}(\nabla^{T^{1,0}M \otimes F}(G \lrcorner \nabla^F s)) \tag{9}$$

More explicitly if $\partial_1, \ldots, \partial_n$ is a local frame of $T^{1,0}M$ and $\ell_1, \ldots \ell_n$ is the dual frame,

$$\Delta^G s = \sum_k \ell_k (\nabla_k^{T^{1,0}M \otimes F} (\sum_{i,j} G_{ij} \partial_i \otimes \nabla_j^F s))$$

where ∇_i is the covariant derivative with respect to ∂_i and $G = \sum G_{ij} \partial_i \otimes \partial_j$.

Proposition 2.2. *Assume that G is a holomorphic section of $S^2(T^{1,0}M)$, then*

$$\bar{\partial} \Delta^G = \sum_{i,j} (2R^F + R^{\mathrm{det}})(\cdot, \partial_i) G_{ij} \nabla_j^F + \theta^F$$

where R^F is the curvature of ∇^F, R^{det} is the curvature of the Chern connection of $\wedge^n T^{1,0}M$ and θ^F is the one-form of M given by

$$\theta^F(\bar{Z}) = \sum_{k,i,j} \ell_k (\nabla_k^{T^{1,0}M} (R^F(\bar{Z}, \partial_i) G_{ij} \partial_j))$$

for any (local) holomorphic section Z of $T^{1,0}M$.

This is a slight generalization of a computation in [20], page 364. The assumption that the metric is Kähler is used for certain symmetries of the curvature tensor of $T^{1,0}M$.

3 Application to Kähler quantization

3.1 Without metaplectic correction

Assume now that M is a symplectic manifold with symplectic form ω, and that $(j_u)_{u\in U}$ is a family of compatible complex structures. So each M_u is a Kähler manifold.

Let $L_M \to M$ be a prequantum bundle, that is a Hermitian line bundle with a connection of curvature $\frac{1}{i}\omega$. For any u, denote by $L_u \to M_u$ the bundle L_M with the holomorphic structure compatible with the connection and j_u. Denote by L the pull-back of L_M by the projection $U \times M \to M$ and endow L with the pull-back connection.

Our aim is to apply the construction of chapter 2.3 to the bundle $F = L^k$. So consider the vector space

$$\mathcal{H}_{k,u} = H^0(M_u, L_u^k).$$

It follows from Kodaira vanishing theorem and Riemann-Roch theorem that for any compact set C of U, if k is sufficiently large, the dimension of $\mathcal{H}_{k,u}$ is constant when u runs over C. Let us do the following global assumption: there exists k_0 such that the dimension of $\mathcal{H}_{k,u}$ does not depend on $u \in U$ when $k \geqslant k_0$. Then for $k \geqslant k_0$, the spaces $\mathcal{H}_{k,u}$ are the fibers of a smooth vector bundle $\mathcal{H}_k$ over U, whose sections are the smooth families of holomorphic sections. In the sequel we always assume that $k \geqslant k_0$.

For any point $u \in U$ and any tangent vector $X \in T_uU$, introduce the section $\mu(X)$ of $\mathrm{Hom}(T^{0,1}M_u, T^{1,0}M_u)$ measuring the variation of the complex structure as in Section 2.2. Let $G(X)$ be the section of $S^2(T^{1,0}M_u)$ such that

$$\sum_{i,j} G_{ij}(X)\omega(\partial_i, \cdot)\partial_j = \mu(X)_u, \qquad G(X) = G_{ij}(X)\partial_i \otimes \partial_j \tag{10}$$

$G(X)$ is symmetric because ω and j_u are compatible.

Theorem 3.1 ([20]). *Assume that M is simply connected, $c_1(M) = \frac{\lambda}{i}[\omega]$ for some real number λ, and the section $G(X)$ defined in (10) is holomorphic* *for any $X \in TU$. Then for any $k \geqslant k_0$, the bundle $\mathcal{H}_k \to U$ admits a connection $\nabla^{\mathcal{H}_k}$ of the form*

$$\nabla^{\mathcal{H}_k}_X := \nabla^{L^k}_X + P_k(X), \qquad X \in \Gamma(U, TU)$$

with $P_k \in D_2^{\mathrm{hol}}(L^k) \otimes p^*(T^*U)$. Furthermore, the principal symbol of $P_k(X)$ is $(4k+2\lambda)^{-1} G(X)$.

Since the exposition in [20] is different, we outline a proof. It is based on the computations pages 365-366 of [20].

Proof. Introduce a familly of Ricci potentials $F \in \mathcal{C}^\infty(U \times M)$ satisfying

$$2i\partial_u \bar{\partial}_u F = R_u^{\mathrm{det}} - \lambda\omega, \qquad \forall u \in U.$$

For any $X \in T_u U$, let

$$\nu(X) = G(X) \lrcorner \partial F = \sum G_{ij}(X)(\partial_i F)\partial_j.$$

Then it follows from Proposition 2.2 that for any section Z of E_u,

$$\big[\bar{\partial}(\Delta^{G(X)} + 2i\nabla^{L^k}_{\nu(X)})\big](\bar{Z}) = i(2k+\lambda)\nabla_{\mu(X)(\bar{Z})} + k\theta(X)(\bar{Z})$$

with $\theta(X)$ a $(0,1)$-form of M_u independent of k. Using that $\bar{\partial}^2(\Delta^{G(X)} + 2i\nabla^{L^k}_{\nu(X)}) = 0$ for any k, we deduce that $\bar{\partial}\theta(X) = 0$. Since M is simply connected, by Hodge decomposition, the Dolbeault cohomology group $H^{1,0}(M_u)$ vanishes for any u. So there exists a unique function $f(X)$ such that

$$\bar{\partial} f(X) = \theta(X), \qquad \int_M f(X)\, \omega^n = 0$$

Finally set $P_k(X) = (4k+2\lambda)^{-1}\big(\Delta^{G(X)} + 2i\nabla^{L^k}_{\nu(X)} - kf(X)\big)$. Then we have

$$\bar{\partial} P_k(X) = \frac{i}{2}\nabla_{\mu(X)}$$

and we conclude with Proposition 2.1. $\square$

3.2 With metaplectic correction

Introduce the same data as in the previous section. Consider furthermore a pair (δ, φ) which consists of a line bundle δ over $U \times M$ with an isomorphism φ from δ^2 to $\wedge^{\mathrm{top}} E^*$. If U is contractible, such a pair exists if and only if the second Stiefel-Whitney class of M vanishes. The restriction δ_u of δ to M_u has holomorphic and Hermitian structures determined by the condition that the isomorphism $\varphi_u : \delta_u^2 \to \wedge^{\mathrm{top}} E_u^*$ is an isomorphism of Hermitian holomorphic bundles. We call $(\delta_u, \varphi_u)_{u \in U}$ a family of half-form bundles.

Let us define a connection on δ. First consider the connection $\nabla^{E \oplus \bar{E}}$ on $E \oplus \bar{E}$ such that its restriction to each slice M_u is the Levi-Civita connection of the Kähler metric of M_u and the covariant derivative in a direction tangent to U is the obvious one. This makes sense because the restriction of $E \oplus \bar{E}$ to $U \times \{x\}$ is the trivial bundle with fiber $T_x M \otimes \mathbb{C}$. Next we consider the following connection on E

$$\nabla^E = \pi \circ \nabla^{E \oplus \bar{E}},$$

where $\pi = \frac{1}{2}(\mathrm{id} - ij)$ is the projection of $E \oplus \bar{E}$ onto E with kernel $\bar{E}$. This defines a connection on the associated bundle $\wedge^{\mathrm{top}} E^*$ and finally a connection ∇^δ on δ.

Instead of L^k, consider the bundle $F = L^k \otimes \delta$. Introduce the vector space

$$\mathcal{H}^m_{k,u} = H^0(M_u, L^k_u \otimes \delta_u).$$

Here the subscript m stands for metaplectic correction. Assume that there exists k_0 such that the dimension of $\mathcal{H}^m_{k,u}$ does not depend on $u \in U$ when $k \geqslant k_0$. Let $\mathcal{H}^m_k \to U$ be the bundle with fibers $\mathcal{H}^m_{k,u}$, $u \in U$.

Theorem 3.2 ([4])**.** *Assume that M is simply connected and the section $G(X)$ defined in (10) is holomorphic for any $X \in TU$. Then for any $k \geqslant k_0$, the bundle $\mathcal{H}^m_k \to U$ admits a connection $\nabla^{\mathcal{H}_k}$ of the form*

$$\nabla^{\mathcal{H}_k}_X := \nabla^{L^k \otimes \delta}_X + \frac{1}{4k}\big(\Delta^{G(X)} - H(X)\big), \qquad X \in \Gamma(U, TU) \tag{11}$$

where $\Delta^{G(X)}$ is given by the equations (9) and $H \in \Gamma(U \times M, p^*(T^*U))$ is independent of k.

We outline the proof, which is similar to the one of Theorem 3.1.

Proof. By proposition 2.2, we have for any $X \in T_u U$ and section Z of E_u,

$$\big[\bar{\partial}\Delta^{G(X)}\big](\bar{Z}) = 2ik\nabla^{L^k \otimes \delta}_{\mu(X)(\bar{Z})} + k\theta^{L_u}(\bar{Z}) + \theta^{\delta_u}(\bar{Z})$$

It is proved in [4] that $4R^\delta(X, \bar{Z}) = \theta^{L_u}(\bar{Z})$. So

$$\big[\bar{\partial}\Delta^{G(X)}\big](\bar{Z}) = 4k\big[\tfrac{i}{2}\nabla^{L^k \otimes \delta}_{\mu(X)(\bar{Z})} + R^\delta(X, \bar{Z})\big] + \theta^{\delta_u}(\bar{Z})$$

For $k=0$, this implies that $\bar{\partial}\theta^{\delta_u}=0$. So M being simply connected, there exists a function $H(X)$ such that

$$\bar{\partial}H(X)=\theta^{\delta_u}, \qquad \int_M H(X)\,\omega^n=0.$$

We conclude with Proposition 2.1. $\square$

4 An algebra of Toeplitz operators

Consider a complex compact manifold M and a family of Hermitian holomorphic line bundles $F=(F_j\to M)_{j\in J}$. For any $j\in J$, let $\mathrm{Op}(F_j)$ be the algebra of holomorphic differential operators acting on the sections of F_j. Consider the subalgebra $\mathrm{Op}_{sc}(F)$ of $\prod_{j\in J}\mathrm{Op}(F_j)$ consisting of the family (P_j) satisfying the following condition: there exists ℓ such that for any complex coordinate system $(U,z^1,\dots,z^n)$, there exists a family $(a_\alpha)_{|\alpha|\leqslant\ell}$ of $\mathcal{C}^\infty(U)$ such that we have over U

$$P_j=\sum_\alpha a_\alpha\big(\nabla_1^{F_i}\big)^{\alpha(1)}\dots\big(\nabla_n^{F_i}\big)^{\alpha(n)}, \qquad \forall j\in J.$$

Here ∇^{F_j} is the Chern connection of F_j, and $\nabla_\ell^{F_j}$ is the covariant derivative with respect to ∂_{z^ℓ}. To check that $\mathrm{Op}_{sc}(F)$ is a subalgebra, it suffices to use that $[\nabla_k^{F_j},\nabla_\ell^{F_j}]=0$.

Let μ be a measure of M. We define a scalar product on $\Gamma(M,F_j)$ by integrating the pointwise scalar product of sections against μ. Denote by Π_j the orthogonal projector of $\Gamma(M,F_j)$ onto its subspace of holomorphic sections $H^0(M,F_j)$.

Proposition 4.1. *For any $(P_j)\in\mathrm{Op}_{sc}(F)$, there exists a function $f\in\mathcal{C}^\infty(M)$ such that for any $j\in J$*

$$\Pi_jP_j\Pi_j=\Pi_jM_f\Pi_j$$

where M_f is multiplication operator of $\Gamma(M,F_j)$ with multiplicator f.

The proof is based on a trick due to Tuynman [32], a similar result was also used in [3].

Proof. Let $(P_j)\in\mathrm{Op}(F)$. Observe that all the P_j have the same order ℓ and the same symbol σ. This symbol is a section of $S^\ell(T^{1,0}M)$. Using a partition of unity, we can write σ under the form

$$\sigma=\sum_{i=1,\dots,r}X_1^i\otimes\dots\otimes X_\ell^i$$

where the X^i_k are smooth sections of $T^{1,0}M$. Since

$$P_j - \sum_{i=1,\dots,r} \nabla^{F_j}_{X^i_1} \dots \nabla^{F_j}_{X^i_\ell}$$

has order $\ell - 1$, it suffices to prove the proposition for each operator

$$\nabla^{F_j}_{X^i_1} \dots \nabla^{F_j}_{X^i_\ell}.$$

Let s_1, s_2 be smooth sections of F_j. Assume that s_2 is holomorphic. Then for any smooth section X of $T^{1,0}M$, we have

$$X.[(s_1, s_2)\mu] = (\nabla^{F_i}_X s_1, s_2)\mu + (s_1, s_2)(\operatorname{div} X)\mu$$

So if we denote by $(\cdot, \cdot)_{F_j}$ the scalar product of sections we obtain

$$(\nabla^{F_j}_X s_1, s_2)_{F_j} = (f s_1, s_2)_{F_j}$$

with $f = -\operatorname{div} X$. So if $X_1, \dots, X_\ell$ are smooth sections of $T^{1,0}M$, then

$$\begin{aligned}
(\nabla^{F_j}_{X_1} \dots \nabla^{F_j}_{X_\ell} s_1, s_2)_{F_j} =& (f_1 \nabla^{F_j}_{X_2} \dots \nabla^{F_j}_{X_\ell} s_1, s_2)_{F_j} \\
=& (\nabla^{F_j}_{f_1 X_2} \dots \nabla^{F_j}_{X_\ell} s_1, s_2)_{F_j} \\
& \dots \\
=& (f s_1, s_2)_{F_j}
\end{aligned}$$

where f is a function which depends only on μ and the vector fields X_i. This proves the result. □

5 Asymptotic flatness

In this part we consider the same data as in section 3.2. We will prove that the curvature of the connection $\nabla^{\mathcal{H}_k}$ defined in theorem 3.2 vanishes in the semi-classical limit $k \to \infty$. For any vector field X of U, denote by $P_k(X)$ the operator

$$P_k(X) = \tfrac{1}{4}(\Delta^{G(X)} - H(X))$$

The curvature of (11) in the directions $X, Y \in \Gamma(U, TU)$ is

$$\begin{aligned}
R_k(X, Y) =& \big[\nabla^{L^k \otimes \delta}_X + k^{-1} P_k(X), \nabla^{L^k \otimes \delta}_Y + k^{-1} P_k(Y)\big] \\
& - \nabla^{L^k \otimes \delta}_{[X,Y]} - k^{-1} P_k([X, Y]).
\end{aligned}$$

Our aim is to prove that the connection is asymptotically flat in the following sense.

Theorem 5.1. *For any vector fields X, Y of U, any compact set K of U, there exists $C > 0$, such that the curvature at any point u of U,*

$$R_k(X,Y)(u) : \mathcal{H}^m_{k,u} \to \mathcal{H}^m_{k,u},$$

has a uniform norm bounded by Ck^{-1}.

Recall that the algebra of differential operators acting on the sections of a holomorphic fiber bundle is the direct sum of the algebra of holomorphic differential operators and the left ideal generated by the anti-holomorphic vector fields. If Q_u is a differential operator acting on $\Gamma(M_u, L_u^k \otimes \delta_u)$, we denote by Q_u^{Hol} its holomorphic part. We use the same notations for families $(Q_u)_{u\in U}$. We denote by Op_{sc} the space of families

$$(Q_{u,k} : \Gamma(M_u, L_u^k \otimes \delta_u) \to \Gamma(M_u, L_u^k \otimes \delta_u))_{u \in U, k \geqslant k_0}$$

consisting of differential operators such that for any k, $Q_{u,k}$ depends smoothly on u and for any u, $(Q_{u,k})_k$ belongs to the algebra $\mathrm{Op}_{sc}(L_u^k \otimes \delta_u, k \geqslant k_0)$ introduced in the previous section.

Theorem 5.2. *For any vector fields X, Y of U, one has*

$$R_k(X,Y)^{\mathrm{Hol}} = k^{-1} P_{1,k}(X,Y) + k^{-2} P_{2,k}(X,Y), \qquad \forall k \geqslant k_0$$

where the families $(P_{1,k}(X,Y))_k$ and $(P_{2,k}(X,Y))_k$ belong to Op_{sc}.

Theorem 5.1 follows from Theorem 5.2. Indeed by proposition 4.1, for any family $(Q_{u,k})$ of Op_{sc}, there exists a continuous function $C : U \to \mathbb{R}$ such that for any k and u, the uniform norm of

$$\Pi_{u,k} Q_{u,k} : H^0(M_u, L_u^k \otimes \delta_u) \to H^0(M_u, L_u^k \otimes \delta_u)$$

is bounded by $C(u)$.

The remainder of this section is devoted to the proof of theorem 5.2. Since L is the pull-back of a bundle over M, its curvature in the directions tangent to U vanishes. For the half-form bundle, the curvature R^δ depends on the derivative of the complex structure. Recall that we denote by $\mu(X)$ the variation of the complex structure, cf. section 2.2.

Proposition 5.3. *For any vector field X, Y of U,*

$$R^\delta(X,Y) = \frac{1}{8} \operatorname{tr}(\mu(X)\overline{\mu}(Y) - \mu(Y)\overline{\mu}(X)).$$

The proof is easy, cf. as instance the proof of theorem 7.2 in [10].

Proposition 5.4. *For any vector fields X, Y of U we have*

$$\left[\nabla_X^{L^k\otimes\delta}, P_k(Y)\right]^{\mathrm{Hol}} = \frac{k}{8}\,\mathrm{tr}(\mu(Y)\overline{\mu}(X)) + P_k(X,Y)$$

where the family $(P_k(X,Y))$ belongs to Op_{sc}.

Proof. Introduce a local frame $\partial_1, \dots, \partial_n$ of the relative holomorphic tangent bundle of $M \times U$. Denote by ∇^k the covariant derivative of $L^k \otimes \delta$ and by ∇_i^k the covariant derivative in the direction of ∂_i. In the sequel, repeated indices i and j are summed over. We have

$$\Delta^{G(Y)} = f_j \nabla_j^k + G_{i,j}\nabla_i^k \nabla_j^k. \tag{12}$$

where the coefficients f_i and G_{ij} do not depend on k. Then

$$\left[\nabla_X^k, f_j\nabla_j^k\right] = (X.f_j)\nabla_j^k + f_j \nabla^k_{[X,\partial_j]} + f_j R^k(X,\partial_j) \tag{13}$$

where R^k is the curvature of ∇^k. The first term of the right hand side clearly belongs to Op_{sc}, the third term also because

$$R^k(X,\partial_j) = R^\delta(X,\partial_j)$$

is independent of k. For the second term, observe that the holomorphic part of $\nabla^k_{[X,\partial_j]}$ is a linear combination of the ∇_j^k with smooth coefficients which do not depend on k. So the holomorphic part of (13) belongs to Op_{sc}. Let us compute the bracket of ∇_X^k with the second term of (12).

$$\left[\nabla_X^k, G_{ij}\nabla_i^k\nabla_j^k\right] = (X.G_{ij})\nabla_i^k\nabla_j^k + G_{ij}\left[\nabla_X^k, \nabla_i^k\right]\nabla_j^k + G_{ij}\nabla_i^k\left[\nabla_X^k, \nabla_j^k\right]$$

The first term of the right hand side belongs to Op_{sc}. The same holds for the holomorphic part of the third term because

$$G_{ij}\nabla_i^k\left[\nabla_X^k, \nabla_j^k\right] = G_{ij}\nabla_i^k\nabla^k_{[X,\partial_j]} + G_{ij}\nabla_i^k R^k(X,\partial_j).$$

and we can argue as we did for (13). The second term is equal to

$$\begin{aligned} G_{ij}\left[\nabla_X^k, \nabla_i^k\right]\nabla_j^k =& G_{ij}\nabla^k_{[X,\partial_i]}\nabla_j^k + G_{ij}R^k(X,\partial_i)\nabla_j^k \\ =& G_{ij}\nabla_j^k\nabla^k_{[X,\partial_i]} + G_{ij}\nabla^k_{[[X,\partial_i],\partial_j]} + G_{ij}R^k([X,\partial_i],\partial_j) \\ &+ G_{ij}R^k(X,\partial_i)\nabla_j^k \end{aligned}$$

All the terms of this last sum have a holomorphic part in Op_{sc} except the third one which is equal to

$$G_{ij}R^k([X,\partial_i],\partial_j) = G_{ij}R^\delta([X,\partial_i],\partial_j) + \frac{k}{i}G_{ij}\omega([X,\partial_i],\partial_j)$$

Since $\mu(Y) = G_{ij}\omega(\partial_i,\cdot)\partial_j$, we have

$$\mathrm{tr}(\mu(Y)\overline{\mu}(X)) = G_{ij}\omega(\partial_i, \overline{\mu}(X)(\partial_j))$$

Using that $[X,\partial_i] = -\frac{i}{2}\overline{\mu}(X)(\partial_i)$ modulo E, it follows that

$$\frac{k}{i}G_{ij}\omega([X,\partial_i],\partial_j) = \frac{k}{2}\,\mathrm{tr}(\mu(Y)\overline{\mu}(X))$$

Collecting the various terms, we obtain the result. $\square$

Let us conclude the proof of theorem 5.2. We have

$$\begin{aligned} R_k(X,Y) =& R^\delta(X,Y) + \frac{1}{k}\big[\nabla_X^{L^k\otimes\delta}, P_k(Y)\big] - \frac{1}{k}\big[\nabla_Y^{L^k\otimes\delta}, P_k(X)\big] \\ &+ \frac{1}{k^2}\big[P_k(X), P_k(Y)\big] - \frac{1}{k}P_k([X,Y]) \end{aligned}$$

By propositions 5.3 and 5.4, the holomorphic part of the sum of the first three terms is in $k^{-1}\mathrm{Op}_{sc}$. The last two terms belong respectively to $k^{-2}\mathrm{Op}_{sc}$ and $k^{-1}\mathrm{Op}_{sc}$.

6 Semi-classical connection

Let (M,ω) be a compact symplectic manifold with a prequantum bundle $L_M \to M$. Consider a manifold U and a smooth family $(j_u,\delta_u,\varphi_u)_{u\in U}$ consisting of compatible complex structures with half-form bundles.

We adopt the same notations and conventions as in sections 2.2 and 3.1. Namely, $M_u = \{u\}\times M$ is endowed with the complex structure j_u, $L_u \to M_u$ is the prequantum bundle with the holomorphic structure induced by j_u. We denote by L, δ and E the bundles over $U\times M$ whose restrictions to each M_u are L_u, δ_u and $T^{1,0}M_u$. Let $\mathcal{H}_k^m$ be the vector bundle over U whose fibers are the Hilbert spaces

$$\mathcal{H}_{k,u}^m = H^0(M_u, L_u^k\otimes\delta_u)$$

and denote by $\Pi_{k,u}$ the orthogonal projector from $\Gamma(M_u, L_u^k\otimes\delta_u)$ onto $\mathcal{H}_{k,u}^m$.

We now define a connection of the bundle $\mathcal{H}_k^m$. Consider the same connections on δ and L as in section 3.1. We set for any vector field X of U and section s of $\mathcal{H}_k^m$

$$(\nabla_X^{\mathrm{Toep},k} s)(u) := \Pi_{k,u}\big((\nabla_X^{L^k\otimes\delta} s)(u)\big)$$

It is easily proved that this is indeed a connection. More generally we shall consider the connections

$$\nabla^{\mathrm{Toep},k} + A_k, \qquad A_k \in \Omega^1(U, \mathrm{End}(\mathcal{H}_k^m))$$

where the family $(A_k, k = 1, 2, \ldots)$ is a Toeplitz operator. This has the following meaning. Let p be the projection from $U \times M$ onto U. Then there exists a sequence $f(\cdot, k)$ of $\Gamma(U \times M, p^*(T^*U \otimes \mathbb{C}))$ admitting an asymptotic expansion of the form $f_0 + k^{-1} f_1 + \ldots$ for the $\mathcal{C}^\infty$-topology on the compact subsets, such that

$$A_k(X)(u) = \Pi_{k,u} M_{f(X)(\cdot,u,k)} : \mathcal{H}_{k,u}^m \to \mathcal{H}_{k,u}^m$$

for any vector field X of M. Here M_g denote the multiplication operator by g. We call f_0 the principal symbol of (A_k). This includes the connection defined in section 3.2. Indeed proposition 4.1 implies the

Proposition 6.1. *There exists $f_1 \in \Gamma(U \times M, \pi^*(T^*U) \otimes \mathbb{C})$ such that for any $k \geqslant k_0$, we have*

$$\nabla_X^{\mathcal{H}_k^m} = \nabla_X^{\mathrm{Toep},k} + k^{-1} \Pi_k M_{f_1(X)}$$

for any vector field X of U.

The following theorem says that the curvature of these connections is a Toeplitz operator in a semi-classical sense.

Theorem 6.2 ([10], Theorem 7.1)**.** *There exists a sequence $g(\cdot, k) \in \Gamma(U \times M, p^*(\wedge^2 T^*U \otimes \mathbb{C}))$ admitting an asymptotic expansion of the form $g_0 + k^{-1} g_1 + \ldots$ for the $\mathcal{C}^\infty$-topology on compact subsets, such that the curvature of $\nabla^{\mathrm{Toep},k} + A_k$ satisfies*

$$R^{A_k}(X,Y)_u = \Pi_{k,u} M_{g(X,Y)(\cdot,u,k)} + O(k^{-\infty})$$

where the $O(k^{-\infty})$ is uniform on compact set of U. Furthermore, g_0 is given by

$$g_0(X,Y) = X.f_0(Y) - Y.f_0(X) - f_0([X,Y])$$

with f_0 the principal symbol of (A_k).

To describe the parallel transport along a path γ in the bundle $\mathcal{H}^m_k$ we introduce the notion of half-form isomorphism. For any $u, u' \in U$ and $x \in M$ denote by $\pi_{u',u,x}$ the projection from $E_{u',x}$ to $E_{u,x}$ with kernel $\overline{E}_{u',x}$. We say that a linear isomorphism Ψ of $\mathrm{Hom}(\delta_{u,x}, \delta_{u',x})$ is a *half-form isomorphism* if its square is the pull-back by $\pi_{u',u,x}$, more precisely

$$\varphi_{u',x} \circ \Psi^2 = \pi^*_{u',u,x} \circ \varphi_{u,x}.$$

Such an isomorphism is unique up to a plus or minus sign. If γ is a path of U, then for any x, there exists a unique continuous path of half-form morphism $\delta_{\gamma(0),x} \to \delta_{\gamma(t),x}$ starting from the identity. We denote by $\Psi(\gamma)$ the morphism $\delta_{\gamma(0)} \to \delta_{\gamma(1)}$ obtained at $t = 1$.

Theorem 6.3 ([10], Theorem 7.1)**.** *Let $u, u' \in U$ and γ be a path from u to u'. For any $k \geqslant k_0$, let $T_k : \mathcal{H}^m_{k,u} \to \mathcal{H}^m_{k,u'}$ be the parallel transport along γ in the bundle $\mathcal{H}^m_k$ for the connection $\nabla^{\mathrm{Toep},k} + A_k$. Then the Schwartz kernels of the operators T_k have the following form*

$$T_k(x, y) = \left(\frac{k}{2\pi}\right)^n F^k(x, y) \otimes f(x, y, k) + O(k^{-\infty})$$

where n is half the dimension of M and

- *F is a section of $L \boxtimes \overline{L}$ such that $|F(x, y)| < 1$ if $x \neq y$, $F(x, x) = v \otimes \bar{v}$ for all x and $v \in L_x$ of norm 1, and $\bar{\partial}_{j_{u'} \times -j_u} F \equiv 0$ modulo a section vanishing to any order along the diagonal.*

- *$f(., k)$ is a sequence of sections of $\delta_{u'} \boxtimes \bar{\delta}_u \to M^2$ which admits an asymptotic expansion in the $\mathcal{C}^\infty$ topology of the form*

  $$h(., k) = h_0 + k^{-1} h_1 + k^{-2} h_2 + ...$$

  whose coefficients satisfy $\bar{\partial}_{j_{u'} \times -j_u} f_i \equiv 0$ modulo a section vanishing to any order along the diagonal.

- *If the principal symbol of (A_k) vanishes, then $h_0(x, x) = \Psi(\gamma).v \otimes \bar{v}$ for any $x \in M$ and $v \in \delta_{u,x}$ with norm 1.*

Theorems 6.2 and Theorems 6.3 are generalizations of theorem 7.1 of [10], where we did not consider the terms (A_k). The proof is an immediate generalization of the one in [10]. Theorems 6.2 provides another proof of theorem 5.1.

Consider an automorphism $\Phi_L : L \to L$ of the prequantum bundle L, that is an automorphism of line bundle preserving the connection and the

Hermitian structure. Φ_L lifts a Hamiltonian diffeomorphism Φ of M. Let $u_0,\ u_1 \in U$ such that $\Phi_*(j_{u_0}) = j_{u_1}$. Then Φ induces a isomorphims from the canonical bundle of M_{u_0} to the canonical bundle of M_{u_1}. Assume that this isomorphism has a square root $\Phi_\delta : \delta_{u_0} \to \delta_{u_1}$. Then the isomorphism $\Phi_L^k \otimes \Phi_\delta$ defines by push-forward a familly of linear map

$$V_k : \mathcal{H}^m_{k,u_0} \to \mathcal{H}^m_{k,u_1}, \qquad k \in \mathbb{N}^*.$$

Consider a path γ in U from u_1 to u_0. For any positive k, let

$$T_k : \mathcal{H}^m_{k,u_1} \to \mathcal{H}^m_{k,u_0}$$

be the parallel transport along γ in the bundle $\mathcal{H}^m_k$ for the connection $\nabla^{\mathrm{Toep},k} + A_k$. The operator

$$U_k = T_k \circ V_k : \mathcal{H}^m_{k,u_0} \to \mathcal{H}^m_{k,u_0}$$

has to be considered as a quantization of the symplectomorphism Φ. Denote by $x \to U_k(x)$ the restriction of the Schwartz kernel of U_k to the diagonal, so that

$$\operatorname{Tr}(U_k) = \int U_k(x) \mu_M(x)$$

where μ_M is the Liouville measure of M. Theorem 6.3 gives us the asymptotic behaviour of $U_k(x)$. The following result has been proved in [15], Theorem 5.3.1.

Theorem 6.4. *([15]) Let $f \in \mathcal{C}^\infty(M)$. Assume that the fixed points x of Φ contained in the support of f are all non-degenerate, meaning that* $\mathrm{id} - T_x\Phi$ *is an isomorphism of T_xM. Then we have*

$$\int_M f(x) U_k(x) \mu_M(x) = \sum_{x \in \operatorname{Supp} f /\ \Phi(x) = x} f(x) \frac{i^{m(x)} u_x^k}{|\det(\mathrm{id} - T_x\Phi)|^{1/2}} + \mathcal{O}(k^{-1}).$$

where for any fixed point x of Φ, $u_x \in \mathbb{C}$ is the trace of the endomorphism $\Phi_{L,x} : L_x \to L_x$ and $m_x \in \mathbb{Z}/4\mathbb{Z}$ is the index of $\Psi(\gamma)_x \circ \Phi_{\delta,x} : \delta_{u_0,x} \to \delta_{u_0,x}$.

The index m_x is defined in [15]. The proof is an application of the stationnary phase lemma, the fact that the fixed points are non-degenerate is equivalent to the fact that the critical point of the phase are non-degenerate.

7 Proof of Theorem 1.1

Moduli spaces of $SU(n)$ bundles on surfaces

Let Σ be a compact connected oriented surface of genus $\geqslant 2$ and whose boundary is a circle, denoted by C. Let n, d be two coprime integers with $n \geqslant 2$. Consider the moduli space $\mathcal{M}$ of flat $\mathrm{SU}(n)$-principal bundles over Σ such that the holonomy of C is equal to $\exp(2i\pi d/n)\,\mathrm{id}$. Since $(n,d)=1$, $\mathcal{M}$ is a smooth compact manifold.

Let $\bar{\Sigma} \supset \Sigma$ be the closed surface obtained by gluing a disk D along C. For any $[P] \in \mathcal{M}$, the bundle associated to P via the adjoint representation is a flat real vector bundle over Σ such that the holonomy of C is trivial. So this associated bundle is the restriction of a flat real vector bundle $\mathrm{Ad}\,P$ over $\bar{\Sigma}$, unique up to isomorphism. The tangent space of the moduli space at $[P]$ is

$$T_{[P]}\mathcal{M} \simeq H^1(\bar{\Sigma}, \mathrm{Ad}\,P).$$

The bundle $\mathrm{Ad}\,P$ has a natural metric coming from the basic scalar product of $\mathfrak{su}(n)$

$$a \cdot b = -\frac{1}{4\pi^2}\,\mathrm{tr}(ab), \qquad a, b \in \mathfrak{su}(n)$$

Atiyah and Bott [5] introduced a symplectic form $\omega_{\mathcal{M}}$ on $\mathcal{M}$. It is given by

$$\omega_{\mathcal{M}}([a],[b]) = 2\pi \int_{\Sigma} a \cdot b \tag{14}$$

where a and b are any closed forms of $\Omega^1(\bar{\Sigma}, \mathrm{Ad}\,P)$.

Prequantum bundle

The following facts are proved in [5], [29], [27]: $\mathcal{M}$ is simply connected, it has no torsion, its second Betti number is one and $n\omega_{\mathcal{M}}$ is a generator of $H^2(\mathcal{M}, \mathbb{Z}) \subset H^2(\mathcal{M}, \mathbb{R})$. So there exists a Hermitian line bundle

$$L \to \mathcal{M}$$

equipped with a connection of curvature $\frac{n}{i}\omega_{\mathcal{M}}$. Since $\mathcal{M}$ is simply connected, L is unique up to isomorphism.

Various explicit constructions of L as a quotient in gauge theory were given in [16], [26] and [9] extending the construction of Ramadas-Singer-Weitsman [28] in the case Σ has no boundary. This is important for our purpose because one can deduce that the mapping class group action on $\mathcal{M}$ lifts to L. The relevant mapping class groups here are

$$\operatorname{Mod}(\Sigma) := \pi_0(\operatorname{Diff}^+(\Sigma, C)) = \pi_0(\operatorname{Diff}^+(\bar{\Sigma}, D))$$

that we denote also by Mod and

$$\operatorname{Mod}(\bar{\Sigma}, p) := \pi_0(\operatorname{Diff}^+(\bar{\Sigma}, p))$$

where p is a point in the interior of D. Recall that $\operatorname{Mod}(\bar{\Sigma}, p)$ is an extension of Mod by $\mathbb{Z}$,

$$1 \to \mathbb{Z} \to \operatorname{Mod}(\Sigma) \to \operatorname{Mod}(\bar{\Sigma}, p) \to 1$$

where the kernel is generated by a Dehn twist on C.

The following facts are explained in detail in the paper [12]. First, $\operatorname{Mod}(\bar{\Sigma}, p)$ acts on $\mathcal{M}$ by symplectomorphism. Second, $\operatorname{Mod}(\Sigma)$ acts on L by automorphisms of prequantum bundles. This action lifts the action of $\operatorname{Mod}(\bar{\Sigma}, p)$ on $\mathcal{M}$. Nevertheless it does not in general factor through an action of $\operatorname{Mod}(\bar{\Sigma}, p)$. Indeed, the Dehn twist on C acts on L by multiplication by $\exp\big(i\pi(n-1)d^2\big)$ in each fiber.

Complex structure

Suppose $\bar{\Sigma}$ is endowed a complex structure compatible with the orientation. Then by Hodge decomposition, for any $[P] \in \mathcal{M}$

$$H^1(\bar{\Sigma}, \operatorname{Ad} P) \otimes \mathbb{C} = H^{0,1}(\bar{\Sigma}, (\operatorname{Ad} P) \otimes \mathbb{C}) \oplus H^{1,0}(\bar{\Sigma}, (\operatorname{Ad} P) \otimes \mathbb{C})$$

$\mathcal{M}$ has a complex structure such that the holomorphic tangent space at $[P]$ is the first summand in the previous decomposition. This complex structure is integrable, compatible with $\omega_{\mathcal{M}}$ and positive. So it makes $\mathcal{M}$ a Kähler manifold. It may also be defined by identifying $\mathcal{M}$ with the moduli space of holomorphic vector bundles of rank n, degree d with a fixed determinant through the Narasimhan-Seshadri theorem.

Let $\mathcal{A}$ be the space of complex structures of Σ and

$$\mathcal{T} := \mathcal{A} / \operatorname{Diff}^+_0(\bar{\Sigma})$$

be the Teichmüller space. As a fact, the complex structure of $\mathcal{M}$ induced by $j \in \mathcal{A}$ only depends on the class of j in the Teichmüller space.

The mapping class group $\operatorname{Mod}(\bar{\Sigma}) := \pi_0(\operatorname{Diff}^+(\bar{\Sigma}))$ acts on $\mathcal{T}$. This action is compatible with the one of $\operatorname{Mod}(\bar{\Sigma}, p)$ on $\mathcal{M}$, meaning that for any $\gamma \in \operatorname{Mod}(\bar{\Sigma}, p)$ and $\sigma \in \mathcal{T}$, the action of γ on $\mathcal{M}$ sends the complex structure induced by σ into the one induced by $\pi(\gamma).\sigma$ where π is the natural map $\operatorname{Mod}(\bar{\Sigma}, p) \to \operatorname{Mod}(\bar{\Sigma})$.

Quantum representation

For any class σ in the Teichmüller space, we let $\mathcal{M}_\sigma$ be $\mathcal{M}$ endowed with the complex structure induced by σ and $L_\sigma \to \mathcal{M}_\sigma$ be the line bundle L with the corresponding holomorphic structure. Define

$$\mathcal{H}_{k,\sigma} := H^0(\mathcal{M}_\sigma, L_\sigma^k), \qquad \sigma \in \mathcal{T}.$$

Since the canonical class of $\mathcal{M}_\sigma$ is negative, the dimension of $\mathcal{H}_{k,\sigma}$ does not depend on σ. So for any $k \in \mathbb{N}^*$, we can consider the bundle $\mathcal{H}_k \to \mathcal{T}$ with fiber $\mathcal{H}_{k,\sigma}$.

Applying Theorem 3.1, we get a connection $\nabla^{\mathcal{H}_k}$ on $\mathcal{H}_k$. Hitchin proved that this connection is projectively flat [20].

The mapping class group Mod acts on the bundle $\mathcal{H}_k$ by preserving the connection. So we can deduce a projective representation of Mod as follows. Choose $\sigma_0 \in \mathcal{T}$. For any $h \in \mathrm{Mod}$, define

$$\rho_k(h) \in \mathrm{PGL}(\mathcal{H}_{k,\sigma_0})$$

as the composition of the action of h from $\mathcal{H}_{k,\sigma_0}$ to $\mathcal{H}_{k,h.\sigma_0}$ with the parallel transport in $\mathcal{H}_k$ along a path from $h.\sigma_0$ to σ_0. The connection being projectively flat, $\rho_k(h)$ does not depend on the choice of the path. The connection being Mod-invariant, $\rho_k(h)\rho_k(h') = \rho_k(h.h')$. This defines the family of projective representation

$$\rho_k : \mathrm{Mod} \to \mathrm{PGl}(\mathcal{H}_{k,\sigma_0}), \qquad k \in \mathbb{N}^*.$$

considered in the introduction (1).

Metaplectic correction

By [29], the canonical class of $\mathcal{M}$ has a square root. The Teichmüller space being contractible, there exist a smooth familly $(\delta_\sigma \to \mathcal{M}_\sigma, \sigma \in \mathcal{T})$ of half-form bundle. So we can introduce the bundle $\mathcal{H}_k^m \to \mathcal{T}$ with fibers

$$\mathcal{H}_k^m = H^0(\mathcal{M}_\sigma, L_\sigma^k \otimes \delta_\sigma), \qquad \sigma \in \mathcal{T}.$$

By Theorem 3.2, this bundle has a natural connection $\nabla^{\mathcal{H}_k^m}$.

Actually, by [26], the canonical class of $\mathcal{M}$ is $-2c_1(L)$. So the fiber bundles δ_σ and L^{-1} are isomorphic. Since the Jacobian variety of each $\mathcal{M}_\sigma$ is trivial, there even exists a holomorphic bundle isomorphism between δ_σ and L_σ^{-1}, unique up to $\mathbb{C}^*$. Using again that $\mathcal{T}$ is contractible, we can choose these isomorphisms in such a way that they depend smoothly on σ and we obtain an isomorphism

$$\mathcal{H}_k^m \simeq \mathcal{H}_{k+1}.$$

So we can compare the two connections $\nabla^{\mathcal{H}_k^m}$ and $\nabla^{\mathcal{H}_{k+1}}$. It follows from the discussion in section 2.4 that these connections differ by $\alpha\,\mathrm{id}$, with $\alpha \in \Omega^1(\mathcal{T})$. As a consequence, $\nabla^{\mathcal{H}_k^m}$ is projectively flat.

The asymptotic representation

As in the introduction, consider the group $\widetilde{\mathrm{Mod}}$ defined as the fiber product of Mod and the group of automorphisms of the (unique) metaplectic structure of $\mathcal{M}$. Any $h \in \widetilde{\mathrm{Mod}}$ determines a continuous family ($\varphi_\sigma : \delta_\sigma \to \delta_{p(h).\sigma}, \sigma \in \mathcal{T}$) of half-form bundle isomorphisms. [1] Here p is the projection from $\widetilde{\mathrm{Mod}}$ to Mod.

So $\widetilde{\mathrm{Mod}}$ acts on the bundle $\mathcal{H}_k^m$ by preserving the connection $\nabla^{\mathcal{H}_k^m}$. Choose $\sigma_0 \in \mathcal{T}$ and for any $h \in \mathrm{Mod}(\bar{\Sigma})$ a path from $h.\sigma_0$ to σ_0. Then composing the $\widetilde{\mathrm{Mod}}$-action with the parallel transport in $\mathcal{H}_k^m$ along the convenient path, we obtain a family of invertible operators

$$\rho_k^m(h) : \mathcal{H}_{k,\sigma_0}^m \to \mathcal{H}_{k,\sigma_0}^m, \qquad h \in \widetilde{\mathrm{Mod}}.$$

We can now procced to the proof of Theorem 1.1. Since the connections $\nabla^{\mathcal{H}_k^m}$ and $\nabla^{\mathcal{H}_{k+1}}$ are the same up to a projective term, we have

$$[\rho_k^m(h)] = \rho_{k+1}(p(h)), \qquad \forall h \in \widetilde{\mathrm{Mod}}.$$

Since the connection $\nabla^{\mathcal{H}_k^m}$ is projectively flat,

$$\rho_k^m(h.h') = c_k(h,h')\rho_k^m(h)\rho_k^m(h'), \qquad \forall h,h' \in \widetilde{\mathrm{Mod}}$$

with $c_k(h,h') \in \mathbb{C}^*$. Since the connection $\nabla^{\mathcal{H}_k^m}$ is asymptotically flat in the sense of Theorem 5.1,

$$c_k(h,h') = 1 + \mathcal{O}(k^{-1}), \qquad \forall h,h' \in \widetilde{\mathrm{Mod}}.$$

So the familly ($\tilde{\rho}_k := \rho_{k+1}^m, k \in \mathbb{N}^*$) satisfies all the conditions of Theorem 1.1.

[1] φ_σ is a half-form bundle isomorphism means that the square of φ_σ is the isomorphism between the canonical bundles of $\mathcal{M}_\sigma$ and $\mathcal{M}_{\gamma.\sigma}$ induced by the action of γ.

8 Proof of Theorems 1.2 and 1.3

8.1 Moduli spaces of mapping tori

Consider the same setting as in the previous section. Let $h \in \widetilde{\mathrm{Mod}}$ and $\phi \in \mathrm{Diff}^+(\Sigma, C)$ be a representative of $p(h)$. The proof consists in applying Theorems 6.4 to $\rho_k(h)$ and reinterpreting the assumption and the invariants involved in the results in terms of the mapping torus of ϕ

$$\Sigma_\phi := (\Sigma \times \mathbb{R})/(x, t+1) \sim (\phi(x), t).$$

For $X = \Sigma$ or Σ_ϕ, let $\mathcal{M}_n(X)$ be the moduli space of flat $SU(n)$-principal bundles with base X. Denote by $j : \Sigma \to \Sigma_\phi$ the map sending x into $[x, 0]$.

Lemma 8.1. *There is a one-to one equivalence between $\mathcal{M}_n(\Sigma_\phi)$ and the set of isomorphism class of pairs (F, ϕ_F) such that $F \in \mathcal{M}_n(\Sigma)$ and ϕ_F is an automorphism of F lifting ϕ.*

Proof. Given $\tilde{F} \in \mathcal{M}_n(\Sigma_\phi)$, we associate to it its restriction $F = j^*\tilde{F}$ to Σ and the automorphism ϕ_F of F obtained by parallel transport in $\tilde{F}$ along the path $[0,1] \ni ct \to [t, x]$. Conversely, given a pair (F, ϕ_F) we define $\tilde{F}$ as the mapping torus of ϕ_F. $\square$

Embedd C into Σ_ϕ through the map $x \to [x, 0]$. Introduce the moduli space $\mathcal{M}' \subset \mathcal{M}_n(\Sigma_\phi)$ consisting of the bundles such that the holonomy of C is $\exp(2i\pi d/n)\,\mathrm{id}$.

Recall that a flat $SU(n)$-principal bundle is said to be irreducible if its gauge automorphism group is the center $\mathbb{Z}_n$ of $SU(n)$. When n and d are coprime, the bundles in $\mathcal{M}$ are all irreducible. So by Lemma 8.1, the fibers of the map

$$p : \mathcal{M}' \to \mathcal{M}, \qquad \tilde{F} \to F = j^*\tilde{F} \tag{15}$$

are the orbits of the $\mathbb{Z}_n$-action given by $u.(F, \phi_F) = (F, \phi_F u)$. By Lemma 8.1, the image of p is the set of fixed point of $\phi_* : \mathcal{M} \to \mathcal{M}$.

Lemma 8.2. *Let $\tilde{F}$ be an irreducible bundle of $\mathcal{M}_n(\Sigma_\phi)$ corresponding to a pair (F, ϕ_F). Then the following sequence is exact*

$$0 \to H^1(\bar{\Sigma}_\phi, \mathrm{Ad}\,\tilde{F}) \xrightarrow{j^*} H^1(\bar{\Sigma}, \mathrm{Ad}\,F) \xrightarrow{\mathrm{id} - L(F)} H^1(\bar{\Sigma}, \mathrm{Ad}\,F)$$

where $L(F)$ is the linear map $(\mathrm{Ad}\,\phi_F)_* : H^1(\bar{\Sigma}, \mathrm{Ad}\,F) \to H^1(\bar{\Sigma}, \mathrm{Ad}\,F)$. If furthermore $H^1(\bar{\Sigma}_\phi, \mathrm{Ad}\,\tilde{F}) = (0)$, then the Reidemeister torsion of the flat vector bundle $\mathrm{Ad}\,\tilde{F}$ satisfies

$$|\tau(\mathrm{Ad}\,\tilde{F})| = |\det(\mathrm{id} - L(F))|^{-1}.$$

Proof. This follows from the standard computation of the cohomology of a mapping torus by using the Mayer-Vietoris sequence, cf. page 87 of [23]. $\square$

Let us apply Theorem 6.4 to the sequence $U_k = \tilde{\rho}_k(h)$. The symplectomorphism Φ is the map $\phi_* : \mathcal{M} \to \mathcal{M}$. The tangent space $T_F\mathcal{M}$ is naturally isomorphic to $H^1(\bar{\Sigma}, \operatorname{Ad} F)$, so that the linear tangent map $T_F\Phi$ corresponds to $L(F)$. So by Lemma 8.2, the transversality assumption of Theorem 6.4 is equivalent to the fact that $H^1(\bar{\Sigma}_\phi, \operatorname{Ad} \tilde{F}) = 0$. Furthermore $|\det(\mathrm{id} - T_F\varphi)|^{-1}$ is equal to the torsion of $\operatorname{Ad} \tilde{F}$. To complete the proof of Theorems 1.2 and 1.3 it remains to express the complex number u_x appearing in Theorem 6.4 in terms of Chern-Simons invariant.

Remark 8.1. Lemma 8.2 may be naturally interpreted with the map p. Indeed the Zariski tangent space $T_{\tilde{F}}\mathcal{M}'$ is isomorphic to $H^1(\bar{\Sigma}_\phi, \operatorname{Ad} \tilde{F})$ so that $T_F p = j^*$.

8.2 Chern-Simons bundle

Let $\mathcal{A}$ be the vector space $\Omega^1(\Sigma, \mathfrak{su}(n))$. Consider the trivial line bundle $\mathcal{A} \times \mathbb{C}$ with connection $\nabla = d + \frac{2\pi}{i} C$ where

$$C_a(\dot{a}) = \frac{1}{2} \int_\Sigma a \cdot \dot{a}, \qquad a, \dot{a} \in \mathcal{A}$$

where the dot stands for the basic scalar product.

The elements of $\mathcal{A}$ may be viewed as connection of the trivial $SU(n)$-principal bundle $\Sigma \times SU(n)$. The curvature is given by $F(a) = da + \frac{1}{2}[a, a]$. The gauge group $\mathcal{G} = \mathcal{C}^\infty(\Sigma, SU(n))$ acts on $\mathcal{A}$ by $a^g = gag^{-1} - dgg^{-1}$. Choose $b \in \Omega^1(C, \mathfrak{su}(n))$ with holonomy $\exp(2i\pi d/n)\,\mathrm{id}$. Denote by $\mathcal{A}_b^\flat$ the subset of $\mathcal{A}$ consisting of the elements which are flat and restrict on C to b. Let $\mathcal{G}_b$ be the subgroup of $\mathcal{G}$ consisting of the elements whose restriction to the boundary fixes b. Then the map sending the flat connection $a \in \mathcal{A}_b^\flat$ to the flat associated bundle factors to a bijection from $\mathcal{A}_b^\flat / \mathcal{G}_b$ to $\mathcal{M}$.

Proposition 8.3 ([12])**.** *There is a unique action of $\mathcal{G}_b$ on $\mathcal{A}_b^\flat \times \mathbb{C}$ lifting the action on $\mathcal{A}_b^\flat$ and such that for any path $[0,1] \to \mathcal{G}_b$, $t \to g_t$ and $a \in \mathcal{A}_b^\flat$, the path $t \to g_t.(a, 1)$ is horizontal. The quotient $\mathcal{A}_b^\flat \times \mathbb{C}/\mathcal{G}_b$ is the Chern-Simons bundle $L_{CS} \to \mathcal{M}$.*

Here there is the subtlety that the group $\mathcal{A}_b$ does not act freely on $\mathcal{M}$. Actually the isotropy group of any $F \in \mathcal{M}$ is the center of $SU(n)$. This isotropy group does not act trivially on $\mathcal{A}_b^\flat \times \mathbb{C}$, so that the quotient L_{CS} is not a genuine line bundle but an orbifold bundle. Nevertheless by Lemma

3.3 of [12], the n-th power of the isotropy group action is trivial, so that $L := L^n_{\mathrm{CS}}$ is a honnest line bundle.

Let us compute the parallel transport in terms of a Chern-Simons invariant. Recall that for any 3-dimensional compact oriented manifold X,

$$\mathrm{CS}_X(a) = \frac{1}{8\pi^2}\int_X \mathrm{tr}(ada) + \frac{2}{3}a^3 = -\frac{1}{2}\int_M \big(a \cdot F(a) - \frac{1}{6} a \cdot [a,a]\big)$$

for all $a \in \Omega^1(X, \mathfrak{su}(n))$.

Proposition 8.4. *Let $(g_t, t \in [0,1])$ be a path of $\mathcal{G}_b$ starting from the identity. Let $a_0 \in \mathcal{A}_b^\flat$ and set $g_t.(a_0, 1) = (a_t, e^{2i\pi\varphi_t})$. Then*

$$\varphi_1 = \mathrm{CS}_X(a^g) + \frac{1}{2}\int_{\partial\Sigma\times[0,1]} b \cdot X_t dt \tag{16}$$

where $X = \Sigma \times [0,1]$, $X_t = \dot{g}_t g_t^{-1}$, $a \in \Omega^1(X, \mathfrak{su}(n))$ is the pull-back of a_0 by the projection $X \to \Sigma$, and $g \in \mathcal{C}^\infty(X, SU(n))$ is such that for any t, its restriction to $\Sigma \times \{t\}$ is g_t.

Proof. Since $t \to (a_t, e^{2i\pi\varphi_t})$ is horizontal, we have $\dot{\varphi}_t = C_{a_t}(\dot{a}_t) = \frac{1}{2}\int_\Sigma a_t \cdot \dot{a}_t$ so that

$$\varphi_1 = \frac{1}{2}\int_0^1 dt \int_\Sigma a_t \cdot \dot{a}_t$$

A straightforward computation shows that $\dot{a}_t = [X_t, a_t] - dX_t$. Consequently,

$$\begin{aligned} a_t \cdot \dot{a}_t =& a_t \cdot [X_t, a_t] - a_t \cdot dX_t \\ =& a_t \cdot [X_t, a_t] - d(a_t \cdot X_t) - da_t \cdot X_t \\ =& -\tfrac{1}{2} a_t \cdot [a_t, a_t] + d(a_t \cdot X_t) \end{aligned}$$

where we have used at the last line that $F(a_t) = 0$. Since $a^g = a_t - X_t dt$, we obtain

$$a_t \cdot \dot{a}_t \; dt = \frac{1}{6} a^g \cdot [a^g, a^g] + d(a_t \cdot X_t \wedge dt)$$

To end the proof, we integrate this expression on $\Sigma \times [0,1]$, apply Stokes Theorem and observe that a^g is flat so that its Chern-Simons simplifies. $\square$

8.3 Automorphisms

Consider now $\phi \in \mathrm{Diff}^+(\Sigma, C)$. Such a map induces an automorphism of the $\mathcal{G}_b$-bundle $\mathcal{A}_b^\flat \times \mathbb{C}$. Taking the quotient we get a prequantum bundle automorphism of $L \to \mathcal{M}$ lifting the action of ϕ on the moduli space.

Consider $F \in \mathcal{M}_{n,d}(\Sigma)$ such that $\phi^* F = F$. Then ϕ_* preserves the fiber of L at F and acts on it by multiplication by a complex number u_F. We can compute this number in terms of the mapping torus Σ_ϕ. Choose an automorphism ϕ_F of F lifting ϕ and consider the mapping torus $\tilde{F}$ of ϕ_F. Recall that the boundary of Σ_ϕ is the torus $T = C \times \mathbb{R}/\mathbb{Z}$.

Proposition 8.5. *Let $\beta \in \Omega^1(\Sigma_\phi, \mathfrak{su}(n))$ be any connection representing $\tilde{F}$ and such that its restriction to T has the form $b - X dt$ with $X \in \mathcal{C}^\infty(T, \mathfrak{su}(n))$. Then $u_F = \exp(2i\pi n\theta)$ with*

$$\theta = \mathrm{CS}_{\Sigma_\phi}(\beta) + \frac{1}{2}\int_T b \cdot X_t dt \tag{17}$$

Proof. Denoting by j the injection $\Sigma \to \Sigma_\phi$ sending x into $[x, 0]$, we set $a_0 = j^*\beta$. Consider $a, \beta' \in \Omega^1(\Sigma \times [0,1], \mathfrak{su}(n))$ defined respectively as the pull-bak of a_0 by the projection $\Sigma \times [0,1] \to \Sigma$ and the pull-back of β by the map $\Sigma \times [0,1] \to \Sigma_\phi$. Since a and β' are flat and their restriction to $\Sigma \times \{0\}$ are equal, there exists $g \in \mathcal{C}^\infty(\Sigma \times [0,1], SU(n))$ such that $\beta' = a^g$. Denote by g_t the restriction of g to $\Sigma \times \{t\}$. We have that $a_0^{g_1} = \phi^* a_0$. So the map ϕ_* from L to itself sends $[a_0^{g_1}, z]$ into $[a_0, z]$. By proposition 8.4, if we choose g in such a way that $g_0 = \mathrm{id}$, we have that $[a_0, z] = [a_0^{g_1}, z\exp(2i\pi n\theta)]$ where

$$\theta = \mathrm{CS}_{\Sigma \times [0,1]}(\beta') + \frac{1}{2}\int_T b \cdot X_t dt$$

Since $\mathrm{CS}_{\Sigma\times[0,1]}(\beta') = \mathrm{CS}_{\Sigma_\phi}(\beta)$, the result follows. ◻

This completes the proof of Theorems 1.2 and 1.3.

8.4 Scalar product in the Chern-Simons bundle of the torus

Proposition 8.5 implies that the number $\exp(2i\pi n\theta)$ with θ given by Equation (17) only depends on the gauge class of β. In this last section we give another proof of this fact by interpreting $\exp(2i\pi\theta)$ as a scalar product in the Chern-Simons bundle of the torus T.

Let $\mathcal{M}_n(T)$ be the moduli space of flat $SU(n)$-principal bundle on T. Introduce the associated Chern-Simons bundle $L_{\mathrm{CS}}(T) \to \mathcal{M}_n(T)$. This bundle has the property that for any compact oriented 3-dimensional manifold X with boundary T, for any $\beta \in \Omega^1(X, \mathfrak{su}(n))$, the class

$$[\beta, \exp(2i\pi\,\mathrm{CS}(\beta))] \in L_{\mathrm{CS}}(T)$$

only depends on the gauge class of β. In particular, the boundary of Σ_ϕ being T, for any flat bundle $\tilde{F} \to \Sigma_\phi$, $\exp(2i\pi\,\mathrm{CS}(\tilde{F}))$ is a well-defined vector in the fiber of $L_{CS}(T)$ at $\tilde{F}|_T$.

Let $\mathfrak{t}$ be the subspace of $\mathfrak{su}(n)$ consisting of diagonal matrices. Let $\Lambda \subset \mathfrak{t}$ be the kernel of the exponential map $\exp : \mathfrak{t} \to SU(n))$. Let W be the Weyl group of $SU(n)$. The space $H_1(T)$ has a preferred basis (ℓ, m) where ℓ and m are respectively the class of C and of $t \to [x,t]$. Consequently $\mathcal{M}_n(T)$ may be naturally identified with the quotient $\mathfrak{t}^2/\Lambda^2 \rtimes W$ by the map sending $(p,q) \in \mathfrak{t}^2$ to the bundle such that the holonomies of ℓ and m are respectively $\exp(p)$, $\exp(q)$. As explained in [11], Section 3.2, the Chern-Simons bundle is naturally isomorphic with the quotient of $\mathfrak{t}^2 \times \mathbb{C}$ by $\Lambda^2 \rtimes W$ where Λ^2 acts by

$$(\dot{p},\dot{q}).(p,q,z) = (p+\dot{p}, q+\dot{q}, z\exp(i\pi k(\dot{p}\dot{q} + \dot{p}q - p\dot{q})))$$

and W by $w.(p,q,z) = (w(p), w(q), z)$.

Let $\Lambda^* \subset \mathfrak{t}$ be the dual lattice with respect to the basic scalar product.

Proposition 8.6. *For any $p, q \in \mathfrak{t}$ with $p \in k^{-1}\Lambda^*$, the fiber of $L^k_{CS}(T)$ at $F = [p,q]$ has a canonical vector given by*

$$v(F,k) = [p, q, \exp(ik\pi pq)].$$

Proof. We have to check that $u(\gamma, k)$ does not depend on p, q. Let $(\dot{p},\dot{q}) \in \Lambda^2$. Since $p \in k^{-1}\Lambda^*$ we have $kp\dot{q} \in \mathbb{Z}$ so that

$$(\dot{p},\dot{q}).(p,q,\exp(i\pi kp\cdot q)) = (p+\dot{p}, q+\dot{q}, \exp(i\pi k(p+\dot{p})(q+\dot{q})))$$

Furthermore $w(p)\cdot w(q) = p \cdot q$. □

Consider $\tilde{F} \in \mathcal{M}'$ and set $F = j^*\tilde{F}$. Then by Proposition 8.6, we can define $v(\gamma, n)$ and the number u_F of Proposition 8.5 is given by

$$u_F = \big\langle \exp(2in\pi\,\mathrm{CS}(\tilde{F})), v(\gamma,n)\big\rangle.$$

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