Asymptotic properties of the quantum representations of the modular group

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Abstract

We study the asymptotic behaviour of the quantum representations of the modular group in the large level limit. We prove that each element of the modular group acts as a Fourier integral operator. This provides a link between the classical and quantum Chern-Simons theories for the torus. From this result we deduce the known asymptotic expansion of the Witten-Reshetikhin-Turaev invariants of the torus bundles with hyperbolic monodromy.

Quantum Chern-Simons theory was introduced twenty years ago by Witten [10] and Reshetikhin-Turaev [8]. It provides among other things invariants of three-dimensional manifold and representation of the mapping class group of surfaces, cf. [1], [9] for an exposition of the theory and [5] for a survey on recent developments. This theory has a semi-classical limit, where the level, an integral parameter denoted by k, plays the role of the inverse of the Planck constant. In this paper, we are concerned with the torus and its mapping class group, $\mathrm{Sl}(2, \mathbb{Z})$. We study the large k behaviour of the quantum representation of the modular group.

The quantum representations may be equivalently defined with algebraic or geometrical methods. Geometrically, we consider a line bundle, called the Chern-Simon bundle, over the moduli space of flat G-principal bundles on the torus. Here G is a compact Lie group that we assume to be simple and simply connected. Then the modular group acts linearly on the space of holomorphic sections of the k-th tensor power of the Chern-Simons bundle.

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A part of this construction is standard in geometric quantization: to any compact Kähler manifold with an integral fundamental form one associates the space of holomorphic sections of a prequantum bundle. In this general context the usual tools of microlocal analysis have been introduced. In particular given a prequantum bundle automorphism, we define a class of operators similar to the Fourier integral operators [2], [4] quantizing it. Our main result, theorem 9.1, says that each element of $\mathrm{Sl}(2, \mathbb{Z})$ acts as a Fourier integral operator on the quantum spaces, the underlying action on the Chern-Simons bundle being defined through gauge theory. This establishes a clear link between the quantum and classical Chern-Simons theories.

As a corollary, we can estimate the character of the quantum representations of the hyperbolic elements of $\mathrm{Sl}(2, \mathbb{Z})$. More generally, under a transversality assumption, one proves that the trace of a Fourier integral operators has an asymptotic expansion, which generalizes in some sense the Lefschetz fixed point formula [4]. The characters of the quantum representation of the modular group are the three-dimensional invariants of the torus bundles. In this way we recover the asymptotic expansion proved by Jeffrey [6], whose leading term is given in terms of the Chern-Simons invariants and the torsion of some flat bundles over the torus bundle. The proof in [6] is completely different and relies on the reciprocity formula for Gauss sum. Our result is slightly more general since we treat any hyperbolic element with any simple simply connected group G. But, what is more important, we hope that our analytic method will work in other cases. In the companion paper [3], we prove similar result for the mapping class group in genus $\geqslant 2$.

Besides the semiclassical results, we also give a careful construction of the quantum representations, comparing the geometric and algebraic methods. Strictly speaking, we do not have representations of the modular group but only projective representations which lift to genuine representations of the appropriate extension of $\mathrm{Sl}(2, \mathbb{Z})$. The extensions appearing naturally are not the same in the geometric and the algebraic approach.

The paper is organised as follows. In section 1, we state our result about the asymptotic expansion of the trace of the quantum representations. In section 3, we introduce the phase space of the Chern-Simons theory for the torus, its symplectic structure and prequantum bundle. The relation with gauge theory is the content of section 4. In section 5, we introduce a complex structures on the phase space and the associated quantum Hilbert spaces. We exhibit basis in terms of theta functions. The modular group acts naturally on the previous datas but does not preserve the complex structure. In section 6 we identify the quantum spaces associated to the various complex structures. This leads to the definition of the quantum representations. In section 7 we compare these representations with the ones defined by algebraic methods. The next two sections are devoted to semi-classical results: section 8 on the identification of the quantum spaces and section 9 on the quantum representations. In a first appendix we prove basic facts on theta functions. In a second appendix we list some notations used in the paper.

1 Characters of the quantum representations

Let G be a compact simple and simply connected Lie group. The phase space of the Chern-Simons theory for an oriented surface Σ and group G is the moduli space of flat G-principal bundles over Σ. For a torus, this moduli space identifies with the quotient T^2/W, where T is a maximal torus of G and W is the Weyl group acting diagonally. The modular group $\mathrm{Sl}(2,\mathbb{Z})$, being the mapping class group of the torus, acts on this moduli space. More explicitly, since $\mathrm{T} = \mathfrak{t}/\Lambda$, with $\mathfrak{t}$ the Lie algebra of T and Λ the integral lattice, we have a bijection $\mathrm{T}^2/W \simeq \mathfrak{t}^2/(\Lambda^2 \rtimes W)$. Identify $\mathfrak{t}^2$ with $\mathbb{R}^2 \otimes \mathfrak{t}$, then an element $A \in \mathrm{Sl}(2,\mathbb{Z})$ acts on the moduli space by sending the class of x to the class of $(A \otimes \mathrm{id}_{\mathfrak{t}}).x$.

Applying geometric quantization, we obtain a family of projective representations of the modular group indexed by a positive integer k. Since the construction is rather long, we only give in this introduction the representation of the generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of the modular group. Let B be the basic inner product of the Lie algebra of G. We choose a set of positive roots and denote by $\mathfrak{A} \subset \mathfrak{t}$ the corresponding open fundamental Weyl alcove. We identify the weight lattice $\Lambda^* \subset \mathfrak{t}^*$ with a lattice of $\mathfrak{t}$ via the basic inner product. The k-th representation has a particular basis indexed by the set $\mathfrak{A} \cap k^{-1}\Lambda^*$. For any $\lambda, \mu \in \mathfrak{A} \cap k^{-1}\Lambda^*$, let

$$t_{\lambda\mu} = \delta_{\lambda,\mu} \exp(i\pi k B(\lambda,\lambda))$$

and

$$s_{\lambda\mu} = i^p k^{-\frac{n}{2}} \,\mathrm{Vol}^{-1}(\mathfrak{t}/\Lambda) \sum_{w\in W} (-1)^{\ell(w)} \exp(-2i\pi k B(\lambda, w(\mu)))$$

where n is the rank of G and p is the integral part of $n/2$.

If the rank of G is even, the map sending S and T to the matrices $s_{\lambda\mu}$ and $t_{\lambda\mu}$ extends to a unitary representation R^k_{ev} of the modular group. If the rank of G is odd, we obtain a representation R^k_{odd} of an extension $\mathrm{Mp}(2,\mathbb{Z})$ of the modular group by $\mathbb{Z}/2\mathbb{Z}$. Recall that the metaplectic group $\mathrm{Mp}(2,\mathbb{R})$ is the connected two-cover of $\mathrm{Sl}(2,\mathbb{R})$. Then $\mathrm{Mp}(2,\mathbb{Z})$ is defined as the subgroup of the metaplectic group consisting of the elements which project onto the modular group. The representation R^k_{odd} of the elements projecting onto S and T is given by the matrices $\pm s_{\lambda\mu}$ and $\pm t_{\lambda\mu}$.

Theorem 1.1. *Assume the rank of G is even, then for any hyperbolic element $A \in \mathrm{Sl}(2,\mathbb{Z})$, we have*

$$\operatorname{tr} R^k_{\mathrm{ev}}(A) = \frac{(-1)^{2\epsilon p}}{|W|} \sum_{\substack{w \in W,\, x \in \mathrm{T}^2/ \\ (A\otimes w).x = x}} (-1)^{\ell(w)} \frac{e^{ik\theta(A\otimes w, x)}}{|\det(\mathrm{id} - A\otimes w)|^{1/2}} + O(k^{-1})$$

where

- $\epsilon = 0$ *if the trace of A is bigger than* 2 *and* $\epsilon = 1$ *otherwise.*
- $\theta(A\otimes w, x) = \pi(B(\mu,p) - B(\gamma,q) + B(\gamma,\mu))$ *if* $x \in \mathrm{T}^2 \simeq (\mathfrak{t}/\Lambda)^2$ *is the class of* $(p,q) \in \mathfrak{t}^2$ *and* $(\gamma,\mu) = (A\otimes w)(p,q) - (p,q)$.

If the rank of G is odd, let $\tilde{A} \in \mathrm{Mp}(2,\mathbb{R})$ projecting onto an hyperbolic element A of the modular group. Then the same result holds for the trace of $R^k_{\mathrm{odd}}(\tilde{A})$ except that the equivalent has to be multiplied by $\exp(i\frac{\pi}{2}\operatorname{ind}(\tilde{A}))$, where $\operatorname{ind}(\tilde{A}) \in \mathbb{Z}$ modulo $4\mathbb{Z}$.

It is a general property of topological quantum field theories that the trace of the quantum representation of an element A of the modular group is the invariant of the mapping torus

$$M_A := \big((\mathbb{R}^2/\mathbb{Z}^2) \times \mathbb{R}\big)/(Ay,t) \sim (y, t+1).$$

Here we ignore the the complications due to the framing of 3-dimensional manifold and the related fact that we only have a projective representation. For any $(x_1,x_2) \in \mathrm{T}^2$ and $w \in W$ such that $(A\otimes w).(x_1,x_2) = (x_1,x_2)$, consider the flat G-principal bundle $P_A \to M_A$ whose holonomies along the paths $\gamma(s) = [s,0,0]$, $[0,s,0]$ and $[0,0,s]$ are respectively x_1, x_2 and w^{-1}. Then $(2\pi)^{-1}\theta(A\otimes w, x)$ is the Chern-Simons invariant of P_A. This is in agreement with the formula obtained by Witten using the Feynman path integral (heuristic) definition of the three-dimensional invariants. We refer the reader to Jeffrey's paper [6] for more details. In particular the factor $|\det(\mathrm{id} - A\otimes w)|^{-1}$ appears as an integral of the torsion of the adjoint bundles over the moduli space of flat G-principal bundle over M_A.

2 Lie group notations

Let $\mathfrak{g}$ be a compact simple Lie algebra, and G the corresponding compact connected and simply-connected Lie group. Choose a maximal torus T of G and denote by $\mathfrak{t}$ its Lie algebra. The integral lattice Λ of $\mathfrak{t}$ is defined as the kernel of the exponential map $\mathfrak{t} \to \mathrm{T}$. Since G is simply-connected, Λ is the lattice of $\mathfrak{t}$ generated by the coroots $\alpha^\vee$ for the (real) roots α.

Let the basic inner product B be the unique invariant inner product on $\mathfrak{g}$ such that for each long root α, $B(\alpha^\vee, \alpha^\vee) = 2$. Through the paper, we will use B to identify $\mathfrak{t}$ with $\mathfrak{t}^*$. The basic inner product has the important property that it restricts to an integer-valued $\mathbb{Z}$-bilinear form on Λ which takes even values on the diagonal.

We fix a set Δ_+ of positive roots and let $\mathfrak{t}_+$ be the corresponding positive open Weyl chamber. Let α_0 be the highest root and $\mathfrak{A}$ be the open fundamental Weyl alcove

$$\mathfrak{A} := \{\lambda \in \mathfrak{t}_+ / \ \alpha_0(\lambda) < 1\}$$

We denote by W the Weyl group of (G, T). Let $\ell : W \to \{\pm 1\}$ be the alternating character of W.

3 The symplectic data

In this section we endow T^2 with a symplectic form ω and a prequantum bundle L, that is a complex Hermitian line bundle together with a connection of curvature $\frac{1}{i}\omega$. Furthermore we introduce commuting actions of the Weyl group and the modular group on L.

3.1 A prequantum bundle on $\mathfrak{t}^2$

Denote by p and q the projections $\mathfrak{t}^2 \to \mathfrak{t}$ on the first and second factor respectively. Let ω be the symplectic form on $\mathfrak{t}^2$ given by

$$\omega = 2\pi B(dp, dq).$$

Consider the trivial complex line bundle $L_{\mathfrak{t}^2}$ over $\mathfrak{t}^2$ with fiber $\mathbb{C}$ and connection

$$d + \frac{\pi}{i}(B(p, dq) - B(q, dp)).$$

Its curvature is $\frac{1}{i}\omega$, so it is a prequantum bundle.

3.2 Heisenberg group and reduction to T^2

Introduce the (reduced) Heisenberg group $\mathfrak{t}^2 \times \mathrm{U}(1)$ with multiplication

$$(x, u).(y, v) = \big(x + y, uv \exp\big(\tfrac{i}{2}\omega(x, y)\big)\big)$$

The same formula defines an action of the Heisenberg group on $L_{\mathfrak{t}^2} = \mathfrak{t}^2 \times \mathbb{C}$. This action preserves the trivial metric and the connection. The lattice Λ^2 embeds into the Heisenberg group

$$\Lambda^2 \to \mathfrak{t}^2 \times \mathrm{U}(1), \qquad (p, q) \to (p, q, \exp(i\pi B(p, q)))$$

Using that B takes integral values on Λ, we prove that this map is a group morphism. Hence we get an action of Λ^2 on $L_{\mathfrak{t}^2}$ by automorphisms of prequantum bundle.

By quotienting, we obtain a symplectic form on $\mathrm{T}^2 = \mathfrak{t}^2/\Lambda^2$ with a prequantum bundle $L := \mathfrak{t}^2 \times \mathbb{C}/\Lambda^2$ over T^2.

3.3 Weyl group

Consider the diagonal action of the Weyl group W on $\mathfrak{t}^2$ and lift this action trivially on the bundle $L_{\mathfrak{t}^2}$. Since W acts on $\mathfrak{t}$ by isometries, W acts on $\mathfrak{t}^2$ by linear symplectomorphisms and on $L_{\mathfrak{t}^2}$ by isomorphisms of prequantum bundle.

The Weyl group preserves the integral lattice Λ. Consider the semi-direct product $W \rtimes \Lambda^2$ where W acts diagonally on Λ^2. Is is easily checked that the actions of Λ^2 and W on the prequantum bundle over $\mathfrak{t}^2$ generate an action of $W \rtimes \Lambda^2$. Then, quotienting by Λ^2, we obtain an action of $W = (W \rtimes \Lambda^2)/\Lambda^2$ on L. Since the action of the Weyl group on the base T^2 is not free, we will not consider the orbifold quotient T^2/W and its prequantum bundle.

3.4 Modular group

Let us consider the symplectic action of the modular group $\Gamma = \mathrm{Sl}(2, \mathbb{Z})$ on $\mathfrak{t}^2$ given by

$$A.(p, q) = (ap + bq, cp + dq), \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The trivial lift to the prequantum bundle $L_{\mathfrak{t}^2}$ preserves the metric and the connection. Furthermore this action together with the action of Λ^2 define an action of the semi-direct product $\Gamma \rtimes \Lambda^2$. To prove this, one has to use that $B(p,q)$ is integral when $p,q \in \Lambda$ and even if furthermore $p = q$. Consequently we get an action of the modular group on L by prequantum bundle isomorphisms. Observe that the Weyl group action on L commute with the modular action.

4 Chern-Simons theory

We explain how the definitions of the previous section can be deduced from gauge theory. Our aim is only to motivate the constructions. No proof in the paper relies on the gauge theoretic considerations.

The phase space of the Chern-Simons theory for an oriented surface Σ is the moduli space of representations of the fundamental group of Σ in G. When Σ is a torus, the fundamental group is the free Abelian group with two generators, so each representation is given by a pair of commuting elements of G unique up to conjugation. In the same way that $G/\operatorname{Ad} G \simeq \mathrm{T}/W$, one shows that these representations are conjugate to a representation in the maximal torus T, uniquely up to the action of the Weyl group. So the moduli space of representation for the torus is T^2/W.

4.1 Gauge theory presentation

Consider the space $\Omega^1(\Sigma, \mathfrak{g})$ of connections of the trivial G-principal bundle with base Σ. It is a symplectic vector space with symplectic product given by

$$\Omega(a,b) = 2\pi \int_\Sigma B(a,b).$$

The gauge group $\mathcal{C}^\infty(\Sigma, G)$ acts on $\Omega^1(\Sigma, \mathfrak{g})$ by symplectic affine isomorphisms:

$$g.a = \operatorname{Ad}_g a - g^*\bar{\theta}$$

where $\bar{\theta} \in \Omega^1(G, \mathfrak{g})$ is the right-invariant Maurer-Cartan form. Each gauge class of flat connections is determined by its holonomy representation. The quotient of the space of flat connections by the gauge group may be viewed as a symplectic quotient which defines a symplectic structure on the moduli space of representations.

In the case Σ is a torus, we can avoid this infinite dimensional quotient proceeding as follows. Represent Σ as the quotient $\mathbb{R}^2/\mathbb{Z}^2$ with coordinates x, y. Then the map

$$\mathfrak{t}^2 \to \Omega^1(\Sigma, \mathfrak{g}), \qquad (p,q) \to p dx + q dy$$

is a symplectic embedding where the symplectic product of $\mathfrak{t}^2$ is the one of section 3.1. This embedding is equivariant with respect to the action of $W \rtimes \Lambda^2$ on $\mathfrak{t}^2$ and the morphism from $W \rtimes \Lambda^2$ to the gauge group sending an element $w \in W$ to the constant gauge transform w and $(\dot{p}, \dot{q}) \in \Lambda^2$ to $\exp(-(x\dot{p}+y\dot{q}))$. Furthermore each gauge class of flat connections intersects the image of the embedding.

4.2 Prequantum bundle

Consider the trivial line bundle with base $\Omega^1(\Sigma, \mathfrak{g})$ and connection $d + \frac{1}{i}\alpha$, where α is the primitive of Ω given by

$$\alpha|_a(b) = \tfrac{1}{2}\Omega(a, b).$$

This is a prequantum bundle and the gauge group actions lifts to it in such a way that it preserves the trivial metric and the connection. Explicitly, the action is given by

$$g.(a, u) = \Big(g.a, \exp\Big(-2i\pi W(g) - i\pi \int_\Sigma B(g^*\theta, a)\Big)u\Big)$$

where $\theta \in \Omega^1(G, \mathfrak{g})$ is the left invariant Maurer-Cartan one-form and $W(g)$ is the Wess-Zumino-Witten term

$$W(g) = \int_M \tilde{g}^*\chi$$

Here M is any three-dimensional compact oriented manifold with boundary Σ, $\tilde{g} \in \mathcal{C}^\infty(M, G)$ any extension of g and χ is the Cartan three-form defined in terms of the left or right-invariant Maurer Cartan forms by

$$\chi = \frac{1}{12}B([\theta, \theta], \theta) = \frac{1}{12}B([\bar{\theta}, \bar{\theta}], \bar{\theta})$$

Since B is the basic inner product, the cohomology class of χ is integral.

Assume now that Σ is a torus and consider the equivariant embedding of $\mathfrak{t}^2$ in $\Omega^1(\Sigma, \mathfrak{g})$ defined in section 4.1. By pulling back, we obtain a prequantum bundle on $\mathfrak{t}^2$ together with an action of $\Lambda^2 \rtimes W$ on it. It is not difficult to check that this bundle and this action are exactly the ones we introduced in section 3.

4.3 Mapping class group

The group of orientation preserving diffeomorphisms of Σ acts symplectically on $\Omega^1(\Sigma, \mathfrak{g})$. The trivial lift to the prequantum bundle preserves the connection and the trivial metric. After quotienting by the gauge group, this defines an action of the mapping class group on the moduli space of representation and its prequantum bundle.

When Σ is a torus, we recover the action of Γ introduced in section 3.4. For any $A \in \Gamma$, define the diffeomorphism of the torus

$$\varphi_A(x, y) = (dx - cy, -bx + ay)$$

where a, b, c, and d are the coefficients of A. On one hand, we recover the usual formula by considering the basis $\alpha = (0, -1)$ and $\beta = (1, 0)$. Indeed $\varphi_A(\alpha) = a\alpha + b\beta$ and $\varphi_A(\beta) = c\alpha + d\beta$. On the other hand,

$$\varphi_A^*(pdx + qdy) = (ap + bq)dx + (cp + dq)dy$$

which corresponds to the action of section 3.4.

5 Quantization

Let us begin with a brief description of the general set-up. Consider a symplectic manifold (M, ω) with a prequantum bundle $L \to M$. Assume (M, ω) is endowed with a compatible positive complex structure, so ω is a $(1, 1)$ form and $-i\omega(Z, \bar{Z}) > 0$ for any non vanishing tangent vector Z of type $(1, 0)$. Then the prequantum bundle has a unique holomorphic structure such that the local holomorphic sections satisfy the Cauchy-Riemann equations:

$$\nabla_{\bar{Z}} s = 0, \qquad \text{for any vector field } Z \text{ of type } (1, 0).$$

The quantum space associated to these data is the space $H^0(M, L^k)$ of holomorphic sections of L^k. It has a natural scalar product obtained by integrating the punctual scalar product of sections against the Liouville measure $|\omega^n|/n!$.

In a first subsection we introduce complex structures on T^2 and define a basis of the quantum space by using theta functions. We also describe the action of the matrices S and T of Γ in these basis. These results are standard. We provide proofs in appendix. Next we we move on to a subspace of equivariant sections with respect to the Weyl group action, the so called alternating sections. We compute the actions of S and T in this space.

5.1 Complex structure and theta functions

Denote by $\mathcal{H}_+$ the Poincaré upper half-plane

$$\mathcal{H}_+ = \{x + iy / x, y \in \mathbb{R},\ y > 0\}.$$

Let $\tau \in \mathcal{H}_+$. Identify $\mathfrak{t}^2$ with $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$ by the isomorphism sending (p, q) to $p + \tau q$. Hence $\mathfrak{t}^2$ becomes a complex vector space and T^2 inherits a complex structure. One may compute the symplectic form ω in terms of the complex coordinate $\zeta = p + \tau q$

$$\omega = \frac{2\pi}{\bar{\tau} - \tau} B(d\zeta, d\bar{\zeta}).$$

It is a positive real form of type $(1,1)$. So the prequantum bundle L has a unique holomorphic structure compatible with the connection. We denote by $H^0_\tau(\mathrm{T}^2, L^k)$ its space of holomorphic sections.

Consider the section of $L_{\mathfrak{t}^2}$

$$s = \exp(i\pi B(\zeta, q))$$

Its covariant derivative is $2i\pi B(d\zeta, q) \otimes s$. Since this form is of type $(1,0)$, s is holomorphic. Furthermore s doesn't vanish anywhere. So the holomorphic sections of L^k identify with the sections over $\mathfrak{t}^2$ of the form fs^k such that $f : \mathfrak{t}^2 \to \mathbb{C}$ is holomorphic and fs^k is Λ^2-invariant.

As previously, we embed Λ^* in $\mathfrak{t}$ via the identification given by the basic inner product. Recall that $\Lambda \subset \Lambda^*$. For any $\mu \in k^{-1}\Lambda^*$, consider the theta function

$$\Theta_{\mu,k}(p,q) = \sum_{\gamma \in \mu + \Lambda} \exp\bigl(2i\pi k \bigl(\tfrac{\tau}{2} B(\gamma,\gamma) - B(\zeta,\gamma)\bigr)\bigr)$$

This series converges uniformly on compact sets to a holomorphic function, it depends only on μ mod Λ.

Theorem 5.1. *For any integer k, the sections $\Theta_{\mu,k} s^k$, where μ runs over $k^{-1}\Lambda^*$ mod Λ, are Λ^2-invariant and form an orthonormal basis of $H^0_\tau(\mathrm{T}^2, L^k)$. Furthermore,*

$$\|\Theta_{\mu,k} s^k\|^2 = \Bigl(\frac{2\pi}{k}\Bigr)^{n/2} \Bigl(\frac{2i\pi}{\tau - \bar{\tau}}\Bigr)^{n/2} \mathrm{Vol}(\mathfrak{t}/\Lambda),$$

where $\mathrm{Vol}(\mathfrak{t}/\Lambda)$ is the Riemannian volume determined by B.

Recall that the modular group acts on T^2 and its prequantum bundle. The induced action on the sections of L^k doesn't preserve the space $H^0_\tau(\mathrm{T}^2, L^k)$, because of the complex structure. Actually, $A \in \Gamma$ acts as a holomorphic map from (T^2, j_τ) to $(\mathrm{T}^2, j_{A\tau})$ with

$$A\tau = \frac{a\tau - b}{-c\tau + d}$$

So for any τ, A acts as an isomorphism

$$H^0_\tau(\mathrm{T}^2, L^k) \to H^0_{A\tau}(\mathrm{T}^2, L^k)$$

One may compute explicitly this isomorphism in the basis of theta functions when A is the matrix S or T. We make explicit the dependence in τ in our notations to avoid any ambiguity.

Theorem 5.2. *For any $\tau \in \mathcal{H}_+$ and $\mu \in k^{-1}\Lambda^*$, one has*

$$S.\big(\Theta^\tau_{\mu,k} s^k_\tau\big) = C \sum_{\mu' \in k^{-1}\Lambda^* \bmod \Lambda} \exp(-2i\pi k B(\mu, \mu'))\Theta^{S.\tau}_{\mu',k}\ s^k_{S.\tau}$$

with $C = \big(S.\tau/i\big)^{n/2} k^{-n/2}\,\mathrm{Vol}(\mathfrak{t}/\Lambda)^{-1}$ *and*

$$T.\big(\Theta^\tau_{\mu,k} s^k_\tau\big) = \exp(i\pi k B(\mu, \mu))\Theta^{T.\tau}_{\mu,k}\ s^k_{T.\tau}$$

Here $(\tau/i)^{n/2}$ is the determination continuous with respect to τ and equal to 1 when $\tau = i$.

5.2 Alternating sections

The action of the Weyl group on T^2 is holomorphic with respect to the complex structure defined by any $\tau \in \mathcal{H}_+$. Let us consider the alternating sections of L^k, i.e. the sections Ψ satisfying

$$w.\Psi = (-1)^{\ell(w)}\Psi, \qquad \forall w \in W.$$

For any $\mu \in k^{-1}\Lambda^*$, let

$$\chi_{\mu,k} = \sum_{w \in W} (-1)^{\ell(w)} \Theta_{w(\mu),k}\ s^k.$$

Recall that we denote by $\mathfrak{A}$ the fundamental open Weyl alcove.

Theorem 5.3. *The family $(\chi_{\mu,k},\ \mu \in \mathfrak{A} \cap k^{-1}\Lambda^*)$ is a basis of the space of alternating holomorphic sections of L^k.*

Proof. Using that the Weyl group action preserves Λ and B, we check that for any $w \in W$,

$$w.(\Theta_{\mu,k}\ s^k) = \Theta_{w(\mu),k}\ s^k.$$

So $\chi_{\mu,k}$ is alternating.

For any root α and integer n, the orthogonal reflexion with respect to the hyperplane $\alpha^{-1}(n)$ belongs to the affine Weyl group $W \rtimes \Lambda$. So for any $\mu \in \alpha^{-1}(n)$, there exists $w \in W$ with $\ell(w) = 1$ such that $w(\mu) = \mu$ modulo Λ. Hence $\chi_{\mu,k} = -\chi_{w(\mu),k} = -\chi_{\mu,k}$, so $\chi_{\mu,k}$ vanishes.

Recall that the affine Weyl group $W \rtimes \Lambda$ acts simply transitively on the set of components of $\mathfrak{t} \setminus \cup_{\alpha,n}\alpha^{-1}(n)$ and that $\mathfrak{A}$ is one of these components. The result follows from theorem 5.1. □

The modular action and the action of the Weyl group on T^2 and L commute. So the representation of the modular group preserves the subspace of alternating sections.

Theorem 5.4. *For any $\tau \in \mathcal{H}_+$ and $\mu \in \mathfrak{A} \cap k^{-1}\Lambda^*$, one has*

$$S.\big(\chi^\tau_{\mu,k} s^k_\tau\big) = C \sum_{\substack{\mu' \in \mathfrak{A}\cap k^{-1}\Lambda^*,\\ w\in W}} (-1)^{\ell(w)} \exp(-2i\pi k B(\mu, w(\mu')))\chi^{S.\tau}_{\mu',k}\ s^k_{S.\tau}$$

with C defined as in theorem 5.2 and

$$T.\big(\chi^\tau_{\mu,k} s^k_\tau\big) = \exp(i\pi k B(\mu,\mu))\chi^{T.\tau}_{\mu,k}\ s^k_{T.\tau}$$

Proof. The second formula follows from theorem 5.2 using that the Weyl group acts isometrically on $\mathfrak{t}$. Let us prove the first one. By theorem 5.2,

$$\begin{aligned}
S.(\chi^\tau_{\mu,k} s^k_\tau) =& C \sum_{\substack{\mu' \in k^{-1}\Lambda^* \bmod \Lambda,\\ w\in W}} (-1)^{\ell(w)} \exp(-2i\pi k B(w(\mu),\mu'))\ \theta^{S.\tau}_{\mu',k} s^k_{S.\tau} \\
=& C \sum_{\substack{\mu' \in k^{-1}\Lambda^* \bmod \Lambda,\\ w\in W}} (-1)^{\ell(w)} \exp(-2i\pi k B(\mu,\mu'))\ \theta^{S.\tau}_{w(\mu'),k} s^k_{S.\tau} \\
=& C \sum_{\mu' \in k^{-1}\Lambda^* \bmod \Lambda} \exp(-2i\pi k B(\mu,\mu'))\ \chi^{S.\tau}_{\mu',k} s^k_{S.\tau} \\
=& C \sum_{\substack{\mu' \in \mathfrak{A}\cap k^{-1}\Lambda^*,\\ w\in W}} \exp(-2i\pi k B(\mu, w(\mu')))\ \chi^{S.\tau}_{w(\mu'),k} s^k_{S.\tau}
\end{aligned}$$

In the last line, we used that the affine Weyl group acts simply transitively on the set of connected components of $\mathfrak{t} \setminus \bigcup \alpha^{-1}(n)$ and that χ_μ vanishes if $\mu \in \alpha^{-1}(n)$. Finally,

$$S.(\chi^\tau_{\mu,k} s^k_\tau) = C \sum_{\substack{\mu' \in \mathfrak{A} \cap k^{-1}\Lambda^* \\ w \in W}} (-1)^{\ell(w)} \exp(-2i\pi k B(\mu, w(\mu')) \, \chi^{S.\tau}_{\mu',k} s^k_{S.\tau}$$

since the $\chi_{\mu,k}$'s are alternating. □

6 Geometric quantum representation

We introduce a representation of the modular group on the quantum spaces. To do this we identify the various spaces $H^0_\tau(\mathrm{T}^2, L^k)$ via the sections $\Theta^\tau_{\mu,k} s^k_\tau$. Unfortunately the norm of these sections and the action of the modular group depend on τ as it appears in theorems 5.1 and 5.2. We introduce half-form bundle to correct this.

6.1 Half-form bundles

Let us begin with some definitions. Consider a symplectic manifold M with a prequantum bundle L and a positive compatible complex structure. Then a half-form bundle is a complex line bundle δ over M with an isomorphism from δ^2 to the canonical bundle of M. A half-form bundle admits a natural metric and a natural holomorphic structure making the isomorphism with the canonical bundle a morphism of Hermitian holomorphic bundle. The quantization of M with metaplectic correction is then the space of holomorphic sections of L^k tensored with δ. The scalar product is defined by integrating the punctual norm of sections against the Liouville measure.

Let us return to our particular situation. Consider $\Omega \in \wedge^n \mathfrak{t}^*_{\mathbb{C}}$ such that for a basis (γ_i) of Λ, one has

$$\Omega(\gamma_1 \wedge \ldots \wedge \gamma_n) = 1.$$

Ω is uniquely defined up to a plus or minus sign. For any $\tau \in \mathcal{H}_+$, we defined a complex structure j_τ on T^2 via the isomorphism

$$\mathrm{T}^2 \to \mathfrak{t}_{\mathbb{C}}/(\Lambda + \tau\Lambda), \qquad [p, q] \to [p + \tau q].$$

So the holomorphic tangent bundle of (T^2, j_τ) is naturally isomorphic to the trivial bundle with fiber $\mathfrak{t}_{\mathbb{C}}$. Consequently the canonical bundle is naturally isomorphic to the trivial bundle with fiber $\wedge^n \mathfrak{t}^*_{\mathbb{C}}$. Denote by Ω_τ the section of the canonical bundle which is sent into the constant section equal to Ω by this trivialization.

Lemme 6.1. *The section Ω_τ is holomorphic and has a constant punctual norm equal to* $\left(\frac{\tau-\bar\tau}{2i\pi}\right)^{n/2}\mathrm{Vol}(\mathfrak{t}/\Lambda)^{-1}$.

Proof. Introduce an orthonormal basis (u_i) of $\mathfrak{t}$ and denote by (p_i) and (q_i) the associated linear coordinates of $\mathfrak{t}^2$. Let ζ_τ^i be the complex coordinate $p_i+\tau q_i$. Then if (γ_i) is a basis of Λ, one has

$$\begin{aligned} d\zeta_\tau^1\wedge\ldots\wedge d\zeta_\tau^n(\gamma_1,\ldots,\gamma_n) =& dp_1\wedge\ldots\wedge dp_n(\gamma_1,\ldots,\gamma_n)\\ =& \pm\,\mathrm{Vol}(\mathfrak{t}/\Lambda)\end{aligned}$$

where the plus or minus sign depends on the orientation of the various basis. Consequently

$$\Omega_\tau=\pm\,\mathrm{Vol}(\mathfrak{t}/\Lambda)^{-1}d\zeta_\tau^1\wedge\ldots\wedge d\zeta_\tau^n \tag{1}$$

Now, by definition the Hermitian product of two tangent vectors X,Y of type $(1,0)$ is given by $\frac{1}{i}\omega(X,\overline{Y})$. Since

$$\omega=2\pi\sum dp_i\wedge dq_i=i\frac{2i\pi}{\tau-\overline{\tau}}\sum d\zeta_\tau^i\wedge d\bar\zeta_\tau^i$$

the vectors $\partial_{\zeta_\tau^i}$ are mutually orthogonal and

$$\left|\partial_{\zeta_\tau^i}\right|^2=\frac{2i\pi}{\tau-\overline{\tau}}.$$

So the $d\zeta_\tau^i$ are mutually orthogonal and

$$\left|d\zeta_\tau^i\right|^2=\frac{\tau-\overline{\tau}}{2i\pi}$$

which implies

$$|d\zeta_\tau^1\wedge\ldots\wedge d\zeta_\tau^n|^2=\left(\frac{\tau-\bar\tau}{2i\pi}\right)^n$$

and concludes the proof. □

Let δ be a complex vector line and φ be an isomorphism $\delta^{\otimes 2}\to\wedge^{\mathrm{top}}\mathfrak{t}_{\mathbb{C}}^*$. Let $\Omega^{1/2}\in\delta$ be such that

$$\varphi(\Omega^{1/2}\otimes\Omega^{1/2})=\Omega.$$

For any $\tau \in \mathcal{H}_+$, the trivial bundle δ_τ with base T^2 and fiber δ is a half-form bundle, with the squaring map sending $\Omega^{1/2}$ into Ω_τ. By the previous lemma, the constant section equal to $\Omega^{1/2}$ is a holomorphic section of δ_τ with constant punctual norm equal to $\left(\frac{\tau-\bar\tau}{2i\pi}\right)^{n/4} \mathrm{Vol}(\mathfrak{t}/\Lambda)^{-1/2}$.

Instead of sections of L^k, consider now sections of $L^k \otimes \delta_\tau$. The multiplication by $\Omega^{1/2}$ is an isomorphism

$$H^0_\tau(\mathrm{T}^2, L^k) \simeq H^0_\tau(\mathrm{T}^2, L^k \otimes \delta_\tau)$$

of vector space. We deduce from theorem 5.1 and lemma 6.1 the

Theorem 6.2. *For any $\tau \in \mathcal{H}_+$, the sections*

$$\Theta^\tau_{\mu,k} s^k_\tau \otimes \Omega^{1/2}, \qquad \mu \in k^{-1}\Lambda^* \ mod\ \Lambda,$$

form a basis of $H^0_\tau(\mathrm{T}^2, L^k \otimes \delta_\tau)$. They are mutually orthogonal and

$$\| \Theta^\tau_{\mu,k} s^k_\tau \otimes \Omega^{1/2} \|^2 = \left(\frac{2\pi}{k} \right)^{n/2}.$$

For any τ_1 and τ_2 in $\mathcal{H}_+$, let $\Psi_{\tau_1,\tau_2,k}$ be the isomorphism from $H^0_{\tau_1}(\mathrm{T}^2, L^k \otimes \delta_{\tau_1})$ to $H^0_{\tau_2}(\mathrm{T}^2, L^k \otimes \delta_{\tau_2})$ defined by

$$\Psi_{\tau_1,\tau_2,k}(\Theta^{\tau_1}_{\mu,k} s^k_{\tau_1} \otimes \Omega^{1/2}) = \Theta^{\tau_2}_{\mu,k} s^k_{\tau_2} \otimes \Omega^{1/2}, \qquad \forall \mu.$$

By the previous theorem, $\Psi_{\tau_1,\tau_2,k}$ is a unitary map.

6.2 Modular action

Let $A \in \Gamma$ be the matrix with coefficients a, b, c and d. The map $\varphi_A(p,q) = (ap+bq, cp+dq)$ is a holomorphic map from (T^2, j_τ) to $(\mathrm{T}^2, j_{A.\tau})$. Using the coordinates introduced in the proof of lemma 6.1, we show that

$$\varphi_A^* \Omega_{A.\tau} = (-c\tau + d)^{-n} \Omega_\tau. \tag{2}$$

We will lift the action of A to the half-form bundles in such a way that it squares to $(\varphi_A^*)^{-1}$. Since $(-c\tau+d)^n$ doesn't admit a preferred square root, we have to pass to an extension of the modular group.

Let Γ_2 be the set of pairs (A, e) where $A \in \Gamma$ and e is a continuous function from $\mathcal{H}_+$ to $\mathbb{C}$ satisfying

$$e(\tau)^2 = (-c\tau + d)^n \quad \text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Γ_2 is a group with the product given by $(A,e).(A',e') = (AA',e'')$ where $e''(\tau) = e(A'\tau)e'(\tau)$. Γ_2 acts on the product $\mathcal{H}_+ \times \mathrm{T}^2 \times \delta$

$$(A,e).\big(\tau,p,q,z\Omega^{1/2}\big) = \Big(\frac{a\tau - b}{-c\tau + d}, ap+bq, cp+dq, ze(\tau)\Omega^{1/2}\Big)$$

Restricting to a particular value τ, we obtain a morphism from δ_τ to $\delta_{A.\tau}$ lifting the action of A on T^2. The important point is that the square of this isomorphism is the natural map between the canonical bundles of (T^2, j_τ) and $(\mathrm{T}^2, j_{A\tau})$. Since the Hermitian and holomorphic structures of a half-form bundle are determined by the corresponding canonical bundle, the morphism $\delta_\tau \to \delta_{A.\tau}$ that we consider is a unitary holomorphic isomorphism.

As previously we lift trivially the action of Γ to the prequantum bundle. Then we obtain for any $(A,e,\tau) \in \Gamma_2 \times \mathcal{H}_+$ an isomorphism

$$\varphi^*_{(A,e)^{-1},\tau} : H^0_\tau(\mathrm{T}^2, L^k \otimes \delta_\tau) \to H^0_{A.\tau}(\mathrm{T}^2, L^k \otimes \delta_{A.\tau})$$

By trivial reason, it is a unitary map. Miraculously, these isomorphisms all fit together.

Proposition 6.3. *For any $\tau_1, \tau_2 \in \mathcal{H}_+$ and any $(A,e) \in \Gamma_2$, the diagram*

$$\begin{array}{ccc} H^0_{\tau_1}(\mathrm{T}^2, L^k \otimes \delta_{\tau_1}) & \xrightarrow{\varphi^*_{(A,e)^{-1},\tau}} & H^0_{A.\tau_1}(\mathrm{T}^2, L^k \otimes \delta_{A.\tau_1}) \\ \downarrow \Psi_{\tau_1,\tau_2} & & \downarrow \Psi_{A.\tau_1, A.\tau_2} \\ H^0_{\tau_2}(\mathrm{T}^2, L^k \otimes \delta_{\tau_2}) & \xrightarrow{\varphi^*_{(A,e)^{-1},\tau}} & H^0_{A.\tau_2}(\mathrm{T}^2, L^k \otimes \delta_{A.\tau_2}) \end{array}$$

commute.

Proof. It is sufficient to prove it for $(A,e) = (T,1)$, (S,e) or $(\mathrm{id}, -1)$ since these elements generate Γ. The actions of $(T,1)$ and (S,e) in the basis $\Theta^\tau_{\mu,k} s^k_\tau \Omega^{1/2}$ are given by the same formulas as in theorem 5.2 except that the constant C has to be replaced by

$$C' = Ce(\tau) = e(i)k^{-n/2}\,\mathrm{Vol}(\mathfrak{t}/\Lambda)^{-1}$$

This proves the result because C' does not depend on τ. □

Corollary 6.4. *For any τ, the map*

$$R_2 : (A,e) \to \varphi^*_{(A,e)^{-1},\tau} \circ \Psi_{\tau, A^{-1}\tau}$$

is a unitary representation of Γ_2 on $H^0_\tau(\mathrm{T}^2, L^k \otimes \delta_\tau)$.

Consider now the diagonal action of the Weyl group on T^2. Its trivial lift to the half-form bundle acts holomorphically and preserves the metric. So we have a unitary representation of W on $H^0_\tau(\mathrm{T}^2, L^k \otimes \delta_\tau)$. It is easy to see that this representation commutes with the one of Γ_2. This gives a representation of Γ_2 on the subspace of alternating sections that we denote by R_2^{alt}. By theorem 5.3, we have

Theorem 6.5. *The coefficients of the matrix of $R_2^{\mathrm{alt}}(S,e)$ and $R_2^{\mathrm{alt}}(T,1)$ in the basis $\chi_{\mu,k} \otimes \Omega^{1/2}$, $\mu \in \mathfrak{A} \cap k^{-1}\Lambda^*$ are respectively*

$$\frac{e(i)^{-1}}{k^{\frac{n}{2}}\,\mathrm{Vol}(\mathfrak{t}/\Lambda)} \sum_{w \in W} (-1)^{\ell(w)} \exp(-2i\pi k B(\mu, w(\mu')))$$

and

$$\delta_{\mu,\mu'} \exp(i\pi k B(\mu,\mu)).$$

7 Algebraic projective representation

Consider the category of representations of the quantum group $U_q(\mathfrak{g})$ for q being the root of unity. This category has a subquotient, called the fusion category, which is a modular tensor category. Following [9], we can define a natural projective representation of the mapping class group of any 2-dimensional surface with marked points on appropriate spaces of morphisms in this category. In particular, for the torus, we get a projective representation of the modular group that we define in this section.

7.1 The representation R_∞

Denote by ρ the half sum of positive roots and by $h^\vee = 1 + \rho(\alpha_0^\vee)$ the dual Coxeter number. Let k be an integer bigger than $h^\vee$. Then the set of admissible weights at level $k - h^\vee$ is

$$C_k := \Lambda^* \cap (k - h^\vee)\overline{\mathfrak{A}}$$

For any $\lambda, \mu \in C_k$, let

$$\tilde{s}_{\lambda\mu} := \Big|\frac{\Lambda^*}{k\Lambda}\Big|^{-1/2} i^{|\Delta_+|} \sum_{w \in W} (-1)^{\ell(w)} \exp\Big(-\frac{2i\pi}{k} B(w(\lambda+\rho), \mu+\rho)\Big)$$

and

$$\tilde{t}_{\lambda\mu} := \delta_{\lambda\mu} \exp\Big(\frac{i\pi}{k} B(\lambda, \lambda+2\rho)\Big).$$

Consider the action of the modular group on the real projective line induced by the standard action on $\mathbb{R}^2$. Choose a base point $L_o \in \mathbb{P}^1(\mathbb{R})$. Let Γ_∞ be the set of pairs (A, γ) where $A \in \Gamma$ and γ is a homotopy class (with fixed endpoint) of a path in $\mathbb{P}^1(\mathbb{R})$ from $A.L_o$ to L_o. Γ_∞ is an extension by $\mathbb{Z}$ of the modular group, the product being given by

$$(A, \gamma).(A', \gamma') = (AA', A(\gamma') \star \gamma).$$

Assume that $L_o = [1, 0]$. Then Γ_∞ is generated by $\hat{s} = (S, [\phi])$, $\hat{t} = (T, [1, 0])$ and $\hat{\gamma} = (\mathrm{id}, [\gamma])$ where ϕ and γ are the paths

$$\phi(t) = [\sin(t\frac{\pi}{2}), \cos(t\frac{\pi}{2})], \quad \gamma(t) = [\cos(t\pi), \sin(t\pi)],$$

with $0 \leqslant t \leqslant 1$.

Theorem 7.1. Γ_∞ *admits a representation* R_∞ *on* $\mathbb{C}^{C_k}$ *determined by*

$$R_\infty(\hat{s}) = (\exp(-i\tfrac{\pi}{4}c)\tilde{s}_{\lambda\mu}), \quad R_\infty(\hat{t}) = (\tilde{t}_{\lambda\mu}), \quad R_\infty(\hat{\gamma}) = (\exp(i\tfrac{\pi}{2}c)\delta_{\lambda\mu}).$$

The reader is referred to the book of Bakalov-Kirillov [1] for a detailed exposition on modular tensor category and the fusion category of $U_q(\mathfrak{g})$. The formulas for $s_{\lambda\mu}$ and $t_{\lambda\mu}$ are given in theorem 3.3.20, that they give a projective representation is the content of chapter 3.1, cf. remark 3.1.9. The definition of the central extension is in chapter 5.7, the case of the torus being treated in example 5.7.7. Note also there is a misprint in formula for $s_{\lambda\mu}$, compare with proposition 3.8 of [7].

7.2 Comparison of the representations R_2^{alt} and R_∞

To compare R_2^{alt} with R_∞, we introduce a third extension Γ_4 of the modular group. By definition, Γ_4 consists of the pairs (A, e) where $A \in \Gamma$ and e is a continuous function from $\mathcal{H}_+$ to $\mathbb{C}$ satisfying $e(\tau)^4 = (-c\tau + d)^{2n}$, with a, b, c, d the coefficients of A. The product is given by the same formula as for Γ_2. Let $\mathbb{Z}_4 = \{\pm 1, \pm i\}$. The morphism

$$\Gamma_2 \times \mathbb{Z}_4 \to \Gamma_4, \qquad (A, e, u) \to (A, eu)$$

is onto with kernel $\{(\mathrm{id}, 1, 1), (\mathrm{id}, -1, -1)\}$. Thus we have a representation R_4 of Γ_4 given by

$$R_4(A, eu) = uR_2^{\mathrm{alt}}(A, e), \qquad (A, e) \in \Gamma_2, \ u \in \mathbb{Z}_4.$$

Recall that Γ_∞ is generated by $\hat{s}$, $\hat{t}$ and $\hat{\gamma}$.

Lemme 7.2. *There exists a morphism Ψ from Γ_∞ onto Γ_4 such that*

$$\Psi(\hat{s}) = (S, e) \quad \textit{with} \quad e(i) = e^{-i\frac{\pi}{4}n}$$
$$\Psi(\hat{\gamma}) = (\mathrm{id}, i^{\dim G}), \qquad \Psi(\hat{t}) = (T, 1).$$

Proof. Consider the groups $\Gamma_\infty^{\mathbb{R}}$ and $\Gamma_4^{\mathbb{R}}$ defined exactly as Γ_∞ and Γ_4 except that we replace the modular group by $\Gamma^{\mathbb{R}} := \mathrm{Sl}(2, \mathbb{R})$. We will construct a morphism from $\Gamma_\infty^{\mathbb{R}}$ onto $\Gamma_4^{\mathbb{R}}$ whose restriction to Γ_∞ is Ψ. Let j be the morphism from the unit circle $\mathrm{U} \subset \mathbb{C}$ into $\Gamma^{\mathbb{R}}$

$$j(u) = \begin{pmatrix} \mathrm{reel}\, u & -\mathrm{im}\, u \\ \mathrm{im}\, u & \mathrm{reel}\, u \end{pmatrix}$$

Since $\Gamma^{\mathbb{R}}$ admits a deformation retract onto $j(\mathrm{U})$, one has a bijective correspondence between the morphisms from $\Gamma_\infty^{\mathbb{R}}$ to $\Gamma_4^{\mathbb{R}}$ covering the identity of $\Gamma^{\mathbb{R}}$ and the morphisms from $j^*\Gamma_\infty^{\mathbb{R}}$ to $j^*\Gamma_4^{\mathbb{R}}$ covering the identity of U. On one hand

$$j^*\Gamma_4^{\mathbb{R}} \simeq \{(u, e) \in \mathrm{U} \times \mathbb{C}^* /\ e^4 = u^{-2n}\} =: \mathrm{U}^4$$

because for $\tau = i$, $-c\tau + d = u^{-1}$ if a, b, c, and d are the coefficients of $j(u)$. On the other hand

$$j^*\Gamma_\infty^{\mathbb{R}} \simeq \{(u, \theta)/\ \exp(i\theta) = u^2\} =: \mathrm{U}^\infty$$

where we send (u, θ) into $(j(u), [\gamma])$ with γ the path

$$\gamma(t) = [\cos(\tfrac{1-t}{2}\theta), \sin(\tfrac{1-t}{2}\theta)], \qquad t \in [0, 1].$$

In particular $\hat{s}$ and $\hat{\gamma}$ are identified with (i, π) and $(1, -2\pi)$. The morphism we are looking for is

$$\mathrm{U}^\infty \to \mathrm{U}^4, \qquad (u, \theta) \to (u, u^{|\Delta_+|} e^{-i\frac{\theta}{4}\dim G})$$

Here $|\Delta_+|$ is the number of positive roots of G. The images of $\hat{s}$ and $\hat{\gamma}$ are given by a straightforward computation using that $\dim G = 2|\Delta_+| + n$. □

Lemme 7.3. *For any k, there is a morphism $\zeta_k : \Gamma_\infty \to \mathbb{C}^*$ such that*

$$\zeta_k(\hat{s}) = e^{i\frac{\pi}{4}x_k}, \quad \zeta_k(\hat{t}) = e^{-i\frac{\pi}{12}x_k}, \quad \zeta_k(\hat{\gamma}) = e^{-i\frac{\pi}{2}x_k}$$

with $x_k = h^\vee \dim G/k$ where $h^\vee$ is the dual Coxeter number of G.

Proof. The value of x_k does not matter for the proof. We only use that Γ_∞ is generated by $\hat{s}$, $\hat{t}$ and $\hat{\gamma}$ with the relations

$$\big(\hat{\gamma}\hat{s}^2\big)^2 = \mathrm{id}, \quad \big(\hat{s}\hat{t}\big)^3 = \hat{s}^2, \quad \hat{\gamma} \text{ and } \hat{s}^2 \text{ are central}$$

as it is asserted in [1], example 5.7.7. □

So by the previous lemmas, we define a representation of Γ_∞ by

$$\tilde{R}_\infty(A,\gamma) = \zeta_k(A,\gamma).R_4(\Psi(A,\gamma))$$

We will prove it is the same as R_∞. To do this we have to identify $\mathbb{C}^{C_k}$ with $H^0_\tau(\mathrm{T}^2, L^k \otimes \delta_\tau)$. We need the following well-known fact.

Lemme 7.4. *For any k, the map sending λ into $\frac{\lambda+\rho}{k}$ is a bijection between C_k and $\mathfrak{A} \cap k^{-1}\Lambda^*$*

The identification is then defined by sending the canonical basis of $\mathbb{C}^{C_k}$ into the basis $\chi_{\mu,k} \otimes \Omega^{1/2}$, $\mu \in \mathfrak{A} \cap k^{-1}\Lambda^*$.

Theorem 7.5. *The matrices of $\tilde{R}_\infty(\hat{s})$, $\tilde{R}_\infty(\hat{t})$, $\tilde{R}_\infty(\hat{\gamma})$ in the basis $\chi_{(\lambda+\rho)/k,k} \otimes \Omega^{1/2}$, $\lambda \in C_k$ are respectively given by*

$$(\exp(-i\tfrac{\pi}{4}c)\tilde{s}_{\lambda\mu}), \qquad (\tilde{t}_{\lambda\mu}), \qquad (\exp(i\tfrac{\pi}{2}c)\delta_{\lambda\mu}).$$

where $(s_{\lambda\mu})$ and $(t_{\lambda\mu})$ are the matrices defined in section 7.

Not only does it prove that $\tilde{R}_\infty = R_\infty$, but it also proves theorem 7.1.

Proof. This follows from theorem 6.5. Since $\Psi(\hat{s}) = (S, e)$ with $e(i) = e^{-i\frac{\pi}{4}}$ and $\zeta_k(\hat{s}) = \exp(i\frac{\pi}{4}\dim G \frac{h^\vee}{k})$, the matrix of $\tilde{R}_\infty(\hat{s})$ is

$$\begin{aligned} &\frac{e^{i\frac{\pi}{4}\left(n+\frac{h^\vee}{k}\dim G\right)}}{k^{\frac{n}{2}}\,\mathrm{Vol}(\mathfrak{t}/\Lambda)} \sum_{w\in W} (-1)^{\ell(w)} \exp(-2i\pi k B(\tfrac{\rho+\mu}{k}, w(\tfrac{\rho+\lambda}{k})) \\ =&e^{i\frac{\pi}{4}\left(n+\frac{h^\vee}{k}\dim G\right)} \Big|\frac{\Lambda^*}{k\Lambda}\Big|^{-1/2} \sum_{w\in W} (-1)^{\ell(w)} \exp(-2i\pi k^{-1} B(\rho+\mu, w(\rho+\lambda)) \end{aligned}$$

because

$$\Big|\frac{\Lambda^*}{\Lambda}\Big| = \frac{\mathrm{Vol}(\mathfrak{t}/\Lambda)}{\mathrm{Vol}(\mathfrak{t}/\Lambda^*)} = \mathrm{Vol}^2(\mathfrak{t}/\Lambda).$$

To conclude it suffices to compare with the formula defining $\tilde{s}_{\lambda\mu}$. Since $\Psi(\hat{t}) = (T, 1)$ and because of the value of $\zeta_k(\hat{t})$, the matrix of $\tilde{R}_\infty(\hat{t})$ is

$$\begin{aligned} & e^{-i\frac{\pi}{12}\dim G \frac{h^\vee}{k}} \delta_{\mu,\lambda} \exp(i\pi k B(\tfrac{\rho+\mu}{k}, \tfrac{\rho+\mu}{k})) \\ =& \delta_{\mu,\lambda} \exp(i\pi k^{-1} B(\mu, 2\rho + \mu)) \end{aligned}$$

where we have used that

$$\frac{B(\rho,\rho)}{2k} = \frac{1}{24}(\dim G - c)$$

which follows from Freudenthal's strange formula. □

8 Complex structure dependence in the semiclassical limit

8.1 Fourier integral operators

Let M be a symplectic compact manifold with a prequantization bundle L. Consider two pairs (j_a, δ_a) and (j_b, δ_b) consisting of a positive complex structure of M together with a half-form bundle. We say that a section φ of $\mathrm{Hom}(\delta_a, \delta_b)$ is a half-form bundle isomorphism if its square at x is given by

$$\varphi_x^{\otimes 2} = \pi^*_{b,a,x} : \wedge^{\mathrm{top}} E^*_{a,x} \to \wedge^{\mathrm{top}} E^*_{b,x}$$

where $\pi_{b,a,x}$ is the projection from $E_{b,x} := T_x^{1,0}(M, j_b)$ to $E_{a,x} := T_x^{1,0}(M, j_a)$ with kernel $\overline{E}_{b,x}$.

For $c = a, b$, denote by $\mathcal{H}_k(c)$ the space of sections of $L^k \otimes \delta_c$ holomorphic with respect to j_c. Consider a sequence (S_k) such that for every k, S_k is an operator $\mathcal{H}_k(a) \to \mathcal{H}_k(b)$. The scalar product of $\mathcal{H}_k(a)$ gives us an isomorphism

$$\mathrm{Hom}(\mathcal{H}_k(a), \mathcal{H}_k(b)) \simeq \mathcal{H}_k(b) \otimes \overline{\mathcal{H}}_k(a).$$

The latter space can be regarded as the space of holomorphic sections of

$$(L^k \otimes \delta_b) \boxtimes (\bar{L}^k \otimes \overline{\delta}_a) \to M^2,$$

where M^2 is endowed with the complex structure $(j_b, -j_a)$. The section $S_k(x, y)$ associated in this way to S_k is its Schwartz kernel.

We say that (S_k) is a Fourier integral operator with symbol the half-form bundle isomorphism φ if

$$S_k(x, y) = \Big(\frac{k}{2\pi}\Big)^n E^k(x, y) f(x, y, k) + O(k^{-\infty}) \qquad (3)$$

where

- E is a section of $L \boxtimes \bar{L} \to M^2$ such that $\|E(x,y)\| < 1$ if $x \neq y$,

$$E(x,x) = u \otimes \bar{u}, \quad \forall u \in L_x \text{ such that } \|u\| = 1,$$

  and $\bar{\partial} E \equiv 0$ modulo a section vanishing to any order along the diagonal.

- $f(.,k)$ is a sequence of sections of $\delta_b \boxtimes \bar{\delta}_a \to M^2$ which admits an asymptotic expansion in the $\mathcal{C}^\infty$ topology of the form

$$f(.,k) = f_0 + k^{-1} f_1 + k^{-2} f_2 + ...$$

  whose coefficients satisfy $\bar{\partial} f_i \equiv 0$ modulo a section vanishing to any order along the diagonal.

- The restriction to the diagonal of the leading coefficient f_0 is equal to φ, if we identify $\delta_b \otimes \overline{\delta}_a$ with $\operatorname{Hom}(\delta_a, \delta_b)$ using the metric of δ_a.

8.2 The maps $\Psi_{\tau_1,\tau_2,k}$ in the semi-classical limit

Recall that for any $\tau \in \mathcal{H}_+$, we defined a complex structure on T^2 together with a half-form bundle δ_τ. Consider the morphism φ_{τ_1,τ_2} form δ_{τ_1} to δ_{τ_2} given by

$$\varphi_{\tau_1,\tau_2}(\Omega^{1/2}) = \Big(\frac{\tau_1 - \overline{\tau}_1}{\tau_2 - \overline{\tau}_1} \Big)^{n/2} \Omega^{1/2}$$

where the square root is determined in such a way that it depends continuously on τ_1, τ_2 and equal to 1 when $\tau_1 = \tau_2$.

Lemme 8.1. *The map φ_{τ_1,τ_2} is a half-form bundle isomorphism.*

Proof. Introduce an orthonormal basis (u_i) of $\mathfrak{t}$ and denote by (p_i) and (q_i) the associated linear coordinates of $\mathfrak{t}^2$. Let ζ^i_τ be the complex coordinate $p_i + \tau q_i$. Let $E_\tau = \operatorname{span}(\partial_{\zeta^1_\tau}, \ldots, \partial_{\zeta^n_\tau})$ be the space of vector of type $(1,0)$ with respect to the complex structure j_τ. Let π_{τ_2,τ_1} be the projection from E_{τ_2} to E_{τ_1} with kernel $\overline{E}_{\tau_2}$. One has

$$d\zeta^i_{\tau_2} = \Big(\frac{\tau_2 - \overline{\tau}_1}{\tau_1 - \overline{\tau}_1} \Big) d\zeta^i_{\tau_1} + \Big(\frac{\tau_1 - \tau_2}{\tau_1 - \overline{\tau}_1} \Big) d\overline{\zeta}^i_{\tau_1}$$

which implies that

$$\pi^*_{\tau_2,\tau_1} d\zeta^i_{\tau_1} = \Big(\frac{\tau_1 - \overline{\tau}_1}{\tau_2 - \overline{\tau}_1} \Big) d\zeta^i_{\tau_2}$$

and then

$$\pi^*_{\tau_2,\tau_1}(d\zeta^1_{\tau_1}\wedge\ldots\wedge d\zeta^n_{\tau_1}) = \Big(\frac{\tau_1-\overline{\tau}_1}{\tau_2-\overline{\tau}_1}\Big)^n d\zeta^1_{\tau_2}\wedge\ldots\wedge d\zeta^n_{\tau_2}$$

Finally recall that by equation (1) the squaring map of δ_τ sends $\Omega^{1/2}$ into $\pm\,\mathrm{Vol}(\mathfrak{t}/\Lambda)d\zeta^1_\tau\wedge\ldots\wedge d\zeta^n_\tau$. □

Theorem 8.2. *For any $\tau_1,\tau_2\in\mathcal{H}_+$, the sequence $(\Psi_{\tau_1,\tau_2,k})_k$ is a Fourier integral operator with symbol φ_{τ_1,τ_2}.*

Section 8.3 is devoted to the proof.

8.3 Proof of theorem 8.2

As previously we lift everything from T^2 to $\mathfrak{t}^2$. Denote by p_2,q_2 (resp. p_1,q_1) the coordinates on the left (resp. right) factor of $\mathfrak{t}^2\times\mathfrak{t}^2$. Let $\zeta_i=p_i+\tau_i q_i$ for $i=1,2$.

Let us first compute the section E and the leading coefficient f_0 of (3). Let $L_{\mathfrak{t}^2}$ be the trivial line bundle over $\mathfrak{t}^2$ with the connection defined in section 3. Consider the bundle $L_{\mathfrak{t}^2}\boxtimes\overline{L}_{\mathfrak{t}^2}$ endowed with the holomorphic structure compatible with the connection and the complex structure $(j_{\tau_2},-j_{\tau_1})$ of $\mathfrak{t}^2\times\mathfrak{t}^2$.

Lemme 8.3. *The section of $L_{\mathfrak{t}^2}\boxtimes\overline{L}_{\mathfrak{t}^2}$*

$$\exp\Big(\frac{-i\pi}{\tau_2-\overline{\tau}_1}B(\zeta_2-\overline{\zeta}_1,\zeta_2-\overline{\zeta}_1)\Big)s_{\tau_2}\boxtimes\overline{s}_{\tau_1}$$

is holomorphic and restricts to the constant section equal to 1 on the diagonal. The morphism φ_{τ_1,τ_2} is sent to the constant section equal to

$$\Big(\frac{2i\pi}{\tau_2-\overline{\tau_1}}\Big)^{n/2}\mathrm{Vol}(\mathfrak{t}/\Lambda)\Omega^{1/2}\otimes\overline{\Omega}^{1/2}$$

by the isomorphism between $\mathrm{Hom}(\delta_{\tau_1},\delta_{\tau_2})$ *and* $\delta_{\tau_2}\otimes\overline{\delta}_{\tau_1}$ *induced by the metric of δ_{τ_1}.*

The first part is proved by a straightforward computation and the second part follows from lemma 6.1. By theorem 6.2, the Schwartz kernel of $\Psi_{\tau_1,\tau_2,k}$ lifts from $\mathrm{T}^2\times\mathrm{T}^2$ to $\mathfrak{t}^2\times\mathfrak{t}^2$ into

$$\begin{aligned} S_k(p_2,q_2,p_1,q_1) &= \sum_{\mu\in(k^{-1}\Lambda^*)/\Lambda}\Theta^{\tau_2}_{\mu,k}(p_2,q_2)\overline{\Theta}^{\tau_1}_{\mu,k}(p_1,q_1)(s^k_{\tau_2}\Omega^{1/2})\boxtimes(\overline{s}^k_{\tau_1}\overline{\Omega}^{1/2}) \\ &= \sum_{\substack{\mu\in(k^{-1}\Lambda^*)/\Lambda \\ \gamma_1,\gamma_2\in\mu+\Lambda}} f_{\gamma_2,\gamma_1}(p_2,q_2,p_1,q_1)(s^k_{\tau_2}\Omega^{1/2})\boxtimes(\overline{s}^k_{\tau_1}\overline{\Omega}^{1/2}) \end{aligned}$$

where the coefficients are given by

$$f_{\gamma_2,\gamma_1} = \exp\bigl(2i\pi k\bigl(\tfrac{\tau_2}{2}B(\gamma_2,\gamma_2) - B(\zeta_2,\gamma_2) - \tfrac{\overline{\tau}_1}{2}B(\gamma_1,\gamma_1) + B(\overline{\zeta}_1,\gamma_1)\bigr)\bigr).$$

Lemme 8.4. *For any compact set K of $\mathfrak{t}$ such that $K\cap\Lambda=\emptyset$, there exists $C>0$ such that for all p_2,q_2,p_1,q_1 satisfying $q_1-q_2\notin K$ one has*

$$\bigl|S_k(p_2,q_2,p_1,q_1)\bigr| \leqslant Ce^{-k/C}$$

This shows that the sequence of Schwartz kernels of $\Psi_{\tau_1,\tau_2,k}$ is a $O(k^{-\infty})$ outside the diagonal. For the proof of the lemma we need the following general estimates.

Lemme 8.5. *For any $C>0$ there exists $C'>0$ such that*

$$\sum_{\gamma\in\mathbb{Z}^n} e^{-kC|\gamma-x|^2} \leqslant C'k^{n/2}, \qquad \forall x\in\mathbb{R}^n$$

For any $C>0$, for any compact K of $\mathbb{R}^n$ and for any subset P of $\mathbb{Z}^n$ such that $K\cap(\mathbb{Z}^n\setminus P)=\emptyset$, there exists $C'>0$ such that

$$\sum_{\gamma\in\mathbb{Z}^n\setminus P} e^{-kC|\gamma-x|^2} \leqslant C'e^{-k/C'}, \quad \forall x\in K.$$

Proof of lemma 8.4. By a straightforward computation we obtain that

$$\bigl|f_{\gamma_2,\gamma_1}s^k_{\tau_2}\boxtimes\overline{s}^k_{\tau_1}\bigr| = \exp\Bigl(-k\pi\Bigl(\frac{\tau_2-\overline{\tau}_2}{2i}|q_2-\gamma_2|^2+\frac{\tau_1-\overline{\tau}_1}{2i}|q_1-\gamma_1|^2\Bigr)\Bigr)$$

Hence for some positive C,

$$\bigl|f_{\gamma_2,\gamma_1}s^k_{\tau_2}\boxtimes\overline{s}^k_{\tau_1}\bigr| \leqslant e^{-kC(|\gamma_2-q_2|^2+|(q_2-q_1)-(\gamma_2-\gamma_1)|^2)}$$

So with $q=q_2-q_1$,

$$\begin{aligned}\sum_{\gamma_1,\gamma_2\in\mu+\Lambda}\bigl|f_{\gamma_2,\gamma_1}s^k_{\tau_2}\boxtimes\overline{s}^k_{\tau_1}\bigr| &\leqslant \sum_{\substack{\gamma_2\in\mu+\Lambda\\ \gamma\in\Lambda}} e^{-kC(|\gamma_2-q_2|^2+|q-\gamma|^2)}\\ &\leqslant\Bigl(\sum_{\gamma\in\Lambda}e^{-kC|\mu+\gamma-q_2|^2}\Bigr)\Bigl(\sum_{\gamma\in\Lambda}e^{-kC|\gamma-q|^2}\Bigr)\end{aligned}$$

By lemma 8.5, there exists $C'>0$ such that the first factor is bounded by $C'k^{n/2}$, the second one by $C'e^{-k/C'}$, and these estimates are uniform with respect to μ, q_1 and q_2 such that $q_1 - q_2 \in K$. Since the cardinal of $(k^{-1}\Lambda^*)/\Lambda$ is $k^n|\Lambda^*/\Lambda|$, we obtain with a larger C' that

$$\sum_{\substack{\mu\in(k^{-1}\Lambda^*)/\Lambda \\ \gamma_1,\gamma_2\in\mu+\Lambda}} \left| f_{\gamma_2,\gamma_1}(p_2,q_2,p_1,q_1) s^k_{\tau_2} \boxtimes \overline{s}^k_{\tau_1} \right| \leqslant C' e^{-k/C'}$$

which proves the result. □

Using exactly the same method and lemma 8.5 with $P = \{0\}$, we show

Lemme 8.6. *There exists $C > 0$ such that*

$$\sum_{\substack{\mu\in(k^{-1}\Lambda^*)/\Lambda \\ \gamma_1,\gamma_2\in\mu+\Lambda,\ \gamma_1\neq\gamma_2}} \left| f_{\gamma_2,\gamma_1}(p_2,q_2,p_1,q_1) s^k_{\tau_2} \boxtimes \overline{s}^k_{\tau_1} \right| \leqslant C e^{-k/C}$$

for all p_2, q_2, p_1 and q_1 such that $|q_1 - q_2| \leqslant R/2$ with $R = \min\bigl\{|\gamma|,\ \gamma \in \Lambda \setminus \{0\}\bigr\}$.

So up to a $O(k^{-\infty})$, $S_k(p_2,q_2,p_1,q_1)$ is given on a neighborhood of the diagonal by the sum of the $f_{\gamma,\gamma}$ where γ runs over $k^{-1}\Lambda^*$.

Lemme 8.7. *We have*

$$\begin{aligned}\sum_{\gamma\in k^{-1}\Lambda^*} f_{\gamma,\gamma}(p_2,q_2,p_1,q_1) =& \Bigl(\frac{ik}{\tau_2 - \overline{\tau}_1}\Bigr)^{n/2} \operatorname{Vol}(\mathfrak{t}/\Lambda) \\ & \times \sum_{\lambda\in\Lambda} \exp\Bigl(\frac{-i\pi k}{\tau_2 - \overline{\tau}_1} B(\lambda + \zeta_2 - \overline{\zeta}_1, \lambda + \zeta_2 - \overline{\zeta}_1)\Bigr)\end{aligned}$$

Proof. Introduce a basis (π_i) of Λ^*. Let $\gamma \in k^{-1}\Lambda^*$, write $k\gamma = \sum x_i\pi_i$. One has

$$\begin{aligned} f_{\gamma,\gamma}(p_2,q_2,p_1,q_1) =& \exp\Bigl(-\Bigl(\frac{\tau_2 - \overline{\tau}_1}{2i}\Bigr)\Bigl(\frac{2\pi}{k}\Bigr) B(k\gamma,k\gamma) - 2i\pi B(\zeta_2 - \overline{\zeta}_1, k\gamma)\Bigr)\Bigr) \\ =& u(x)\exp(-i\langle\eta, x\rangle)\end{aligned}$$

where u is the complex valued function of $\mathbb{R}^n$ given by

$$u(x) = \exp\Bigl(-\frac{1}{2}\Bigl(\frac{\tau_2 - \overline{\tau}_1}{i}\Bigr)\Bigl(\frac{2\pi}{k}\Bigr)\sum_{i,j} B(\pi_i,\pi_j)x_ix_j\Bigr)$$

and

$$\eta_i = 2\pi B(\zeta_2 - \overline{\zeta}_1, \pi_i).$$

Let (γ_i) be the dual basis of (π_i). Then using that $B(\gamma_i, \gamma_j)$ is the inverse of $B(\pi_i, \pi_j)$ and that the determinant of $\big(B(\gamma_i, \gamma_j)\big)_{i,j}$ is the square of the volume of $\mathfrak{t}/\Lambda$, we prove that the Fourier transform of u is

$$\hat{u}(\xi) = \Big(\frac{ik}{\tau_2 - \overline{\tau}_1}\Big)^{n/2} \operatorname{Vol}(\mathfrak{t}/\Lambda) \exp\Big(-\frac{1}{2}\Big(\frac{i}{\tau_2 - \overline{\tau}_1}\Big)\Big(\frac{k}{2\pi}\Big) \sum_{i,j} B(\gamma_i, \gamma_j)\xi_i\xi_j\Big).$$

The Fourier transform of $v(x) = u(x)\exp(-i\langle \eta, x\rangle)$ is

$$\hat{v}(\xi) = \hat{u}(\xi + \eta)$$

for real η. This is also verified for any $\eta \in \mathbb{C}^n$ by analytic prolongation. By Poisson's summation formula,

$$\sum_{x \in \mathbb{Z}^n} v(x) = \sum_{\xi \in \mathbb{Z}^n} \hat{v}(2\pi\xi)$$

Let us compute $\hat{v}(2\pi\xi)$. Since $\sum \eta_i \gamma_i = 2\pi(\zeta_2 - \overline{\zeta}_1)$, we have

$$\sum_{i,j} B(\gamma_i, \gamma_j)(2\pi\xi_i + \eta_i)(2\pi\xi_j + \eta_j) = (2\pi)^2 B(\lambda + \zeta_2 - \overline{\zeta}_1, \lambda + \zeta_2 - \overline{\zeta}_1)$$

where $\lambda = \sum \xi_i \gamma_i \in \Lambda$. So $\hat{v}(2\pi\xi)$ which is equal to $\hat{u}(2\pi\xi + \eta)$ is given by

$$\hat{v}(2\pi\xi) = \Big(\frac{ik}{\tau_2 - \overline{\tau}_1}\Big)^{n/2} \operatorname{Vol}(\mathfrak{t}/\Lambda) \exp\Big(\frac{-i\pi k}{\tau_2 - \overline{\tau}_1} B(\lambda + \zeta_2 - \overline{\zeta}_1, \lambda + \zeta_2 - \overline{\zeta}_1)\Big)$$

which concludes the proof. □

Lemme 8.8. *There exists $\epsilon > 0$ and $C > 0$ such that*

$$\sum_{\lambda \in \Lambda \setminus \{0\}} \Big| \exp\Big(\frac{-i\pi k}{\tau_2 - \overline{\tau}_1} B(\lambda + \zeta_2 - \overline{\zeta}_1, \lambda + \zeta_2 - \overline{\zeta}_1)\Big) s^k_{\tau_2} \boxtimes \overline{s}^k_{\tau_1} \Big| \leqslant C e^{-k/C}$$

for all p_2, q_2, p_1 and q_1 satisfying $|q_1 - q_2| \leqslant \epsilon$ and $|p_1 - p_2| \leqslant \epsilon$.

Proof. With a straightforward computation, we obtain that

$$\Big| \exp\Big(\frac{-i\pi}{\tau_2 - \overline{\tau}_1} |\lambda + \zeta_2 - \overline{\zeta}_1|^2\Big) s_{\tau_2} \boxtimes \overline{s}_{\tau_1} \Big|^2 = e^{-(a|\mu|^2 + 2bB(\mu, q) + c|q|^2)}$$

where $q = q_2 - q_1$ $\mu = \lambda + p_2 - p_1$ and a, b and c are the real numbers

$$a = \frac{-i\pi}{|\tau_2 - \overline{\tau}_1|^2}(\tau_2 - \overline{\tau}_2 + \tau_1 - \overline{\tau}_1)$$

$$b = \frac{i\pi}{|\tau_2 - \overline{\tau}_1|^2}(\overline{\tau}_2\overline{\tau}_1 - \tau_2\tau_1)$$

$$c = \frac{-i\pi}{|\tau_2 - \overline{\tau}_1|^2}(|\tau_2|^2(\tau_1 - \overline{\tau}_1) + |\tau_1|^2(\tau_2 - \overline{\tau}_2))$$

Write

$$a|\mu|^2 + 2bB(\mu, q) + c|q|^2 = a|\mu + \tfrac{b}{a}q|^2 + (c - \tfrac{b^2}{a})|q|^2.$$

Using that a is positive, one proves that there exists $\epsilon > 0$ such that

$$|q| \leqslant \epsilon \text{ and } |p_2 - p_1| \leqslant \epsilon \Rightarrow \tfrac{a}{2}|\mu + \tfrac{b}{a}q|^2 + \big(c - \tfrac{b^2}{a}\big)|q|^2 \geqslant 0$$

for any non-vanishing $\lambda \in \Lambda$. So

$$\sum_{\lambda \in \Lambda \setminus \{0\}} \Big| \exp\Big(\frac{-ik\pi}{\tau_2 - \overline{\tau}_1}|\lambda + \zeta_2 - \overline{\zeta}_1|^2\Big) s^k_{\tau_2} \boxtimes \overline{s}^k_{\tau_1} \Big| \leqslant \sum_{\lambda \in \Lambda \setminus \{0\}} e^{-\frac{a}{4}k\left|\lambda + p_2 - p_1 + \frac{b}{a}q\right|^2}$$

when $|q|$ and $|p_2 - p_1|$ are smaller than ϵ. We conclude with lemma 8.5. □

Collecting together the previous lemmas, we obtain

$$S_k(p_2, q_2, p_1, q_1) = \Big(\frac{k}{2\pi}\Big)^n \Big(\frac{2i\pi}{\tau_2 - \overline{\tau}_1}\Big)^{n/2} \operatorname{Vol}(\mathfrak{t}/\Lambda)$$
$$\times \exp\Big(-\frac{i\pi k}{\tau_2 - \overline{\tau}_1} B(\zeta_2 - \overline{\zeta}_1, \zeta_2 - \overline{\zeta}_1)\Big)(s^k_{\tau_2}\Omega^{1/2}) \boxtimes (\overline{s}^k_{\tau_1}\overline{\Omega}^{1/2}) + R_k(p_2, q_2, p_1, q_1)$$

where the remainder satisfies for some $\epsilon > 0$ and $C > 0$,

$$|q_1 - q_2|, |p_1 - p_2| \leqslant \epsilon \Rightarrow |R_k(p_2, q_2, p_1, q_1)| \leqslant Ce^{-k/C}.$$

Using lemma 8.3, this proves theorem 8.2.

9 Asymptotic properties of the quantum representations

9.1 Definitions

Let M be a symplectic compact manifold with a positive complex structure j, a prequantization bundle L and a half-form bundle δ. Consider a symplectomorphism $\Phi : M \to M$ together with automorphisms Φ_L and φ of the bundles L and δ respectively which lift Φ. We assume that Φ_L preserves the connection and metric of L.

Let $\mathcal{H}_k$ be the space of holomorphic sections of $L^k \otimes \delta$. Consider a family (S_k) such that for every k, S_k is an operator $\mathcal{H}_k \to \mathcal{H}_k$. The Schwartz kernel of S_k is a holomorphic section of

$$(L^k \otimes \delta) \boxtimes (\overline{L}^k \otimes \overline{\delta}) \to M^2,$$

where M^2 is endowed with the complex structure $(j, -j)$. We say that (S_k) is a Fourier integral operator associated to Φ_L with symbol φ if the Schwartz kernel sequence is of the form

$$S_k(x,y) = \Big(\frac{k}{2\pi}\Big)^n F^k(x,y) g(x,y,k) + O(k^{-\infty})$$

where

- F is a section of $L \boxtimes \bar{L} \to M^2$ such that $\|F(x,y)\| < 1$ if $x \neq \Phi(y)$,

  $$F(\Phi(x), x) = \Phi_L(u) \otimes \bar{u}, \quad \forall u \in L_x \text{ such that } \|u\| = 1,$$

  and $\bar{\partial} F \equiv 0$ modulo a section vanishing to any order along the graph of Φ^{-1}.

- $g(.,k)$ is a sequence of sections of $\delta \boxtimes \overline{\delta} \to M^2$ which admits an asymptotic expansion in the $\mathcal{C}^\infty$ topology of the form

  $$g(.,k) = g_0 + k^{-1} g_1 + k^{-2} g_2 + ...$$

  whose coefficients satisfy $\bar{\partial} g_i \equiv 0$ modulo a section vanishing to any order along the graph of Φ^{-1}.

- The restriction to the diagonal of the leading coefficient g_0 is equal to φ, if we identify $\delta \otimes \overline{\delta}$ with $\mathrm{Hom}(\delta, \delta)$ using the metric of δ.

Let us explain the relation with the Fourier integral operators of section 8.1. Let $\Phi(j)$ be the complex structure obtained by pushing forward j with Φ. Consider a half-form bundle δ' of the complex manifold $(M, \Phi(j))$ together with an isomorphism $\varphi_1 : \delta \to \delta'$ whose square is equal to

$$\varphi_{1,x}^{\otimes 2} = ((T_x \Phi)^*)^{-1} : \wedge_j^{\mathrm{top},0} T_x^* M \to \wedge_{\Phi(j)}^{\mathrm{top},0} T^*_{\Phi(x)} M$$

Then the isomorphisms Φ_L and φ_1 induce a linear isomorphism Φ_* from $\mathcal{H}_k$ to the space $\mathcal{H}'_k$ consisting of the sections of $L^k \otimes \delta'$ holomorphic with respect to $\Phi(j)$. Now suppose that

$$S_k = T_k \circ \Phi_* : \mathcal{H}_k \to \mathcal{H}_k, \qquad k = 1, 2, \ldots$$

for an endomorphism $T_k : \mathcal{H}'_k \to \mathcal{H}_k$. Then comparing the definition of section 8.1 with the previous one, we prove that (T_k) is a Fourier integral operator with symbol φ_2 in the sense of section 8.1 if and only if (S_k) is a Fourier integral operator associated to Φ_L with symbol $\varphi = \varphi_2 \circ \varphi_1$. This applies to the representation R_2 defined in corollary 6.4. Indeed for any $(A, e) \in \Gamma_2$, $R_2(A, e)$ is the composition of a pull-back with the map $\Psi_{A\tau,\tau}$, which is a Fourier integral operator by theorem 8.2.

Since the half-form bundle δ_τ is the trivial bundle, we can identify its automorphisms with functions on T^2, the correspondence being given by $\varphi_x(\Omega^{1/2}) = f(x)\Omega^{1/2}$. We use this convention in the sequel for the symbols of the Fourier integral operators.

Theorem 9.1. *For any τ and $(A, e) \in \Gamma_2$ the sequence*

$$R_2(A, e) : H^0_\tau(\mathrm{T}^2, L^k \otimes \delta) \to H^0_\tau(\mathrm{T}^2, L^k \otimes \delta), \qquad k = 1, 2, \ldots$$

is a Fourier integral operator associated to the prequantum lift of A to L. Its symbol is the constant function equal to

$$\sigma(A, e) = e(\tau) \left(\frac{A\tau - \overline{A\tau}}{\tau - \overline{A\tau}} \right)^{n/2}.$$

Applying theorem 8.2 with $\tau_1 = \tau_2$, the representation of the Weyl group W is also given by Fourier integral operators.

Theorem 9.2. *For any τ and $w \in W$ the sequence*

$$w : H^0_\tau(\mathrm{T}^2, L^k \otimes \delta) \to H^0_\tau(\mathrm{T}^2, L^k \otimes \delta), \qquad k = 1, 2, \ldots$$

is a Fourier integral operator associated to the prequantum lift of w to L. Its symbol is the constant function equal to 1.

9.2 Metaplectic group

Let S be a symplectic vector space with a positive compatible complex structure j. Denote by $E = \ker(\mathrm{id} + ij)$ the space of vectors with type $(1, 0)$. Let $\mathrm{Sp}(S)$ be the symplectic group of S. Using the complex structure we introduce a group $\mathrm{Mp}(S, j)$, isomorphic to the metaplectic group of S. $\mathrm{Mp}(S, j)$ consists of the pairs (A, z) such that $A \in \mathrm{Sp}(S)$ and z is a complex number satisfying

$$z^2 = \det(g^{-1}\pi_{E,gE} : E \to E).$$

Here $\pi_{E,gE}$ is the projection from E onto gE with kernel $\overline{E}$. The product of $\mathrm{Mp}(S,j)$ is determined by the condition that the projection onto the symplectic group is a group morphism and that the identity is the pair $(\mathrm{id}, 1)$. We shall also consider an extension $\mathrm{Mp}_2(S,j)$ by $\mathbb{Z}_4 = \{\pm 1, \pm i\}$ of the symplectic group. It is defined as the set of pairs (A, z) such that $z^4 = \det^2(g^{-1}\pi_{E,gE} : E \to E)$. The product is determined by the condition that the map

$$\mathrm{Mp}(S,j) \times \mathbb{Z}_4 \to \mathrm{Mp}_2(S,j)$$

sending (A, z, u) into (A, zu) is a group morphism.

We apply these constructions to $S = \mathfrak{t} \oplus \mathfrak{t} = \mathbb{R}^2 \otimes \mathfrak{t}$ with the complex structure given by some τ in the upper half-plane. Then the symbols of the operators defining the representation R_2 belongs to the metaplectic group.

Proposition 9.3. *We have a group morphism from* Γ_2 *to* $\mathrm{Mp}(\mathfrak{t} \oplus \mathfrak{t}, j_\tau)$ *sending* (A, e) *into* $(A \otimes \mathrm{id}_{\mathfrak{t}}, \sigma(A, e))$ *with*

$$\sigma(A,e) = e(\tau)\big(\tfrac{A\tau - \overline{A\tau}}{\tau - \overline{A\tau}}\big)^{n/2}$$

Proof. Observe that for any $g \in \mathrm{Sp}(S)$, the endomorphism

$$\pi^*_{E,gE} \circ (g^{-1})^* : \wedge^{\mathrm{top}} E^* \to \wedge^{\mathrm{top}} (gE)^* \to \wedge^{\mathrm{top}} E^*$$

is the multiplication by $\det(g^{-1}\pi_{E,gE} : E \to E)$.

Let us apply this to $g = A \otimes \mathrm{id}_{\mathfrak{t}}$. By equation (2) and the condition $e(\tau)^2 = (-c\tau + d)^n$, the pull-back by g^{-1} is multiplication by $e^2(\tau)$. By lemma 8.1, $\pi^*_{E,gE}$ is the multiplication by $\big(\frac{A\tau - \overline{A\tau}}{\tau - \overline{A\tau}}\big)^n$. This implies that $(A \otimes \mathrm{id}_{\mathfrak{t}}, \sigma(A, e))$ belongs to the metaplectic group $\mathrm{Mp}(\mathfrak{t} \oplus \mathfrak{t}, j_\tau)$.

One shows that the map is a group morphism by extending it to the group defined as Γ_2 by replacing $\mathrm{Sl}(2, \mathbb{Z})$ with $\mathrm{Sl}(2, \mathbb{R})$ and using a continuity argument. □

For any $A \in \mathrm{Sl}(2, \mathbb{R})$, let $d(A, \tau) = \det(A^{-1}\pi_{E,AE} : E \to E)$ with E the complex polarization determined by the complex structure $p + \tau q$. Then in the proof of the previous proposition we showed that

$$d(A,\tau) = (-c\tau + d)\tfrac{A\tau - \overline{A\tau}}{\tau - \overline{A\tau}}. \tag{4}$$

We will use this equation several times in the sequel. Assume that the rank of G is even, so $n = 2p$. We have a morphism from Γ into Γ_2 sending A into

$(A, (-c\tau + d)^p)$. Composed with the morphism provided by proposition 9.3, we obtain the group morphism

$$\Gamma \to \mathrm{Mp}(\mathfrak{t} \oplus \mathfrak{t}, j_\tau), \qquad A \to (A \otimes \mathrm{id}_{\mathfrak{t}}, d(A, \tau)^p). \tag{5}$$

Assume now that the rank of G is odd, $n = 2p + 1$. Introduce the subgroup $\mathrm{Mp}(\mathbb{Z}, \tau)$ of $\mathrm{Mp}(\mathbb{R}^2, \tau)$ consisting of the pairs $(A, z) \in \Gamma \times \mathbb{C}^*$ such that $z^2 = d(A, \tau)$. Using again (4), we prove that this group is isomorphic to Γ_2, the isomorphism being given by

$$(A, z) \in \mathrm{Mp}(\mathbb{Z}, \tau) \to \Big(A, z(-c\tau + d)^p \big(\tfrac{A\tau - \overline{A\tau}}{\tau - \overline{A\tau}} \big)^{-1/2} \Big) \in \Gamma_2$$

Finally composing this morphism with the one of proposition 9.3, we obtain the group morphism

$$\mathrm{Mp}(\mathbb{Z}, \tau) \to \mathrm{Mp}(\mathfrak{t} \oplus \mathfrak{t}, j_\tau), \qquad (A, z) \to (A \otimes \mathrm{id}_{\mathfrak{t}}, z d(A, \tau)^p). \tag{6}$$

Considering the representation of the Weyl group, we obtain a morphism into the extension $\mathrm{Mp}_2(\mathfrak{t} \oplus \mathfrak{t}, j_\tau)$ of the symplectic group.

Proposition 9.4. *We have a group morphism from W to $\mathrm{Mp}_2(\mathfrak{t} \oplus \mathfrak{t}, j_\tau)$ sending w into $(\mathrm{id}_{\mathbb{R}^2} \otimes w, 1)$*

More generally, if w is an element of the orthogonal group of $\mathfrak{t}$, then $(\mathrm{id}_{\mathbb{R}^2} \otimes w, u)$ belongs to the metaplectic group (resp. the extension by $\mathbb{Z}_4$) if and only if $u^2 = \det w$ (resp. $u^4 = 1$).

9.3 Index computation

As previously consider the metaplectic group $\mathrm{Mp}(S, j)$ of a symplectic vector space endowed with a complex structure. Let $\mathrm{Mp}_*(S, j)$ be the subset consisting of the pairs (g, z) such that 1 is not an eigenvalue of g. Then we defined in [4] an index map

$$\mathrm{ind} : \mathrm{Mp}_*(S, j) \to \mathbb{Z}/4\mathbb{Z}$$

It is continuous and takes distinct values on each of the four components of $\mathrm{Mp}_*(S, j)$. To compute it, we only need the two following properties. If E has dimension 2, then

$$\mathrm{ind}(g, z) = k + \tfrac{1}{2}(1 - (-1)^{k+\epsilon}) \tag{7}$$

where $k \in \mathbb{Z}$ is such that the argument of z belongs to $[\frac{\pi}{2} k, \frac{\pi}{2}(k+1)[$ and ϵ is equal to 0 if the trace of g is bigger that 2 and to 1 otherwise. Furthermore if

$(g_1, z_1) \in \mathrm{Mp}_*(S_1, j_1)$ and $(g_2, z_2) \in \mathrm{Mp}_*(S_2, j_2)$ then $(g_1 \oplus g_2, z_1 z_2)$ belongs to $\mathrm{Mp}_*(S_1 \oplus S_2, j_1 \oplus j_2)$ and

$$\mathrm{ind}(g_1 \oplus g_2, z_1 z_2) = \mathrm{ind}(g_1, z_1) + \mathrm{ind}(g_2, z_2) \tag{8}$$

The elements (g, z) of $\mathrm{Mp}_2(S, j)$ such that 1 is not an eigenvalue of g, also have an index defined modulo $4\mathbb{Z}$. It is such that

$$\mathrm{ind}(g, i^k z) = k + \mathrm{ind}(g, z)$$

if $(g, z) \in \mathrm{Mp}_*(S, j)$. In the following we compute the index of some elements of the metaplectic group of $S = \mathfrak{t} \oplus \mathfrak{t}$ endowed with the complex structure determined by $\tau \in \mathcal{H}_+$.

Lemme 9.5. *For any hyperbolic $A \in \mathrm{Sl}(2, \mathbb{R})$ and w in the orthogonal group of $\mathfrak{t}$, we have*

$$\mathrm{ind}(A \otimes w, z) = \mathrm{ind}(A \otimes \mathrm{id}_{\mathfrak{t}}, z)$$

where z is any complex number such that $(A \otimes \mathrm{id}_{\mathfrak{t}}, z)$ belongs to $\mathrm{Mp}_2(\mathfrak{t} \oplus \mathfrak{t}, j_\tau)$.

Proof. A being hyperbolic, 1 is not an eigenvalue of $A \otimes w$. Since

$$(A \otimes \mathrm{id}_{\mathfrak{t}}, z).(\mathrm{id}_{\mathbb{R}^2} \otimes w, u) = (A \otimes w, zu)$$

The fact that $(A \otimes \mathrm{id}_{\mathfrak{t}}, z)$ belongs to $\mathrm{Mp}_2(\mathfrak{t} \oplus \mathfrak{t}, j_\tau)$ implies that $(A \otimes w, z)$ also belongs to $\mathrm{Mp}_2(\mathfrak{t} \oplus \mathfrak{t}, j_\tau)$. Let us prove that they have the same index. Since the index is locally constant, the result is straightforward if w belongs to the special orthogonal group. Otherwise we may assume that w is a reflexion and that A is the diagonal matrix with coefficient 2, 1/2. Let us decompose $\mathfrak{t}$ as a direct sum of orthogonal lines. The complex structure j_τ preserves the associated decomposition of $\mathbb{R}^2 \otimes \mathfrak{t}$. So using (8) it is sufficient to prove that

$$\mathrm{ind}(A, u) = \mathrm{ind}(-A, u)$$

One may assume that $(A, u) \in \mathrm{Mp}(\mathbb{R}^2, \tau)$ so that $(-A, iu) \in \mathrm{Mp}(\mathbb{R}^2, \tau)$. Then the result follows from formula (7). □

We can give explicit formulas for the index of $(A \otimes \mathrm{id}_{\mathfrak{t}}, z)$ by decomposing $\mathbb{R}^2 \otimes \mathfrak{t}$ into a direct sum of $\mathbb{R}^2$'s as we did in the previous proof.

Lemme 9.6. *If $n = 2p$, for any $A \in \mathrm{Sl}_*(2, \mathbb{R})$, we have*

$$\mathrm{ind}(A \otimes \mathrm{id}_{\mathfrak{t}}, d(A)^p) = 2\epsilon p$$

where ϵ is equal to 0 *if the trace of* A *is bigger than 2 and to 1 otherwise. If* $n = 2p+1$, *for any* $(A,z) \in \mathrm{Mp}_*(\mathbb{R}^2, \tau)$, *we have*

$$\mathrm{ind}(A \otimes \mathrm{id}_{\mathfrak{t}}, d(A)^p z) = 2\epsilon p + \mathrm{ind}(A, z)$$

where ϵ is defined as previously.

In the second case, the index of (A, z) is given by (7). With these formulas we obtain the index of any element in the images of the morphisms (5) and (6).

Proof. Working with the decomposition of $\mathbb{R}^2 \otimes \mathfrak{t}$, we only have to consider $n = 2$. We have

$$\mathrm{ind}(A \oplus A, d(A)) = 2\,\mathrm{ind}(A, z)$$

where $z^2 = d(A)$. We conclude with formula (7). □

9.4 Trace estimates

Under a transversality condition, the trace of a Fourier integral operator admits an asymptotic expansion and we can explicitly compute the leading term in terms of the symbol. Next theorem has been proved in [4]. We restrict to the case of T^2 to simplify the statement.

Theorem 9.7. *Let Φ be a symplectomorphism of* T^2 *whose graph intersects transversally the diagonal. Let Φ_L be a prequantum bundle isomorphism of L lifting Φ and* $(T_k : H^0_\tau(\mathrm{T}^2, L^k \otimes \delta_\tau) \to H^0_\tau(\mathrm{T}^2, L^k \otimes \delta_\tau))$ *be a Fourier integral operator associated to Φ_L with symbol f. Then for any fixed point x of Φ, there exists a sequence $(a_{\ell,x})$ of complex numbers such that for any N,*

$$\mathrm{tr}(T_k) = \sum_{x/\ \Phi(x)=x} u_x^k \big(a_{x,0} + a_{x,1} k^{-1} + \ldots + a_{x,N} k^{-N} + O(k^{-N-1}) \big)$$

where for any x, u_x is the trace of $\Phi_L(x) : L_x \to L_x$. Furthermore, if $(T_x\Phi, f(x))$ *is an element of* $\mathrm{Mp}_2(\mathfrak{t} \oplus \mathfrak{t}, j_\tau)$, *then*

$$a_{x,0} = \frac{i^{\mathrm{ind}(T_x\Phi, f(x))}}{|\det(\mathrm{id} - T_x\Phi)|^{1/2}}.$$

We apply this to estimate the character of the representation R_2^{alt}. First we have

$$\mathrm{tr}(R_2^{\mathrm{alt}}(A, e)) = \frac{1}{|W|} \sum_{w \in W} (-1)^{\ell(w)}\, \mathrm{tr}(w.R_2(A, e))$$

Then by theorems 9.1 and 9.2, $w.R_2(A,e)$ is a Fourier integral operator associated to the prequantum lift $A \otimes w$. Its symbol is the constant map equal to $\sigma(A,e)$. By 9.5, the index of $(A\otimes w, \sigma(A,e))$ doesn't depend on w. We compute easily the action of the prequantum lift of $A \otimes w$ at the fixed points and obtain the

Theorem 9.8. *For any $(A,e) \in \Gamma_2$ such that A is hyperbolic, we have*

$$\operatorname{tr}(R_2^{\mathrm{alt}}(A,e)) \sim \frac{i^{n(A,e)}}{|W|} \sum_{\substack{w\in W \\ u \in \mathrm{T}^2/\ (A\otimes w).x = x}} (-1)^{\ell(w)} \frac{e^{ik\theta(A\otimes w, x)}}{|\det(\mathrm{id} - A\otimes w)|^{1/2}}$$

where

- *$n(A,e)$ is the index of $(A \otimes \mathrm{id}_{\mathfrak{t}}, \sigma(A,e))$*
- *$\theta(A\otimes w, x) = \pi(B(\mu,p) - B(\gamma,q) + B(\gamma,\mu))$ if x is the class of $(p,q) \in \mathfrak{t}^2$ and $(\gamma,\mu) = (A\otimes w)(p,q) - (p,q)$.*

We can explicitly compute the indices with lemma 9.6. For the statement in the introduction we used the two morphisms (5) and (6).

A Proofs of theorem 5.1 and 5.2

A.1 The basis of $H^0_\tau(\mathrm{T}^2, L^k)$

Recall that the holomorphic sections of L^k identify with the sections over $\mathfrak{t}^2$ of the form fs^k such that $f : \mathfrak{t}^2 \to \mathbb{C}$ is holomorphic and fs^k is Λ^2-invariant. As shows a straightforward computation, this invariance is equivalent to

$$f(p+\dot p, q+\dot q) = f(p,q)\exp\big(-2i\pi k\big(B(\zeta,\dot q) + \tfrac{\tau}{2}B(\dot q,\dot q)\big)\big)$$

for all $\dot p, \dot q$ in Λ. Then to prove that the sections $\Theta_{\mu,k}s^k$ form a basis of the holomorphic sections of L^k, we decompose the functions f as a Fourier series in the p variable with coefficients depending on q. Then the holomorphy and the equivariance in the q-directions translate into a condition on the coefficients, leading to the result.

Let us shows that the sections $\Theta_{\mu,k}s^k$ are mutually orthogonal and compute their norms. Let m be the Riemannian volume of $\mathfrak{t}$. The Liouville measure $|\omega^n|/n!$ of $\mathfrak{t}^2$ is equal to $(2\pi)^n m(p)\otimes m(q)$. The scalar product of fs^k and $f's^k$ is given by

$$(2\pi)^n \int_{(p,q)\in D^2} f(p,q)\overline{f'(p,q)}\ |s(p,q)|^{2k} m(p)\otimes m(q)$$

Here D is the fundamental domain $\{x_1\mu_1+\ldots+x_n\mu_n/\ (x_i)\in[0,1]^n\}$, where (μ_i) is a basis of the lattice Λ. Now a straightforward computation shows that

$$\Theta_{\mu,k}\overline{\Theta}_{\mu',k}|s(p,q)|^{2k} = \sum_{\substack{\gamma\in\mu+\Lambda\\ \gamma'\in\mu'+\Lambda}} c_{\gamma,\gamma'}(q)\exp(2i\pi kB(p,\gamma'-\gamma))$$

where the diagonal coefficients are given by

$$c_{\gamma,\gamma}(q)=\exp(i\pi k(\tau-\bar{\tau})B(q-\gamma,q-\gamma)).$$

Integrating with respect to p, one deduces that the scalar product of $\Theta_{\mu,k}$ and $\Theta_{\mu',k}$ vanishes when $\mu\neq\mu'$ mod Λ. Furthermore,

$$\|\Theta_{\mu,k}s^k\|^2=(2\pi)^n\operatorname{Vol}(\mathfrak{t}/\Lambda)\sum_{\gamma\in\mu+\Lambda}\int_D c_{\gamma,\gamma}(q)m(q)$$

Using that $\mathfrak{t}$ is the disjoint union of the $-\gamma+D$ when γ runs over $\mu+\Lambda$, we obtain

$$\begin{aligned}\sum_{\gamma\in\mu+\Lambda}\int_D c_{\gamma,\gamma}(q)m(q) &= \int_{\mathfrak{t}}\exp(i\pi k(\tau-\bar{\tau})B(q,q))\ m(q)\\ &=\Big(\frac{i}{k(\tau-\bar{\tau})}\Big)^{n/2}\end{aligned}$$

which ends the computation of the norm.

A.2 The action of S and T in the basis of Theta functions

Let us prove theorem 5.2. Denote by φ_A the map sending (p,q) into $(ap+bq,cp+dq)$. Recall that $\zeta_\tau=p+\tau q$ and $s_\tau=\exp(i\pi B(\zeta_\tau,q))$. Then

$$\varphi_A^*\zeta_{A.\tau}=\frac{\zeta_\tau}{-c\tau+d},\quad \varphi_A^*s_{A.\tau}=\exp\Big(i\pi\frac{cB(\zeta_\tau,\zeta_\tau)}{-c\tau+d}\Big)s_\tau$$

So

$$\varphi_A^*(f(\zeta_{A.\tau})s_{A.\tau}^k)=f\Big(\frac{\zeta_\tau}{-c\tau+d}\Big)\exp\Big(i\pi k\frac{cB(\zeta_\tau,\zeta_\tau)}{-c\tau+d}\Big)s_\tau^k$$

In particular, for $A=T^{-1}$, this gives

$$\varphi_A^*(f(\zeta_{A.\tau})s_{A.\tau}^k)=f(\zeta_\tau)s_\tau^k$$

so that

$$\varphi_A^*(\Theta_{\mu,k}^{A.\tau} s_{A.\tau}^k) = \sum_{\gamma\in\mu+\Lambda} \exp\big(2i\pi k\big(\tfrac{\tau+1}{2} B(\gamma,\gamma) - B(\zeta_\tau,\gamma)\big)\big) s_\tau^k$$

Using that B take integral even values on the diagonal of Λ^2, one shows that $\exp(ik\pi B(\gamma,\gamma)) = \exp(ik\pi B(\mu,\mu))$ for any $\gamma \in \mu + \Lambda$. This implies that

$$\varphi_A^*(\Theta_{\mu,k}^{A.\tau} s_{A.\tau}^k) = \sum_{\gamma\in\mu+\Lambda} \exp(ik\pi B(\mu,\mu)) \Theta_{\mu,k}^{\tau} s_\tau^k$$

which proves the second formula of the theorem.

Assume now that $A = S^{-1}$, then

$$\varphi^*(f(\zeta_{A.\tau}) s_{A.\tau}^k) = f\Big(\frac{\zeta_\tau}{\tau}\Big) \exp\Big(-\frac{i\pi k}{\tau} B(\zeta_\tau,\zeta_\tau)\Big) s_\tau^k$$

Applying to the theta functions, we have

$$\varphi^*(\Theta_{\mu,k}^{A.\tau} s_{A.\tau}^k) = \sum_{\gamma\in\mu+\Lambda} \exp\Big(-\frac{i\pi k}{\tau} B(\gamma+\zeta_\tau, \gamma+\zeta_\tau)\Big)\; s_\tau^k$$

Applying Poisson summation formula, we obtain after some computations that $\varphi^*(\Theta_{\mu,k}^{A.\tau} s_{A.\tau}^k)$ is equal to

$$\Big(\frac{\tau}{ik}\Big)^{n/2} \mathrm{Vol}(\mathfrak{t}/\Lambda)^{-1} \sum_{\gamma\in k^{-1}\Lambda^*} \exp\big(2i\pi k(\tfrac{\tau}{2} B(\gamma,\gamma) - B(\mu+\zeta_\tau,\gamma))\big) s_\tau^k$$

Since $k^{-1}\Lambda^* = \bigcup(\mu'+\Lambda)$, where μ' runs over $k^{-1}\Lambda^*$ mod Λ, this is equal to

$$\Big(\frac{\tau}{ik}\Big)^{n/2} \mathrm{Vol}(\mathfrak{t}/\Lambda)^{-1} \sum_{\substack{\mu'\in k^{-1}\Lambda^* \\ \mathrm{mod}\,\Lambda}} \exp(-2i\pi k B(\mu,\mu')) \Theta_{\mu',k}^{\tau}\; s_\tau^k$$

which ends the proof.

B Index

Lie group notations:

$\mathfrak{g}$, B	Lie algebra of G and its basic inner product; 2
T, $\mathfrak{t}$, Λ	maximal torus, its Lie algebra and integral Lattice; 2
$\mathfrak{A}$, W	open fundamental Weyl alcove and Weyl group; 2
$\ell : W \to \{\pm 1\}$	alternating character; 2

Moduli space and its quantization:

$p, q,\ \zeta = p + \tau q$	projections from $\mathfrak{t}^2$ onto $\mathfrak{t}$, complex coordinates; 3.1, 5.1
$\omega,\ L_{\mathfrak{t}^2}$	symplectic form and prequantum bundle of $\mathfrak{t}^2$; 3.1
L	prequantum bundle of T^2; 3.2
$s,\ \Theta_{\mu,k}$	section of $L_{\mathfrak{t}^2}$ and theta function; 5.1
$(\chi_{\mu,k})_\mu$	basis of alternating sections of $H^0_\tau(\mathrm{T}^2, L^k)$; 5.2

Modular group extensions and their representation

$\Gamma_2,\ R_2$	extension by $\mathbb{Z}_2$ of Γ, representation in $H^0_\tau(\mathrm{T}^2, L^k)$; 6.2
R_2^{alt}	representation in the subspace of alternating sections; 6.2
$\Gamma_\infty,\ R_\infty$	extension by $\mathbb{Z}$ of Γ and its representation; 7.1

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