Complex elliptic curves

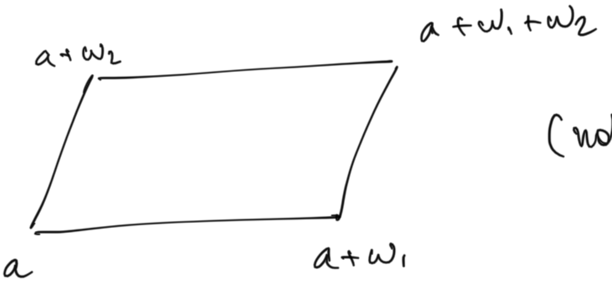
<u>déf</u>: 1) A <u>lattice</u> (réseau) $\Lambda \subseteq \mathbb{C}$ is an abelian subgroup of finite rank 2 which contains a $\mathbb{R}$-basis of $\mathbb{C}$

2) A fundamental domain for a lattice $\Lambda \subseteq \mathbb{C}$ is a subset $D \subseteq \mathbb{C}$ of the form

$$D = \{ a + \lambda_1 \omega_1 + \lambda_2 \omega_2 \in \mathbb{C} : \lambda_1, \lambda_2 \in [0,1[\}$$

where ω_1, ω_2 is a basis of Λ ($\Rightarrow$ $\mathbb{R}$-basis of $\mathbb{C}$).

and $a \in \mathbb{C}$.



(not unique!!)

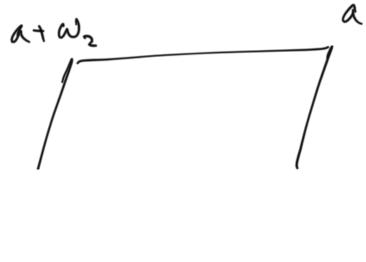
3) An <u>elliptic function</u> (for a lattice Λ) is a meromorphic function $f: \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ t.q.

$$f(z+\lambda) = f(z) \quad \text{for all } \lambda \in \Lambda.$$

Let $\Lambda \subseteq \mathbb{C}$ be a lattice.

<u>Prop.</u> An elliptic function f without poles (or zeros) is constant.

<u>démo.</u> Let $D \subseteq \mathbb{C}$ be a fundamental domain.



Since f is Λ-periodi[c]

$$\sup |f(z)| = \sup |f(z)| < +\infty$$

a ——— $a+\omega_1$ $\quad z \in \mathbb{C} \quad\quad z \in \bar{D}$

$\underset{\text{Liouville's Thm}}{\Longrightarrow}$ f is constant. $\square$

Thm: Let f be a non constant elliptic function. Let D be a fundamental domain for Λ. Then,

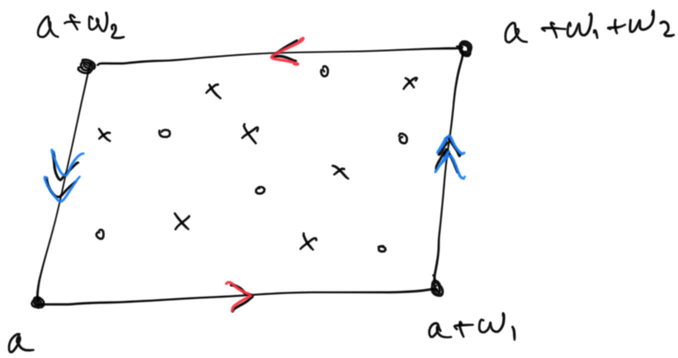
1) $\sum_{x \in D} \operatorname{res}_x(f) = 0$;

2) $\sum_{x \in D} \operatorname{ord}_x(f) = 0$;

3) $\sum_{x \in D} \operatorname{ord}_x(f) \cdot x \in \Lambda$.

the sum does not depend on the fundament domain : $\sum_{x \in \mathbb{C}/\Lambda} \operatorname{res}_x(f)$ or $\sum_{x \in \mathbb{C}/\Lambda} \operatorname{ord}_x(f)$

__démo.__ Since the statements are independent of D, one may take another fundamental domain. Since Λ is discrete, and so is the set of zeroes and poles of f, we may assume that the boundary of D does not contain any zeroes or poles :



1) By the residue theorem,

$$\frac{1}{2\pi i}\oint_{\partial D} f(z)\,dz = \sum_{x\in D} \operatorname{res}_x(f) = 0$$

They sum to 0 (blue)

$$\int_a^{a+\omega_1} f + \int_{a+\omega_1}^{a+\omega_1+\omega_2} f + \int_{a+\omega_1+\omega_2}^{a+\omega_2} f + \int_{a+\omega_2}^{a} f$$

they sum to 0 (red)

2) f elliptic $\Rightarrow$ f' elliptic.

$$0 \overset{(\cdot)}{=} \sum_{x\in D} \operatorname{res}_x\left(\frac{f'}{f}\right) = \sum_{x\in D} \operatorname{ord}_x(f)$$

3) We may suppose that $0 \notin \overline{D}$. We want to apply the residue theorem to the function $z\frac{f'}{f}$.

$$\frac{1}{2\pi i}\oint_{\partial D} z\cdot\frac{f'(z)}{f(z)}\,dz = \sum_{x\in D} \operatorname{ord}_x(f)\cdot x.$$

exo: $\operatorname{res}_x\left(z\frac{f'}{f}\right) = x\cdot \operatorname{ord}_x(f)$ $(x\neq 0)$

$$\oint_{\partial D} z\cdot\frac{f'}{f}\,dz = \int_a^{a+\omega_1} z\frac{f'}{f} + \int_{a+\omega_1}^{a+\omega_1+\omega_2} z\frac{f'}{f} + \int_{a+\omega_1+\omega_2}^{a+\omega_2} z\frac{f'}{f} + \int_{a+\omega_2}^{a} z\frac{f'}{f}$$

$t = z-\omega_2$

$$-\omega_2\int_{a+\omega_1}^{a}\frac{f'}{f} - \int_a^{a+\omega_1}\frac{f'}{f}$$

Rmq: $z\frac{f'}{f}$ not elliptic

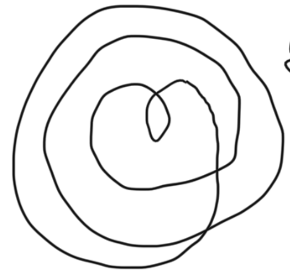
$$= \omega_2\int_a^{a+\omega_1}\frac{f'}{f} - \omega_1\int_a^{a+\omega_2}\frac{f'}{f} \quad \text{(check signs!)}$$

Want to show: this is in Λ.

To check: $\int_a^{a+\omega_1}\frac{f'}{f}$, $\int_a^{a+\omega_2}\frac{f'}{f} \in 2\pi i\mathbb{Z}$.

⌐Rmq: $a,b\in\mathbb{C}$ et g is a holomorphic function on a neighbourhood of $[a,b]$ then $\frac{1}{2\pi i}\int_a^b \frac{g'}{g}$ is the winding number of the path $t \mapsto g(ta + (1-t)b)$

If $g(a) = g(b)$, then it is an integer. ⌋



$g(t)$, $t \in [0,1]$

f elliptic

$\Downarrow$

$f(a) = f(a+\omega_1) = f(a+\omega_2)$

$\Downarrow$

$\int_a^{a+\omega_1} \frac{f'}{f}, \int_a^{a+\omega_2} \frac{f'}{f} \in 2\pi i \mathbb{Z}$.

<u>Recall:</u> The quotient $E = \mathbb{C}/\Lambda$ is endowed with the following structure of a Riemann surface:

$$\mathcal{A} = \{ (\underset{\text{open}}{V \subseteq \mathbb{C}}, \underset{\text{open}}{U \subseteq E}, \underset{\text{homeomorphism}}{\varphi: U \to V}) \} \text{ holomorphic atlas.}$$

Take $V \subseteq \mathbb{C}$ open subset s.t. $\pi: \mathbb{C} \to E = \mathbb{C}/\Lambda$ is bijective on V: $V \xrightarrow{\sim} \pi(V)$. Set $U := \pi(V)$.

and $\varphi := \pi^{-1}: \underset{U}{\pi(V)} \to V$.

<u>Check:</u> It is a holomorphic atlas.

By periodicity, an elliptic function $f: \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ define a continuous $\tilde{f}: E \to \mathbb{P}^1(\mathbb{C})$ which is holomorphic.

$$\left\{ \begin{matrix} \text{meromorphic functions} \\ \text{on } E \end{matrix} \right\} \longleftrightarrow \{\text{elliptic functions}\}$$

1) in the previous statement is the residue theorem for the compact Riemann surface $E = \mathbb{C}/\Lambda$ and the meromorphic 1-form $f\,dz$.

2) on a Riemann surface a non constant meromorphi

function has "as many zeroes as poles".

Cor: A non-constant elliptic function has at least two zeroes (or two poles) counted with multiplicity.

Pf: If there is a single pole of order 1:
$\mathrm{res}_x(f) = 0$ ↯ □

Construction of elliptic functions.

Let $\Lambda \subseteq \mathbb{C}$ be a lattice.

Prop/def: 1) For all integer $k \geq 3$, the series

$$G_k = \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^k}$$ Eisenstein series of weight k

converges normally, and it vanishes for odd k.

2) The series

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$ Weierstrass $\wp$-function

converges absolutely on the compacts of $\mathbb{C} \setminus \Lambda$, and defines a meromorphic functions on $\mathbb{C}$, holomorphic on $\mathbb{C} \setminus \Lambda$ with double poles in each $\lambda \in \Lambda$.

3) $\wp$ is an even elliptic function
↳ $\wp(-z) = \wp(z)$.

Pf: 1) $\# \{ \omega \in \Lambda : N \leq |\omega| < N+1 \} \leq C \cdot N$
↑
there exists $C > 0$ s.t., for all $N \in \mathbb{N}$, ($C \sim$ volume of a fund domain)

$$\sum \frac{1}{\quad} \leq \sum^{\infty} \frac{\# \{ \omega \in \Lambda : N \leq |\omega| < N+1 \}}{\quad}$$

$$\Rightarrow \sum_{w \in \Lambda \setminus \{0\}} |w|^k \quad \sum_{N=1} \quad N^k$$

$$\left[+ \sum_{0 < |w| < 1} \frac{1}{|w|^k} \right]$$ we don't care

$$\leq C \sum_{N=1}^{\infty} \frac{N}{N^k} < +\infty \quad k \geq 3.$$

$$= N^{-k+1}$$

By symmetry it vanishes for odd k.

2) $|w| > |2z|$

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = \frac{|z(2w-z)|}{|w|^2 \, |z-w|^2} \leq 10 \cdot \frac{|z|}{|w|^3}.$$

$$\uparrow \quad |z-w| \geq \frac{|w|}{2}$$

$$|2w - z| \leq \frac{5}{2} |w|$$

$$\rightarrow \wp(z) = \frac{1}{z^2} + \sum_{\substack{w \neq 0 \\ |w| \leq |2z|}} [\quad] + \sum_{|w| > 2|z|} [\quad]$$

finite sum of holomorphic function. (away from Λ)

absolutely convergent

holomorphic away from Λ.

3) $\wp'(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}$ elliptic odd.

Fix $w \in \Lambda$. Then the function

$f(z) = \wp(z+w) - \wp(z)$ constant because $f' = 0$.

$$f\left(-\frac{w}{2}\right) = \wp\left(\frac{w}{2}\right) - \wp\left(-\frac{w}{2}\right) = 0.$$

$\wp\left(-\frac{w}{2}\right) = \wp\left(\frac{w}{2}\right)$ by the formula

$\wp(z)$ by ... in the definition.

$\Longrightarrow \wp$ is elliptic. $\square$

<u>Rule</u>: • $\wp$ has double pole in each $\lambda \in \Lambda$

• $\wp'$ has triple pole in each $\lambda \in \Lambda$.

<u>Theorem</u>: Every elliptic function is a rational combinati of $\wp, \wp'$: $\dfrac{P(\wp, \wp')}{Q(\wp, \wp')}$ $P, Q \in \mathbb{C}[x,y]$.

<u>Proof</u>: Let f be an elliptic function:

$$f(z) = \underbrace{\frac{f(z) + f(-z)}{2}}_{\text{even elliptic}} + \underbrace{\frac{f(z) - f(-z)}{2}}_{\text{odd elliptic.}}$$

g odd elliptic $\Longrightarrow$ $\wp' g$ is even elliptic.

We may suppose that f is an even elliptic function.

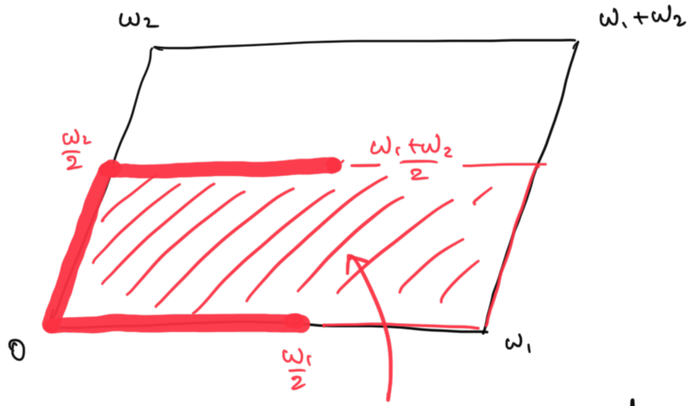
- $x \in \mathbb{C}$ $\operatorname{ord}_x(f) = \operatorname{ord}_{-x}(f)$.
- $\omega \in \Lambda$ $\operatorname{ord}_{\frac{\omega}{2}}(f)$ is even.

[$f(z) = f(-z) \longrightarrow f^{(i)}(z) = (-1)^i f^{(i)}(-z)$.

$$f^{(i)}\left(\tfrac{\omega}{2}\right) = (-1)^i \underbrace{f^{(i)}\left(-\tfrac{\omega}{2}\right)}_{f^{(i)}\left(\omega - \frac{\omega}{2}\right)} = (-1)^i f^{(i)}$$

i odd $\Rightarrow f^{(i)}\left(\frac{\omega}{2}\right) = 0$.]

D fundamental domain for Λ. Consider the following subset



H is a fundamental domain for $E/\{\pm 1\}$.

For $z \in \mathbb{C}$ such that $\mathrm{ord}_z(f) \neq 0$.

$$\mathbb{Z} \ni n_z = \begin{cases} \mathrm{ord}_z(f) & \text{if } z \notin \frac{1}{2}\Lambda. \\ \frac{1}{2}\mathrm{ord}_z(f) & \text{if } z \in \frac{1}{2}\Lambda. \end{cases}$$

$$g(z) = \prod_{w \in H \setminus \{0\}} \left[\wp(z) - \wp(w) \right]^{n_w}$$ elliptic even.

$w \in H \setminus \{0\} \rightarrow \mathrm{ord}_w(f) = \mathrm{ord}_w(g) = \mathrm{ord}_{-w}(g) = \mathrm{ord}_{-w}(f)$

$\xrightarrow{\text{Prop.}} \mathrm{ord}_0(f) = \mathrm{ord}_0(g).$

$\longrightarrow \frac{f}{g}$ elliptic, no zero, no poles,

$\Rightarrow$ constant. $\square$

PROJECTIVE EMBEDDINGS

$\Lambda \subseteq \mathbb{C}$ a lattice.

Theorem: i) The Laurent series for $\wp$ around $z=0$ is

$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1)\, G_{2k+2}\, z^{2k}.$

$\wp(z) = \frac{1}{z^2} + \sum_{k=1} \ldots z^{2k+2}$

2) for $z \in \mathbb{C} \setminus \Lambda$,

$$\wp'(z)^2 = 4\wp(z)^3 - 60 G_4 \wp(z) - 140 G_6.$$

Proof: 1) for $|z| < |w|$, $w \neq 0$,

$$\frac{1}{(z-w)^2} - \frac{1}{w^2} = \frac{1}{w^2}\left[\frac{1}{\left(1-\frac{z}{w}\right)^2} - 1\right]$$

$$= \left[\sum_{n=1}^{\infty} (n+1)\left(\frac{z}{w}\right)^n\right] \cdot \frac{1}{w^2}.$$

$$\sum_{w \neq 0}\left[\frac{1}{(z-w)^2} - \frac{1}{w^2}\right] = \sum_{w\neq 0}\left[\sum_{n=1}^{\infty}(n+1)\left(\frac{z}{w}\right)^n\right]\frac{1}{w^2}$$

$$= \sum_{n=1}^{\infty} \underbrace{\left[\sum_{w\neq 0}\frac{1}{w^{n+2}}\right]}_{G_{n+2}} (n+1) z^n$$

$G_{n+2} = 0$ for odd n $\longrightarrow$ $\underset{n=2k}{=} \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}$.

2) $\wp'(z)^2 = \frac{4}{z^6} - 24 \cdot \frac{G_4}{z^2} - 80 G_6 + \ldots$

$\wp(z)^3 = \frac{1}{z^6} + 9\frac{G_4}{z^2} + 15 G_6 + \ldots$

$\wp(z) = \frac{1}{z^2} + 3 G_4 + \ldots$

The function

$$f(z) = \wp'(z)^2 - 4\wp(z)^3 + 60 G_4 \wp(z) + 140 G_6$$

is elliptic, and it has no poles.

$\Longrightarrow$ f is constant.

and $f(0) = 0$ $\Longrightarrow$ $f = 0$. $\square$

<u>Rmk:</u> In particular, we have a holomorphic mapping

$$(\wp, \wp') : \mathbb{C} \setminus \Lambda \xrightarrow{\pi} E \setminus \{[0]\} \longrightarrow \mathbb{C}^2$$
$$z \longmapsto (\wp(z), \wp'(z))$$

The image is contained in the set $(x, y) \in \mathbb{C}^2$ such that

$$y^2 = 4x^3 - g_2 x - g_3 \qquad \begin{aligned} g_2 &= 60\, G_4 \\ g_3 &= 140\, G_6. \end{aligned}$$

This holomorphic function extends to a holomorphic mapping

$$\varphi : \mathbb{C} \xrightarrow{\pi} E \xrightarrow{\tilde{\varphi}} \mathbb{P}^2(\mathbb{C})$$
$$\Lambda \not\ni z \longmapsto [1 : \wp(z) : \wp'(z)] \qquad \wp(z) \sim \tfrac{1}{z^2}, \quad \wp'(z) \sim \tfrac{1}{z^3}$$
$$z \in \Lambda \longmapsto [0 : 0 : 1]$$

<u>Prop.</u> 1) The polynomial $4x^3 - g_2 x - g_3$ has three pairwise distinct roots;

2) The "algebraic curve"

$$C = \{ [x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{C}) : x_0 x_2^2 = 4x_1^3 - g_2 x_0^2 x_1 - g_3 x_0^3 \}$$

is non-singular, hence carries a structure of a compact Riemann surface and the map $\tilde{\varphi} : E \to C$ is a biholomorphism.

<u>Proof:</u> 1) Let $\omega_1, \omega_2 \in \Lambda$ be a basis. Set $\omega_3 = \omega_1 + \omega_2$.

[illegible]

$\frac{1}{2}\Lambda/\Lambda = \{[0], [\frac{\omega_1}{2}], [\frac{\omega_2}{2}], [\frac{\omega_1+\omega_2}{2}]\}$

the set of 2-torsion points of the abelian group "$\mathbb{C}/\Lambda$"

$\wp'(\frac{\omega_i}{2}) = -\wp'(-\frac{\omega_i}{2}) = -\wp'(\omega_i - \frac{\omega_i}{2}) = -\wp$

↑ $\wp'$ is odd ↑ $\wp'$ elliptic

$\Rightarrow \wp'(\frac{\omega_i}{2}) = 0. \quad i=1,2,3.$

Therefore the roots of the polynomial $4x^3 - g_2 x - g_3$ are $\wp(\frac{\omega_i}{2})$ $i=1,2,3$.

To prove: $\wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_3}{2})$ are pairwise distinct

$f(z) = \wp(z) - \wp(\frac{\omega_i}{2})$. f is elliptic, even, it has 1 pole of order 2

$\longrightarrow$ f has two zeroes (counted with multiplicity).

$f(\frac{\omega_i}{2}) = 0$ and $f'(\frac{\omega_i}{2}) = \wp'(\frac{\omega_i}{2}) = 0.$

$\longrightarrow$ $\frac{\omega_i}{2}$ is the only zero of f.

$\longrightarrow$ $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ are pairwise distinct.

2) Let us show that $\tilde{\varphi}$ is bijective.

<u>Injective:</u> let $z_1, z_2 \in \mathbb{C}/\Lambda$ s.t. $\varphi(z_1) = \varphi(z_2)$

$\longrightarrow \begin{cases} \wp(z_1) = \wp(z_2) \\ \wp'(z_1) = \wp'(z_2) \end{cases}$

We already analyzed the case $\wp'(z_1) = \wp'(z_2) = 0$. We may suppose $\wp'(z_i) \neq 0$.

It follows that the function

$f(z) = \wp(z) - \wp(z_1)$ has two distinct zeroes $z_1,$

$(\sum \text{ord}_x(f)\, x \in \Lambda)$ ⅃

There are two possibilities:

$z_1 + z_1' \in$

* $z_1 \equiv z_2 \pmod{\Lambda}$ $\longrightarrow$ we are done
* $z_1 \equiv -z_2 \pmod{\Lambda}$ $\longrightarrow$ $\wp'(z_1) = \wp'(-z_2)$

$$= -\wp'(z_2)$$

$$\longrightarrow \varphi(z_1) \neq \varphi(z_2). \,!$$