<u>Yesterday II</u>: k field. A = finitely generated k-algebra.

$\mathrm{Max}(A)$ = maximal spectrum = $\{ m \subseteq A \text{ max ideal} \}$

<u>topology</u>: $V(I) = \{ m \in \mathrm{Max}(A) : I \subseteq m \}$ closed
I ideal $\qquad\qquad\qquad \mathrm{Max}(A/I)$

$f \in A \quad D(f) = \{ m \in \mathrm{Max}(A) : f \notin m \} = \mathrm{Max}(A_f)$.

<u>structural sheaf</u>: is the unique sheaf $\mathcal{O}_X$ of k-algebras s.t. $\mathcal{O}_X(D(f)) = A_f$. $(f \in A)$.

Suppose A an integral domain. Any localization of A can be seen as a subring of the fraction field $K = \mathrm{Frac}(A)$: $S \subseteq A$ multiplicative subset

$$S^{-1}A = \left\{ \frac{a}{s} : s \in S, a \in A \right\} \subseteq K$$

- $1 \in S$,
- $s, t \in S \Rightarrow st \in S$.

In particular

- $f \in A$: $A_f = \left\{ \frac{a}{f^n} : a \in A, n \in \mathbb{N} \right\}$
- $m \in \mathrm{Max}(A)$: $A_m = \left\{ \frac{a}{f} : a \in A, f \in A \setminus m \right\}$.

For an open subset $U \subseteq \mathrm{Max}(A)$

$$\mathcal{O}_X(U) = \bigcap_{m \in U} A_m. \qquad \text{(exercise!)}$$

I think at K as rational functions on $X = \mathrm{Max}(A)$ and $\mathcal{O}(U)$ are simply the rational functions without poles on U.

"Abstract" smooth projective curves

Let k be a perfect field. Let K/k be a field of finite type with transcendence degree 1.

There exists $x \in K$ s.t. transcendant over k

$$\begin{array}{c} K \\ | \\ k(x) \end{array} \text{ is finite,}$$

⚠ x is by no means unique!

- $C_K = C = \{ v : K^\times \twoheadrightarrow \mathbb{Z} \ ; \ \text{surjective valuation s.t. } v|_{k^\times} = 0 \}$

[<u>Example</u>: $K = k(t)$ with k alg. closed.

$v : k(t) \longrightarrow \mathbb{Z} \cup \{\infty\}$ surjective valuation s.t. $v|_{k^\times} = 0$.

there are two possibilities

- either $v(t) < 0$
- or $v(t) \geq 0$

1) In the first case $v(f) = \mathrm{ord}_\infty(f)$.

$$f = \sum_{i=0}^{d} a_i t^i \qquad v(t) < 0. \qquad a_d \neq 0.$$

$$v(a_i t^i) = i\, v(t) > d\, v(t).$$

$$\Longrightarrow \quad v(f) = \deg(f)\, v(t).$$

For $\underset{k(t)}{\underset{\cap}{F}} = \frac{f}{g} \to v\left(\frac{f}{g}\right) = [\deg(f) - \deg(g)]\, v(t)$

<u>But</u>: v is surjective $\longrightarrow v(t) = -1$.

So v measures the order of vanishing at infinity

$$f(t) = \sum_{i=0}^{d} a_i t^i = \sum_{i=0}^{d} a_i \left(\frac{1}{t}\right)^{-i}$$ $\to$ f has a pole of order d at infinity.

$v(f) = -d$.

$v\left(\frac{1}{t}\right) = 1$ $\quad$ $\frac{1}{t}$ has a simple zero at infinity.

2) $v(t) \geq 0$ This implies that the ring of valuation of v

$$A_v = \{ f \in k(t) : v(f) \geq 0 \}$$

contains $k[t]$.

$$k[t] \longrightarrow \underset{\rotatebox{90}{$\subseteq$}}{A_v} \xrightarrow{v} \mathbb{Z} \cup \{\infty\}$$

$$\underset{m:=}{\{ f \in k[t] : v(f) > 0 \}} \subseteq \{ f \in A_v : v(f) > 0 \}$$

$\uparrow$ maximal (otherwise v would be trivial).

k alg closed $\Rightarrow$ $m = (t-a)$ $\quad a \in k$

<u>To show</u>: $\underset{f \in k[t]}{v(f) = \operatorname{ord}_a(f)}$.

$$\Rightarrow \quad C_{k(t)} = k \cup \{\infty\} = \mathbb{P}^1(k)$$ ⌟

Let's go back to the general picture:

$C_K = \{$ surj valuations on K, trivial on $k\}$

- topology: the closed subsets are the finite ones.
  The open subsets are the complement of finite ones.

- $v \in C \qquad \mathcal{O}_{C,v} = \{ f \in K : v(f) \geq 0 \}$

$$\begin{aligned} &\cup \\ \mathfrak{m}_{C,v} &= \{ f \in K : v(f) > 0 \}. \end{aligned}$$

$k(v) = \mathcal{O}_{C,v} / \mathfrak{m}_{C,v}$. residue field.

- $U \subseteq C_K$ open, set

$$\mathcal{O}_C(U) := \bigcap_{v \in U} \mathcal{O}_{C,v}$$

$$= \{ f \in K : v(f) \geq 0 \text{ for all } v \in U \}.$$

"Rational functions without poles on U".

[Example: $U \subseteq \mathbb{P}^1(k)$ open subset

$k = \bar{k}$ alg. closed $\qquad \mathbb{P}^1(k) \setminus U = \{ \infty, a_1, \dots, a_n \}$

Check: $\mathcal{O}(U) = \left\{ \frac{f}{g} : \underset{k}{z \in} \neq a_i \Rightarrow g(z) \neq 0 \right\}$.

$$\cup$$

$$\frac{f \in k[t]}{(t-a_1)^{m_1} \cdots (t-a_n)^{m_n}}$$ ⌟

We constructed a topological space C with a sheaf of k-algebras on it.

Want to show: it is an algebraic variety.

def: A k-locally ringed space $(X, \mathcal{O}_X)$ is an **algebraic variety** if there are open subsets $U_1, \dots, U_n \subseteq X$, s.t. $X = \bigcup_{i=1}^n U_i$, finite type k-algebras $A_1, \dots, A_n$ and isomorphisms of k-locally ringed space

$$(U_i, \underbrace{\mathcal{O}_X|_{U_i}}) \longrightarrow (\mathrm{Max}(A_i), \mathcal{O}_{\mathrm{Max}(A_i)}).$$

$V \subseteq U_i$ open subset

$\mathcal{O}_X|_{U_i}(V) := \mathcal{O}_X(V)$.

Construction: Let $v \in C$ and let $f \in K$ be such that $v(f) > 0$. Since v vanishes on k^*, it vanishes also on (the invertible elements of) the algebraic closure of k in K. Therefore f is transcendant.

Since $\mathrm{trdeg}(K/k) = 1$, the extension $K/k(f)$ is finite.

$$\begin{array}{cccccl} f & K & \supseteq & A & = & \text{integral closure of} \\ \uparrow & | & & | & & k[t] \text{ in } K. \\ t & k(t) & \supseteq & k[t] & & \\ & | & & & & \\ & k & & & & \end{array}$$

By the Theorem of yesterday, A is a finite type $k[t]$-module and a Dedekind ring.

↳ A is a finitely generated k-algebra

Lemma: i) $U = \{ w \in C : w(f) \geq 0 \}$ is an open subset;

2) $(U, \mathcal{O}_{C|U}) = (\mathrm{Max}(A), \mathcal{O}_{\mathrm{Max}(A)})$.

Proof: Start by remarking the following:

$w \in U \longleftrightarrow w(f) \geq 0 \longrightarrow k[t] \subseteq A \subseteq \mathcal{O}_{C,w}$

(because $\mathcal{O}_{C,w}$ normal and A is the integral closure of $k[t]$ in K).

Let $m = \{ a \in A : w(a) > 0 \}$. Since w is non trivial, the ideal m is maximal: $m \in \mathrm{Max}(A)$.

Claim: $\mathcal{O}_{C,w} = A_m$.

Proof: The inclusion $A_m \subseteq \mathcal{O}_{C,w}$ is clear because m is contained in the max ideal of $\mathcal{O}_{C,w}$.

Suppose by contradiction there is $g \in \mathcal{O}_{C,w} \setminus A_m$. Since A is Dedekind, A_m is a DVR, thus $\frac{1}{g} \in A_m$. In particular,

$g, \frac{1}{g} \in \mathcal{O}_{C,w}$ so $w(g) = w\left(\frac{1}{g}\right) = 0$.

$\Longrightarrow \frac{1}{g} \notin m A_m \underset{A_m \text{ local}}{\Rightarrow} g = \left(\frac{1}{g}\right)^{-1} \in A_m$. ▯

We saw:

- $U \subseteq \mathrm{Max}(A)$: this an equality because if $m \subseteq A$ is maximal, A_m is a DVR with valuation $w : A_m \to \mathbb{N} \cup \{\infty\}$ and

  $f \in A \Rightarrow w(f) \geq 0 \Rightarrow w \in U$;

  $m = \{ g \in K : w(g) > 0 \}$

- $w \in U \Rightarrow A_m = \mathcal{O}_{C,w}$

  $m = \{g \in K : w(g) > 0\}$.

$$A = \bigcap_{m \in Max(A)} A_m = \{ g \in Frac(A) : v_m(f) \geq 0 \text{ for all } m \in Max(A)\}$$

$m \in Max(A)$

$v_m : K \to \mathbb{Z} \cup \{\infty\}$ the associated discrete valuation

1) In order to see that U is an open subset, one has to prove that the complement is finite. Up to replacing f by $\frac{1}{f}$ this amounts to show that the subset

$$\{w \in C : w(f) > 0\} \text{ is finite.}$$

$$\{m \in Max(A) : f \in m\} = Max(A/fA).$$

Now the ring A/fA is Noetherian and of Krull dimension 0 (prime ideals in A/fA are the max ideals of containing f). Also, A/fA is a finitely generated k-algebra:

fin. gen. k-alg + dim 0 $\Rightarrow$ there are only finitely many max ideals.

$\Rightarrow \{w \in C : w(f) > 0\}$ is finite.

2) We have to show: for all open subsets $V \subseteq U$ we have

$$\mathcal{O}_C(V) = \bigcap_{w \in V} \mathcal{O}_{C,w}.$$

(We have seen that for $U = V$). Since $\mathcal{O}_C$ and $\mathcal{O}_{\mathrm{Max}(A)}$ are sheaves, it suffices to prove it when $V = D(g)$ for some $g \in A$.

$$\mathcal{O}_{\mathrm{Max}(A)}(D(g)) = A_g = A[\tfrac{1}{g}].$$

$$w \in D(g) \longleftrightarrow g \notin \{h \in K : w(h) > 0\}$$

$$\{m \subseteq A : g \notin m\}$$

We know that $w(g) \geq 0$, so

$$D(g) = \{w \in U : w(g) = 0\}$$

We conclude because

$$A_g = \bigcap_{\substack{m \subseteq A_g \\ \max}} (A_g)_m = \bigcap_{\substack{m \subseteq A \\ \max \\ g \notin m}} A_m.$$

$$= \bigcap_{\substack{w \in U \\ w(g)=0}} \mathcal{O}_{C,w} = \bigcap_{w \in V} \mathcal{O}_{C,w}.$$

$$\Rightarrow \quad \mathcal{O}_{\mathrm{Max}(A)}(D(g)) = \bigcap_{w \in D(g)} \mathcal{O}_{C,w} = \mathcal{O}_C(D(g)). \quad \square$$

We have proved that $(C, \mathcal{O}_C)$ is an algebraic variety. Indeed, for $f \in K$ transcendant, C is covered by the open subsets

$$U = \{ v \in C : v(f) \geq 0 \} \simeq (\mathrm{Max}(A), \mathcal{O}_{\mathrm{Max}(A)}).$$

$$V = \{ v \in C : v(1/f) \geq 0 \} \simeq (\mathrm{Max}(B), \mathcal{O}_{\mathrm{Max}(B)}).$$

Projective space: V finite dimensional k-vector space. $n = \dim V$

$\mathrm{Sym}(V^*)$ = symmetric algebra of V^*

For $d \in \mathbb{N}$

$$\left\{ \begin{matrix} \text{alternating} \\ \text{tensors} \end{matrix} \right\} \longrightarrow (V^*)^{\otimes d} \longrightarrow \mathrm{Sym}^d(V^*) \longrightarrow 0.$$

$$\| $$

$$< v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n - v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v$$

If $\varphi_1, \dots, \varphi_n$ is a basis of V^*, then

$$\mathrm{Sym}^d(V^*) \cong \left\{ \begin{matrix} \text{homogeneous polynomials} \\ \text{in the variable} \\ \varphi_1, \dots, \varphi_n \text{ of degree } d \end{matrix} \right\}$$

$$\bigoplus_{d=0}^{\infty} \mathrm{Sym}^d(V^*) =: \mathrm{Sym}(V^*) \simeq k[\varphi_1, \dots, \varphi_n].$$

$\mathbb{P}(V) = \{ \mathfrak{p} \subseteq \mathrm{Sym}(V^*)$: prime homogeneous ideal, not containing the ideal $\left[(\varphi_1, \dots, \varphi_d) \simeq\right] \bigoplus_{d=1}^{\infty} \mathrm{Sym}^d(V^*)$, maximal for this property

$f \in \mathrm{Sym}^d(V^*)$

$V(f) = \{ \mathfrak{p} \in \mathbb{P}(V) : f \in \mathfrak{p} \}$

$V_+(f) = \{$

$$D_+(f) = \{ \mathfrak{p} \in \mathbb{P}(V) : f \notin \mathfrak{p} \}$$

$I \subseteq \operatorname{Sym} V^*$ homogeneous ideal

$$V_+(I) = \{ \mathfrak{p} \in \mathbb{P}(V) : I \subseteq \mathfrak{p} \}.$$

<u>topology on $\mathbb{P}(V)$</u>: declare the closed subsets to be those of the form $V_+(I)$ for I homogeneous. In this way, $D_+(f)$ for $f \in \operatorname{Sym} V^*$ form a basis of the open sets.

<u>structural sheaf:</u> there is a unique sheaf of k-algebras $\mathcal{O}_{\mathbb{P}(V)}$ s.t., for all $f \in \operatorname{Sym} V^*$,

$$\mathcal{O}_{\mathbb{P}(V)}(D_+(f)) = (\operatorname{Sym} V^*)_{(f)}$$

$$= \left\{ \frac{g}{f^d} : g \in \operatorname{Sym}^{d \deg(f)}(V^*) \right\}$$

$$\deg(g) = d \deg(f).$$

<u>Exercise</u>:

$$\left(D_+(f), \mathcal{O}_{\mathbb{P}(V)}|_{D_+(f)} \right) \text{ is isomorphic as a}$$

locally ringed space to

$$\left(\operatorname{Max}(A), \mathcal{O}_{\operatorname{Max}(A)} \right).$$

$$A = (\operatorname{Sym} V^*)_{(f)}$$

This makes the projective space an algebraic variety.

<u>Lemma</u>: (Valuative Criterion of properness). Let K be a field of trdeg 1. Let $(C, \mathcal{O}_C)$ be the

associated "abstract" algebraic curve. Let $U \subseteq C$ be an open subset, and $x \in U$. Let $f : U \setminus \{x\} \to \mathbb{P}^n_k$ a morphism of alg. varieties (i.e. of k-locally ringed spaces). Then, there exits a unique morphism of alg. varieties $\tilde{f} : U \longrightarrow \mathbb{P}^n_k$ that extends f.

<u>Notation</u>: $\mathbb{P}^n_k := \mathbb{P}(k^{n+1})$. $\quad$ Sym $V^* = k[x_0, \dots, x_n]$.

<u>Rmq!</u> Over $\mathbb{C}$, this the following. Let U be compact a Riemann surface and let $x \in U$. Let $f : U \setminus \{x\} \to \mathbb{P}^n(\mathbb{C})$ be holomorphic mapping, meromorphic at x. Then f extends uniquely to a holomorphic mapping

$$\tilde{f} : U \to \mathbb{P}^n(\mathbb{C}).$$

To prove this: up to taking a complex chart, we may assume $U = \{ |z| < 1 \}$, $x = 0$, and that f is a holomorphic mapping

$$f : U \setminus \{x\} \to \mathbb{C}^n.$$
$$z \longmapsto (f_1(z), \dots, f_n(z)).$$

Meromorphic at x means that f_i has at worst a polar singularity.

$$f_i(z) = \frac{1}{z^{m_i}} g_i(z). \quad \text{with } g_i \text{ holomorphic and } g_i(0) \neq 0$$

Let $i_0 \in \{1, \dots, n\}$ be such that

$m_{i_0} = \max\{m_1, \dots, m_n\}$.

$$[1 : f_1(z) : \dots : f_n(z)] = [z^{m_{i_0}} : z^{m_{i_0}} f_1(z) : \dots : z^{m_{i_0}} f_n$$

$$\downarrow z \to 0$$

$$p = [0 : \underset{\mathbb{C}}{a_1} : \dots : \underset{\text{non zero.}}{g_{i_0}(0)} : \underset{\mathbb{C}}{a_{i_0+1}} \dots : \underset{\mathbb{C}}{a}$$

Set

$$\tilde{f}(0) := p.$$

We will do exactly the same proof for algebraic curves.