0.1. **Artinian rings.**

Definition 0.1. A ring A is *Artinian* if it satisfies the DCC (Descending Chain Condition): every descending chain of ideals of A,

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

is stationary.

Exercise 0.2. Let k be a field. A finite-dimensional k-algebra is Artinian.

Exercise 0.3. An Artinian ring has only finitely many maximal ideals.

Proof. An ideal of a k-algebra is a k-vector space. In a descending chain of ideals of a finite-dimensional k-algebra, the dimension stabilizes eventually. $\square$

Exercise 0.4. The Jacobson radical (that is, the intersection of all maximal ideals) of an Artinian ring is nilpotent. (Hint. Use the following form of Nakayama's Lemma: if A is a ring, I ideal of A contained in the Jacobson radical of A and M is a finitely generated A-module such that $M = IM$, then $M = 0$.)

Proof. Let I be the Jacobson radical of A. Since A is Artinian, there is $N \in \mathbb{N}$ such that $I^n = I^{n+1}$ for all $n \geqslant N$. We want to show that I^N is 0. Define

$$J := \{f \in A : fI^N = 0\}.$$

Then we have to show that $J = A$. Suppose by contradiction this is not the case. Then, since A/J is Artinian, there is a minimal ideal J' strictly containing J. Take $x \in J' \smallsetminus J$. Then $J' = J + Ax$.

Claim 0.5. $J' = J + xI$.

Proof of the Claim. The ideal $J + xI$ is contained in J' by definition. On the other hand, xI is not contained in J: otherwise, xI^{N+1} would be contained in $JI^N = 0$, thus x would belong to J. By minimality, it follows $J' = J + xI$. $\square$

Consider the finitely generated A-module $M := J'/J$. Then

$$IM = I(J + Ax)/J = (IJ + Ix + J)/J = (J + Ix)/J = M.$$

By Nakayama's Lemma, one has $M = 0$, that is $IJ' \subseteq J$. This implies

$$xI^{N+1} \subseteq JI^N = 0.$$

Thus $x \in J$. Contradiction. $\square$

Exercise 0.6. Any ring with finitely many maximal ideals and nilpotent Jacobson radical is the product of its localizations at its maximal ideals. Also, all primes are maximal.

Proof. Let I be the Jacobson ideal of A. The nilpotent radical of a ring is the intersections of all its prime ideals. Since I is nilpotent, it coincides with the nilradical of A. It follows that every prime ideal contains I. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be the maximal ideals of A. By the Chinese Remainder Theorem,

$$A/I \simeq \prod_{i=1}^{n} A/\mathfrak{m}_i.$$

Since the prime ideals of a product of fields is maximal, we see that every prime ideal of A is maximal.

Claim 0.7. *Let B be a ring and J a nilpotent ideal of B. Let $\bar{e} \in B/I$ be an idempotent. Then there is an idempotent $e \in B$ lifting $\bar{e}$.*

Proof of the Claim. If $\tilde{e} \in B$ is any lift of B, then $\tilde{e}^2 - \tilde{e} \in J$ is nilpotent. Suppose having found a lift e of $\bar{e}$ such that $e^2 - e$ belongs to J^k for some $k \in \mathbb{N}$. Then

$$e' := e - x$$

with $x = (2e-1)(e^2 - e)$ is such that

$$\begin{aligned} e'^2 - e' &= (e^2 - e) + x(x - (2e-1)) \\ &= (e^2 - e)(1 + (2e-1)(x - 2e + 1)) \\ &= (e^2 - e)((2e-1)x + 4e - 4e^2) \\ &= (e^2 - e)^2((2e-1)^2 - 4) \in J^{2k}. \end{aligned}$$

Therefore, by induction on k, we find a lift e of $\bar{e}$ such that $e^2 - e \in J^{2^k}$. Since J is nilpotent, for k big enough, e will be idempotent. □

Consider the idempotent $\bar{e} = (1, 0, \ldots, 0)$ of $A/I \simeq \prod_{i=1}^n A/\mathfrak{m}_i$. By the Claim, the idempotent $\bar{e}$ lifts to an idempotent e. Then $e' = 1 - e$ is also idempotent:

$$(1-e)^2 - (1-e) = e^2 - e = 0.$$

It follows that A as a ring is isomorphic to $Ae \times Ae'$. (Note that multiplication by an idempotent element is a ring map.) The idempotent e belongs to $\mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n$ because its image in $\prod_{i=2}^n A/\mathfrak{m}_i$ is 0. It follows that the map $Ae \to A/\mathfrak{m}_1$ has kernel

$$Ie = (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n)e = \mathfrak{m}_1 e.$$

The therefore the ideal Ie is maximal and nilpotent in the ring Ae.

Claim 0.8. *Let B be a ring having a nilpotent maximal ideal $\mathfrak{n}$. Then $\mathfrak{n}$ is the unique prime ideal of B.*

Proof of the Claim. Indeed a prime ideal $\mathfrak{p}$ of B contains the nilradical of B, which contains $\mathfrak{n}$ since the latter is nilpotent. Since $\mathfrak{n}$ is maximal, we have $\mathfrak{p} = \mathfrak{n}$. □

It follows that the ring Ae is local with maximal ideal $\mathfrak{m}_1 e$. It follows that, if $a \in A \smallsetminus \mathfrak{m}_1$, then $ae \in Ae$ is invertible. The surjective map $A \to Ae$ therefore factors through a surjective map $\varphi\colon A_{\mathfrak{m}_1} \to Ae$. The map φ is also injective: indeed if $a \in A$ is such that $\varphi(a) = ae = 0$ then $a = 0$ in $A_{\mathfrak{m}_1}$ because $e \notin \mathfrak{m}_1$.

Summing up, we proved that A is isomorphic to $A_{\mathfrak{m}_1} \times A'$. By induction on n we conclude the proof. □

Definition 0.9. Let A be a ring. The *length* of A-module M is the supremum of the $n \in \mathbb{N}$ for which there exists a chain of A-submodules

$$\{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M.$$

Exercise 0.10. A ring A is Artinian if and only if it has finite length as a module over itself. Any such ring A is both Artinian and Noetherian, any prime ideal of A is a maximal ideal, and A is equal to the (finite) product of its localizations at its maximal ideals.

Proof. A ring which has finite length as a module over itself satisfies both the ascending and descending chain conditions.

Suppose that A is Artinian. Then by the previous exercises the Jacobson radical is nilpotent and it has finitely many prime ideals. It follows that A is isomorphic to the product of the localization at its maximal ideals. In order to prove that it has finite length over itself (thus it is Noetherian), we may assume that A is local with maximal ideal $\mathfrak{m}$ and residue field k. It suffices to exhibit a chain of ideals whose successive quotients have finite length.

The maximal ideal $\mathfrak{m}$ coincides with the Jacobson radical of A, therefore it is nilpotent. Let $n \in \mathbb{N}$ be such that $\mathfrak{m}^n = 0$. Consider the chain of ideals

$$0 = \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1} \subseteq \cdots \subseteq \mathfrak{m} \subseteq A.$$

We conclude because, for each $i \in \mathbb{N}$, the k-vector space $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is finite-dimensional (otherwise, it would furnish an infinite descending chain of ideals). □

Remark 0.11. The above argument permits to show the following variation. A Noetherian ring A of dimension 0 is of finite length (hence Artinian).

Indeed, since A is a Noetherian ring, it has finitely minimal prime ideals. These ideals are maximal because A is of dimension 0. In particular, the Jacobson radical coincides with the nilradical, hence it is nilpotent (because A is Noetherian). Therefore A is the product of the localization at its prime ideals. In order to conclude that A has finite length over itself, we may assume that A is local with maximal ideal $\mathfrak{m}$ and residue field k. It suffices to exhibit a descending chain of ideals whose successive quotients are of finite length. Let $n \in \mathbb{N}$ be such that $\mathfrak{m}^n = 0$. Consider the chain of ideals

$$0 = \mathfrak{m}^n \subseteq \mathfrak{m}^{n-1} \subseteq \cdots \subseteq \mathfrak{m} \subseteq A.$$

We conclude that, for each $i \in \mathbb{N}$, the k-vector space $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is finite-dimensional (because A is Noetherian).

0.2. **Morphisms between curves.**

Exercise 0.12. Let A be a DVR with fractions field K. Let K' be a finite extension of K such that the integral closure A' of A in K' is finitely generated as an A-module. Let $\mathfrak{m}$ be the maximal ideal of A. Show the following assertions:

(1) The ring A' is Dedekind with only finitely many maximal ideals.
(2) For each maximal ideal $\mathfrak{m}'$ of A', there is an integer $e(\mathfrak{m}') \geqslant 1$ such that $\mathfrak{m}A'_{\mathfrak{m}'} = \mathfrak{m}'^{e(\mathfrak{m}')}$.
(3) For each maximal ideal $\mathfrak{m}'$ of A', the field $k' = A'/\mathfrak{m}'$ is a finite extension of $k = A/\mathfrak{m}$. Let $f(\mathfrak{m}') = [k' : k]$.
(4) The following formula holds

$$[K' : K] = \sum_{\mathfrak{m}' \in \mathrm{Max}(A')} e(\mathfrak{m}')f(\mathfrak{m}').$$

Proof. (1) The ring A' is normal by hypothesis and Noetherian because it is a finite module over the Noetherian ring A. Let $\mathfrak{p} \subseteq A'$ be a nonzero prime ideal. Then $\mathfrak{p} \cap A$ is a nonzero prime ideal (it contains the norms of the elements of $\mathfrak{p}$). Therefore $\mathfrak{m} := \mathfrak{p} \cap A$ is the unique maximal ideal of A. In particular, the ring $A'/\mathfrak{p}$ is an integral domain and a finite-dimensional $A/\mathfrak{m}$-vector space. It follows that $A'/\mathfrak{p}$ is a field, that is, the prime ideal $\mathfrak{p}$ is maximal. The ring $A'/\mathfrak{m}A'$ is thus Noetherian and 0-dimensional, hence it has only finitely many maximal ideals.

(2) This is true because $A'_{\mathfrak{m}'}$ is a DVR.

(3) This follows from the finiteness of A' as an A-module.

(4) The A-module A' is flat (because it is without torsion) and finitely generated. Therefore it is free of rank $[K' : K]$. Let $k = A/\mathfrak{m}$. Then $A' \otimes_A k = A'/\mathfrak{m}A'$ is a k-vector space of dimension $[K' : K]$.

On the other hand, the ring $A'/\mathfrak{m}A'$ is Artinian and therefore it is the product of the localization at its maximal ideals:

$$A'/\mathfrak{m}A' \simeq \prod_{\mathfrak{m}' \in \mathrm{Max}(A')} A'_{\mathfrak{m}'}/\mathfrak{m}A'_{\mathfrak{m}'} = \prod_{\mathfrak{m}' \in \mathrm{Max}(A')} A'_{\mathfrak{m}'}/\mathfrak{m}'^{e(\mathfrak{m}')}.$$

In particular,

$$[K' : K] = \dim_k(A'/\mathfrak{m}A') = \sum_{\mathfrak{m}' \in \mathrm{Max}(A')} \dim_k(A'_{\mathfrak{m}'}/\mathfrak{m}'^{e(\mathfrak{m}')}).$$

Let $\mathfrak{m}'$ be a maximal ideal of A'. In order to simplify notation, write $B = A'$, $\mathfrak{n} = \mathfrak{m}'$ and $e = e(\mathfrak{n}')$. For $i \in \mathbb{N}$, consider the short exact squence of B-modules

$$0 \longrightarrow \mathfrak{n}^i/\mathfrak{n}^{i+1} \longrightarrow B/\mathfrak{n}^{i+1} \longrightarrow B/\mathfrak{n}^i \longrightarrow 0.$$

The k'-vector space $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ is of dimension 1. One deduces, by induction,

$$\dim_k(B/\mathfrak{n}^e) = e \dim_k(k') = e(\mathfrak{m}')f(\mathfrak{m}').$$

This concludes the proof. □

Exercise 0.13. Let k be a perfect field. Let K, K' be finitely generated fields over k of transcendence degree 1. Let C, C' be the associated algebraic curves. Let $\pi\colon C' \to C$ be a morphism of algebraic varieties (*i.e.* of k-locally ringed spaces). Show the following assertions:

(1) The image of π is either a single point (in which case we say that π is constant) or the whole C.
(2) If f is non-constant, then it induces a k-algebra homomorphism $K \to K'$ making K' a finite extension of K. For each $x \in C$ show that the following formula holds

$$\deg(\pi) := [K' : K] = \sum_{\pi(x')=x} e(x')[k(x') : k(x)],$$

where, for $x' \in \pi^{-1}(x)$, the integer $e(x')$ is the ramification index at x':

$$\mathfrak{m}_x \mathcal{O}_{C',x'} = \mathfrak{m}_{x'}^{e(x')}.$$

(3) Conversely, for each k-algebra homomorphism $\varphi\colon K \to K'$ show that there is a unique non-constant morphism of algebraic varieties $\pi_\varphi\colon C' \to C$ inducing φ.
(4) Let $V \subseteq C'$ be a non-empty open subset. Let $f\colon V \to C$ be a non-constant morphism of algebraic varieties. Show that f extends uniquely to a morphism $C' \to C$.

Proof. (1) Let $U = \mathrm{Spm}(A)$ be a non-empty open subset of X for some finitely generated k-algebra A. Let $U' \subseteq \pi^{-1}(U)$ be a non-empty open subset with such that $U' = \mathrm{Spm}(A')$ for some finitely generated k-algebra A'. The morphism $\pi\colon U' \to U$ corresponds to a homomorphism of k-algebras $\varphi\colon A \to A'$.

Suppose φ not injective. Since A' is an integral domain. Then $\mathfrak{m} := \mathrm{Ker}(\varphi)$ is a nonzero prime ideal of A, hence it is maximal. Therefore the homomorphism φ factors as $A \to A/\mathfrak{m} \to A'$. Let x the point of U corresponding to $\mathfrak{m}$. It follows that the image of $\pi\colon U' \to U$ is $\{x\}$. By continuity π is constant of value x.

Suppose φ injective. In this case $\varphi\colon A \to A'$ extends to a injective homomorphism of k-algebras $\varphi\colon K \to K'$, making K' a finite extension of K. Let $v\colon K \to \mathbb{Z} \cup \{\infty\}$ be a surjective valuation whose restriction to k is trivial. Then v can be extended to K'.[1]

(2) Let $U = \mathrm{Spm}(A)$ be a non-empty open subset of X for some finitely generated k-algebra A. Let A' be the normalization of A in K'.

Claim 0.14. $\mathrm{Spm}(A') = \pi^{-1}(U)$.

[1]There are many ways to see this. Here is one. Let K_v be the completion of K with respect to v. Let $\bar{K}_v$ be an algebraic closure of K. Then the valuation v extends in a unique way to a valuation $\bar{v}$ on $\bar{K}_v$ by setting, for $\alpha \in \bar{K}_v$ with minimal polynomial $x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in K[x]$, the value to be $\bar{v}(\alpha) = v(a_0)/d$. The choice of an embedding $K' \to \bar{K}_v$ gives the wanted valuation.

Proof of the Claim. ($\subseteq$) Clear. Given a maximal ideal $\mathfrak{m}'$ of B, the prime ideal $\mathfrak{m}' \cap A$ is maximal. ($\supseteq$) Let $v' \colon K' \to \mathbb{Z} \cup \{\infty\}$ be a valuation such that $v := v' \circ \varphi$ belongs to U. It follows that the integral closure of A' of A is contained in the valuation ring of v'. Therefore v' corresponds to a maximal ideal of A'. □

The formula is just the preceding exercise.

□

Exercise 0.15 (Valuative criterion of properness)**.** Let k be a perfect field. Let K be finitely generated fields over k of transcendence degree 1. Let C be the associated algebraic curve. Let $U \subseteq C$ be a open subset, and $x \in U$. Let $f \colon U \smallsetminus \{x\} \to \mathbb{P}^n_k$ be a morphism of algebraic varieties. Show that f extends uniquely to a morphism $U \to \mathbb{P}^n_k$.

0.3. **Some algebraic geometry.** Let k be a field. For a finitely generated k-algebra A let

$$\mathrm{Spm}(A) := (\mathrm{Max}(A), \mathcal{O}_{\mathrm{Max}(A)}).$$

Exercise 0.16. Let X be an algebraic variety. Let $Y \subseteq X$ be a closed subset. For each open subset $U \subseteq X$, consider

$$I(U) := \{f \in \mathcal{O}_X(U) : f(y) = 0, y \in Y \cap U\}.$$

(1) Show that $I(U)$ is an ideal of $\mathcal{O}_X(U)$ and $U \mapsto I(U)$ is a sheaf on X.
(2) For each *affine* open subset $V \subseteq X$, set $Q(V) := \mathcal{O}_X(V)/I(V)$. Show that Q extends uniquely to a sheaf on X. (Hint: start by considering X affine and then conclude by taking an affine cover.)
(3) Show that the k-locally ringed space $(Y, Q_{|Y})$ is an algebraic variety. It called the *reduced* structure on Y.
(4) Take $X = \mathbb{A}^2_k \smallsetminus \{(0,0)\}$ and $Y = V(y)$ where x, y are the coordinates on $\mathbb{A}^2_k$. Compute $\mathcal{O}_X(X)$ and $Q(X)$, and conclude that the natural map $\mathcal{O}_X(X) \to Q(X)$ is not surjective.

Exercise 0.17. Let X, Y be algebraic varieties over k.

(1) Show that there exists a unique (up to a unique isomorphism) an algebraic variety Z, called the product of X and Y and denoted $X \times_k Y$, endowed with morphisms $p \colon Z \to X$ and $q \colon Z \to Y$ with the following universal property: given algebraic variety S and morphisms of algebraic varieties $f \colon S \to X$ and $g \colon S \to Y$, there exists a unique morphism of algebraic varieties $h \colon S \to Z$ such that $p \circ h = f$, $q \circ h = g$. (Hint: If $X = \mathrm{Spm}(A)$, $Y = \mathrm{Spm}(B)$ for finitely generated k-algebras A, B, then $Z = \mathrm{Spm}(A \otimes_k B)$. Then glue.)
(2) Show that $\mathbb{A}^m_k \times_k \mathbb{A}^n_k = \mathbb{A}^{m+n}_k$.
(3) Show that $\mathbb{P}^1 \times_k \mathbb{P}^1_k$ is not isomorphic to $\mathbb{P}^2$.

Exercise 0.18 (Segre embedding)**.** Let $m, n \in \mathbb{N}$. Consider the morphism of algebraic varieties $s \colon \mathbb{P}^m_k \times_k \mathbb{P}^n_k \to \mathbb{P}^{(m+1)(n+1)-1}_k$ defined by

$$[x_0 : \cdots : x_m], [y_0 : \cdots : y_n] \longmapsto [x_i y_j : i = 0, \ldots, m, j = 0, \ldots, n].$$

(1) Show that the image X of s is closed, by writing the (quadratic) equations that it satisfies.
(2) Show that s induced an isomorphism of $\mathbb{P}^m_k \times_k \mathbb{P}^n_k$ with X endowed with its reduced structure. (That is, s is a closed embedding.)

0.4. **Embeddings.**

Definition 0.19. Let k be a field. An algebraic variety X is said to be reduced if, for each open subset U of X, the k-algebra $\mathcal{O}_X(U)$ is reduced.

Let X be a reduced algebraic variety. A morphism $f\colon X \to Y$ of algebraic varieties is said to be a closed immersion if the image Z of f is closed and f induces an isomorphism of X with Z endowed of its reduced structure.

Exercise 0.20. Let k be a perfect field. Let K be a finitely generated field over k of degree of transcendance 1. Let C be the associated algebraic curve.

(1) Show that C can be covered by non-empty open subsets $U_1, \dots, U_n$ with $U_i = \mathrm{Spm}(A_i)$ for a finitely generated k-algebra A_i.
(2) By choosing generators of A_i, define a closed embedding $\varepsilon_i\colon U_i \to \mathbb{A}_k^{d_i}$.
(3) Show that the intersection $U := U_1 \cap \cdots \cap U_n$ is a non-empty open subset of C and the morphism
$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)\colon U \longrightarrow \mathbb{A}_k^{d_1} \times_k \cdots \times_k \mathbb{A}_k^{d_1}$$
extends to a morphism $\varepsilon\colon C \to \mathbb{P}_k^{d_1} \times_k \cdots \times_k \mathbb{P}_k^{d_1}$.
(4) Show that ε is a closed embedding.
(5) By composing with Segre's embedding, show that ε yields a closed embedding of C in a projective space.