Differential forms

Let A, B rings. Let $A \to B$ be a homomorphism of rings.

Def: 1) Let M be a B-module. An A-linear derivation (on B) is a A-linear map $\delta: B \to M$ satisfying the Leibniz rule

$$\delta(fg) = f\,\delta(g) + g\,\delta(f) \qquad f, g \in B.$$

$$\mathrm{Der}_A(B, M) = \{ A\text{-linear derivations } B \to M \}.$$

2) The module of (Kähler) differentials is the unique (up to unique isomorphism) B-module $\Omega_{B/A}$ endowed with an A-linear derivation $d_{B/A}: B \to \Omega_{B/A}$ s.t. for each B-Module M and A-linear derivation $\delta: B \to M$ there is a unique B-linear map $\varphi: \Omega_{B/A} \to M$ s.t.

$$\delta = \varphi \circ d_{B/A}.$$

Prop $(\Omega_{B/A}, d_{B/A})$ exists.

Proof:
$$\Omega_{B/A} = \Big(\bigoplus_{b \in B} B\, db \Big) \Big/ \big(d(bb') - (b\,db' + b'\,db) \; ; \; b, b' \in B \big)$$

$$d_{B/A}: B \to \Omega_{B/A}, \quad b \mapsto db. \qquad \square$$

Example: 1) $B = A[x] \ni f = \sum_{i=0}^{d} a_i x^i$

$\delta: A[x] \to M$ A-linear derivation

$$\delta(f) \underset{\text{A-lin}}{=} \sum_{i=0}^{d} a_i\, \delta(x^i) \underset{\text{Leibniz}}{=} \sum_{i=0}^{d} a_i i x^{i-1} \delta(x).$$

In this case

$$\Omega_{B/A} = B\,dx = A[x]\,dx$$

$$\varphi: \Omega_{B/A} \to M$$
$$dx \mapsto \delta(x).$$

2) $B = A[x_1, \dots, x_n] \to \Omega_{B/A} = \bigoplus_{i=1}^{n} B\,dx_i.$

Properties:

$S \subset B$ multiplicative subset $\quad S^{-1}B$ localization w.r.t S

- $S \subseteq B$ multiplicative subset,

$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A} := \Omega_{B/A} \otimes_B S^{-1}B.$$

$$\left[\; d\left(\frac{b}{s}\right) = \frac{s\,db - b\,ds}{s^2}. \;\right]$$

- $A' =$ A-algebra, $B' = A' \otimes_A B$

$$\Omega_{B'/A'} = B' \otimes_B \Omega_{B/A}.$$

- For a ring homomorphism $B \to C$, we have the following exact sequence of C-modules:

$$(*) \qquad C \otimes_B \Omega_{B/A} \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

<u>Proof:</u> Let M be a C-module. By taking $\mathrm{Hom}_C(-, M)$ the exactness of the sequence $(*)$ is equivalent to that of

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_C(\Omega_{C/B}, M) & \to & \mathrm{Hom}_C(\Omega_{C/A}, M) & \to & \mathrm{Hom}_C(C \otimes_B \Omega_{B/A}, M) \\ & \| & & \| & & \| \\ 0 \to & \mathrm{Der}_B(C, M) & \longrightarrow & \mathrm{Der}_A(C, M) & \longrightarrow & \mathrm{Hom}_B(\Omega_{B/A}, M) \\ & & & & & \| \\ & & & & & \mathrm{Der}_A(B, M) \\ & & & \delta & \longmapsto & \delta|_B \end{array}$$

(clearly injective: B-linear $\Rightarrow$ A-linear)

Suppose $\delta : C \to M$ is an A-lin derivation such that $\delta|_B = 0$. For $b \in B$ and $c \in C$,

$$\delta(bc) = b\,\delta(c) + c\,\underbrace{\delta(b)}_{=0} = b\,\delta(c) \Rightarrow \delta \text{ is } B\text{-lin.}$$

$\Rightarrow$ the sequence is exact. $\square$

- $C = B/I$, $I \subseteq B$ ideal $\Rightarrow$ $\Omega_{C/B} = 0$ and the sequence of B/I-modules

$$\begin{array}{lcccccc} (**) & I/I^2 & \longrightarrow & B/I \otimes_B \Omega_{B/A} & \longrightarrow & \Omega_{C/A} & \longrightarrow 0 \\ & f & \longmapsto & 1 \otimes df. & & & \end{array}$$

is exact.

<u>Proof:</u> Since $B \to C$ is surjective, every B-linear derivation on C is zero. Take M to be a B/I-module. By Applying $\mathrm{Hom}_{B/I}(-, M)$ We see that the exactness of $(**)$ is equivalent to that of

($I \otimes_B B/I = \ldots$)

$\ldots M) \to \mathrm{Hom}(\ldots, M) \to \mathrm{Hom}(I/\ldots, M\ldots$

$$0 \to \mathrm{Hom}_{B/I}(\Omega_{B/I/A}, M) \to \mathrm{Hom}_{B/I}(B/I \otimes_B \Omega_{B/A}, M) \to \mathrm{Hom}_{B/I}(I/I^2, \ldots$$

$$\mathrm{Hom}_{B/I}(\Omega_{B/I/A}, M) = \mathrm{Der}_A(B/I, M)$$

$$\mathrm{Hom}_{B/I}(B/I \otimes_B \Omega_{B/A}, M) = \mathrm{Hom}_B(\Omega_{B/A}, M) = \mathrm{Der}_A(B, M) \ni \delta : B \to M$$

$$\mathrm{Hom}_{B/I}(I/I^2, \ldots) = \mathrm{Hom}_B(I, M) \ni \delta_{|I} : I \to M$$

We have to show that it is exact in the center.
Let $\delta : B \to M$ be such that $\delta_{|I}$ is zero. Then it induces an A-linear derivation $\bar{\delta} : B/I \to M$, and
$$\delta : B \to B/I \xrightarrow{\bar{\delta}} M.$$
□

Rmk: $M = B$-module, $M \otimes_B B/I = M/IM$. Apply this to B-module I

Cor: $B = A[x_1, \ldots, x_n]$, $I = (f_1, \ldots, f_r) \subseteq B$ ideal.

$$\Omega_{B/I/A} = \frac{B/I \otimes_B \Omega_{B/A}}{1 \otimes dI} = \frac{\bigoplus_{i=1}^n B/I \, dx_i}{(df_1, \ldots, df_r)}.$$

$$\Omega_{B/A} = \bigoplus_{i=1}^n B \, dx_i \Rightarrow B/I \otimes_B \Omega_{B/A} = \bigoplus_{i=1}^n B/I \, dx_i.$$

Cor: If B is a A-algebra of finite type, then $\Omega_{B/A}$ is a B-module of finite type

Example: $A = k$ field, $n = 2$, $r = 1$: $I = (f) \subseteq k[x,y]$.

$$R = k[x,y]/(f)$$

$$\Omega_{R/k} = \frac{R\,dx \oplus R\,dy}{\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)}.$$

Sub-example: $f(x,y) = y^2 - p(x)$ $\quad p(x) \in k[x,y]$ of degree 3.

$$\Omega_{R/k} = \frac{R\,dx \oplus R\,dy}{(2y\,dy - p'(x)\,dx)}.$$

- Let K be a field and L a finite extension of K. Then L is separable iff $\Omega_{L/K} = 0$.

Proof: First remark that, if L' is an extension of L, then

$$L' \otimes_K \Omega_{L/K} \to \Omega_{L'/K} \to \Omega_{L'/L} \to 0$$

If we know that $\Omega_{L/K} = 0$ then $\Omega_{L'/K} \to \Omega_{L'/L}$ ...

is an isomorphism. By doing this we may suppose $L = K[x]/(f)$ with $f \in K[x]$ irreducible. | + the fact a compositum of separable ext is separable

$$\Omega_{L/K} = \frac{K[x]/(f)\, dx}{(f'(x)\, dx)} \simeq K[x]/(f(x), f'(x))$$

$$\Omega_{L/K} = 0 \Leftrightarrow f'(x) \neq 0 \text{ in } K[x] \Leftrightarrow f \text{ separable}$$

($\uparrow$ f irreducible) $\Leftrightarrow L/K$ is separable. □

<u>Prop.</u> Let k be a field. Let A be a local k-alg s.t. A/m is a finite separable extension of k. Then, the map

$$m/m^2 \longrightarrow A/m \otimes_A \Omega_{A/k} \text{ is an isomorphism.}$$

<u>Proof:</u> The exact sequence (**) in this case is

$$m/m^2 \longrightarrow A/m \otimes_A \Omega_{A/k} \longrightarrow \Omega_{(A/m)/k} \longrightarrow 0$$

(surjective; $\Omega_{(A/m)/k} = 0$ since A/m finite separable)

Need to show that it is injective. This is equivalent to show that, for a A/m-module V, that the map

$$\mathrm{Hom}_{A/m}(A/m \otimes_A \Omega_{A/k}, V) \longrightarrow \mathrm{Hom}_{A/m}(m/m^2, V)$$

is surjective.

$$\| \qquad\qquad\qquad \|$$

$$\mathrm{Der}_k(A, V) \longrightarrow \mathrm{Hom}_A(m, V)$$

$$\delta \longmapsto \delta|_m.$$

We have to show that every A-linear map $\varphi: m \to V$ extends to a derivation $\delta: A \to V$. So let $\varphi: m \to V$ be an A-linear map.

Pick $f \in A$. Then $\bar{f} \in A/m$ is algebraic over k and its minimal polynomial $P \in k[x]$ is separable.

$$A \ni P(f) \longmapsto 0 \in A/m \implies P(f) \in m.$$

Suppose there is a derivation δ extending φ.

$$P(x) = \sum_{i=0}^{d} a_i x^i$$

$$P(f) = \sum_{i=0}^{d} a_i f^i$$

$$\varphi(P(f)) = \delta(P(f)) = \left[\sum^{d} i a_i f^{i-1}\right] \delta(f)$$

$\varphi(\ldots) = \ldots$ [cut off at top]

$\uparrow$ $p(f) \in m$ $\qquad$ $\underbrace{[\,i=1 \ldots\,]}_{p'(f)}$

$A \ni p(f) \longmapsto \neq 0$ in A/m because p is separable

$\Longrightarrow p(f)$ is invertible in A

We can define

$$\delta(f) := \frac{\varphi(p(f))}{p'(f)} .$$

This will be the derivation extending φ.]

Prop. Let K be a function field of a curve over a perfect field k.
Then $\dim_K \Omega_{K/k} = 1$.

Lemma: $+\infty > \dim_K \Omega_{K/k} \geq 1$.

Proof: Pick f in K transc.

K
| finite
$k(f)$
|
k

finite dimensional
$\downarrow$

$$\Omega_{k(f)/k} \otimes_{k(f)} K \to \Omega_{K/k} \to \Omega_{K/k(f)} \to 0$$

If $K/k(f)$ is not separable, then we're done.
Suppose $K/k(f)$ separable.

Therefore $K = k(t)[x]/(f(x))$, $f_t(x) \in k(t)[x]$ is separable.

We can assume $f(t,x) \in k[t,x]$ (by clearing denominators).

$$A = k[t,x]/(f(t,x)).$$

$$\Omega_{A/k} = \frac{A\,dx \oplus A\,dt}{\left(\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx\right)} \qquad K = \mathrm{Frac}(A)$$

$$\Omega_{K/k} = K \otimes_A \Omega_{K/k} = \frac{K\,dx \oplus K\,dt}{\left(\frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx\right)}$$

$$\Longrightarrow \dim_K (\Omega_{K/k}) = 1. \qquad \square$$

Prop. Let K be the function field of a curve C over a perfect field k.
Let $x \in C$. Then $\Omega_{\mathcal{O}_{C,x}/k}$ is a free $\mathcal{O}_{C,x}$-module of rk 1.
$f \in \mathcal{O}_{C,x}$ is a uniformizer $\Longleftrightarrow$ df is a generator.

Proof: Since A is a DVR an k is perfect,

$$\Omega_{\mathcal{O}_{C,x}/k} - \underbrace{\Omega_{C,x}} \otimes_{\mathcal{O}_{C,x}} k(x) = m_x/m_x^2.$$

On the other hand,

$$K \otimes_{\mathcal{O}_{C,x}} \Omega_{C,x} = \Omega_{K/k}$$ this is a finite dimensional K-vector space of dim ≥ 1.

<u>Claim</u>: $\Omega_{\mathcal{O}_{C,x}/k}$ is a $\mathcal{O}_{C,x}$-module of finite type.

<u>Proof of the Claim</u>: Pick $t \in K$ transc. s.t. $\mathrm{ord}_x(t) \geq 0$.

$$\begin{array}{ccc} K & \supseteq & A \text{ normalization of } k[t] \\ | & & | \\ k(t) & \supseteq & k[t] \\ | & & | \\ k & & k \end{array}$$

We know that A is a k-alg. of finite type.

$\Rightarrow \Omega_{A/k}$ is an A-module of finite type.

Let $m \subseteq A$ be the max ideal corresponding to x. Then

$$\mathcal{O}_{C,x} = A_m. \text{ and}$$

$$\Omega_{C,x} = \Omega_{A/k} \otimes_A A_m \leftarrow A_m\text{-module of f.t.}$$ □

Nakayama $\Rightarrow$ that dt generates $\Omega_{C,x}$ because

$$\Omega_{C,x} \otimes_{\mathcal{O}_{C,x}} k(x) \simeq m_x/m_x^2$$
$$\ni t \neq 0 \Rightarrow \text{generator.}$$

$\Rightarrow dt \in \Omega_{C,x}$ generator.

Since $\dim_K (\Omega_{K/k}) \geq 1$, $\Omega_{C,x}$ cannot be torsion.

and we have $\underbrace{\Omega_{C,x} = \mathcal{O}_{C,x}\, dt.}$ □

$\Downarrow$

$$\Omega_{K/k} = \Omega_{C,x} \otimes_{\mathcal{O}_{C,x}} K$$

is of dimension 1.

<u>Prop.</u> Let K be a function field of a curve over a perfect field k. Let L/K be a finite extension. Then

$$L \otimes_K \Omega_{K/k} \xrightarrow{\alpha} \Omega_{L/k}$$

is an isomorphism iff L is a finite separable extension.

is an ...

Proof: The exact sequence (*) gives

$$L \otimes_K \Omega_{K/k} \to \Omega_{L/k} \to \Omega_{L/K} \to 0.$$

($\Rightarrow$) α isomorphism $\Rightarrow$ $\Omega_{L/K} = 0$ $\Rightarrow$ L/K separable

($\Leftarrow$) $\Omega_{L/K} = 0$ $\Rightarrow$ α is surjective

$$\dim_L \left(L \otimes_K \Omega_{K/k} \right) = \dim_K \left(\Omega_{K/k} \right) = 1$$

$$\dim_L \Omega_{L/k} = 1.$$

$\Rightarrow$ α is bijective. □

Let k be a perfect field.

Prop. Let K, K' be function fields of curves $/k$. Let $K \to K'$ be an injective homomorphism. Let $\pi : C' \to C$ be the corresponding morphism. Let $x' \in C'$ and $\pi(x') =: x \in C$.

$\mathcal{O}_{C',x'} / \frac{\pi^* dt}{dt'} \mathcal{O}_{C',x'}$.

$$\underbrace{\Omega_{C,x} \otimes_{\mathcal{O}_{C,x}} \mathcal{O}_{C',x'}}_{\mathcal{O}_{C',x'} \cdot \pi^* dt} \to \underbrace{\Omega_{C',x'}}_{\mathcal{O}_{C',x'}} \longrightarrow \Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}} \to 0$$

($dt \otimes 1 = \pi^* dt$)

Let $t \in \mathcal{O}_{C,x}$ be a uniformizer an $t' \in \mathcal{O}_{C',x'}$ be a uniformizer.

Suppose K'/K separable. Then:

$$+\infty > \mathrm{length}_{\mathcal{O}_{C',x'}} \left(\Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}} \right) \geq \overbrace{\mathrm{ord}_{x'}(\pi^* t)}^{\mathrm{ord}_{x'}(\pi)} - 1$$

$t \mapsto \pi^* t$

$K \to K'$

with equality iff $\underbrace{\mathrm{ord}_{x'}(\pi^* t)}_{\in \mathbb{Z}} \neq 0$ in k.

$\Leftrightarrow$ $\mathrm{char}(k) = p \nmid \mathrm{ord}_{x'}(\pi^* t)$).

Proof. Write $t = u t'^e$ with $u \in \mathcal{O}_{C',x'}$ a unit. Then

$$dt = u \cdot e\, t'^{e-1} dt' + t'^e du.$$

Since the extension K'/K is separable,

$$\Omega_{K/k} \otimes_K K' \xrightarrow{\sim} \Omega_{K'/k}.$$

This implies that the $\mathcal{O}_{C',x'}$-module $\Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}}$

is torsion and it is a $\mathcal{O}_{C',x'}$-module of finite type.

$\Rightarrow$ dt is non zero in $\Omega_{C',x'}$

$$\text{length}_{\mathcal{O}_{C',x'}}(\Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}}) = \text{ord}_{x'}\left(\frac{dt}{dt'}\right) = \text{ord}_{x'}\left(ue\,t'^{e-1} + t'^{e}\frac{du}{dt}\right)$$

$\geq e-1$ with equality iff $e \neq 0$ in k. ☐

<u>Cor</u>: If $\pi : C' \to C$ surjective separable (K'/K separable) morphism then

$$\#\{x' \in C' : \text{ord}_{x'}(\pi) > 1\} < +\infty.$$

<u>Proof</u>:
$$\begin{array}{ccc} K' & \supseteq & A' \\ | & & | \\ K & \supseteq & A \end{array}$$
A' normalization of A in K'.

$$(0 \to)\ A' \otimes_A \Omega_{A/k} \to \Omega_{A'/k} \longrightarrow \Omega_{A'/A} \to 0$$

I don't need this

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$-\otimes K'$ $\quad 0 \to K' \otimes_K \Omega_{K/k} \xrightarrow{\sim} \Omega_{K'/k} \longrightarrow 0 = \Omega_{A'/A} \otimes_{A'} K'$

K'/K separable

$\Longrightarrow$ $\Omega_{A'/A}$ is an torsion f.t. A'-module.
because A' is a finite type A-module (A-alg in part.)

It suffices to show that

$$\#\{m' \subseteq A' : \boxed{\Omega_{A'/A} \otimes_{A'} A'/m'} \neq 0\} < +\infty$$

$A'/m' = \mathcal{O}_{C',x'}/m'_{x'}$ $\qquad$ $\Omega_{A'/A} \otimes_{A'} \mathcal{O}_{C',x'} = \Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}}$

↑ $x' \in C'$ is the point corresponding to m'.

$$\Omega_{A'/A} \otimes_{A'} A'/m' = \Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}} \otimes_{\mathcal{O}_{C',x'}} k(x').$$

$\cong \mathcal{O}_{C',x'} / \frac{dt}{dt'}\mathcal{O}_{C',x'}$ $\quad$ $\text{ord}_{x'}(\pi) := \text{ord}_{x'}(t)$

$\mathcal{O}_{C,x} \ni t$ = uniformizer

$\mathcal{O}_{C',x'} \ni t'$ = uniformizer

<u>Rmk</u>: $\{x' \in \text{Max}(A') : \text{ord}_{x'}(\pi) > 1\} = \{m' \subseteq A' : \Omega_{A'/A} \otimes_{A'} A'/m' \neq 0\}$

$\Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}} \otimes k(x') \neq 0 \iff \mathrm{ord}_{x'}\left(\frac{dt}{dt'}\right) \geq 1$

$\Vert \leftarrow$ if $\mathrm{ord}_{x'}(\pi) \neq 0$ in k

$\mathrm{ord}_{x'}(\pi) - 1$

Suppose $\mathrm{ord}_{x'}\left(\frac{dt}{dt'}\right) = 0$. Then $\mathrm{ord}_{x'}(\pi) = 1$.

$t = ut'^e$ $\quad u$ unit in $\mathcal{O}_{C',x'}$ $\quad \frac{dt}{dt'} = eut'^{e-1} + t'^e \frac{du}{dt'}$ (valuation $\geq$)

$e \geq 1$.

$\uparrow$ this is a unit iff $e=1$.

Rmk: $\mathrm{ord}_{x'}\left(\frac{dt}{dt'}\right) = 0 \iff \Omega_{\mathcal{O}_{C',x'}/\mathcal{O}_{C,x}} \otimes k(x') = 0$

$\iff \mathrm{ord}_x(\pi) = 1$.

Recall: A Noeth ring, $M = A$-module of f.t.

$\mathrm{Ann}(M) = \{a \in A : am = 0 \ \forall m \in M\}$ ideal

Lemma: For a prime ideal $p \subseteq A$, the following are equivalent:

1) $\mathrm{Ann}(M) \subseteq p$

2) $M \otimes_A A_p/pA_p \neq 0$.

Proof: Exercise. $\square$

In particular,

$\{m' \in A' : \Omega_{A'/A} \otimes_{A'} A'/m' \neq 0\} = V(\mathrm{Ann}(\Omega_{A'/A}))$

$= \{m' \in A' : \mathrm{Ann}(\Omega_{A'/A}) \subseteq m'\}$

$\mathrm{Ann}(\Omega_{A'/A}) \neq 0$ because it is torsion.

$\Rightarrow V(\mathrm{Ann}(\Omega_{A'/A}))$ is a finite set.