Last time: C smooth projective curve over a perfect field k.

D a divisor on C s.t. for each $x \in X$ there is $f \in H^0(D)$ s.t.

$$\mathrm{ord}_x(f) + \mathrm{mult}_x(D) = 0$$

I called this a globally generated divisor.

Kodaira map: If $k = \bar{k}$,

$$i_D : C \longrightarrow \mathbb{P}(H^0(D)^*)$$
$$x \longmapsto H_x = \{ f \in H^0(D) : \mathrm{mult}_x(D) + \mathrm{ord}_x(f) > 0 \}$$

Look at the exact sequence ($r = \mathrm{mult}_x(D)$)

$$\begin{tikzcd} 0 \arrow[r] & H^0(D-[x]) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H^0(D) \arrow[r] & \underbrace{m_x^{-r}/m_x^{-r+1}}_{\shortparallel} \arrow[r] & 0 \\ & H_x & & \end{tikzcd}$$

$\uparrow$ D is globally generated $\Rightarrow$ surjective

k-vector space of dim 1

When k is not necessarily algebraically closed

$$\mathrm{Sym}^d H^0(D) \xrightarrow{\mu} H^0(dD) \longrightarrow m_x^{-rd}/m_x^{-rd+1} \qquad (d \in \mathbb{N})$$
$$[g_1] \cdots [g_d] \longmapsto g_1 \cdots g_d$$
$$g_i \in H^0(D)$$

$$\bigoplus_{d\in\mathbb{N}} \mathrm{Sym}^d H^0(D) \xrightarrow{\mu} \bigoplus_{d\in\mathbb{N}} H^0(dD) \longrightarrow \bigoplus_{d\in\mathbb{N}} m_x^{-rd}/m_x^{-rd+1}$$

$\shortparallel$ $\qquad\qquad\qquad\qquad$ $\cong$ $t \in \mathcal{O}_{C,x}$ uniformizer

$$\mathrm{Sym}\ H^0(D) \xrightarrow{\qquad \eta_x \qquad} k(x)[t^{-r}]$$

Def: $i_D(x) = \mathrm{Ker}(\eta_x)$. It does not depend on the choice of the uniformizer, and it is a prime, homogeneous ideal which is <u>maximal</u> among those not containing $\bigoplus_{d\geq 1} \mathrm{Sym}^d H^0(D)$.

↳ (we use that D is globally generated.)

For $f \in \mathrm{Sym}^d H^0(D)$, then $\mu(f) \in H^0(dD)$

$$U_f = \{ x \in C : d\,\mathrm{mult}_x D + \mathrm{ord}_x(\mu(f)) = 0 \}$$

open set (it contains X, D) if $\neq \emptyset$,

Rmq: $U_f = i_D^{-1}(D_+(f))$

because the $x \in U_f \iff 0 \neq \eta_x(\mu(f)) \in \mathfrak{m}_x / \mathfrak{m}_x^{-rd+1}$

$$\Updownarrow$$

$$\mathrm{ord}_x(\mu(f)) + d\, \mathrm{mult}_x(D) = 0.$$

At the level of sheaves the map is defined as

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(H^0(D)^*)}(D_+(f)) \arrow[r] \arrow[d, equal] & i_{D*}\mathcal{O}_C(D_+(f)) = \mathcal{O}_C(U_f) \\
{[\mathrm{Sym}\, H^0(D)]_{(f)}} &
\end{tikzcd}$$

$$g \in \mathrm{Sym}^{d_i} H^0(D),\ \frac{g}{f^i} \longmapsto \frac{\mu(g)}{\mu(f^i)} \in \mathcal{O}_C(U_f)$$

(Suppose $\mu(f) \neq 0$.)

$$x \in U_f \implies \mathrm{ord}_x\left(\frac{\mu(g)}{\mu(f^i)}\right) \geq 0$$

$$\begin{array}{c}
\| \\
\underset{\underset{-\mathrm{mult}_x(d_i D)}{\rotatebox{90}{$\leq$}}}{\mathrm{ord}_x(\mu(g))} - \underset{\underset{\mathrm{mult}_x(d_i D)}{\|}}{\mathrm{ord}_x(\mu(f)^i)} = 0.
\end{array}$$

In this way we constructed a morphism of algebraic varieties

$$i_D : C \longrightarrow \mathbb{P}(H^0(D)^*).$$

Def. 1) i_D is called the Kodaira map.

2) D is very ample is the Kodaira map i_D is a closed embedding.

Th: Let C be a smooth projective curve of genus g. Let D be a divisor of degree $\geq 2g+1$. Then D is very ample.

Proof (k alg. closed). We have to show two things:

A) i_D induces a homeomorphism with $i_D(C)$ and $i_D(C)$ is closed in $\mathbb{P}(H^0(D)^*)$.

B) For each $x \in C$, the map $\mathcal{O}_{\mathbb{P}(H^0(D)^*), i(x)} \to \mathcal{O}_{C,x}$ is surjective.

i_D is injective ("D separates points") Let $x, y \in C$, $x \neq y$. We have to show that there is $f \in H^0(D)$ s.t.

$$\left.\begin{array}{l}
\mathrm{ord}_x(f) + \mathrm{mult}_x(D) = 0 \\
\mathrm{ord}_y(f) + \mathrm{mult}_y(D) > 0.
\end{array}\right| \Rightarrow \begin{array}{l} x \in D_+(f) \text{ and} \\ y \in V_+(f) \subseteq \mathbb{P}(H^0(D)^*) \end{array}$$

Indeed consider the divisor $D' = D - [x] - [y]$

Indeed, consider

$$\deg D' = \deg D - 2 \geq 2g+1-2 \geq 2g-1 > \deg K_C$$

$$\Longrightarrow \quad \deg(K_C - D') < 0$$

$$\Longrightarrow \quad h^0(K_C - D') = 0.$$

RR: $$h^0(D-[x]-[y]) - \overbrace{h^0(K_C-(D-[x]-[y]))}^{\overset{0}{\shortparallel}} = \deg D - 2 + 1 - g$$

$$h^0(D-[x]) - \underbrace{h^0(K_C-(D-[x]))}_{\underset{0}{\shortparallel}} = \deg D - 1 + 1 - g$$

$$\Longrightarrow \quad h^0(D-[x]) = h^0(D-[x]-[y]) + 1.$$

$$\Rightarrow \quad \exists f \in H^0(D-[x]) \setminus H^0(D-[x]-[y])$$

This f does the job.

<u>i_D is closed:</u> Let $C' \subseteq \mathbb{P}(\overbrace{H^0(D)^*}^{V})$ be the closure of $i_D(C)$ in $\mathbb{P}(V)$. C' inherits a unique structure of algebraic variety with sheaf of structure $\mathcal{O}_{\mathbb{P}(V)}/\mathcal{I}$ where $\mathcal{I}$ is the kernel of the sheaf homomorphism

$$i_D^\# : \mathcal{O}_{\mathbb{P}(V)} \longrightarrow i_{D*}\mathcal{O}_C$$

$$\mathcal{I} = \ker(i_D^\#).$$

We see that, for $x \in C$, the map $(i = i_D)$

$$i_{D,x}^\# : \boxed{\mathcal{O}_{C', i(x)}} \longrightarrow \mathcal{O}_{C,x} \quad \text{is injective.}$$

This means that C' is of dimension 1, it is integral: it a possibly singular curve with fraction field K'. The map $i_{D,x}^\#$ gives an injection $K' \longrightarrow K$ = function field of C

$$\begin{tikzcd} & \tilde{C}' \arrow[d, "\pi"] \\ C \arrow[ur, "\alpha"] \arrow[r, "i_D"'] \arrow[ur, phantom, shift right=4, "\circlearrowright"] & C' \end{tikzcd}$$

$\tilde{C}'$ smooth projective curve associated with K'

map given by the inclusion $K' \longrightarrow K$ $\longrightarrow$ α

Since α is non constant, then it is surjective

We are left with proving that π is surjective.

Pick $x' \in C'$. Look at the maximal ideal $m \subseteq \mathcal{O}_{C',x'}$

$m = (f_1, \dots, f_r)$ look at $k[f_1, \dots, f_r] \subseteq K'$: it is a k-algebra of dimension 1, so by Noether's Normalisation there is $f \in k[f_1, \dots, f_r]$ s.t. $k[f_1, \dots, f_r]$ is a finite over $k[f]$

Let $v \in \tilde{C}'$ be a valuation such that $v(f) > 0$.

$$\Longrightarrow v(f_i) > 0 \text{ for } i = 1, \dots, r$$

$$\mathfrak{m} \subseteq \text{the max ideal in } \mathcal{O}_{\tilde{C}, v}.$$

$$\Longrightarrow$$

$$\Longrightarrow \pi(v) = x'.$$

<u>Rmq</u>: We proved that $i_D : C \to C'$ is surjective

$$C' = \overline{i_D(C)} \longrightarrow i_D(C) \text{ is closed.}$$

If $F \subseteq C$ is finite $\Rightarrow i_D(F) \subseteq C'$ is finite $\underset{\substack{\uparrow \\ i_D \text{ is surj.}}}{\Rightarrow}$ i_D is a closed map

$\Longrightarrow$ i_D is a homeomorphism with a closed subset of $\mathbb{P}(V)$.

B) <u>Lemma</u>: Let A, B be Noeth. local rings. Let $\varphi : A \to B$ be a local ring homomorphism. Assume:

1) B is a finitely generated A-module;

2) $A/\mathfrak{m}_A \xrightarrow{\sim} B/\mathfrak{m}_B$

3) $\mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.

Then $A \to B$ is surjective.

<u>Proof</u>: Exercise, apply Nakayama. □

We want to apply it to $A = \mathcal{O}_{C', i(x)} \longrightarrow B = \mathcal{O}_{C, x}$. $(x \in C)$.

2) is clear $A/\mathfrak{m}_A = B/\mathfrak{m}_B = k$ $(k = \bar{k})$.

1) We know that $i_D^{-1}(i_D(x)) = \{x\}$. It follows that $\mathcal{O}_{C,x}$ is the normalization of $\mathcal{O}_{C', i(x)}$ (because there are no other valuations over $i(x)$). Therefore

$$\mathcal{O}_{C,x} \text{ is a } \mathcal{O}_{C', i(x)}\text{-module of finite type.}$$

3) "D separates tangent vectors": it suffices to show that there is $f \in H^0(D)$ s.t.

$$\mathrm{ord}_x(f) + \mathrm{mult}_x(D) = 1.$$

(Indeed if a is s.t. $\mathrm{ord}_x(f) + \mathrm{mult}_x(D) = 0$ which exists

(Indeed, if g
by hypothesis,

$$\mathrm{ord}_x(f/g) = 1 \implies f/g \text{ is a uniformizer at } x.\;)$$

$$\deg(D - 2[x]) = \deg D - 2 \geq 2g+1-2 > \deg K_C$$

$$\implies \deg\left(K_C - (D - 2[x])\right) < 0$$

$$\implies h^0(K_C - (D - 2[x])) = 0.$$

RR: $\quad h^0(D - 2[x]) = \deg D - 2 + 1 - g$

$$h^0(D - [x]) = \deg D - 1 + 1 - g = h^0(D - 2[x]) + 1$$

$$\implies \exists f \in H^0(D - [x]) \setminus H^0(D - 2[x])$$

This f does the job. $\square$

<u>Rmk</u>: $\quad x \in C \qquad \mathfrak{m}_{i(x)} \subseteq \mathrm{Sym}\, H^0(D)$ corresponding to $i_D(x)$

$$\mathfrak{m}_{i(x)} = H_x\, \mathrm{Sym}\, H^0(D)$$
$$H_x = H^0(D - [x])$$

If $\quad f \in H^0(D - [x]) \setminus H^0(D - 2[x]) \Rightarrow 0 \neq f \in \mathfrak{m}_{i(x)}/\mathfrak{m}_{i(x)}^2$

$$\mathfrak{m}_{i(x)}/\mathfrak{m}_{i(x)}^2 \longrightarrow \underbrace{\mathfrak{m}_x/\mathfrak{m}_x^2}_{\text{dimension } 1} \quad \text{surjective}$$

: it suffices to show that image of is not zero

$$\begin{tikzcd} H^0(D) \arrow[r, "\eta_x"] \arrow[d, hook] & \mathfrak{m}_x^{-r}/\mathfrak{m}_x^{-r+1} \\ {[\mathrm{Sym}\, H^0(D)]_{(g)}} \arrow[r] & \mathcal{O}_{C,x} \\ \frac{f}{g} \arrow[r, mapsto] & \frac{f}{g} \text{ uniformizer} \\ & \rotatebox{90}{$\in$} \\ & \mathfrak{m}_x \setminus \mathfrak{m}_x^2 \end{tikzcd}$$

$r = \mathrm{mult}_x(D)$

$\mathrm{ord}_x(g) + \mathrm{mult}_x(D) = 0$

$i(x) \in D_+(g)$

The maximal ideal at $i(x)$ is

$$\left[(\ker \eta_x)\, \mathrm{Sym}\, H^0(D)\right]_g \cap [\mathrm{Sym}\, H^0(D)]_{(g)}$$

$$\frac{f}{g} \in \mathfrak{m}_{i(x)} \subseteq \mathcal{O}_{\mathbb{P}(V), i(x)}$$

$$\begin{tikzcd} \mathfrak{m}_{i(x)}/\mathfrak{m}_{i(x)}^2 \arrow[r] & \mathfrak{m}_x/\mathfrak{m}_x^2 \\ \frac{f}{g} \arrow[r, mapsto] & \frac{f}{g} \circ \end{tikzcd}$$

k arbitrary

<u>Example:</u> Take E to be a smooth proj curve of genus 1.

Let $e \in E(k)$ be a k-rational point.

$\Leftrightarrow k(e) = k$.

Look at the divisor $[e]$.

By RR, if $\deg D > 0$, then $\deg(K_E - D) < 0$ (because $\deg K_E = 2g-2$.

$$h^0(D) = \deg D + 1 - g = \deg D.$$

$H^0(0 \cdot [e]) = k$ "Elliptic functions without poles are constant"

$H^0([e]) = k$ "Non-constant Elliptic functions have at least a double pole"

$H^0(2[e]) = k \oplus kf$ with $\mathrm{ord}_e(f) = -2$. (" $f = \wp$")

$H^0(3[e]) = k \oplus kf \oplus kg$ with $\mathrm{ord}_e(g) = -3$ (" $g = \wp'$")

$H^0(4[e]) = \langle 1, f, f^2, g \rangle$

$H^0(5[e]) = \langle 1, f, f^2, fg, g \rangle$

$H^0(6[e]) = \langle 1, f, f^2, f^3, fg, g, g^2 \rangle$ (*)

$$3 = 2 \cdot 1 + 1 = 2g + 1$$

$$\Longrightarrow \quad i = i_{3[e]} : E \hookrightarrow \mathbb{P}(H^0(3[e])^*) \underset{1, f, g}{\simeq} \mathbb{P}^2$$

Because of (*) : $\exists\ b_1, b_2, a_0, a_1, a_2, a_3 \in k$ s.t.

$$g^2 + b_1 gf + b_2 g = a_3 f^3 + a_2 f^2 + a_1 f + a_0.$$

char$(k) \neq 2,3$ $\Big\downarrow$ complete the square (char$(k) \neq 2$)

$$\underbrace{\left(g + \frac{b_1}{2} f + \frac{b_2}{2}\right)}_{g' \,\in\, H^0(3[e]) \setminus H^0(2[e])}^2 = a_3' f^3 + a_2' f^2 + a_1' f + a_0'$$

Up to changing g as above and the coefficients a_i, we may assume

$$g^2 = \underbrace{a_3 f^3 + a_2 f^2 + a_1 f + a_0}_{\text{char}(k) \neq 3 \quad \downarrow \text{ complete the cube}}$$

$$= a_3 \Big(\underbrace{f - \frac{a_2}{3a_3}}_{f' \in H^0(2[e]) \setminus H^0([e])}\Big)^3 + a_1' f + a_0'$$

Up to changing f as above we may assume

$$g^2 = a_3 f^3 + a_1 f_1 + a_0 \qquad \text{with } a_3, a_1, a_0 \in k.$$

<u>Def</u>: This is called a Weierstrass equation for E.

This means that the image of $i_{3[e]}$

$$i_{3[e]} : E \longrightarrow \mathbb{P}(H^0(3[e])^*) \underset{1,f,g}{\cong} \mathbb{P}^2_k$$

is given by the equation

$$y^2 = a_3 x^3 + a_1 x + a_0 .$$

<u>Rmk</u>: If $\text{char}(k) = 2$, the equation will be of the form

$$y^2 + b_1 xy + b_2 y = a_3 x^3 + a_1 x + a_0$$

If $\text{char}(k) = 3$, then the equation will be

$$y^2 = a_3 x^3 + a_2 x^2 + a_1 x + a_0 .$$

<u>Smooth planes projective curves</u>

Let $f \in k[x,y]$ be a non constant polynomial.

$$V(f) = C = \{ p \in \mathbb{A}^2_k : f(p) = 0 \}$$

This is endowed with the sheaf of functions $\mathcal{O}_{\mathbb{A}^2} / f \mathcal{O}_{\mathbb{A}^2}$.

<u>def</u>: A point $p \in C$ is said to be <u>non-singular</u> if

$$\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p) \right) \neq (0,0) \qquad \text{It is singular otherwise.}$$

<u>Lemma</u>: If f is irreducible and C has no singular points, Then C is an open subset of a smooth projective curve.

<u>Proof</u>: $A = k[x,y]/(f)$

$$A\,dx \oplus A\,dy$$

$$\Omega_{A/k} = \frac{}{\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)}$$

$$p \in C \implies \underset{\shortparallel}{\Omega_{C,p}} = \mathcal{O}_{C,p} \underset{A}{\otimes} \Omega_{A/k}$$

$$\Omega_{\mathcal{O}_{C,p}/k}$$

$$m_p/m_p^2 = \Omega_{C,p} \underset{\mathcal{O}_{C,p}}{\otimes} k(p) = \frac{k(p)\,dx \oplus k(p)\,dy}{\left(\frac{\partial f}{\partial x}(p)\,dx + \frac{\partial f}{\partial y}(p)\,dy\right)}.$$

We see that p is non singular iff $\dim_{k(p)} m_p/m_p^2 = 1$

$\iff$ $\mathcal{O}_{C,p}$ is a regular local ring of dimension 1

$\iff$ $\mathcal{O}_{C,p}$ is a DVR.

It follows that C is an open subset of the smooth proj. curve $\tilde{C}$ associated with the function field Frac (A). $\square$

$C \subseteq \tilde{C}$ is open because C contains

$$\{p \in \tilde{C} : \mathrm{ord}_p(x), \mathrm{ord}_p(y) \geq 0\}.$$

Let's look at the projective completion of C:

$\tilde{f}(x_0, x_1, x_2) = x_0^{\deg f} f\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$ homogeneous polynomial of deg $d = \deg(f)$.

$$\tilde{C} = V_+(\tilde{f}) = \{p \in \mathbb{P}^2_k : \tilde{f}(p) = 0\}$$

$$\tilde{C} = \bigcup_{i=0}^{2} \tilde{C} \cap D_+(x_i)$$

Def: A point $p \in \tilde{C}$ is non singular if $\exists\, i \in \{0,1,2\}$ s.t. $p \in \tilde{C} \cap D_+(x_i)$ and it is non singular in the previous sense.

We say that $\tilde{C}$ is smooth (and irreducible) if every point is non singular.

Lemma: If $\tilde{C}$ is smooth, then $\tilde{C}$ is a smooth projective curve.

Proof: We showed that C is contained in $\overline{C}$ the projective smooth curve associated with Frac (A).

$\overline{C}$

$\cup\!|$

$$C \hookrightarrow \mathbb{A}^2 \subseteq \mathbb{P}^2$$

By continuity we have a morphism $\overline{C} \longrightarrow \tilde{C}$.
It is surjective because the image of $\overline{C}$ is closed and $\tilde{C}$ is the closure of C in $\mathbb{P}^2_k$. Since every point of $\tilde{C}$ is non singular, we see that $\overline{C} \to \tilde{C}$ is an isomorphism (it comes from the identity on the function field). $\square$

<u>Exercise:</u> Let $f \in k[x,y]$ irreducible of degree $d \geq 3$ st. $\tilde{C}$ is smooth. Look at the meromorphic differential form

$$\omega = x^i y^j \frac{dx}{\partial f / \partial y} \qquad 0 \leq i+j \leq d-3 .$$

Show that $\mathrm{ord}_p(\omega) \geq 0$ for all $p \in \tilde{C}$. Conclude that the genus of $\tilde{C}$ is $\geq \frac{(d-2)(d-1)}{2}$.

If $\deg(f) = 3$, then $\mathrm{ord}_p\left(\frac{dx}{\partial f/\partial y}\right) = 0$ $\forall p \in \tilde{C}$. Conclude that $\tilde{C}$ has genus 1.