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Exceptional Tannaka groups only arise from cubic threefolds

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Abstract. We show that, under mild assumptions, the Fano surfaces of lines on smooth cubic threefolds are the only smooth subvarieties of abelian varieties whose Tannaka group for the convolution of perverse sheaves is an exceptional simple group. This in particular leads to a considerable strengthening of our previous work on the Shafarevich conjecture. A key idea is to control the Hodge decomposition on cohomology by a cocharacter of the Tannaka group of Hodge modules, and to play this off against an improvement of the Hodge number estimates for irregular varieties by Lazarsfeld–Popa and Lombardi.

1. Introduction

Any subvariety of an abelian variety gives rise to a reductive group via the convolution of perverse sheaves. For smooth subvarieties, these Tannaka groups have recently been used to obtain arithmetic finiteness results for varieties over number fields [25, 28] and the big monodromy criterion in [20]. For both applications, one has to exclude exceptional Tannaka groups. Such exceptional groups are known to occur for Fano surfaces of smooth cubic threefolds, which have Tannaka group E_6 by [23]. We show that, in the range relevant for the above applications, these Fano surfaces are the only examples with exceptional Tannaka groups; this leads to a considerable strengthening of [20, 25]. The key idea is of a general nature and goes far beyond the application in this paper: We upgrade the Tannaka formalism from perverse sheaves to complex Hodge modules in the sense of Sabbah–Schnell [35]. In particular, we prove a comparison theorem between the Tannaka groups for perverse sheaves and for Hodge modules which implies that the Hodge decomposition is defined by a cocharacter of the perverse Tannaka group. This puts a strong restriction on the arising Tannaka groups that will be useful also for other applications.

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1.1. Tannaka groups of subvarieties. Let A be a g-dimensional abelian variety over an algebraically closed field k of characteristic zero, and consider an integral smooth subvariety $X \subset A$ of dimension $d < g$. Our results apply both in an analytic and in an algebraic setup: for complex varieties, we use perverse sheaves with coefficients in $\mathbb{F} = \mathbb{C}$, while for varieties over general fields k, we use ℓ-adic perverse sheaves with coefficients in $\mathbb{F} = \overline{\mathbb{Q}}_\ell$ for a prime ℓ. We define the perverse intersection complex

$$\delta_X := i_* \mathbb{F}_X[d]$$

as the pushforward of the constant sheaf under the closed immersion $i\colon X \hookrightarrow A$, shifted in cohomological degree $-d$ so that it becomes an object of the abelian category $\mathrm{Perv}(A, \mathbb{F})$ of perverse sheaves on A as in [3]. The group law on the abelian variety induces a convolution product on perverse sheaves so that the convolution powers of δ_X generate a neutral Tannaka category $\langle \delta_X \rangle$; see for instance [20, Section 3]. In what follows, we fix a fiber functor

$$\omega\colon \langle \delta_X \rangle \to \mathrm{Vect}(\mathbb{F}).$$

The automorphisms of this fiber functor are represented by a reductive algebraic group

$$G_{X,\omega} := G_\omega(\delta_X) \subset \mathrm{GL}(\omega(\delta_X))$$

which we call the *Tannaka group of the subvariety* $X \subset A$. If X is invariant under translation by a nonzero point of A, then it descends to the quotient of A modulo the subgroup generated by that point. Hence, in what follows, we assume that X is *nondivisible* in the sense that it is not stable under the translation by any nonzero point of A. In this case, the connected component of the identity $G^\circ_{X,\omega} \subset G_{X,\omega}$ acts irreducibly on the representation $\omega(\delta_X)$ by [20, Proposition 3.3]. We are interested in its derived group

$$G^*_{X,\omega} := [G^\circ_{X,\omega}, G^\circ_{X,\omega}],$$

which is a connected semisimple algebraic group. The smoothness of X implies that $\omega(\delta_X)$ is a minuscule representation (see [24, Corollary 1.10], [20, Corollary 5.15]) of dimension equal to the absolute value of the topological Euler characteristic of X,

$$\dim \omega(\delta_X) = |\chi_{\mathrm{top}}(X)|.$$

This leaves only very few possibilities because the Tannaka group is known to be simple in most cases. Indeed, suppose that $g \geqslant 3$ and that the normal bundle of $X \subset A$ is ample. Then, by [20, Theorem A], the group $G^*_{X,\omega}$ fails to be simple if and only if there exist smooth positive-dimensional subvarieties $X_1, X_2 \subset A$ such that the sum morphism induces an isomorphism $X_1 \times X_2 \xrightarrow{\sim} X = X_1 + X_2$. Beyond the standard representations of classical groups, there are only few other minuscule representations of simple algebraic groups: wedge powers of the standard representation of SL_n, (half)spin representations of spin groups, and the smallest irreducible representations of the exceptional groups E_6 and E_7. In [20], we completely characterized wedge powers and ruled out spin representations; the main goal of this paper is to understand when exceptional groups occur.

1.2. Exceptional Tannaka groups. The group E_6 does occur as the Tannaka group of the Fano surface of lines on a smooth cubic threefold:

Example. Let $Y \subset \mathbb{P}^4$ be a smooth cubic threefold. The Fano variety of lines on Y is a smooth projective surface X whose Albanese variety $A = \mathrm{Alb}(X)$ has dimension 5. Moreover, for any chosen base point, the corresponding Albanese morphism is a closed embedding $X \hookrightarrow \mathrm{Alb}(X)$. In what follows, we fix a base point and view X as a subvariety of its Albanese variety. By [23, Theorem 2], then

$$(G^*_{X,\omega}, \omega(\delta_X)) \simeq (E_6, V),$$

where V is one of the two irreducible representations of E_6 with $\dim V = 27$. Via the Tannaka formalism, the trilinear form defining E_6 corresponds to an irreducible fiber of the sum map $X \times X \times X \to A$ of dimension 3 parametrizing triples of coplanar lines in Y. It is clear from the description of X as a Fano variety of lines that the classical Gauss map $X \hookrightarrow \mathrm{Gr}_2(\mathrm{Lie}\, A)$ is an embedding; hence $X \subset A$ is nondivisible. The difference morphism $X \times X \to X - X \subset A$ is generically finite of degree 6 and its fibers are given by classical "double-six" configurations of lines on smooth cubic surfaces [8, Section 13]. Its image is a theta divisor of the principally polarized abelian variety A and permits to recover the cubic threefold (see [8, Theorem 13.4], [2]): the theta divisor has an isolated singularity at the origin and its tangent cone C there can be identified with the cubic threefold Y in suitable linear coordinates on $\mathrm{Lie}\, A$. In the same coordinates, Y is also identified with the image of the projection $\mathbb{P}(T_X) \to \mathbb{P}(\mathrm{Lie}\, A)$.

The goal of this paper is to show that, under mild assumptions, the above is the only example of exceptional Tannaka groups. In particular, in the dimension range relevant for [20, 25], we obtain the following result (see Corollary 7.2).

Theorem A. *Let $X \subset A$ be a smooth irreducible subvariety with ample normal bundle and dimension $< g/2$. Then the following are equivalent.*

(1) *$X \subset A$ is nondivisible with Tannaka group $G^*_{X,\omega} \simeq E_6$.*

(2) *X is isomorphic to the Fano surface of lines on a smooth cubic threefold, and the canonical morphism* $\mathrm{Alb}(X) \to A$ *is an isogeny.*

The Fano surface X of lines on a smooth cubic threefold $Y \subset \mathbb{P}^4$ does not always have ample normal bundle in its Albanese variety: this fails for example for the cubic threefold with affine equation $y^2 - f(x) + z^2 t + t^3 = 0$, where $f(x)$ is a degree 3 polynomial without multiple roots. In fact, it is easy to see that the normal bundle of an Albanese embedding $X \hookrightarrow \mathrm{Alb}(X)$ fails to be ample if and only if there is a hyperplane $H \subset \mathbb{P}^4$ such that the cubic surface $H \cap Y \subset H$ is a cone over an elliptic curve. Such hyperplanes are in bijection with the vertices of the corresponding cones, which are the *Eckardt points* of Y. There are at most finitely many such Eckardt points and their absence is equivalent to ampleness of the cotangent bundle of X. See [34, Section 2] for details.

To include cubic threefolds with Eckardt points, we relax the ampleness of the normal bundle to a weaker positivity property: we say that an integral subvariety $Z \subset A$ of an abelian variety A is *nondegenerate* if, for every surjective morphism of abelian varieties $\varphi: A \to B$, we have $\dim \varphi(Z) = \min\{\dim Z, \dim B\}$. The Fano surface of lines on any smooth cubic threefold is nondegenerate inside its Albanese variety by [39, Theorem 1], since it is a summand of a theta divisor; in fact, [39, Theorem 1] refers to a stronger notion of nondegeneracy, which implies the above one by [33, Lemma II.12]. Our proof of Theorem A proceeds by reduction to the case of surfaces. In this case, we only need to assume nondegeneracy; hence we obtain the following result which now includes the Fano surfaces of *all* smooth cubic threefolds (see Theorem 7.1).

Theorem B. *Suppose $g \geqslant 5$. For any smooth irreducible surface $X \subset A$, the following three properties are equivalent.*

(1) *$X \subset A$ is nondivisible and nondegenerate with Tannaka group $G^*_{X,\omega} \simeq E_6$.*

(2) *$X \subset A$ is nondivisible and nondegenerate with*

$$\chi(X, \mathcal{O}_X) = 6, \quad c_2(X) = 27,$$

the difference morphism $X \times X \to X - X$ has generic degree $\geqslant 6$, and the sum morphism $X \times X \times X \to A$ has an irreducible fiber of dimension $\geqslant 3$.

(3) *X is isomorphic to the Fano surface of lines on a smooth cubic threefold, and the canonical morphism $\mathrm{Alb}(X) \to A$ is an isogeny.*

A different characterization has been obtained by Casalaina-Martin, Popa and Schreieder in [7, Theorem 6.1]: they show that if X is a subvariety of minimal cohomology class of a principally polarized abelian fivefold A and $X - X$ is a theta divisor, then X is the Fano surface of lines on a smooth cubic threefold. Note that, in both Theorems A and B, the canonical morphism $\mathrm{Alb}(X) \to A$ need not be an isomorphism.

Example. Let X be the Fano surface of lines on a smooth cubic threefold, and let

$$\Theta = X - X \subset \mathrm{Alb}(X).$$

be its theta divisor. The upper bound on the number of 2-torsion points on theta divisors in [32] shows that, for any theta divisor on a principally polarized abelian variety, there exists a 2-torsion point outside that theta divisor. Applying this to our case, we can find a point $a \in \mathrm{Alb}(X)$ of order two with $a \notin \Theta = X - X$. We then have $X \cap (X + a) = \varnothing$ and it follows that the isogeny

$$p\colon \mathrm{Alb}(X) \to A := \mathrm{Alb}(X)/\langle a \rangle$$

maps $X \subset \mathrm{Alb}(X)$ isomorphically onto its image $\bar{X} := p(X) \subset A$. So this image is still an integral smooth nondivisible subvariety. The pushforward of perverse sheaves under an isogeny does not change the derived connected component of their Tannaka groups [20, Corollary 3.5], so we still have

$$(G^*_\omega(\bar{X}), \omega(\delta_{\bar{X}})) \simeq (E_6, V)$$

although the canonical morphism $p\colon \mathrm{Alb}(\bar{X}) \to A$ is not an isomorphism.

1.3. The Hodge filtration. A key ingredient in our proof of Theorems A and B is the theory of complex Hodge modules by Sabbah–Schnell [35]. To explain the idea, let A be a complex abelian variety and $\mathrm{HM}(A)$ the category of direct sums of complex Hodge modules of arbitrary weight on it; see Section 2 for details. The category $\mathrm{HM}(A)$ is semisimple $\mathbb{C}$-linear abelian and comes equipped with a faithful exact $\mathbb{C}$-linear functor

$$\mathrm{DR}\colon \mathrm{HM}(A) \to \mathrm{Perv}(A, \mathbb{C}).$$

To any $M \in \mathrm{HM}(A)$, one may attach a Tannaka group $G_\omega(M)$ in the same way as for perverse sheaves; in Corollary 2.5, we show a comparison theorem between this group and the Tannaka group $G_\omega(P)$ of the perverse sheaf $P = \mathrm{DR}(M)$. In particular, the derived groups of their connected components of the identity are the same: $G_\omega^*(M) = G_\omega^*(P)$. This allows us to enrich with Hodge theoretic data all the previous Tannaka constructions for perverse sheaves, so for the rest of this section, we will work only with Hodge modules. It is now time to be more specific about our choice of fiber functors. Fix a Hodge module $M \in \mathrm{HM}(A)$, and let L be a unitary local system of rank one on A with the property that all perverse subquotients of $\mathrm{DR}(M) \otimes L$ have their cohomology concentrated in degree zero; there are plenty of such local systems by generic vanishing for perverse sheaves on abelian varieties [27,37]. Then the functor

$$\omega\colon \langle M \rangle \to \mathrm{Vect}(\mathbb{C}), \quad N \mapsto \mathrm{H}^0(A, \mathrm{DR}(N) \otimes L)$$

is a fiber functor. Now L underlies a complex variation of Hodge structures $\mathbb{L}$, so from the natural identification $\mathrm{DR}(M) \otimes L = \mathrm{DR}(M \otimes \mathbb{L})$, we obtain a Hodge decomposition

$$\mathrm{H}^0(A, \mathrm{DR}(M) \otimes L) = \bigoplus_{(p,q) \in \mathbb{Z}^2} \mathrm{H}^{p,q}(M \otimes \mathbb{L})$$

We show that this Hodge decomposition is compatible with the convolution product, which allows to enrich ω to a tensor functor $\omega^H\colon \langle M \rangle \to \mathrm{Vect}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C})$ with values in the category of finite-dimensional bigraded complex vector spaces. This leads to the following result (see Corollary 3.2).

Theorem C. *There is a natural morphism $\lambda\colon \mathbb{G}_m^2 \to G_\omega(M)$ such that the Hodge decomposition*

$$\omega^H(M) = \bigoplus_{(p,q) \in \mathbb{Z}^2} \mathrm{H}^{p,q}(M \otimes \mathbb{L})$$

is the decomposition in weight spaces for the characters $(p,q) \in \mathbb{Z}^2 = \mathrm{Hom}(\mathbb{G}_m^2, \mathbb{G}_m)$.

We call λ the *Hodge cocharacter* of M. The fact that this cocharacter takes values in the subgroup $G_\omega^*(M) \subset \mathrm{GL}(\omega(M))$ imposes strong restrictions on the possible Hodge filtrations if the subgroup is known to be small. This in particular applies to the Tannaka group of smooth subvarieties $X \subset A$: the perverse intersection complex $\delta_X \in \mathrm{Perv}(A, \mathbb{C})$ naturally lifts to a complex Hodge module $\delta_X^H \in \mathrm{HM}(A)$ of weight $d = \dim X$ and its Hodge numbers are

$$\dim \mathrm{H}^{p,d-p}(\delta_X^H \otimes \mathbb{L}) = (-1)^{d-p} \chi(X, \Omega_X^p);$$

see Section 3.2. Comparing this with an improvement to the Hodge estimates by Lazarsfeld–Popa [30] and Lombardi [31] explained in Appendix A, we obtain the following result (see Proposition 4.6).

Theorem D. *Let $X \subset A$ be a smooth nondivisible irreducible subvariety with ample normal bundle and dimension d such that $G_{X,\omega}^*$ is a simple exceptional group. Then either*

(1) $G_{X,\omega}^* \simeq E_6$, $|\chi_{\mathrm{top}}(X)| = 27$ *and* $d \in \{2,4,6\}$, *or*

(2) $G_{X,\omega}^* \simeq E_7$, $|\chi_{\mathrm{top}}(X)| = 56$ *and* $d \in \{3,\dots,15\}$ *is odd.*

Moreover, $g \leqslant g_{\max}$ for the following upper bound $g_{\max}$ depending on d:

d	2	4	6
$g_{\max}$	7	6	8

(1)

d	3	5	7	9	11	13	15
$g_{\max}$	9	10	12	13	15	16	18

(2)

In each case, we also know the Hodge numbers $h^p(X) := (-1)^{d-p}\chi(X, \Omega_X^p)$, as they must be among those listed in Propositions 4.3 and 4.5. This essentially reduces the proof of Theorem A to the case of surfaces treated in Theorem B. For the proof of the latter, we apply the representation theory of E_6 to show that the difference morphism

$$X \times X \to D := X - X \subset A$$

is generically finite of degree $\geqslant 6$ over its image (see Section 4.6), and then use this numerical information to show by direct geometric arguments that the projective tangent cone to $D \subset A$ at the origin is a smooth cubic threefold whose associated Fano surface of lines is isomorphic to X (see Section 7).

1.4. No country for E_7. Although Theorem D imposes strong restrictions, it is not enough to rule out completely the existence of subvarieties X with $G^*_{X,\omega} \simeq E_7$ in the dimension range $d < g/2$. With the E_6 case in mind, one might think that a natural candidate for such an X would be the Fano variety of lines on a quartic double solid, that is, a double cover of $\mathbb{P}^3$ branched along a smooth quartic. Indeed, the (-1)-curves on a del Pezzo surface of degree 3 and 2 give rise to a root system of type E_6 and E_7 respectively. A cubic threefold can be thought of as a one-parameter family of del Pezzo surfaces of degree 3 since these are cubic surfaces. Similarly, a quartic double solid is a one-parameter family of del Pezzo surfaces of degree 2 since the anticanonical divisor realizes them as double covers of $\mathbb{P}^2$ branched along smooth quartic curves.

However, the pieces of the jigsaw puzzle do not quite match together in the E_7 case: for instance, lines on a quartic double solid are parametrized by a surface, while Theorem D states that the sought-for example must be a threefold. This parity issue is hardwired in the representation theory of E_7 and lies at the heart of the following negative result (see Corollary 8.2).

Theorem E. *For any smooth irreducible subvariety $X \subset A$ with ample normal bundle and dimension $< g/2$, we have $G^*_{X,\omega} \not\simeq E_7$.*

The proof of this will be given in Section 8. It does not require Hodge modules and is largely independent from the rest of this paper: The idea is to control the decomposition of the tensor square $\delta_X * \delta_X$ by looking at the fibers of the sum morphism $X \times X \to A$ and using Kashiwara's estimate for characteristic varieties of direct images. Once we know how the tensor square decomposes, we can then conclude by a version of Larsen's alternative.

1.5. Big monodromy. As an application of the above results, we can significantly strengthen the big monodromy criterion in [20]. More precisely, let S be a smooth complex variety. Let A be a complex abelian variety of dimension g and $\mathcal{X} \subset A_S := A \times S$ a subvariety such that the projection $f\colon \mathcal{X} \to S$ is a smooth morphism with connected fibers of dimension d. The situation is summarized in the following diagram:

$$\begin{array}{ccccc} & \overset{\pi}{\swarrow} & \mathcal{X} & \overset{f}{\searrow} & \\ & & \Vert & & \\ A & \xleftarrow{\mathrm{pr}_A} & A_S & \xrightarrow{\mathrm{pr}_S} & S. \end{array}$$

For a character $\chi\colon \pi_1(A,0) \to \mathbb{C}^\times$, let L_χ denote the associated rank-one local system on A. Given an n-tuple of such characters $\underline{\chi} = (\chi_1, \dots, \chi_n)$, we consider the local system

$$V_{\underline{\chi}} := R^d f_* \pi^* L_{\underline{\chi}}, \quad \text{where } L_{\underline{\chi}} := L_{\chi_1} \oplus \cdots \oplus L_{\chi_n}.$$

The fiber at s of this local system comes with a linear action of the group $\pi_1(S,s)$ via the monodromy representation. With terminology as in [20], we obtain the following strengthening of the big monodromy result in [20, Section 1.1].

Big monodromy criterion. *Suppose $X := \mathcal{X}_{\bar{\eta}} \subset A_{S,\bar{\eta}}$ has ample normal bundle and dimension $d < (g-1)/2$. If $\mathcal{X}$ is symmetric up to translation, assume that* (1.1) *below holds. Then the following are equivalent:*

(1) *X is nondivisible, not constant up to translation, not a symmetric power of a curve and not a product;*

(2) *$V_{\underline{\chi}}$ has big monodromy for most torsion n-tuples of characters $\underline{\chi}$.*

Here $\mathcal{X}$ is said to be *symmetric up to translation* if there is a morphism $a\colon S \to A$ with the property that $\mathcal{X}_t = a(t) - \mathcal{X}_t$ for all $t \in S(\mathbb{C})$. Only in this case, we need to exclude some very specific numerics by assuming that the topological Euler characteristic $e = \chi_{\mathrm{top}}(X)$ of X has absolute value

$$\text{(1.1)} \qquad |e| \neq 2^{2m-1} \quad \text{if } d \geqslant (g-1)/4 \text{ and } m \in \{3, \dots, d\},\ m \equiv d \text{ modulo } 2.$$

This assumption is empty for $d < (g-1)/4$. Apart from curves, in [20], we had no control on the dimension and irregularity of varieties with exceptional Tannaka group, so we assumed $|e| \neq 27, 56$ for all (d,g) in the given range. Theorems A and E now permit to remove these assumptions completely. For applications outside this range, we only need to exclude the finite list of (d,g) in Theorem D.

Conventions. A *variety* over a field k is a separated k-scheme of finite type, and a *subvariety* is a closed subvariety. For a vector bundle $\mathcal{V}$ on a variety, we denote by $\mathbb{P}(\mathcal{V})$ the projective bundle of lines in $\mathcal{V}$.

2. Tannaka groups of Hodge modules

In this section, we introduce Tannaka categories of Hodge modules on complex abelian varieties and relate their Tannaka groups to those for perverse sheaves.

2.1. Complex Hodge modules. For a complex manifold X and $w \in \mathbb{Z}$, we denote by $\mathrm{HM}(X,w)$ the category of polarizable complex Hodge modules of weight w on X in the sense of [35, Definition 14.2.2]. This is a semisimple abelian $\mathbb{C}$-linear category, and it comes with a faithful exact $\mathbb{C}$-linear functor $\mathrm{DR}\colon \mathrm{HM}(X,w) \to \mathrm{Perv}(X)$ to the category $\mathrm{Perv}(X)$ of perverse sheaves on X with complex coefficients. We denote by

$$\mathrm{HM}(X) := \bigoplus_{w\in\mathbb{Z}} \mathrm{HM}(X,w)$$

the category whose objects are the formal direct sums $M = \bigoplus_{w\in\mathbb{Z}} M_w$ of Hodge modules $M_w \in \mathrm{HM}(X,w)$ such that $M_w = 0$ for all but finitely many weights w, with morphisms

$$\mathrm{Hom}_{\mathrm{HM}(X)}(M,N) := \bigoplus_{w\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{HM}(X,w)}(M_w,N_w).$$

This is again a semisimple abelian $\mathbb{C}$-linear category with a faithful exact $\mathbb{C}$-linear functor

$$\mathrm{DR}\colon \mathrm{HM}(X) \to \mathrm{Perv}(X), \quad M \mapsto \bigoplus_{w\in\mathbb{Z}} \mathrm{DR}(M_w).$$

Example 2.1. If $X = \{\mathrm{pt}\}$ is a point, then $\mathrm{HM}(\{\mathrm{pt}\})$ is the category $\mathrm{Vect}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C})$ of finite-dimensional bigraded complex vector spaces

$$V = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$$

and $\mathrm{HM}(\{\mathrm{pt}\},w)$ is its subcategory of *pure* $\mathbb{C}$*-Hodge structures of weight* w, by which we mean those bigraded vector spaces that satisfy $V^{p,q} = 0$ for all $p,q\in\mathbb{Z}$ with $p+q \neq w$. Moreover,

$$\mathrm{DR}\colon \mathrm{HM}(\{\mathrm{pt}\}) = \mathrm{Vect}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C}) \to \mathrm{Perv}(\{\mathrm{pt}\}) = \mathrm{Vect}(\mathbb{C})$$

is the functor that forgets the grading.

By a *variation of* $\mathbb{C}$*-Hodge structures* of weight w on X, we mean a $\mathcal{C}^\infty$-bundle $\mathcal{V}$ together with a flat connection and a decomposition into $\mathcal{C}^\infty$-subbundles

$$\mathcal{V} = \bigoplus_{p+q=w} \mathcal{V}^{p,q}$$

that satisfies Griffiths transversality. In what follows, we only consider *polarizable* variations of $\mathbb{C}$-Hodge structures in the sense of [35, Definition 4.1.9]; by [35, Theorem 14.6.1], any such polarizable variation of $\mathbb{C}$-Hodge structures can be viewed as an object of $\mathrm{HM}(X, w+\dim X)$. More generally, for any integral subvariety $Z \subset X$ and any pure polarizable variation of $\mathbb{C}$-Hodge structures $\mathbb{V}$ of weight w on an open dense $U \subset Z^{\mathrm{reg}}$, there is a unique simple Hodge module $M = \mathrm{IC}_Z(\mathbb{V}) \in \mathrm{HM}(X, w+\dim Z)$ with $M_{|U} = \mathbb{V}$. Conversely, every simple Hodge module $M \in \mathrm{HM}(X)$ arises like this for a unique germ of a simple pure polarizable variation of $\mathbb{C}$-Hodge structures on a dense open subset of an integral subvariety of X (see [35, Theorem 16.2.1]).

Example 2.2. For any integral subvariety $Z \subset X$, the trivial variation $\mathbb{C}_U$ of Hodge structures of rank one and weight zero on the smooth locus $U = Z^{\mathrm{reg}}$ defines a pure Hodge module $\delta_Z^H = \mathrm{IC}_Z(\mathbb{C}_U) \in \mathrm{HM}(X,d)$ of weight $d = \dim Z$. Its image under the functor DR is the perverse intersection complex $\delta_Z := \mathrm{DR}(\delta_Z^H) \in \mathrm{Perv}(X)$. If Z is smooth, then this intersection complex is given by $\delta_Z = \mathbb{C}_Z[d]$.

2.2. Hodge structure on cohomology. Passing from perverse sheaves to Hodge modules will enrich the cohomology groups with $\mathbb{C}$-Hodge structures. Indeed, with the notation of [35, Definition 12.7.28], we have for any proper morphism $f\colon X \to Y$ direct image functors

$$ {}_T f_*^{(k)}\colon \mathrm{HM}(X) \to \mathrm{HM}(Y) $$

for $k \in \mathbb{Z}$. These lift the perverse direct image functors ${}^p\mathcal{H}^k \circ Rf_*$ in the sense that the following diagram commutes:

$$ \begin{array}{ccc} \mathrm{HM}(X) & \xrightarrow{\;{}_T f_*^{(k)}\;} & \mathrm{HM}(Y) \\ \downarrow{\scriptstyle \mathrm{DR}} & & \downarrow{\scriptstyle \mathrm{DR}} \\ \mathrm{Perv}(X) & \xrightarrow{\;{}^p\mathcal{H}^k \circ \mathrm{R}f_*\;} & \mathrm{Perv}(Y). \end{array} $$

Taking $f\colon X \to Y = \{\mathrm{pt}\}$, we see that, for any $M \in \mathrm{HM}(X)$, the groups $\mathrm{H}^k(X, \mathrm{DR}(M))$ come with a natural bigrading. We call this the *Hodge decomposition* and denote it by

$$ \mathrm{H}^k(X, \mathrm{DR}(M)) = \bigoplus_{p,q \in \mathbb{Z}} \mathrm{H}^{p,q}(M). $$

Note that this is a pure $\mathbb{C}$-Hodge structure of weight $w + k$ if $M \in \mathrm{HM}(X, w)$.

2.3. Tannaka categories. Now let $X = A$ be an abelian variety. In this case, every perverse sheaf has nonnegative Euler characteristic by [15, Corollary 1.4], so inside the abelian category of perverse sheaves, those of Euler characteristic zero form a Serre subcategory

$$ \mathrm{S}_{\mathrm{Perv}}(A) := \{P \in \mathrm{Perv}(A) \mid \chi(A, P) = 0\} \subset \mathrm{Perv}(A). $$

The objects of this subcategory are called *negligible* perverse sheaves. On the abelian quotient category $\overline{\mathrm{Perv}}(A) = \mathrm{Perv}(A)/\mathrm{S}_{\mathrm{Perv}}(A)$, the group law $\sigma\colon A \times A \to A$ gives rise to a convolution product

$$ *\colon \overline{\mathrm{Perv}}(A) \times \overline{\mathrm{Perv}}(A) \to \overline{\mathrm{Perv}}(A), \quad P_1 * P_2 := {}^p\mathcal{H}^0(R\sigma_*(P_1 \boxtimes P_2)) $$

making $\overline{\mathrm{Perv}}(A)$ a neutral Tannaka category as recalled in [20, Section 3.1]. For any fiber functor $\omega\colon \overline{\mathrm{Perv}}(A) \to \mathrm{Vect}(\mathbb{C})$, the dimension of objects is the Euler characteristic

$$ \dim_{\mathbb{C}} \omega(P) = \chi(A, P); \tag{2.1} $$

see [27, proof of Corollary 4.2]. The same constructions work also for the category of Hodge modules. Consider the Serre subcategory

$$ \mathrm{S}_{\mathrm{HM}}(A) := \{M \in \mathrm{HM}(A) \mid \chi(A, \mathrm{DR}(M)) = 0\} \subset \mathrm{HM}(A). $$

Then the functor DR descends to an exact faithful $\mathbb{C}$-linear functor on the abelian quotient category

$$ \mathrm{DR}\colon \overline{\mathrm{HM}}(A) := \mathrm{HM}(A)/\mathrm{S}_{\mathrm{HM}}(A) \to \overline{\mathrm{Perv}}(A), $$

and this functor naturally underlies a tensor functor with respect to the convolution product

$$ *\colon \overline{\mathrm{HM}}(A) \times \overline{\mathrm{HM}}(A) \to \overline{\mathrm{HM}}(A), \quad M_1 * M_2 := {}_T\sigma_*^{(0)}(M_1 \boxtimes M_2). $$

So, for $M_1, M_2 \in \overline{\mathrm{HM}}(A)$, we have isomorphisms

$$\mathrm{DR}(M_1 * M_2) \xrightarrow{\sim} \mathrm{DR}(M_1) * \mathrm{DR}(M_2)$$

compatible with the associativity, commutativity and unit of the respective tensor categories; note that, to avoid sign issues in the commutativity, we will always use right $\mathcal{D}$-modules for the definition of $\mathrm{HM}(A)$.

Corollary 2.3. $\overline{\mathrm{HM}}(A)$ *is again a neutral Tannaka category.*

Proof. Pick any fiber functor on the neutral Tannaka category $\overline{\mathrm{Perv}}(A)$. Composing it with the faithful exact $\mathbb{C}$-linear functor DR, which is a tensor functor by the above, we get a fiber functor on $\overline{\mathrm{HM}}(A)$. □

2.4. An exact sequence of Tannaka groups. We want to compare the Tannaka groups for Hodge modules with those for the underlying perverse sheaves. For the rest of this section, we fix a full abelian tensor subcategory $\mathcal{C} \subset \overline{\mathrm{HM}}(A)$ and denote by $\mathrm{DR}(\mathcal{C}) \subset \overline{\mathrm{Perv}}(A)$ the smallest full abelian tensor subcategory containing all subquotients of $\mathrm{DR}(M)$ for $M \in \mathcal{C}$. Suppose that, on this subcategory, we are given a fiber functor, i.e. a faithful exact $\mathbb{C}$-linear tensor functor $\omega\colon \mathrm{DR}(\mathcal{C}) \to \mathrm{Vect}(\mathbb{C})$. Precomposing with the exact faithful $\mathbb{C}$-linear tensor functor $\mathrm{DR}\colon \mathcal{C} \to \mathrm{DR}(\mathcal{C})$, we get a fiber functor $\omega \circ \mathrm{DR}$ on the original category,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ \omega\circ\mathrm{DR}\ } & \mathrm{Vect}(\mathbb{C}). \\ & {\scriptstyle \mathrm{DR}}\searrow \quad \nearrow{\scriptstyle \omega} & \\ & \mathrm{DR}(\mathcal{C}) & \end{array}$$

Let

$$G_\omega(\mathcal{C}) = \mathrm{Aut}^{\otimes}(\omega \circ \mathrm{DR}) \quad \text{and} \quad G_\omega(\mathrm{DR}(\mathcal{C})) = \mathrm{Aut}^{\otimes}(\omega)$$

be the corresponding Tannaka groups. It follows from the definition of the category $\mathrm{DR}(\mathcal{C})$ that DR induces an embedding $G_\omega(\mathrm{DR}(\mathcal{C})) \hookrightarrow G_\omega(\mathcal{C})$ as a closed subgroup [11, Proposition 2.21b]. In fact, this subgroup is normal. To understand this, let

$$\mathcal{C}_{\{0\}} := \{M \in \mathcal{C} \mid \mathrm{Supp}(M) = \{0\}\} \subset \mathcal{C}$$

be the full subcategory of objects in $\mathcal{C}$ supported at the origin; it can be seen as a tensor subcategory of the category of bigraded vector spaces via Example 2.1; hence its Tannaka group

$$G_\omega(\mathcal{C}_{\{0\}}) := \mathrm{Aut}^{\otimes}(\omega \circ \mathrm{DR}\,|\mathcal{C}_{\{0\}})$$

is a quotient of a torus of rank two; in particular, it is a torus. We then have the following Hodge theoretic analog of the Galois sequence in [20, Theorem 4.3].

Proposition 2.4. *We have an exact sequence*

$$1 \to G_\omega(\mathrm{DR}(\mathcal{C})) \to G_\omega(\mathcal{C}) \to G_\omega(\mathcal{C}_{\{0\}}) \to 1.$$

Proof. We have an embedding $\mathcal{C}_{\{0\}} \hookrightarrow \mathcal{C}$ as a full abelian tensor subcategory which is stable under subobjects. By Tannaka duality, any such embedding corresponds to an epimorphism $G_\omega(\mathcal{C}) \twoheadrightarrow G_\omega(\mathcal{C}_{\{0\}})$ (see [11, Proposition 2.21a]). The exactness of the sequence of Tannaka groups then follows by the same argument as in the proof of [20, Theorem 4.3]. Indeed, by [9, Proposition A.13 and Lemma A.4 (1)], we only need to show that, for any $M \in \mathcal{C}$, the following two properties hold.

(1) Every rank-one subobject of $\mathrm{DR}(M)$ is a direct summand in a semisimple object $\mathrm{DR}(N)$ with $N \in \mathcal{C}$.

(2) The maximal trivial subobject of $\mathrm{DR}(M)$ lies in the essential image of the functor

$$\mathrm{DR}\colon \langle \mathcal{C} \rangle \to \mathrm{DR}(\mathcal{C}).$$

Both properties are trivial in this case: by [41, Section 10] or [24, Lemma 3.5], the rank-one objects of the tensor category $\mathcal{C}$ are precisely those whose underlying perverse sheaf is a skyscraper sheaf of rank one, and the maximal trivial subobject is the direct sum of all skyscraper subsheaves supported at the origin. Hence both claims follow from the fact that, for any $M \in \mathrm{HM}(A)$, every simple subobject of the perverse sheaf $\mathrm{DR}(M)$ (or equivalently, of the underlying holonomic $\mathcal{D}$-module) has the property that a direct sum of twists of that subobject underlies a Hodge submodule of M (see [35, Proposition 16.3.1]). □

For $M \in \mathcal{C}$ with underlying perverse sheaf $P = \mathrm{DR}(M)$, let $V = \omega(P)$. Then the groups

$$G_\omega(M) := \mathrm{Im}(G_\omega(\mathcal{C}) \to \mathrm{GL}(V)),$$
$$G_\omega(P) := \mathrm{Im}(G_\omega(\mathrm{DR}(\mathcal{C})) \to \mathrm{GL}(V))$$

can be identified with the Tannaka groups of the full tensor subcategories

$$\langle M \rangle \subset \mathcal{C} \quad \text{and} \quad \langle P \rangle \subset \mathrm{DR}(\mathcal{C})$$

generated by M and P respectively. These Tannaka groups are algebraic groups; by construction, they are subgroups of $\mathrm{GL}(V)$. They are reductive since all our Hodge modules and perverse sheaves are semisimple. So the derived group of their connected component of the identity is a connected semisimple group which we denote by

$$G^*_\omega(-) := [G^\circ_\omega(-), G^\circ_\omega(-)].$$

The groups for Hodge modules and for perverse sheaves are then related as follows.

Corollary 2.5. *For $M \in \mathcal{C}$, consider the perverse sheaf $P = \mathrm{DR}(M)$. Let $V = \omega(P)$ as above. Then*

$$G_\omega(P) \subset G_\omega(M) \subset N_{\mathrm{GL}(V)}(G_\omega(P)) \quad \text{and} \quad G^*_\omega(P) = G^*_\omega(M).$$

Proof. Applying Proposition 2.4 to the tensor category $\mathcal{C} = \langle M \rangle$ generated by M, we get an exact sequence $1 \to G_\omega(P) \to G_\omega(M) \to G_\omega(\mathcal{C}_{\{0\}}) \to 1$. So $G_\omega(P) \trianglelefteq G_\omega(M)$ is a normal subgroup. Since both are subgroups of $\mathrm{GL}(V)$, this implies the claim about the containment in the normalizer. For the claim about the derived groups of the connected component, recall that $G_\omega(\mathcal{C}_{\{0\}})$ is a torus, so the reductive Lie algebras $\mathrm{Lie}\, G_\omega(M)$ and $\mathrm{Lie}\, G_\omega(P)$ have the same maximal semisimple Lie subalgebra; hence $G_\omega(M)^\circ$ and $G_\omega(P)^\circ$ share the same derived subgroup. □

2.5. Character twists. Any unitary local system L of rank one on A naturally underlies a unique pure polarizable variation $\mathbb{L}$ of Hodge structures of weight zero, concentrated in bidegree $(0,0)$. We then have an endofunctor $\mathrm{HM}(A) \to \mathrm{HM}(A)$, $M \mapsto M \otimes \mathbb{L}$ which descends to the quotient categories modulo negligibles, where $M \in \mathrm{HM}(A)$ is called *negligible* if $\mathrm{DR}(M)$ is a negligible perverse sheaf in the sense of Section 2.3. The induced functor $\overline{\mathrm{HM}}(A) \to \overline{\mathrm{HM}}(A)$ is a tensor functor with respect to the convolution product: for $M_1, M_2 \in \overline{\mathrm{HM}}(A)$, we have natural isomorphisms

$$\begin{aligned}(M_1 * M_2) \otimes \mathbb{L} &\simeq {}_T\sigma_*^{(0)}(M_1 \boxtimes M_2) \otimes \mathbb{L} \\ &\simeq {}_T\sigma_*^{(0)}((M_1 \boxtimes M_2) \otimes \sigma^{-1}(\mathbb{L})) \quad \text{by the projection formula} \\ &\simeq {}_T\sigma_*^{(0)}((M_1 \otimes \mathbb{L}) \boxtimes (M_2 \otimes \mathbb{L})) \quad \text{since } \sigma^{-1}(\mathbb{L}) \simeq \mathbb{L} \boxtimes \mathbb{L} \\ &\simeq (M_1 \otimes \mathbb{L}) * (M_2 \otimes \mathbb{L}).\end{aligned}$$

Let $\mathcal{C}$ and ω be as in the previous section. Then, up to noncanonical isomorphism, the Tannaka group of a Hodge module in $\mathcal{C}$ does not change under twists.

Lemma 2.6. *For $M \in \mathcal{C}$ and $\mathbb{L}$ a unitary rank-one local system with $M \otimes \mathbb{L} \in \mathcal{C}$, we have $G_\omega(M \otimes \mathbb{L}) \simeq G_\omega(M)$.*

Proof. As in [27, Proposition 4.1], twisting by $\mathbb{L}$ gives rise to an equivalence of tensor categories

$$\varphi\colon \langle M \rangle \xrightarrow{\sim} \langle M \otimes \mathbb{L} \rangle, \quad N \mapsto N \otimes \mathbb{L}.$$

Both the source and the target of this equivalence are tensor subcategories of $\mathcal{C}$; hence ω restricts to a fiber functor on both of them. The equivalence φ need not be compatible with these fiber functors, but over an algebraically closed field of characteristic zero, any two fiber functors are noncanonically isomorphic; hence the same holds for the corresponding Tannaka groups. □

3. The Hodge cocharacter

We now show that the Hodge decomposition on cohomology is induced by a cocharacter of the Tannaka group, and we gather some general estimates for the Hodge level of intersection complexes that will be useful later.

3.1. The Hodge decomposition. Let $\mathrm{Perv}_0(A) \subset \mathrm{Perv}(A)$ be the full subcategory of all perverse sheaves P with the property that all the perverse subquotients Q of P satisfy

$$\mathrm{H}^i(A, Q) = 0 \quad \text{for all } i \neq 0.$$

As in [20, Section 4.3], its image $\overline{\mathrm{Perv}}_0(A) \subset \overline{\mathrm{Perv}}(A)$ is a full abelian tensor subcategory which is equivalent to $\mathrm{Perv}_0(A)/S_0(A)$, where $\mathrm{S}_0(A) \subset \mathrm{Perv}_0(A)$ is the full subcategory of perverse sheaves P with the property that all the subquotients Q of P satisfy $H^\bullet(A, Q) = 0$. We then get an exact functor

$$\omega_0\colon \overline{\mathrm{Perv}}_0(A) = \mathrm{Perv}_0(A)/S_0(A) \to \mathrm{Vect}(\mathbb{C}), \quad Q \mapsto H^0(A, Q).$$

This is a tensor functor. In particular, for all perverse sheaves $P_1, P_2 \in \overline{\mathrm{Perv}}_0(A)$, we have natural isomorphisms

$$\vartheta_{P_1,P_2}\colon \omega_0(P_1 * P_2) \xrightarrow{\sim} \omega_0(P_1) \otimes \omega_0(P_2)$$

which are defined as follows: let $f\colon A \to \{\mathrm{pt}\}$ be the map to a point; then

$$\omega_0(P) = \mathcal{H}^0(\mathrm{R}f_*(P)) \quad \text{for all } P \in \mathrm{Perv}_0(A).$$

Now, the complex $\mathrm{R}f_*(P)$ has cohomology at most in degree 0, so we identify it with $\omega(P)$. With this in mind, using the Künneth isomorphism and the fact that $f \circ \sigma = f \times f$, we obtain

$$\begin{aligned}\omega_0(P_1 * P_2) &\simeq Rf_*(P_1 * P_2)\\ &\simeq Rf_* R\sigma_*(P_1 \boxtimes P_2)\\ &\simeq R(f \times f)_*(P_1 \boxtimes P_2)\\ &\simeq Rf_*(P_1) \otimes Rf_*(P_2)\\ &\simeq \omega_0(P_1) \otimes \omega_0(P_2).\end{aligned}$$

These constructions can be upgraded to Hodge modules. Let $\mathrm{HM}_0(A) \subset \mathrm{HM}(A)$ be the full subcategory of all Hodge modules M with $\mathrm{DR}(M) \in \mathrm{Perv}_0(A)$, and let $\overline{\mathrm{HM}}_0(A) \subset \overline{\mathrm{HM}}(A)$ be its image in the Tannaka category from Section 2.3. Then ω_0 induces the functor

$$\omega_0^H\colon \overline{\mathrm{HM}}_0(A) \to \mathrm{Vect}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C}), \quad M \mapsto \mathrm{H}^0(A, \mathrm{DR}(M)),$$

where the bigrading on the values of the functor is the Hodge decomposition from Section 2.2,

$$\mathrm{H}^0(A, \mathrm{DR}(M)) = \bigoplus_{p,q\in\mathbb{Z}} \mathrm{H}^{p,q}(M).$$

Proposition 3.1. *For $M_1, M_2 \in \overline{\mathrm{HM}}_0(A)$, the isomorphism $\vartheta_{\mathrm{DR}(M_1),\mathrm{DR}(M_2)}$ is compatible with the Hodge decomposition. Hence the functor ω_0^H is a tensor functor.*

Proof. The construction is the same as for perverse sheaves, but working on the level of Hodge modules. For a morphism g, we denote by ${}_Tg_*$ the pushforward functor of [35, Definition 12.7.35], which replaces the derived pushforward Rg_* of constructible sheaves. As above, let $f\colon A \to \{\mathrm{pt}\}$ be the map to a point; then

$$\omega_0^H(M) = {}_Tf_*^{(0)}(M) \quad \text{for all } M \in \overline{\mathrm{HM}}_0(A).$$

As above, the complex of bigraded vector space ${}_Tf_*(M)$ has cohomology at most in degree 0, so we identify it with $\omega_0^H(M)$. Let $\sigma\colon A \times A \to A$ be the group law. Then $f \circ \sigma = f \times f$; hence the Leray spectral sequence [35, Corollary 12.7.38] gives ${}_Tf_* \circ {}_T\sigma_* = {}_T(f \times f)_*$. Thus, in $\mathrm{HM}(\{\mathrm{pt}\}) = \mathrm{Vect}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C})$, we obtain

$$\begin{aligned}\omega_0^H(M_1 * M_2) &\simeq {}_Tf_*(M_1 * M_2)\\ &\simeq {}_Tf_{*T}\sigma_*(M_1 \boxtimes M_2)\\ &\simeq {}_T(f \times f)_*(M_1 \boxtimes M_2)\\ &\simeq {}_Tf_*(M_1) \otimes {}_Tf_*(M_2)\\ &\simeq \omega_0^H(M_1) \otimes \omega_0^H(M_2),\end{aligned}$$

where in the third step we have used the Künneth isomorphism. □

The above can be applied to any Hodge module $M \in \overline{\mathrm{HM}}(A)$ after twisting by a suitable rank-one local system. More precisely, by generic vanishing on abelian varieties [27, 37], we can find a unitary local system L of rank one on A such that $M \otimes \mathbb{L} \in \overline{\mathrm{HM}}_0(A)$. Fixing M and $\mathbb{L}$, we obtain a tensor functor

$$\omega^H : \langle M \rangle \to \mathrm{Vect}_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C}), \quad N \mapsto \mathrm{H}^0(A, N \otimes \mathbb{L})$$

by precomposing ω_0^H with the tensor functor $N \mapsto N \otimes \mathbb{L}$; see [27, Proposition 4.1].

Corollary 3.2. *There is a morphism $\lambda : \mathbb{G}_m^2 \to G_\omega(M)$ of algebraic groups such that the Hodge decomposition*

$$\omega^H(M) = \bigoplus_{(p,q)\in\mathbb{Z}^2} \mathrm{H}^{p,q}(M \otimes \mathbb{L})$$

is the decomposition in weight spaces for the characters $(p,q) \in \mathbb{Z}^2 = \mathrm{Hom}(\mathbb{G}_m^2, \mathbb{G}_m)$.

In classical Hodge theory, the bigrading is given by a representation of $\mathbb{S}_\mathbb{C} = \mathbb{G}_m^2$ for $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. By analogy, we give the following definition.

Definition 3.3. The morphism $\lambda : \mathbb{G}_m^2 \to G_\omega(M)$ in the above corollary is called the *Hodge cocharacter*.

The fact that λ factors over the Tannaka group will allow us to find information about Tannaka groups from information about Hodge numbers.

3.2. Hodge level of intersection complexes. Any object of $\mathrm{HM}(A)$ is a direct sum of simple objects, and any simple Hodge module is the intersection complex $\mathrm{IC}_Z(\mathbb{V})$ of a variation of Hodge structures $\mathbb{V}$ on an open dense subset of an integral subvariety $Z \subset A$. We are interested in the twisted Hodge numbers

$$h^p(M) := h^{p,w-p}(M) := \dim \mathrm{H}^{p,w-p}(M \otimes \mathbb{L}),$$

where the twist is by a unitary local system L of rank one with $\mathrm{H}^i(A, \mathrm{DR}(M) \otimes L) = 0$ for all $i \neq 0$; we drop the local system from the notation since it does not affect the Hodge numbers. If $\mathbb{V} = \mathbb{C}_U$ is the trivial variation of Hodge structures on the smooth locus $U \subset X$ of an integral subvariety $X \subset A$ of dimension d, we also write

$$\delta_X^H := \mathrm{IC}_X(\mathbb{C}_U) \in \mathrm{HM}(A, d).$$

The underlying perverse sheaf is the perverse intersection complex $\delta_X := \mathrm{DR}(\delta_X^H)$ of the subvariety, and we write

$$h^p(X) := h^p(\delta_X^H).$$

For smooth X, these twisted Hodge numbers have the following properties.

Lemma 3.4. *If $X \subset A$ is a smooth subvariety of dimension d, then for all $p \in \mathbb{Z}$, we have*

$$h^p(X) = (-1)^{d-p}\chi(X, \Omega_X^p) = h^{d-p}(X).$$

If moreover $X \subset A$ has ample normal bundle and $d < g = \dim A$, then these Hodge numbers satisfy

$$h^p(X) \geqslant \begin{cases} g - d + 1 & \text{for } p \in \{0, d\}, \\ 2 & \text{for } p \in \{1, d-1\}, \\ 1 & \text{for } 2 \leqslant p \leqslant d-2. \end{cases}$$

Proof. If X is smooth, then $h^p(X) = (-1)^{d-p}\chi(X, \Omega_X^p)$ by [25, Appendix B]; thus $h^p(X) = h^{d-p}(X)$ by Serre duality. For the inequalities, see Corollary A.2. □

Corollary 3.5. *For any smooth subvariety $X \subset A$ with nonzero ample normal bundle, its dimension is equal to the level of the Hodge decomposition, i.e.*

$$\dim X = \max\{|2p - d| \mid h^p(X) \neq 0\}.$$

For perverse intersection complexes of singular subvarieties $X \subset A$, we still have an upper bound. More generally, for intersection complexes of variations $\mathbb{V}$ of Hodge structures, denote by $\ell(\mathbb{V}) := \max\{|p - q| \mid \mathbb{V}^{p,q} \neq 0\}$ the *level* of the Hodge structure. We then obtain the following lemma.

Lemma 3.6. *Let $X \subset A$ be an integral subvariety of dimension d. Then, for any pure polarizable variation of $\mathbb{C}$-Hodge structures on an open dense subset $U \subset X^{\mathrm{reg}}$, we have*

$$h^{p,q}(\mathrm{IC}_X(\mathbb{V})) = 0 \quad \text{for } |p - q| > \ell(\mathbb{V}) + d.$$

Proof. By definition, $h^{p,q}(\mathrm{IC}_X(\mathbb{V})) = \dim \mathrm{H}^{p,q}(\mathrm{IC}_X(\mathbb{V}) \otimes \mathbb{L})$, where $\mathbb{L}$ is a unitary local system of rank one. Since $\mathrm{IC}_X(\mathbb{V}) \otimes \mathbb{L} = \mathrm{IC}_X(\mathbb{V} \otimes \mathbb{L})$, we may replace $\mathbb{V}$ by the variation of $\mathbb{C}$-Hodge structures $\mathbb{V} \otimes \mathbb{L}$ which has the same level. Hence, in what follows, we will assume that $\mathbb{L}$ is the trivial local system.

Now $\mathbb{V}$ embeds into a variation of pure polarizable $\mathbb{R}$-Hodge structures of the same level. To see this, write $\mathcal{V}$ for the flat $\mathcal{C}^\infty$-bundle underlying $\mathbb{V}$. Define $\bar{\mathbb{V}}$ to be the variation of $\mathbb{C}$-Hodge structures whose underlying flat $\mathcal{C}^\infty$-bundle is the complex conjugate of $\mathcal{V}$ and whose component of bidegree (p, q) is the complex conjugate of $\mathcal{V}^{q,p}$. This defines a variation of $\mathbb{C}$-Hodge structures by [35, Section 4.1.a]. Note that a polarization on $\mathbb{V}$ induces a natural polarization on $\bar{\mathbb{V}}$. Therefore, $\mathbb{W} = \mathbb{V} \oplus \bar{\mathbb{V}}$ is the complexification of the sought-for polarizable variation of $\mathbb{R}$-Hodge structures; see also [35, Remark 4.1.12]. Replacing $\mathbb{V}$ by $\mathbb{W}$, we assume in what follows that $\mathbb{V}$ comes from a polarizable variation of $\mathbb{R}$-Hodge structures.

Now let $k = \max\{p \in \mathbb{Z} \mid F^p\mathbb{V} \neq 0\}$, so that $F^k\mathbb{V} \subset \mathbb{V}$ is the smallest piece in the Hodge filtration. If the variation of Hodge structures has weight w, then by Hodge symmetry, the Hodge numbers of its fibers can be nonzero at most in bidegrees $(i, w - i)$ with $w - k \leqslant i \leqslant k$. So the level of the Hodge structure on the fibers is $\ell(\mathbb{V}) = 2k - w$. Now, by the conventions of [35, Appendix A.7], the right $\mathcal{D}_U$-module $\mathcal{M}_U := \omega_U \otimes \mathbb{V}$ associated to the variation of Hodge structures comes with the increasing Hodge filtration

$$F_p\mathcal{M}_U := \omega_U \otimes F^{-p-d}\mathbb{V}$$

which starts in degree $p = -k - d$. By the theory of Hodge modules, this filtration naturally extends to an increasing filtration on the right $\mathcal{D}_A$-module $\mathcal{M}$ underlying the Hodge module

$M = \mathrm{IC}_X(\mathbb{V})$. Moreover, the lowest piece $F_p\mathcal{M}$ of this filtration is still in degree $p = -k - d$; see for instance [38, Section 2.4] in the context of Saito's Hodge modules. Now consider the de Rham complex

$$\mathrm{DR}(\mathcal{M}) = [\cdots \to \mathcal{M} \otimes \wedge^2 \mathcal{T}_A \to \mathcal{M} \otimes \mathcal{T}_A \to \mathcal{M}],$$

where the last term $\mathcal{M}$ is placed in degree zero. It comes with the increasing filtration

$$F_p\,\mathrm{DR}(\mathcal{M}) = [\cdots \to F_{p-2}(\mathcal{M}) \otimes \wedge^2 \mathcal{T}_A \to F_{p-1}(\mathcal{M}) \otimes \mathcal{T}_A \to F_p(\mathcal{M})]$$

which by the above starts in degree $p = -k - d$. Hence, on de Rham cohomology, the filtration

$$F_p\mathrm{H}^0(A, \mathrm{DR}(M)) = \mathrm{Im}\big(\mathrm{H}^0(A, F_p\,\mathrm{DR}(M)) \to \mathrm{H}^0(A, \mathrm{DR}(M))\big)$$

can only start in degree $p = -k - d$ or higher. Since M is a pure Hodge module of weight $w + d$ by [35, Theorem 14.6.1], the de Rham cohomology $\mathrm{H}^0(A, \mathrm{DR}(M))$ in degree zero is a pure Hodge structure of weight $w + d$ by [35, Theorem 14.3.2 (2)]. Its classical decreasing Hodge filtration goes at most up to degree $-p = k + d$ by the above. So, again by Hodge symmetry, its level is

$$\ell(\mathrm{H}^0(A, \mathrm{DR}(M)) \leqslant 2(k + d) - (w + d) = (2k - w) + d = \ell(\mathbb{V}) + d,$$

which concludes the proof. □

The above then also gives an estimate on the level of the Hodge structures arising from the decomposition theorem applied to intersection complexes.

Lemma 3.7. *Let Y be a smooth variety with a proper morphism $f\colon Y \to A$. Then, for every direct summand $M \subset {}_T f_*^{(0)}(\delta_Y^H)$, its twisted Hodge numbers satisfy*

$$h^{p,q}(M) = 0 \quad \textit{for } |p - q| > \dim f^{-1}(\mathrm{Supp}\, M).$$

Proof. Recall that, by S-decomposability of Hodge modules [35, Theorem 14.2.19 (1)], we have

$${}_T f_*^{(0)}(\delta_Y^H) = \bigoplus_{X \subset A} M_X,$$

where $X \subset A$ runs through all integral subvarieties and $M_X \subset {}_T f_*^{(0)}(\delta_Y^H)$ denotes the maximal Hodge submodule with strict support X. It thus suffices to prove the lemma when $M = M_X$ for some $X \subset A$. Note that the above strict support decomposition is compatible with the natural functor from Saito's $\mathbb{Q}$-Hodge modules to complex Hodge modules, since under the faithful de Rham functor, it corresponds to the strict support decomposition of the underlying perverse sheaves. Thus the summand $M \subset {}_T f_*^{(0)}(\delta_Y^H)$ arises from a direct summand in the category of $\mathbb{Q}$-Hodge modules; this will allow us to embed its generic stalk cohomology in the mixed Hodge structure of a singular variety even though the mixed case is not yet written for complex Hodge modules [35].

Since M has strict support X, we have $M = \mathrm{IC}_X(\mathbb{V})$ for a polarizable variation $\mathbb{V}$ of $\mathbb{C}$-Hodge structures on an open dense subset of X. Let $d = \dim X$; then, at a general point $x \in X$, the fiber of the local system V underlying the variation $\mathbb{V}$ is a direct summand

$$V_x = \mathcal{H}^{-d}(\mathrm{DR}(M))_x \hookrightarrow \mathcal{H}^{-d}(Rf_*(\delta_Y))_x = \mathrm{H}^{\dim Y - d}(f^{-1}(x)).$$

By the discussion in the first paragraph, this inclusion is induced by an inclusion of mixed $\mathbb{Q}$-Hodge structures. The fibers $f^{-1}(x)$ may be singular, but since, for any singular variety, the mixed Hodge structure on its cohomology has level at most the dimension of the variety, our variation of Hodge structures has level

$$\ell(\mathbb{V}) \leqslant \ell(\mathrm{H}^\bullet(f^{-1}(x)) \leqslant \dim f^{-1}(x) = \dim f^{-1}(X) - d,$$

where the first inequality comes from the inclusion $V_x \hookrightarrow \mathrm{H}^\bullet(f^{-1}(x))$ of Hodge structures and the last equality uses that $x \in X$ is general. Since $M = \mathrm{IC}_X(\mathbb{V})$, Lemma 3.6 gives

$$h^{p,q}(M) = 0 \quad \text{for } |p-q| > (\dim f^{-1}(X) - d) + d = \dim f^{-1}(X). \qquad \square$$

4. Subvarieties with exceptional Tannaka groups

We now discuss the restrictions which the above results impose on subvarieties with Tannaka group E_6 and E_7. As above, we work over the complex numbers.

4.1. Reduction to semisimple groups. For Hodge modules of nonzero weight, the Tannaka group is not semisimple: the weight is given by a central cocharacter of this group. But for fixed weight, we can still describe the Hodge decomposition by a cocharacter with values in a connected semisimple group as follows. To fix notation, suppose we are given

- a finite-dimensional vector space V over $\mathbb{C}$,
- a connected semisimple subgroup $G \subset \mathrm{GL}(V)$,
- a cocharacter $\lambda\colon \mathbb{G}_m^2 \to Z \cdot G$, where $Z = \mathbb{G}_m \cdot \mathrm{id}_V \subset \mathrm{GL}(V)$,

such that the weight spaces

$$V^{p,q} := \{v \in V \mid \lambda(x,y)(v) = x^p y^q \cdot v \text{ for all } (x,y) \in \mathbb{G}_m^2\}$$

satisfy Hodge symmetry in the sense that $\dim V^{p,q} = \dim V^{q,p}$ for all p, q. The Hodge symmetry implies $\lambda(z, z^{-1}) \in \mathrm{SL}(V)$ for all $z \in \mathbb{G}_m$. On the other hand, we have

$$G = (Z \cdot G) \cap \mathrm{SL}(V)$$

since G is a connected semisimple group. So λ restricts on the antidiagonal to a cocharacter

$$\rho\colon \mathbb{G}_m \to G, \quad z \mapsto \lambda(z, z^{-1})$$

with values in the connected semisimple subgroup $G \subset \mathrm{GL}(V)$. For $v \in V^{p,q}$, we have

$$\rho(z)(v) = z^{p-q} \cdot v = z^{2p-w} \cdot v \quad \text{for the weight } w := p+q.$$

We will apply this in the following situation.

Example 4.1. Let $M \in \mathrm{HM}(A, w)$ be a simple Hodge module on a complex abelian variety A. Assume that the Hodge decomposition

$$V := \mathrm{H}^0(A, \mathrm{DR}(M \otimes \mathbb{L})) = \bigoplus_{p+q=w} V^{p,q}$$

from Section 3.1 satisfies Hodge symmetry in the sense that $\dim V^{p,q} = \dim V^{q,p}$ for all $p, q \in \mathbb{Z}$. Recall that this decomposition is given by the weights for the Hodge cocharacter

$$\lambda: \mathbb{G}_m^2 \to G_\omega(M) \subset \mathrm{GL}(V) \quad \text{with } \lambda(z,w)_{|V^{p,q}} = z^p w^q \cdot \mathrm{id}_{V^{p,q}}.$$

Since M is a simple Hodge module, the group $G_\omega(M)$ acts on V by an irreducible representation. By Schur's lemma, its center then acts on V by scalars. Since any connected reductive group is the almost direct product of its center and its derived group, it follows that

$$G_\omega^\circ(M) \subset Z \cdot G \quad \text{for } Z := \mathbb{G}_m \cdot \mathrm{id}_V \subset \mathrm{GL}(V).$$

So we are in the situation described above.

4.2. Cocharacters with Hodge properties. Let G be a complex semisimple algebraic group, and let V be a finite-dimensional representation of G with finite kernel. Any cocharacter $\lambda: \mathbb{G}_m \to G$ induces a weight space decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V^n(\lambda),$$

where λ acts on $V^n(\lambda)$ via the character $n \in \mathbb{Z} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$. We want to determine all choices of the cocharacter λ such that the Hodge numbers $\dim V^n(\lambda)$ have the following properties.

(H1) Hodge symmetry: $\dim V^n(\lambda) = \dim V^{-n}(\lambda)$ for all $n \in \mathbb{Z}$.

(H2) Gap freeness: There exists an integer $\ell \geqslant 0$ such that

$$V^n(\lambda) \neq 0 \iff n = 2i - \ell \text{ with } i \in \{0, 1, \dots, \ell\}.$$

(H3) Outer Hodge number estimate: For ℓ as above, we have

$$\dim V^{\ell-2}(\lambda) \geqslant 2 \quad \text{and} \quad \dim V^{\ell}(\lambda) \geqslant 3.$$

The set of such cocharacters is a finite union of conjugacy classes.

Lemma 4.2. *For any given (G, V), there exist up to G-conjugacy only finitely many cocharacters λ that satisfy property* (H2).

Proof. Fix a maximal torus $T \subset G$. Any cocharacter is conjugate to one with values in this torus, so it suffices to consider cocharacters $\lambda \in X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$. The weight space decomposition for λ is then obtained as follows. Consider the weight space decomposition

$$V = \bigoplus_{\chi \in X^*(T)} V(\chi)$$

for our maximal torus, where the action of the torus on the subspace $V(\chi) \subset V$ is given by $\chi \in X^*(T) = \mathrm{Hom}(T, \mathbb{G})$. Then the weight n space for the cocharacter λ is the direct sum

$$V^n(\lambda) = \bigoplus_{\langle \lambda, \chi \rangle = n} V(\chi),$$

where the sum is over all characters $\chi \in X^*(T)$ with $\langle \chi, \lambda \rangle = n$. To make use of this, let

$$S := \{\chi \in X^*(T) \mid V(\chi) \neq 0\}$$

be the set of characters that enter in the given representation. Condition (H2) then implies $|\langle \lambda, \chi \rangle| \leqslant \ell < \dim V$ for all $\chi \in S$. This condition leaves only finitely many choices of

$$\lambda \in X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

in the cocharacter lattice, since in the dual character lattice, the occurring weights $\chi \in S$ span a subgroup $\langle S \rangle_{\mathbb{Z}} \subset X^*(T)$ of finite index. Indeed, if the subgroup spanned by those weights had smaller rank than $X^*(T)$, then we could find $\mu \in X_*(T) \smallsetminus \{0\}$ with $\langle \mu, \chi \rangle = 0$ for all $\chi \in S$. But then the image of $\mu \colon \mathbb{G}_m \to G$ would be a subgroup of positive dimension acting trivially on the representation V, which is impossible since V has finite kernel. □

The above proof allows to enumerate all cocharacters λ with (H1), (H2), (H3) for a given (G, V) by an exhaustive computer search. Below, we will list the results for the exceptional groups E_6 and E_7 in their minimal representations.

4.3. Cocharacters of $G = E_6$. Consider the simply connected group $G = E_6$. Fix a system of positive roots for a maximal torus $T \subset G$. We label the fundamental dominant weights $\varpi_1, \dots, \varpi_6 \in X^*(T)$ as in [4, Chapter VI, Section 4.12], and the dual basis by $\varpi_1^\vee, \dots, \varpi_6^\vee \in X_*(T)$. An exhaustive computer search gives the following.

Proposition 4.3. *Let V be a 27-dimensional irreducible representation of E_6; then, for any $\lambda \in X_*(E_6)$ with* (H1), (H2), (H3)*, the numbers $h^i(\lambda) := \dim V^{2i-\ell}(\lambda)$ for $0 \leqslant i \leqslant \ell$ are given by one of the rows of the following table:*

h^0	h^1	h^2	h^3	h^4	h^5	h^6
6	15	6				
3	6	9	6	3		
3	3	3	9	3	3	3

Moreover, up to conjugacy, the only cocharacter giving the Hodge numbers 6, 15, 6 *in the first row of the table is the cocharacter $\lambda = \varpi_1^\vee$.*

Remark 4.4. On the adjoint representation $W = \mathrm{Lie}\, E_6$, the cocharacter $\lambda = \varpi_1^\vee$ induces a grading of length $\ell = 4$ with

$$\dim W^{2i-\ell}(\lambda) = \begin{cases} 1 & \text{for } i = 0, 4, \\ 20 & \text{for } i = 1, 3, \\ 36 & \text{for } i = 2. \end{cases}$$

This information will be useful in the proof of Proposition 4.8 below.

4.4. Cocharacters of $G = E_7$. Now take the simply connected group $G = E_7$; then, as above, an exhaustive computer search gives the following.

Proposition 4.5. *Let V be a 56-dimensional irreducible representation of E_7; then, for any $\lambda \in X_*(E_7)$ with* (H1), (H2), (H3) *whose associated Hodge decomposition has odd length ℓ, the numbers $h^i(\lambda) = \dim V^{2i-\ell}(\lambda)$ for $0 \leqslant i \leqslant \ell$ are given by one of the rows of the following table:*

h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8	h^9	h^{10}	h^{11}	h^{12}	h^{13}	h^{14}	h^{15}
7	21	21	7												
6	7	15	15	7	6										
6	6	1	15	15	1	6	6								
5	3	10	10	10	10	3	5								
5	2	6	5	10	10	5	6	2	5						
4	3	3	12	6	6	12	3	3	4						
5	2	5	1	5	10	10	5	1	5	2	5				
4	2	3	5	8	6	6	8	5	3	2	4				
4	2	2	5	1	8	6	6	8	1	5	2	2	4		
4	2	2	4	1	1	8	6	6	8	1	1	4	2	2	4

4.5. Hodge numbers of exceptional subvarieties. Putting together the above, we obtain the following restriction on smooth subvarieties $X \subset A$ of complex abelian varieties A such that the Tannaka group $G^*_{X,\omega} := G^*_\omega(\delta_X)$ of their perverse intersection complex is a simple exceptional group.

Proposition 4.6. *Let A be an abelian variety of dimension g, and $X \subset A$ a smooth nondivisible irreducible subvariety of dimension d with ample normal bundle such that $G^*_{X,\omega}$ is a simple exceptional group. Then either*

(1) $G^*_{X,\omega} \simeq E_6$, $\dim V_X = 27$ *and* $d \in \{2, 4, 6\}$, *or*

(2) $G^*_{X,\omega} \simeq E_7$, $\dim V_X = 56$ *and* $d \in \{3, \dots, 15\}$ *is odd.*

Moreover, $g \leqslant g_{\max}$ for the following upper bound $g_{\max}$ depending on d:

d	2	4	6
$g_{\max}$	7	6	8

(1)

d	3	5	7	9	11	13	15
$g_{\max}$	9	10	12	13	15	16	18

(2)

Proof. Since $X \subset A$ is a smooth subvariety, the Tannaka group $G^*_{X,\omega}$ acts on V_X via a minuscule representation [24, Corollary 1.10]. For the simple exceptional groups, the only minuscule representations are the irreducible representations of dimension 27 for E_6 and of dimension 56 for E_7. The latter representation admits a nondegenerate bilinear form which is alternating, so in this case, X must have odd dimension by [20, Section 1.2]; hence, for E_7, we only need to consider cocharacters whose associated Hodge decomposition has odd length d. Also, when X is a divisor, the topological Euler characteristic of X is divisible by $g!$, hence cannot be 27 or 56 up to sign. Therefore, we can assume $d \leqslant g-2$. Since $X \subset A$ has ample normal bundle, Lemma 3.4 and Corollary 3.5 imply that the Hodge cocharacter induces a cocharacter of $G^*_{X,\omega}$ satisfying properties (H1), (H2) and (H3). Lemma 3.4 also implies the following bound on the codimension:

$$g - d + 1 \leqslant h^d(X) := \chi(X, \Omega^d_X).$$

So it follows that the Hodge numbers $h^p(X) = (-1)^{d-p}\chi(X, \Omega^p_X)$ must be among those in Propositions 4.3 and 4.5. □

Corollary 4.7. *If in the above proposition we moreover assume that $d < g/2$, then the topological Euler characteristic $\chi_{\mathrm{top}}(X) = (-1)^d \dim V_X$ and the Hodge numbers*

$$h^p(X) = (-1)^{d-p}\chi(X, \Omega^p_X)$$

are among one of the following cases:

G^*_X	$\dim V_X$	d	g	h^0	h^1	h^2	h^3
E_6	27	2	5, 6, 7	6	15	6	
E_7	56	3	7, 8, 9	7	21	21	7

In the rest of this paper, we will show that the E_6 case in the above table occurs only when X is the Fano surface of lines on a smooth cubic threefold, and A is isogenous to its Albanese variety (which has $g = 5$). We will also show that the E_7 case never occurs.

4.6. Surfaces with group E_6 and the difference morphism. Let us take a closer look at the first row of the table in Corollary 4.7. The corollary in particular says that if $X \subset A$ is a smooth nondivisible irreducible subvariety with ample normal bundle and dimension $< g/2$ such that $G^*_{X,\omega} \simeq E_6$, then X must be a surface with

$$\chi(X, \mathcal{O}_X) = 6, \quad \chi(X, \Omega^1_X) = -15, \quad \chi(X, \Omega^2_X) = 6.$$

We also know the Chern numbers: the top Chern class is $c_2(X) = \dim \omega(\delta_S) = 27$, so we get $c_1(X)^2 = 45$ by Noether's formula. Moreover, by the representation theory of E_6, there is a unique one-dimensional subrepresentation inside $V_X \otimes V_X \otimes V_X$. Since by [41, Section 10] or [24, Lemma 3.5] the one-dimensional representations of the Tannaka group correspond to skyscraper sheaves, there is a unique skyscraper direct summand inside $\delta_X * \delta_X * \delta_X$. By base change, this means that the sum morphism $X^3 \to A$ has a unique three-dimensional fiber and that this fiber is irreducible. To show that X is the Fano surface of lines on a smooth cubic threefold (in which case the above fiber parametrizes coplanar triples of lines), we will need one more piece of numerical information: the degree of the difference morphism $d\colon X \times X \to X - X \subset A$. The representation theory of the exceptional group E_6 implies the following.

Proposition 4.8. *Let A be an abelian variety of dimension $g \geqslant 5$ and let $X \subset A$ be a smooth nondegenerate nondivisible surface with $G^*_{X,\omega} \simeq E_6$. Then the difference morphism $d\colon X \times X \to X - X$ is generically finite of degree* $\deg(d) \geqslant 6$.

Proof. Let $D := X - X$. Since X is nondegenerate, the morphism $d\colon X \times X \to D$ is generically finite; hence, over some open dense subset $U \subset D^{\mathrm{reg}}$, it restricts to a finite étale cover of degree $\deg(d)$. The direct image $\delta_X * \delta_{-X} = Rd_*(\delta_{X\times X})$ therefore restricts on U to a local system of rank $\deg(d)$, placed in cohomological degree $-\dim D = -4$. By adjunction, that local system contains the trivial local system of rank one as a direct summand. The decomposition theorem [3, Theorem 6.2.5] therefore shows that

(4.1a) $$\delta_X * \delta_{-X} \simeq \delta_D \oplus P,$$

where P is a semisimple perverse sheaf on D which restricts on $U \subset D$ to a local system of rank $\deg(d) - 1$, placed in cohomological degree -4. Passing to the clean characteristic cycles of these perverse sheaves in the sense of [20, Section 5.3], we get

(4.1b) $$\mathrm{cc}(\delta_D) = \Lambda_D + \sum_Z m_Z(\delta_D) \cdot \Lambda_Z,$$

(4.1c) $$\mathrm{cc}(P) = (\deg(d) - 1) \cdot \Lambda_D + \sum_Z m_Z(P) \cdot \Lambda_Z$$

with certain integers $m_Z(\delta_D), m_Z(P) \geqslant 0$. On the right-hand side of these equations, the sums are taken over all integral proper subvarieties $Z \subsetneq D$, and $\Lambda_Z \subset T^*A$ denotes the conormal variety to Z inside A. Hence we only need to show that the conormal variety Λ_D enters in $\mathrm{cc}(\delta_X * \delta_{-X})$ with multiplicity at least six.

To interpret this in terms of representation theory, we need some notation. Fix a maximal torus $T \subset E_6$. We use the fundamental weights $\varpi_1, \dots, \varpi_6 \in X^*(T)$ from Section 4.3. The dominant integral weights are precisely those of the form

$$\lambda = a_1\varpi_1 + \cdots + a_6\varpi_6 \quad \text{with integers } a_1, \dots, a_6 \geqslant 0,$$

and for any such weight, we will denote by V_λ the irreducible representation of E_6 of highest weight λ. The classes of these irreducible representations form a basis for the representation ring $\mathrm{R}(E_6)$, the Grothendieck ring of the tensor category of all finite-dimensional representations of E_6. We identify the representation ring $\mathrm{R}(E_6)$ with the Weyl group invariants in the group algebra $\mathbb{Z}[X^*(T)]$ by the map sending a representation V to its character

$$\mathrm{char}(V) \in \mathbb{Z}[X^*(T)].$$

In these terms, the representation ring $\mathrm{R}(E_6)$ has a basis consisting of the Weyl group orbits of the dominant integral weights. For a weight $\lambda \in X^*(T)$, we denote by $[\lambda] \in \mathrm{R}(E_6)$ its Weyl group orbit. The character of a representation is a single Weyl group orbit only if the representation is minuscule, so the basis given by Weyl group orbits differs from the one given by the irreducible representations. In what follows, we are interested in the irreducible representations that appear in the tensor product of the 27-dimensional representation $V = V_{\varpi_1}$ with its dual $V^\vee = V_{\varpi_6}$:

(4.2a) $$V \otimes V^\vee \simeq V_\alpha \oplus V_\beta \oplus \mathbf{1} \quad \text{with} \begin{cases} \alpha = \varpi_2, \\ \beta = \varpi_1 + \varpi_6. \end{cases}$$

Here V_α is the adjoint representation of E_6 and V_β is an irreducible representation of dimension 650. Their characters are

(4.2b) $$\mathrm{char}(V_\alpha) = 6 \cdot [0] + 1 \cdot [\alpha],$$

(4.2c) $$\mathrm{char}(V_\beta) = 20 \cdot [0] + 5 \cdot [\alpha] + 1 \cdot [\beta].$$

Now we come back to geometry. Fix an isomorphism $G^*_{X,\omega} \simeq E_6$. By Corollary 2.5, we have $G^*_\omega(\delta_X^H) = G^*_{X,\omega}$, so we may work as well with Hodge modules. Since $\omega(\delta_X^H)$ is a minuscule representation, it must be one of the two irreducible representations of dimension 27, which are dual to each other. So we may assume $\omega(\delta_X^H) \simeq V$. Then equation (4.2a) translates to

$$\delta_X^H * \delta_{-X}^H \simeq M_\alpha \oplus M_\beta \oplus \delta_0^H, \tag{4.3a}$$

where for $\lambda \in \{\alpha, \beta\}$, we denote by $M_\lambda \in \langle \delta_X^H \rangle$ the simple Hodge module with associated representation $\omega(M_\lambda) \simeq V_\lambda$. The characteristic cycles of these Hodge modules can be computed as follows. By [20, Theorem 5.11, Lemma 5.14], there is a natural way to attach to any Weyl group orbit $[\lambda]$ an effective Lagrangian cycle Λ_λ on the cotangent bundle T^*A such that if we extend this map additively to the representation ring $\mathrm{R}(E_6)$, then it sends the character of any representation to the characteristic cycle of the corresponding perverse sheaf. Thus equations (4.2b) and (4.2c) translate to

$$\mathrm{cc}(M_\alpha) = 6 \cdot \Lambda_0 + 1 \cdot \Lambda_\alpha, \tag{4.3b}$$

$$\mathrm{cc}(M_\beta) = 20 \cdot \Lambda_0 + 5 \cdot \Lambda_\alpha + 1 \cdot \Lambda_\beta, \tag{4.3c}$$

where all summands on the right-hand side are effective Lagrangian cycles.

Comparing equations (4.1a) and (4.3a), we see that $\delta_D^H \simeq M_\lambda$ for some $\lambda \in \{\alpha, \beta\}$. We claim that $\delta_D^H \simeq M_\alpha$. Indeed, if $\delta_D^H \simeq M_\beta$, then $\mathrm{cc}(M_\beta)$ would contain the irreducible component Λ_D with multiplicity one. By equation (4.3c), this component would then have to enter in the summand Λ_β, since the other summands enter with higher multiplicity. Then, by equation (4.1b), the other summand Λ_α could involve only conormal varieties Λ_Z with

$$\dim(Z) < \dim(D) = 4.$$

But then, by equation (4.3b), the cycle $\mathrm{cc}(M_\alpha)$ would also only involve conormal varieties Λ_Z with $\dim(Z) < 4$. This would imply

$$\dim \operatorname{Supp} M_\alpha < \dim \operatorname{Supp} D = 4;$$

hence $\dim d^{-1}(\operatorname{Supp} M_\alpha) < 4$ because $d\colon X \times X \to D$ is dominant and $X \times X$ is irreducible. By Lemma 3.7, then $h^{p,q}(M_\alpha) = 0$ for $|p - q| > 3$. But $\omega(M_\alpha) \simeq V_\alpha$ is the adjoint representation of E_6, and by Remark 4.4, the Hodge cocharacter induces on the adjoint representation a Hodge decomposition of level 4, a contradiction. This proves that $\delta_D^H \simeq M_\alpha$ as claimed. But then, by equation (4.3b), the conormal variety Λ_D enters as a component in the cycle Λ_α. It then follows that the cycle

$$\mathrm{cc}(\delta_X^H * \delta_{-X}^H) = 26 \cdot \Lambda_0 + 6 \cdot \Lambda_\alpha + 1 \cdot \Lambda_\beta$$

contains Λ_D with multiplicity at least six and we are done. □

The above numerical results in fact characterize Fano surfaces of smooth cubic threefolds, as we will see in the classification of subvarieties with Tannaka group E_6 in Theorem 7.1. In particular, the degree of the morphism $d\colon X \times X \to X - X$ is precisely 6. However, for this, we will not use any representation theory; instead, we will show by direct geometric arguments that any smooth projective surface with the above numerical properties is isomorphic to the Fano surface of a smooth cubic threefold. As a preparation for this, we gather in the next two sections some general facts about cubic hypersurfaces and Fano varieties of lines.

5. Nonnormal cubic hypersurfaces

In this section, we give a classification of nonnormal cubic hypersurfaces over an arbitrary algebraically closed field k.

5.1. Statement. In order to understand the Fano variety of lines on a singular cubic threefold, it will be useful to have a complete classification of the nonnormal ones. Examples of these are cones over nonnormal cubic plane curves and cones over nonnormal cubic surfaces, where by a cone we mean the following.

Definition 5.1. Let $\pi\colon V \to W$ be a linear map of finite-dimensional k-vector spaces and $Y \subset \mathbb{P}(W)$ a subvariety. If π is surjective but not an isomorphism, then the scheme-theoretic closure in $\mathbb{P}(V)$ of the preimage of Y under the induced morphism

$$\mathbb{P}(V) \smallsetminus \mathbb{P}(\operatorname{Ker}\pi) \to \mathbb{P}(W)$$

is called the *cone* over Y with respect to π.

The result of the classification of nonnormal cubic threefolds will be that, apart from cones, the only one is the following.

Example 5.2. A subvariety $X \subset \mathbb{P}^4$ is said to be a *twisted plane* if it is defined by the equation

$$x_0^2 x_1 + x_0 x_2 x_4 + x_3 x_4^2 = 0$$

for some homogeneous coordinates $x_0, \dots, x_4$ on $\mathbb{P}^4$ (see [14, Section 2.2.9]). The singular locus of X is the plane $x_0 = x_4 = 0$; thus X is nonnormal.

For cubic surfaces, we correct some arguments proposed by Dolgachev [12] in the discussion after [12, Theorem 9.2.1]. To this end, let $\pi\colon S \to \mathbb{P}^2$ be the blow-up in a point o, with exceptional divisor $E \subset S$. Consider the closed embedding

$$S \hookrightarrow \mathbb{P}(V^\vee), \quad \text{where } V = \mathrm{H}^0(S, \mathcal{O}_S(2)(-E)),$$

and let $\mathcal{O}_S(i) := \pi^*\mathcal{O}_{\mathbb{P}^2}(i)$ for $i \geqslant 0$. The purpose of this section is to prove the following result, which is probably well known but for which we could not find a complete proof in the literature.

Proposition 5.3. *Let $X \subset \mathbb{P}^{n+1}$ be an integral nonnormal cubic hypersurface.*

- *If $n = 1$, then X is a nodal or a cuspidal curve.*
- *If $n = 2$, then X is either a cone over a nonnormal cubic integral plane curve or a projection of $S \subset \mathbb{P}(V^\vee)$ from a point not in S.*
- *If $n = 3$, then X is either a cone over a nonnormal cubic surface or a twisted plane as in Example 5.2.*
- *If $n \geqslant 4$, then X is a cone over a nonnormal integral cubic threefold.*

The rest of the section will be devoted to the proof of this statement.

5.2. Reduction to surfaces. We begin with the following.

Lemma 5.4. *Let $X \subset \mathbb{P}^{n+1}$ be a nonnormal integral cubic hypersurface with $n \geqslant 1$; then X^{sing} is a linear subspace of dimension $n-1$.*

Proof. We argue by induction on $n \geqslant 1$. For $n = 1$, the only nonnormal integral cubic plane curves are the nodal and the cuspidal plane cubics, which both have only one singular point. Suppose $n \geqslant 2$. By Serre's criterion of normality, the singular locus of X has dimension $n-1$. Let $Y \subset X^{\mathrm{sing}}$ be an irreducible component of dimension $n-1$ and suppose that Y is not a linear subspace. Pick a point $y \in Y$ such that the line through y and a general point $y' \in Y \smallsetminus \{y\}$ is not contained in Y. As $\deg X = 3$ and $Y \subset X^{\mathrm{sing}}$, every such line is contained in X. An open subset of X is thus covered by lines going through y. It follows that X is a cone with vertex y over an integral cubic hypersurface $Z \subset \mathbb{P}^n$; thus Y^{sing} is a cone with vertex y over Z^{sing}. In particular, we have $\dim Z^{\mathrm{sing}} = n-2$; hence the cubic hypersurface Z is not normal. By the induction hypothesis, Z^{sing} is a linear subspace of dimension $n-2$; thus Y^{sing} is a linear subspace of dimension $n-1$. □

Lemma 5.5. *Let $X \subset \mathbb{P}^{n+1}$ be a nonnormal integral cubic hypersurface with $n \geqslant 3$.*

(1) *If $n = 3$, then X is either a cone over a nonnormal cubic surface or the twisted plane from Example* 5.2*.*

(2) *If $n \geqslant 4$, then X is a cone over a nonnormal integral cubic threefold.*

Proof. By Lemma 5.4, the singular locus is a linear subspace of dimension $n-1$, which we may assume to have equations $x_0 = x_1 = 0$. In such coordinates, the hypersurface X is defined by an equation of the form $x_0^2 f_1 + x_0 x_1 f_2 + x_1^2 f_3 = 0$ for some linear forms f_1, f_2, f_3. Let $m+2$ be the dimension of the linear span of the linear forms x_0, x_1, f_1, f_2 and f_3. If $n > m$, then X is the cone over a cubic hypersurface in $\mathbb{P}^{m+1}$ which is necessarily integral and nonnormal. Since $m \leqslant 3 \leqslant n$, the equality $n = m$ holds only for $m = 3$, in which case we may choose f_1, f_2, f_3 as coordinate functions and obtain that X is the twisted plane in Example 5.2. □

With the above lemma, the proof of Proposition 5.3 will be reduced at the end of the next section to the study of nonnormal cubic surfaces given below.

5.3. Nonnormal cubic surfaces. In order to conclude the proof of Proposition 5.3, we need to describe nonnormal cubic surfaces which are not cones over nonnormal plane cubic curves. To do this, for a point $v \in \mathbb{P}(V^\vee) \smallsetminus S$, let $W_v \subset V$ be the subspace of linear forms vanishing at v. Let $p_v \colon \mathbb{P}(V^\vee) \smallsetminus \{v\} \to \mathbb{P}(W_v^\vee)$ be the corresponding projection. The geometry of the projection $p_v \colon S \to p_v(S)$ depends on the image of v via the projection with center E that we denote by

$$q \colon \mathbb{P}(V^\vee) \smallsetminus E \to \mathbb{P}(V'^\vee), \quad \text{where } V' := \mathrm{H}^0(S, \mathcal{O}_S(2)(-2E)).$$

More precisely, the line bundle $\mathcal{L} := \mathcal{O}_S(1)(-E)$ is globally generated on S; hence the restriction of q to $S \smallsetminus E$ extends to a morphism $q \colon S \to \mathbb{P}(V'^\vee)$. The latter is the composite morphism

$$S \to \mathbb{P}(\mathrm{H}^0(S, \mathcal{L})^\vee) \hookrightarrow \mathbb{P}(\mathrm{H}^0(S, \mathcal{L}^{\otimes 2})^\vee) = \mathbb{P}(V'^\vee),$$

where the second map is the second Veronese embedding. Note that the first map can be identified with the projection $S \to E$ onto the exceptional divisor. Let $C := q(S) \subset \mathbb{P}(V'^\vee)$ be the smooth conic which is the image of the previous composite morphism. For any $v' \in \mathbb{P}(V'^\vee)$, the closure of $q^{-1}(v')$ in $\mathbb{P}(V')$ is a plane $P_{v'}$ which intersects S in the following way.

- If $v' \in C$, then $S \cap P_{v'} = L \cup E$, where $L \subset S$ is the strict transform of a line in $\mathbb{P}^2$ passing through the point o.
- If $v' \notin C$, then $S \cap P_{v'} = E$, because $C = q(S)$ and E is the center of the projection q.

Recall that the strict transform of a line $L \subset \mathbb{P}^2$ is mapped to a line if L contains the point o, and to a smooth conic otherwise. With this in mind, we have the following proposition.

Proposition 5.6. *The image $X_v := p_v(S) \subset \mathbb{P}(V'^\vee)$ is a nonnormal cubic surface and its singular locus $X_v^{\text{sing}} \subset \mathbb{P}(V'^\vee)$ is a line. The morphism $p\colon S \to X_v$ is finite birational, and on the ramification locus $B \subset S$, it is of degree* 2*. More precisely, we have the following.*

- *If $q(v) \in C$, then $B = L \cup E$, where $L \subset S$ is the strict transform of a line in $\mathbb{P}^2$ passing through o and $p_v\colon L \cup E \to X_v^{\text{sing}}$ identifies the two lines.*
- *If $q(v) \notin C$, then B is a smooth conic and $q\colon B \to X_v^{\text{sing}}$ is a degree* 2 *map.*

Proof. The nonnormality of X_v follows from the description of $p_v\colon S \to X_v$. Indeed, if X_v were to be normal, then p_v would be an isomorphism because p_v is finite birational and S is smooth. But we will show that p_v is not an isomorphism; thus the surface X_v cannot be normal. The line bundle $\mathcal{O}_S(2)(-E)$ has self-intersection 3; thus $S \subset \mathbb{P}(V^\vee)$ has degree 3 and the same is true for $X_v \subset \mathbb{P}(W_v^\vee)$. By Lemma 5.4, the singular locus X_v^{sing} is a line.

Suppose $q(v) \in C$. In this case, $B = P \cap S$, where P is the closure of $q^{-1}(q(v))$. Indeed, as recalled above, P meets S in $L \cup E$, where L is the strict transform of a line in $\mathbb{P}^2$ passing through o. When $x \in L \cup E$, the line D joining x and v is contained in the plane P. Therefore, the scheme-theoretic intersection $D \cap (L \cup E)$ is the cycle $[x] + [y]$, where $y \in E$ is equal to x if and only if x is the intersection point of L and E. This shows that $p_v\colon L \cup E \to X_v$ identifies the lines L and E. To conclude, it suffices to prove that $p_v\colon S \smallsetminus B \to X_v$ is injective and takes values in $X_v \smallsetminus p_v(B)$. Indeed, we show that, given $x \in S \smallsetminus (L \cup E)$, the line D joining x and v meets S only in x. To see this, first notice that $D \cap E = \varnothing$: otherwise, the line D would lie in the closure P' of $q^{-1}(q(x))$ and we would have $q(x) = q(v)$. Now $D \cap S$ is contained in the preimage of $q(D) \cap C$. The line $q(D)$ meets the smooth conic C in the two distinct points $q(x)$ and $q(v)$; thus

$$D \cap S \subset q^{-1}(q(D) \cap C) = L \cup L',$$

where $P' \cap S = L' \cup E$ and $L' \neq L$ is the strict transform of a line in $\mathbb{P}^2$ passing through o. The lines D and L do not meet (otherwise v would belong to S) and D meets L' only in x.

Suppose $q(v) \notin C$. We claim that there is a unique plane $P \subset \mathbb{P}(V^\vee)$ containing v and the strict transform Q of a line in $\mathbb{P}^2$ not passing through o. For the uniqueness, let P and P' distinct be planes in $\mathbb{P}(V^\vee)$ meeting S in the strict transform Q and Q' respectively of distinct lines in $\mathbb{P}^2$ not passing through o. Then $P \cap P'$ is the meeting point of Q and Q', and in particular, it is contained in S. To see this, notice that $P \cup P'$ is not contained in any hyperplane. If it were contained in some hyperplane H say, then we would have

$$Q \cup Q' = (P \cup P') \cap S \subset H \cap P.$$

But by construction, $H \cap S$ is the strict transform of a (possibly singular) conic in $\mathbb{P}^2$ that *passes* through o. It follows that $P \cap P' = \{v\}$ is contained S, which contradicts the hypothesis $v \notin S$. For the existence, note that every line L in $\mathbb{P}^2$ determines a plane $P_L \subset \mathbb{P}(V^\vee)$ and these planes cover $\mathbb{P}(V^\vee)$. Moreover, by construction, the points in the cone in $\mathbb{P}(V^\vee)$ over C are exactly the points lying in P_L for some L passing through o. Since v is not one of those, we obtain the wanted plane.

We are now ready to conclude the proof. Let P be the plane containing v and the strict transform B of a line in $\mathbb{P}^2$ not passing through o. Then $B \subset P$ is a smooth conic and any line through v meets B in two points (taking in account multiplicities). This shows that $B \to X_v$ is of degree 2 onto its image. To conclude, it suffices to show that $p_v\colon S \smallsetminus B \to X_v$ takes values in $X_v \smallsetminus p_v(B)$ and is injective. To do this, let $x, y \in S$ be distinct points such that $p_v(x) = p_v(y)$ and let D be the line joining x, y and v. Note that x and y cannot lie both in the exceptional divisor; otherwise, we would have $D = E$; hence $v \in S$. It follows that x and y have distinct images under the blow-up map $\pi\colon S \to \mathbb{P}^2$. Let $L \subset \mathbb{P}^2$ be the line joining $\pi(x)$ and $\pi(y)$, and let $Q \subset S$ be its strict transform. If the blow-up point o were to belong to L, then Q would be mapped to a line in $\mathbb{P}(V^\vee)$ containing x and y, implying $D = Q$ and $v \in S$ again. Therefore, L does not contain o, so Q is embedded as a smooth conic in $\mathbb{P}(V^\vee)$. To conclude, notice that the plane P' determined by Q must be P: otherwise, as argued above, P and P' would meet in a single point which belongs to S, but $v \in P \cap P'$, so that would imply $v \in S$ once more. Finally, we have $P = P'$; hence $Q = B$. □

Proof of Proposition 5.3. We choose homogeneous coordinates x_0, x_1, x_2 on $\mathbb{P}^2$ and identify S with the blow-up of $\mathbb{P}^2$ in $o = [1:0:0]$. Then a basis of V is given by the monomials $u_{ij} = x_i x_j$ with $0 \leqslant i \leqslant j \leqslant 2$ and $(i,j) \neq (0,0)$, and S is cut out by the vanishing of the polynomials $h_1 = u_{02}u_{11} - u_{01}u_{12}$, $h_2 = u_{01}u_{22} - u_{02}u_{12}$ and $h_3 = u_{11}u_{22} - u_{12}^2$. With this notation, the vector space V' is spanned by u_{11}, u_{12} and u_{22} and the conic C has equation $h_3 = 0$. If $v \in \mathbb{P}(V)$ is defined by the vanishing of u_{01}, u_{12}, u_{22} and $u_{02} + u_{11}$, then $q(v) \in C$ and

$$X_v\colon u_{12}^3 + u_{22}^2 u_{01} - u_{12}u_{22}(u_{02} + u_{11}) = 0. \tag{5.1}$$

If instead v is defined by the vanishing of u_{ij} for $(i,j) \neq (1,2)$, then $q(v) \notin C$ and

$$X_v\colon u_{02}^2 u_{11} - u_{01}^2 u_{22} = 0. \tag{5.2}$$

The discussion preceding [12, Theorem 9.2.1] shows that, up to a linear change of variables, any nonnormal integral cubic surface $X \subset \mathbb{P}^3$ which is not a cone over a nonnormal plane cubic curve has equation (5.1) or (5.2). Together with Lemma 5.5 and Proposition 5.6, this concludes the proof. □

6. On Fano varieties of lines

We now prove some facts about Fano varieties of lines that will be used for Theorem B. Let k be an algebraically closed field. For a finite-dimensional k-vector space V, the Fano variety of lines on a subvariety $X \subset \mathbb{P}(V)$ is the k-scheme F_X whose points with values in a k-scheme S are rank 2 vector bundles $\mathcal{V} \subset V \otimes_k \mathcal{O}_S$ on S with locally free quotient such that $\mathbb{P}(\mathcal{V}) \subset \mathbb{P}(V) \times S$ is contained in $X \times S$. Equivalently, F_X is the Hilbert scheme parametrizing subvarieties of X with Hilbert polynomial $P(z) = z + 1$ with respect to line bundle $\mathcal{O}(1)$.

6.1. Rational normal scrolls. For an integer vector $a = (a_1, \dots, a_r) \in \mathbb{Z}^r$, we put $\mathcal{O}_{\mathbb{P}^1}(a) = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$. Let $p_a\colon \mathbb{P}(a) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a)) \to \mathbb{P}^1$ be the projective bundle of lines in $\mathcal{O}_{\mathbb{P}^1}(-a)$. If $0 \leqslant a_1 \leqslant \cdots \leqslant a_r$ with $a_r > 0$, then the line bundle $\mathcal{O}_{\mathbb{P}(a)}(1)$ on $\mathbb{P}(a)$ is generated by global sections. This defines a morphism

$$f_a\colon \mathbb{P}(a) \to \mathbb{P}(V(a)^\vee) \quad \text{with } V(a) = V(a_1) \oplus \cdots \oplus V(a_r),$$

where $V(a_i) = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i))$. The morphism f_a is birational and embeds each fiber F of p_a in such a way that $f_a^*\mathcal{O}(1) \simeq \mathcal{O}_F(1)$. Moreover, f_a is a closed immersion if and only if $a_i > 0$ for all $i = 1, \dots, r$. The image of f_a is called the *rational normal scroll* of type a and will be denoted $S(a)$. The rational normal scroll $S(a)$ is smooth in the following cases [13, p. 6]:

- $a_i > 0$ for all i, in which case $f_a\colon \mathbb{P}(a) \to S_a$ is an isomorphism;
- $a = (0, \dots, 0, 1)$, in which case $S(a)$ is a projective r-plane in $\mathbb{P}(V(a)^\vee)$.

Let $F(a) = F_{S(a)}$ be the Fano variety of lines in $S(a)$, and denote by $\mathrm{Gr}_2(\mathcal{O}_{\mathbb{P}^1}(-a))$ the Grassmannian of planes in $\mathcal{O}_{\mathbb{P}^1}(-a)$.

Proposition 6.1. *Suppose $a_i > 0$ for all $i = 1, \dots, r$. Then $F(a) = F(a)_v \sqcup F(a)_h$ with*

- $F(a)_v = \mathrm{Gr}_2(\mathcal{O}_{\mathbb{P}^1}(-a))$,
- $F(a)_h = \mathbb{P}(W^\vee)$, *where* $W = V(1)^n$ *and* $n = |\{i \mid a_i = 1\}|$.

Moreover, $F(a)_h \subset \mathrm{Gr}_2(V(a)^\vee)$ is linearly embedded via the Plücker embedding.

Proof. As f_a is a closed embedding, a line in $S(a)$ is the image of a closed embedding $\varepsilon\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}(a)$ such that the line bundle $\varepsilon^*\mathcal{O}_{\mathbb{P}(a)}(1)$ has degree 1. The morphism

$$g := p_a \circ \varepsilon\colon \mathbb{P}^1 \to \mathbb{P}^1$$

is either dominant or constant. We say that the corresponding line is *horizontal* in the first case, and *vertical* in the second. The Fano variety of lines decomposes correspondingly as

$$F(a) = F(a)_h \sqcup F(a)_v,$$

where the subscripts h and v stands for horizontal and vertical respectively. Vertical lines are parametrized by the relative Grassmannian $\mathrm{Gr}_2(\mathcal{O}_{\mathbb{P}^1}(-a))$. For horizontal lines, the condition $\deg \varepsilon^*\mathcal{O}_{\mathbb{P}(a)}(1) = 1$ implies that the morphism g is an isomorphism; hence, up to precomposing ε by g^{-1}, we may assume $g = \mathrm{id}$. It follows that horizontal lines in $S(a)$ are given by degree 1 line bundles on $\mathbb{P}^1$ which are quotients of $\mathcal{O}_{\mathbb{P}^1}(a)$. Such quotients are necessarily quotients of $\mathcal{O}_{\mathbb{P}^1}(a_1, \dots, a_n) = \mathcal{O}_{\mathbb{P}^1}(1)^n$, where n is as in the statement. Finally, twisting by $\mathcal{O}_{\mathbb{P}^1}(-1)$, this discussion shows that vertical lines in $S(a)$ are given by surjective morphisms $\mathcal{O}_{\mathbb{P}^1}^n \to \mathcal{O}_{\mathbb{P}^1}$. This identifies $F(a)_h$ with $\mathbb{P}(W^\vee)$ for $W = V(1)^n$. For the second claim, write $V := V(a) = W \oplus V'$ with

$$V' = V(a_{n+1}) \oplus \cdots \oplus V(a_r).$$

The Plücker embedding of $\mathbb{P}(W^\vee) \hookrightarrow \mathbb{P}(\mathrm{Alt}^2 V^\vee)$ is given by $L \mapsto L \otimes \mathrm{Alt}^2(H^\vee)$ which is clearly linear. □

Example 6.2. The case when $S(a)$ is smooth with $a_1 + \cdots + a_r = 3$ will be relevant to us. Because of the previous characterization, there are three possibilities.

- $a = (3)$. In this case, $S(3) \subset \mathbb{P}^3$ is the twisted cubic, i.e. the triple Veronese embedding of the projective line. Hence $F(3) = \varnothing$.
- $a = (1, 2)$. In this case, $S(1, 2) \subset \mathbb{P}^4$ is the blow-up $\pi\colon S \to \mathbb{P}^2$ of $\mathbb{P}^2$ in a point and it is embedded via the line bundle $\pi^*\mathcal{O}_{\mathbb{P}^2}(2)(-E)$, where $E \subset S$ is the exceptional divisor. Thus
$$F(1,2)_v = \{\text{fibers of the projection } S \to E\}, \quad F(1,2)_h = \{E\}.$$
- $a = (1, 1, 1)$. In this case, $S(1, 1, 1)$ is $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in $\mathbb{P}^5$ via the Segre embedding. Then the Plücker embedding
$$F(1,1,1)_v \hookrightarrow \mathbb{P}(\mathrm{Alt}^2 \mathcal{O}_{\mathbb{P}^1}(-1,-1,-1)) = \mathbb{P}(2,2,2) \cong \mathbb{P}^1 \times \mathbb{P}^2$$
is an isomorphism and $F(1,1,1)_h = \mathbb{P}^2$.

6.2. The Fano variety of a cone. Let $\pi\colon V \to W$ be a surjective linear map between finite-dimensional k-vector spaces. For $i = 0, 1, 2$, consider the following closed subvariety of the Grassmannian of planes in V:
$$\mathrm{Gr}_2^i(V,\pi) := \{E \subset V \mid \dim(E \cap \operatorname{Ker}\pi) \geqslant i\} \subset \mathrm{Gr}_2(V).$$
We have $\mathrm{Gr}_2^0(V,\pi) = \mathrm{Gr}_2(V)$ and $\mathrm{Gr}_2^2(V,\pi) = \mathrm{Gr}_2(\operatorname{Ker}\pi)$. For $i = 0, 1$, the induced morphism
$$\pi_i\colon \mathrm{Gr}_2^i(V,\pi) \smallsetminus \mathrm{Gr}_2^{i+1}(V,\pi) \to \mathrm{Gr}_{2-i}(W), \quad E \mapsto \pi(E)$$
is surjective and smooth. Moreover, given a plane $E \subset W$, the closure of the fiber of π_0 in the corresponding point $[E] \in \mathrm{Gr}_2(W)$ is $\mathrm{Gr}_2(\pi^{-1}(E))$.

Let $Y \subset \mathbb{P}(W)$ be a nonempty subvariety and $X \subset \mathbb{P}(V)$ the cone over Y. Consider the respective Fano varieties of lines $F_X \subset \mathrm{Gr}_2(V)$ and $F_Y \subset \mathrm{Gr}_2(W)$. At the set-theoretical level, we have
$$F_X = \pi_0^{-1}(F_Y) \sqcup \pi_1^{-1}(Y) \sqcup \mathrm{Gr}_2(\operatorname{Ker}\pi).$$
At the topological level, note that $\mathrm{Gr}_2(\operatorname{Ker}\pi)$ is contained in the closure of any fiber of π_1. For the other components, we have the following proposition.

Proposition 6.3. *Let the notation be as above.*

(1) *If Y is covered by lines, then $\pi_0^{-1}(F_Y)$ is dense in F_X and π_0 sets up a bijection between the irreducible components of F_X and those of F_Y.*

(2) *If $Y' \subset Y$ is an irreducible component that is not covered by lines, then the closure of $\pi_1^{-1}(Y')$ is an irreducible component of F_X.*

(3) *If Y contains no lines, then π_1 sets up a bijection between the irreducible components of F_X and those of Y.*

Proof. (1) The subset $\pi_1^{-1}(Y) \subset F_X$ parametrizes lines contained in the closure of fibers of $X \dashrightarrow Y$. By hypothesis, any $y \in Y$ belongs to some line $L \subset Y$. Therefore, lines in the closure of fiber of $X \dashrightarrow Y$ in y are contained in

$$\mathrm{Gr}_2(\pi^{-1}(E)) = \overline{\pi_0^{-1}([E])},$$

where $L = \mathbb{P}(E)$ and $[E] \in \mathrm{Gr}_2(W)$ is the point defined by E.

(2) Lines in the closure of fiber of $X \dashrightarrow Y$ in a point $y \in Y'$ belonging to no line contained in Y form a subset of $\pi_1^{-1}(Y')$ not lying in the closure of $\pi_0^{-1}(F_Y)$.

(3) This is clear. □

It will be useful to combine Proposition 6.3 with the following remark: for each irreducible component $F' \subset F_Y$ and $Y' \subset Y$, we have

$$\dim \pi_0^{-1}(F') = \dim F' + 2(\dim V - \dim W),$$
$$\dim \pi_1^{-1}(Y') = \dim Y' + 2(\dim V - \dim W - 1).$$

Example 6.4. Suppose $\dim W = 3$ and that Y is an integral cubic plane curve. Then the preceding statement says that the Fano variety of lines in the cone X is irreducible of dimension $2(\dim V - 4) + 1$.

Example 6.5. Suppose $\dim W = 4$ and that $Y \subset \mathbb{P}(W)$ is an integral cubic surface with only finitely many lines. This is the case if Y is smooth, or if it has only isolated singularities and is not a cone over an elliptic curve in $\mathbb{P}^2$ (see [6, 36]). Then F_X is pure of dimension $2(\dim V - 4)$ and its irreducible components (with the reduced structure) are the following: the closure of $\pi_1^{-1}(Y)$ in $\mathrm{Gr}_2(V)$ and, for each line in Y, one component isomorphic to $\mathbb{P}^{2(\dim V - 4)}$ linearly embedded with respect to the Plücker embedding.

Example 6.6. Suppose $\dim W = 4$ and that $Y \subset \mathbb{P}(W)$ is a nonnormal integral cubic surface that is not a cone over a cubic plane curve. Borrow notation from Section 5.3. By Proposition 5.3, the cubic surface Y is obtained from the blow-up S of $\mathbb{P}^2$ in a point,

$$S = S(1,2) \subset \mathbb{P}(V^\vee), \quad \text{where } V = V(1,2),$$

by projecting it from a point $v \in \mathbb{P}(V^\vee)$ not in S. Recall from Example 6.2 that

$$F_S = F_{S,h} \sqcup F_{S,v} \quad \text{with } F_{S,h} = \{\mathrm{pt}\},\ F_{S,v} \cong \mathbb{P}^1.$$

It follows from Proposition 5.6 that such a projection induces a morphism between Fano varieties of lines $f \colon F_S \to F_Y$. The morphism f is a closed embedding once restricted to irreducible components. Moreover, following the case distinction in Proposition 5.6, there are two possibilities:

- if $q(v) \in C$, then $F_{Y,\mathrm{red}} = f(F_{S,v})$ has one irreducible component;
- otherwise,
$$F_{Y,\mathrm{red}} = f(F_{S,v}) \sqcup f(F_{S,h}) \sqcup \{Y^{\mathrm{sing}}\}$$
has three irreducible components (recall by Lemma 5.4 that the singular locus Y^{sing} of Y is a line).

The surface Y is covered by lines; thus, by Proposition 6.3, the morphism π_0 sets up a bijection between the irreducible components of the Fano variety of X and that of Y. Continuing with the above distinction, we have the following cases:

- if $q(v) \in C$, then F_X is irreducible of dimension $1 + 2(\dim V - 4)$;
- otherwise, F_X has one irreducible component of dimension $1 + 2(\dim V - 4)$ and two irreducible components isomorphic to $\mathbb{P}^{2(\dim V-4)}$ linearly embedded with respect to the Plücker embedding.

6.3. The Fano variety of singular cubic hypersurfaces. Let $X \subset \mathbb{P}^{n+1}$ be an integral cubic hypersurface and $F = F_X$ its Fano variety of lines. Let $\mathcal{O}_F(1)$ be the restriction to $F \subset \mathrm{Gr}_2(V)$ of the line bundle defining the Plücker embedding.

Proposition 6.7. *If $n = 3$ and $\dim F \leqslant 2$, then F is a locally complete intersection of pure dimension 2, and for all $i \geqslant 0$,*

$$\chi(F, \mathcal{O}_F(i)) = 45\binom{i+1}{2} - 45i + 6.$$

Proof. With the notation in [1], [1, Theorem 3.3 (iv)] states that F is the zero locus of a regular section of the locally free sheaf $\mathrm{Sym}^3 Q$. Note that the statement of [1, Theorem 3.3 (iv)] gives a somewhat weaker statement: for each $x \in F$ at which F has dimension at most 2, the subvariety F is the zero subscheme of a section s_x of $\mathrm{Sym}^3 Q$ which is regular at x. Anyway, it is clear from its proof that the section s_x can be taken to be the same for all points $x \in F$. When X has isolated singularities, the identity in the statement is [1, equation (1.21.1)], whose proof relies on [1, Corollary 1.4, Lemma 1.14 and Proposition 1.15 (i)]. However, the proof of these facts only uses that F is the zero locus of a regular section of $\mathrm{Sym}^3 Q$, hence holds under our more general assumptions. □

The main goal of this section is to complement [1] by describing precisely the cases where F is not of dimension 2. When X is a cone over a cubic surface with finitely many lines, F has dimension 2; see Example 6.5. On the other hand, in Examples 6.6 and 6.4, we saw that the Fano variety of a cone over a nonnormal cubic surface or over an integral cubic plane curve has dimension 3. The twisted plane in Example 5.2 also furnishes such an example.

Example 6.8. Let $X \subset \mathbb{P}^4$ be the twisted plane from Example 5.2, given by

$$x_0^2 x_1 + x_0 x_2 x_4 + x_3 x_4^2 = 0.$$

The discussion in [14, Section 2.2.9] shows that the Fano variety of lines $F = F_X$ on X has three irreducible components:

- one 3-dimensional component is given by the lines in the 1-dimensional family of planes in X with equations, for $[t_0 : t_1] \in \mathbb{P}^1$,

$$t_1 x_0 - t_0 x_4 = 0, \quad t_0^2 x_1 + t_0 t_1 x_2 + t_1^2 x_3 = 0;$$

- one 2-dimensional component is the dual of the plane X^{sing} and is linearly embedded via the Plücker embedding;

- one 2-dimensional component obtained as the closure of the image of the morphism

$$\mathbb{P}^2 \smallsetminus \{[0:1:0],[0:0:1]\} \to \mathrm{Gr}_2(5)$$

associating to $[t_0 : t_1 : t_2]$ the line joining the points

$$[t_0^2 : t_1t_2 : -t_0t_2 : 0 : t_0t_1],\ [0 : t_1^2 : -2t_0t_1 : t_0^2 : 0] \in \mathbb{P}^4.$$

This component has degree 4 with respect to the Plücker embedding.

The main result of this section states that these are the only cases of 3-dimensional Fano varieties. The argument goes through in any dimension and yields the following.

Proposition 6.9. *For $n = 3$, we have $\dim F \leqslant 3$ and equality holds if and only if X is either*

- *the twisted plane in Example* 6.8*, or*
- *a cone over a nonnormal integral cubic surface, or*
- *a cone over a smooth cubic plane curve.*

For $n \geqslant 4$, we have $\dim F \leqslant 2n-3$, and if equality holds, then X is a cone.

This is the combination of the following two lemmas with the classification of nonnormal cubic hypersurfaces in Proposition 5.3.

Lemma 6.10. *Let $x \in X$ and let $F_{X,x} \subset F_X$ be the subvariety of lines passing through x. If X is not a cone, then $\dim F_{X,x} \leqslant n-2$.*

Proof. Suppose there is an irreducible component $Z \subset F_{X,x}$ of dimension $\geqslant n-1$. Consider the universal line $\pi\colon P \to Z$ and the evaluation morphism $f\colon P \to X$. The morphism $f\colon P \smallsetminus f^{-1}(x) \to X$ is injective because there is a unique line passing through two distinct points. As $P \neq f^{-1}(x)$, this implies

$$n = \dim X \geqslant \dim P \smallsetminus f^{-1}(x) = \dim Z + 1 \geqslant n;$$

hence $\dim Z = n-1$ and f is surjective. It follows that the lines in Z cover X, contradicting the assumption that X is not a cone. □

Lemma 6.11. *If $\dim X^{\mathrm{sing}} \leqslant n-2$, then either $\dim F \leqslant 2n-3$ or X is a cone.*

Proof. Suppose that X is not a cone. By [1, Theorem 4.2], the variety F is smooth of dimension $2n-4$ at the point corresponding to any line not meeting the singular locus. Therefore, it remains to treat the case of lines going through singular points. Since $\dim X^{\mathrm{sing}} \leqslant n-2$ by hypothesis, Lemma 6.10 implies that F at a line going through a singular point has dimension at most $2(n-2)$. This concludes the proof. □

Proof of Proposition 6.9. By Lemma 6.11, it remains to treat the case when X is not normal. We then conclude by Proposition 5.3 and Examples 6.6, 6.4 and 6.8. □

6.4. A construction. Let X be a smooth projective variety over k, V a finite-dimensional k-vector space and let

$$0 \to \mathcal{T} \to V \otimes_k \mathcal{O}_X \to \mathcal{N} \to 0 \tag{6.1}$$

be a short exact sequence of vector bundles over X with $\operatorname{rk} \mathcal{T} = 2$. Consider the composite morphism $\pi \colon \mathbb{P}(\mathcal{T}) \hookrightarrow \mathbb{P}(V) \times X \to \mathbb{P}(V)$. Let $Y \subset \mathbb{P}(V)$ be the image of $\mathbb{P}(\mathcal{T})$ and $F = F_Y$ the Fano variety of lines of Y. The short exact sequence (6.1) defines a morphism $\gamma \colon X \to F$. Assuming the generic finiteness of π, the main result of this section states that the generic line of Y parametrized by X cannot be deformed into a line that is not of the same nature.

Proposition 6.12. *Suppose* $\dim V > \dim X + 2$ *and* π *is generically finite. Then there is a nonempty open subset* $X' \subset X$ *such that, for every* $x \in X'$*, the tangent map of* γ *is surjective, i.e.* $\mathrm{T}_x\gamma \colon \mathrm{T}_x X \twoheadrightarrow \mathrm{T}_{\gamma(x)} F$*. If* γ *is generically finite, then the image of* γ *is an irreducible component of* F*.*

Proof. Let $Y' \subset Y$ be a nonempty open subset over which π is finite flat. We claim that $X' = p(\pi^{-1}(Y'))$ does the job, where $p \colon \mathbb{P}(\mathcal{T}) \to X$ is the projection. To show this, given $x \in X'$, let us consider a tangent vector to $\gamma(x)$ in F. By interpreting tangent vectors as points of F with values in $R = k[\varepsilon]$, this consists in a free R-submodule $\mathcal{V} \subset V \otimes_k R$ of rank 2 with free cokernel and such that $\mathbb{P}(\mathcal{V})$ is contained in Y_R. Moreover, saying that such a tangent vector lies over $\gamma(x)$ means that $\mathbb{P}(\mathcal{V})_{\mathrm{red}} = \mathbb{P}(\mathcal{T}_x) \subset \mathbb{P}(L)$. Finally, the fact that x belongs to X' boils down to saying that $\mathbb{P}(\mathcal{V})_{\mathrm{red}}$ meets Y'. It follows that the fiber product

$$P := \mathbb{P}(\mathcal{T})_R \times_{Y_R} (\mathbb{P}(\mathcal{V}) \cap V_R)$$

is nonempty and flat over R. Better, an irreducible component of P_{red} is the open subset

$$\mathbb{P}(\mathcal{V})_{\mathrm{red}} \cap Y' = \mathbb{P}(\mathcal{T}_x) \cap Y' \subset \mathbb{P}(\mathcal{T}_x).$$

In particular, the open subset $P \smallsetminus \overline{P \smallsetminus \mathbb{P}(\mathcal{T}_x)}$ is nonempty and flat over R. Consider its scheme-theoretic closure P' in $\mathbb{P}(\mathcal{T})_R$. Then P' is flat over R and $P'_{\mathrm{red}} = \mathbb{P}(\mathcal{T}_x)$. It follows that the morphism $\pi_{|P'} \colon P' \to \mathbb{P}(\mathcal{V})$ is an isomorphism. Indeed, the morphism $\pi_{|P'}$ is finite; hence the line bundle $\mathcal{L} = \pi^* \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is ample on P'. Since π induces an open immersion

$$P \smallsetminus \overline{P \smallsetminus \mathbb{P}(\mathcal{T}_x)} \hookrightarrow \mathbb{P}(\mathcal{V}),$$

for each $d \geqslant 0$, the morphism

$$\pi^* \colon \mathrm{H}^0(\mathbb{P}(\mathcal{V}), \mathcal{O}(d)) \to \mathrm{H}^0(P', \mathcal{L}^{\otimes d})$$

is injective. By Nakayama's lemma, it is actually an isomorphism because it is an isomorphism modulo ε. Therefore, it induces an isomorphism

$$\pi \colon P' = \operatorname{Proj}\Big(\bigoplus_{d \geqslant 0} \mathrm{H}^0(P', \mathcal{L}^{\otimes d})\Big) \xrightarrow{\sim} \mathbb{P}(\mathcal{V}) = \operatorname{Proj}\Big(\bigoplus_{d \geqslant 0} \mathrm{H}^0(\mathbb{P}(\mathcal{V}), \mathcal{O}(d))\Big).$$

Now $p \colon P' \to X$ factors through a morphism $\operatorname{Spec} k[\varepsilon] \to X$ with the unique point of $\operatorname{Spec} k$ being sent to x: indeed, the coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A} = p_* \mathcal{O}_{P'}$ is set-theoretically supported only on x, so $\mathcal{A}$ is the skyscraper sheaf on x with value

$$\mathrm{H}^0(X, p_* \mathcal{O}_{P'}) = \mathrm{H}^0(P', \mathcal{O}_{P'}) = k[\varepsilon].$$

Seeing this R-valued point of X as a tangent vector to X in x gives the desired lifting and concludes the proof. □

As a sanity check, note that, for $X = \mathbb{P}^1$, the hypotheses of Proposition 6.12 are never fulfilled; hence this is not in contradiction to Proposition 6.1.

7. Subvarieties with Tannaka group E_6

Let A be an abelian variety of dimension g over an algebraically closed field k of characteristic zero. Putting together the results from the previous sections, we now complete our classification of smooth subvarieties with Tannaka group E_6.

7.1. Main result. As recalled in the introduction, the Fano surfaces of lines on cubic threefolds are examples of subvarieties $X \subset A$ with Tannaka group E_6. The main result of this section is that these are the only possible examples among nondegenerate surfaces.

Theorem 7.1. *Suppose $g \geqslant 5$. For any smooth irreducible surface $X \subset A$, the following three properties are equivalent.*

(1) *$X \subset A$ is nondivisible and nondegenerate with Tannaka group $G^*_{X,\omega} \simeq E_6$.*

(2) *$X \subset A$ is nondivisible and nondegenerate with*

$$\chi(X, \mathcal{O}_X) = 6, \quad c_2(X) = 27,$$

the difference morphism $X \times X \to X - X$ has generic degree $\geqslant 6$, and the sum morphism $X \times X \times X \to A$ has an irreducible fiber of dimension $\geqslant 3$.

(3) *X is isomorphic to the Fano surface of lines on a smooth cubic threefold, and the canonical morphism* $\mathrm{Alb}(X) \to A$ *is an isogeny.*

Combining this result with Corollary 4.7, we obtain the following classification of subvarieties with Tannaka group E_6.

Corollary 7.2. *Let $X \subset A$ be a smooth irreducible subvariety with ample normal bundle and dimension $< g/2$. Then the following are equivalent.*

(1) *$X \subset A$ is nondivisible with Tannaka group $G^*_{X,\omega} \simeq E_6$.*

(2) *X is isomorphic to the Fano surface of lines on a smooth cubic threefold, and the canonical morphism* $\mathrm{Alb}(X) \to A$ *is an isogeny.*

The rest of the section is devoted to the proof of Theorem 7.1. By the Lefschetz principle and [20, Corollary 4.4], we may assume $k = \mathbb{C}$. In this case, we already know from Section 4.6 that (1) ⇒ (2), and (3) ⇒ (1) is clear from the example in Section 1.2 of the introduction. So the only remaining point is (2) ⇒ (3), which gives a numerical characterization of Fano surfaces of smooth cubic threefolds.

7.2. Strategy of the proof. Before coming to the details, let us outline the main steps in the proof of (2) ⇒ (3). The idea is to recover the cubic threefold as in the example in

Section 1.2. By assumption, the subvariety $X \subset A$ is a surface with

$$\chi(X, \mathcal{O}_X) = \chi(X, \Omega_X^2) = 6, \quad c_2(X) = 27.$$

By Noether's formula, we then also know $c_1(X)^2 = 45$, and by the Hirzebruch–Riemann–Roch theorem, we have $\chi(X, \Omega_X^1) = -15$. Now we can use the short exact sequence

$$0 \to \mathcal{T} = T_X \to \operatorname{Lie} A \otimes_k \mathcal{O}_X \to \mathcal{N} = \mathcal{N}_{X/A} \to 0 \tag{7.1}$$

to compute the Chern classes of the normal bundle $\mathcal{N}$ and find

$$c_1(\mathcal{N})^2 = 45, \quad c_2(\mathcal{N}) = c_1(X)^2 - c_2(X) = 18.$$

Borrow notation from Section 6.4 with the short exact sequence (6.1) being (7.1) and with $V = \operatorname{Lie} A$. Since X is nondivisible, it is not invariant by translation under any nonzero abelian variety; thus the Gauss map $\gamma\colon X \to \operatorname{Gr}_2(\operatorname{Lie} A)$ is a finite morphism [10, Proposition 3.1]. By interpreting $c_2(\mathcal{N})$ as the top Segre class of T_X, the positivity of $c_2(\mathcal{N})$ implies that the morphism $\pi\colon \mathbb{P}(T_X) \to \mathbb{P}(\operatorname{Lie} A)$ is generically finite onto its image. The bootstrap result is the computation of the degree of π. To do this, we use that the difference morphism

$$d\colon X \times X \to X - X \subset A$$

is generically finite of degree $\geqslant 6$ by assumption. It is then a completely general fact that $\deg \pi \geqslant \deg d$ within this framework. More precisely, we have the following lemma.

Lemma 7.3. *Let $Z \subset A$ be an integral smooth subvariety, and suppose that the difference morphism $d_Z\colon Z \times Z \to A$ and the morphism $\pi_Z\colon \mathbb{P}(T_Z) \to \mathbb{P}(\operatorname{Lie} A)$ are generically finite onto their images. Then $\deg \pi_Z \geqslant \deg d_Z$.*

In particular, we have $\deg \pi \geqslant 6$. On other hand, by construction, the degree of π has to divide $c_2(\mathcal{N}) = 18$. Therefore, there are only three possibilities: $\deg \pi \in \{6, 9, 18\}$. This constrains also the degree of γ, as it must divide the degree of π. Moreover, the pullback on X of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\operatorname{Alt}^2 \operatorname{Lie} A)$ via the composite of γ and of the Plücker embedding is $\operatorname{Alt}^2(T_X^\vee) = \Omega_X^2$. It follows that the degree of γ needs to divide also $c_1(\Omega_X^2) = c_1(X)^2 = 45$. As a consequence, we have

$$\deg \gamma \in \{1, 3, 9\}. \tag{7.2}$$

As in Section 6.4, let $Y \subset \mathbb{P}(\operatorname{Lie} A)$ be the image of π. Then Y has dimension 3 and degree

$$\deg Y = c_2(\mathcal{N})/\deg \pi \in \{1, 2, 3\}.$$

On the other hand, the nondegeneracy of X implies that X generates the abelian variety A. In particular, Y is not contained in any hyperplane, which rules out the possibility $\deg Y = 1$. Furthermore, the classical lower bound (see [13, Proposition 0] for instance) $\deg Y \geqslant 1 + \operatorname{codim} Y$ implies that Y has codimension 1 or 2. Moreover, if $\operatorname{codim} Y = 2$, then equality holds in the above inequality. In [13, Proposition 0], subvarieties for which the above inequality is an identity are completely classified. We rule out the codimension 2 by comparing the Fano variety in each of these cases with the Fano variety F of lines on Y.

Proposition 7.4. *We have $g = 5$ and Y is a cubic hypersurface.*

In particular, $\deg \pi = 6$ and hence $\deg \gamma \in \{1, 3\}$. It remains to show that Y is smooth. For this, we will show F is a smooth surface and apply [1, Theorem 4.2]. We begin with the following proposition.

Proposition 7.5. *The Fano variety F is an integral surface and the finite morphism $\gamma \colon X \to F$ is birational.*

To conclude, a comparison of the Hilbert polynomials with respect to the Plücker embedding of X and of the Fano variety F of lines on Y will show the following proposition.

Proposition 7.6. *The hypersurface Y is smooth and $\gamma \colon X \to F$ is an isomorphism.*

The rest of this section is devoted to the proof of Lemma 7.3 and Propositions 7.4, 7.5 and 7.6.

7.3. The projection to the tangent space: Proof of Lemma 7.3. Let us write $d = d_Z$ and $\pi = \pi_Z$. Let $D := Z - Z$ be the image of the difference morphism. Consider the blow-up $\widetilde{A}$ of A in 0, the blow-up $\widetilde{D}$ of D in 0 and the blow-up $\widetilde{P}$ of $P = Z \times Z$ in the diagonal. The normal bundle of the diagonal in $Z \times Z$ is by definition the tangent bundle of Z. It follows that the exceptional divisor of $\widetilde{P} \to P$ is $\mathbb{P}(T_Z)$. The exceptional divisor of $\widetilde{A} \to A$ is $\mathbb{P}(\mathrm{Lie}\, A)$ and that of $\widetilde{D} \to D$ is the tangent cone C of D at 0. Let $\tilde{d} \colon \widetilde{P} \to \widetilde{D}$ be the morphism induced by the difference map. The situation is summarized in the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{P}(T_Z) & \longrightarrow & C & \hookrightarrow & \mathbb{P}(\mathrm{Lie}\, A) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\widetilde{P} & \xrightarrow{\ \tilde{d}\ } & \widetilde{D} & \hookrightarrow & \widetilde{A} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
P = Z \times Z & \xrightarrow{\ d\ } & D = Z - Z & \hookrightarrow & A.
\end{array}
\tag{7.3}
$$

Lemma 7.7. *The composite morphism $\mathbb{P}(T_Z) \to \mathbb{P}(\mathrm{Lie}\, A)$ is π.*

Proof. We reduce to the case $Z = A$ by noticing that d is the restriction to $Z \times Z$ of the difference morphism $A \times A \to A$ and π is the restriction to $\mathbb{P}(T_Z)$ the projection

$$\mathbb{P}(T_A) = \mathbb{P}(\mathrm{Lie}\, A) \times A \to \mathbb{P}(\mathrm{Lie}\, A).$$

Write d as the composite of the automorphism $(x, y) \mapsto (x - y, y)$ of $A \times A$ and the second projection $\mathrm{pr}_2 \colon A \times A \to A$. In this way, we reduce to the situation where we replace d by the second projection pr_2 and $\widetilde{P}$ by the blow-up of $A \times A$ in $A \times \{0\}$. In this case, the result to prove is that the morphism $A \times \mathbb{P}(\mathrm{Lie}\, A) \to \mathbb{P}(\mathrm{Lie}\, A)$ induced by pr_2 is the second projection. This is of course clear and concludes the proof. □

To conclude the proof of Lemma 7.3, notice first that the top-left square in diagram (7.3) is cartesian. Indeed, this boils down to the fact that the scheme-theoretic fiber of d in 0 is the diagonal $\Delta_Z \colon Z \to Z \times Z$. To see this, notice that the difference morphism $A \times A \to A$ is smooth and the scheme-theoretic preimage of 0 is the diagonal $\Delta_A \colon A \hookrightarrow A \times A$ with its reduced structure; the claim then follows because the following square is cartesian:

$$\begin{array}{ccc} Z & \xrightarrow{\Delta_Z} & Z \times Z \\ \downarrow & & \downarrow \\ A & \xrightarrow{\Delta_A} & A \times A. \end{array}$$

Lemma 7.8. *Let $f: S \to T$ be a proper surjective morphism between varieties and $T' \subset T$ a subvariety whose scheme-theoretic preimage $S' = S \times_T T'$ is integral. If the morphisms f and $f_{|S'}: S' \to T'_{\mathrm{red}}$ are generically finite, then* $\deg(f_{|S'}) \geqslant \deg(f)$.

Proof. This follows from upper semicontinuity of the rank of $f_* \mathcal{O}_S$ on the locus in T where f is finite. □

We apply the preceding lemma with $S = \widetilde{P}$, $T = \widetilde{D}$, $f = \widetilde{d}$ and $T' = C$. The hypotheses are fulfilled because $\widetilde{d}$ is generically finite, the scheme-theoretic preimage of C is $\mathbb{P}(T_Z)$, and the restriction of $\widetilde{d}$ to $\mathbb{P}(T_Z)$ is π by Lemma 7.7, hence generically finite by hypothesis. Moreover, the morphism $\widetilde{d}$ is generically finite of same degree of d because blow-ups are birational maps. □

7.4. The cubic hypersurface: Proof of Proposition 7.4. By hypothesis, X is nondegenerate; thus it generates the abelian variety A. It follows that Y cannot be contained in any hyperplane. Therefore, the lower bound $\deg Y \geqslant 1 + \operatorname{codim} Y$ holds by [13, Proposition 0]. As already argued, we have $\deg \pi \in \{3, 9\}$; hence

$$\deg Y = c_2(\mathcal{N}) / \deg \pi \in \{2, 3\}.$$

We start by excluding the case $\deg Y = 2$. In this case, we have $\operatorname{codim} Y = 1$; hence $g = 5$ and Y is a 3-dimensional quadric. To rule out the case, let F be the Fano variety of lines in Y and recall by Proposition 6.12 that the image of $\gamma: X \to F$ is a 2-dimensional irreducible component of F. The rank r of the quadric Y is necessarily 3, 4 or 5 because Y is integral. We proceed now case by case.

- If $r = 3$, then Y is a cone over a smooth plane conic Q and F is irreducible of dimension $\dim Q + 2(5 - 3 - 1) = 3$ by Proposition 6.3. Contradiction.
- If $r = 4$, then Y is a cone over the rational normal scroll $S(1, 1) = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ via the Segre embedding [18, p. 285]. By Proposition 6.1, the Fano variety of lines in $S(1, 1)$ is $\mathbb{P}^1 \sqcup \mathbb{P}^1$, so F has two irreducible components of dimension $\dim \mathbb{P}^1 + 2(5 - 4) = 3$ by Proposition 6.3. Contradiction.
- If $r = 5$, then Y is a smooth quadric threefold in $\mathbb{P}^4$ and $F \cong \mathbb{P}^3$; see for instance [18, Example 22.6]. Contradiction.

The above discussion shows $\deg Y = 3$. The inequality $\deg Y \geqslant 1 + \operatorname{codim} Y$ implies that $Y \subset \mathbb{P}(\operatorname{Lie} A)$ has codimension 1 or 2. It remains to rule out the codimension 2 case. If this is the case, the variety Y is of minimal degree; hence, by [13, Theorem 1], it is either a cone over a smooth rational scroll or a cone over the Veronese surface, that is, $\mathbb{P}^2$ embedded in $\mathbb{P}^5$ by the second Veronese embedding. Now taking a cone with respect to some linear projection preserves the codimension. This rules out at once the case of the Veronese surface since it has codimension 3, while Y has codimension 2. Suppose Y is a smooth rational normal scroll $S = S(a)$ or a cone over such a scroll. By Proposition 6.12, the scheme-theoretic image of the finite morphism $\gamma\colon X \to F$ is one of the irreducible components of F. A case by case distinction shows that this is a contradiction. More precisely, the tuple $a = (a_1, \dots, a_r)$ satisfies $1 \leqslant a_1 \leqslant \cdots \leqslant a_r$ and $a_1 + \cdots + a_r = 3$. This leaves us with the 3 possibilities described in detail in Example 6.2:

$$a = (3), (1, 2), (1, 1, 1).$$

Suppose $\dim S = 1$. In this case, the scroll S is the image of the triple Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. As S contains no lines, Proposition 6.3 implies that F is irreducible of dimension 3. But X dominates a component of F. Contradiction.

Suppose $\dim S = 2$. In this case, the Fano variety is $\mathbb{P}^1 \sqcup \{\mathrm{pt}\}$. By Proposition 6.3, the component of F over $\mathbb{P}^1$ has dimension $\dim \mathbb{P}^1 + 2(6-5) = 3$; hence X cannot dominate it. Instead, again by Proposition 6.3, the component of F over the singleton is a projective plane embedded linearly via the Plücker embedding. If X were to dominate it, we would have $\deg \gamma = 45$, contradicting (7.2).

Suppose $\dim S = 3$. In this case, $Y = S$ and $F \simeq (\mathbb{P}^1 \times \mathbb{P}^2) \sqcup \mathbb{P}^2$. It follows that X has to dominate the component isomorphic to the projective plane. But this component is linearly embedded via the Plücker embedding. As above, if X were to dominate this component, then $\deg \gamma = 45$, which contradicts (7.2). □

7.5. Birationality of the Gauss map: Proof of Proposition 7.5. By Proposition 6.9, we have $\dim F \leqslant 3$, with equality if and only if Y is either the twisted plane in Example 6.8 or a cone over a nonnormal cubic surface or a cone over a smooth cubic plane curve. These cases are excluded by looking at the Fano variety.

- If Y is a cone over an integral cubic curve, then F is irreducible and has dimension 3; see Example 6.4.
- If Y is a cone over a nonnormal cubic surface that is not itself a cone over a nonnormal cubic curve, then the 2-dimensional irreducible components of F are linearly embedded via the Plücker embedding; see Example 6.6. This would imply $\deg \gamma = 45$, which contradicts (7.2).
- If Y is the twisted plane, then by Example 6.8, the 2-dimensional irreducible components of F have degree 1 and 4 with respect the Plücker embedding; this again contradicts (7.2).

The above shows that we can assume $\dim F = 2$. To conclude that F is integral, it suffices to show that the Gauss map $\gamma\colon X \to F$ is birational onto its image. Indeed, assuming this, let $\mathcal{L}$ be the line bundle on $F \subset \mathrm{Gr}_2(\mathrm{Lie}\, A)$ which is the restriction of $\mathcal{O}(1)$ on $\mathbb{P}(\mathrm{Alt}^2 \mathrm{Lie}\, A)$ via the Plücker embedding. Then $\mathcal{L}$ and $\gamma^*\mathcal{L}$ have self-intersection

$$\mathcal{L}.\mathcal{L} = 45 = \gamma^*\mathcal{L}.\gamma^*\mathcal{L}.$$

The first identity holds by Proposition 6.7; the second is $c_1(X)^2 = 45$ since $\gamma^*\mathcal{L}$ is the canonical bundle of X by definition of γ. Since γ is birational onto its image and $\mathcal{L}$ is ample, this forces F to be irreducible. Proposition 6.12 then implies that F is generically reduced, hence satisfies condition R_0 because its singular locus has codimension $\geqslant 1$. But Proposition 6.7 implies that F is a locally complete intersection, thus satisfies condition S_1. Therefore, F is reduced by [17, Proposition 5.8.5].

We now show that $\gamma\colon X \to F$ is birational onto its image. To do this, we mimic the relation between incidence divisors and the canonical bundle of the Fano surface of a smooth cubic threefold [8, p. 326]. For any $x \in X$, consider the line $L_x := \mathbb{P}(T_x X)$ in $\mathbb{P}(\mathrm{Lie}\, A)$. Inside $X \times X$, consider the locus of couples (x, y) with $L_x \cap L_y \neq \varnothing$ and let

$$D \subset X \times X$$

be the union of its top dimensional components. For $x \in X$, let $D_x = p(q^{-1}(x))$ be the fiber of the second projection $q\colon D \to X$, considered as a subvariety of X via the first projection $p\colon D \to X$. By [33, Lemma II.12], the nondegeneracy of X implies $D_x \neq X$. On the other hand, since $\pi\colon \mathbb{P}(T_X) \to Y$ is generically finite of degree 6 and $\deg(\gamma) \in \{1, 3\}$, there are at least two distinct lines L_x, $L_{x'}$ through a generic $y \in Y$. Thus D_x is a divisor on X for generic $x \in X$.

Lemma 7.9. *Let $U \subset X$ be the open subset where $q\colon D \to X$ is flat.*

(1) *There exist $x, y, z \in U$ such that the lines $L_x, L_y, L_z \subset \mathbb{P}(\mathrm{Lie}\, A)$ are pairwise distinct and span a plane.*

(2) *For any such x, y, z, the sum $D_x + D_y + D_z$ is a canonical divisor on X.*

Proof. (1) By assumption, there exists an integral fiber $I \subset X^3$ of dimension $\geqslant 3$ of the sum morphism $X^3 \to A$. We first claim that, for generic $(x, y, z) \in I$, the three lines L_x, L_y, L_z are pairwise distinct: otherwise, the fiber I would be contained in the preimage of the big diagonal $\Delta \subset \mathrm{Gr}_2(\mathrm{Lie}\, A)^3$ under the morphism

$$\gamma^3\colon X^3 \to \mathrm{Gr}_2(\mathrm{Lie}\, A)^3.$$

Now Δ has three irreducible components which are permuted under the action of the symmetric group S_3. Since I is irreducible and stable under the action of S_3, it must lie in the preimage of the intersection the irreducible components of Δ. This intersection is the small diagonal of $\mathrm{Gr}_2(\mathrm{Lie}\, A)^3$ and its preimage via γ^3 has dimension 2 because γ is finite and $\dim X = 2$, thus contradicting $\dim I \geqslant 3$.

We next claim that, for any $(x, y, z) \in I$, the lines L_x, L_y, L_z are coplanar. Since coplanarity is a closed condition, it suffices to show this for generic $(x, y, z) \in I$. The tangent map of the sum morphism $X^3 \to A$ has generic rank at most 3 on I because it contracts I. Since the sum map on A induces the sum on tangent spaces, this implies $\dim(T_x X + T_y X + T_z X) \leqslant 3$ for generic $(x, y, z) \in I$. Passing to projective spaces, this means that $L_x, L_y, L_z \subset \mathbb{P}(\mathrm{Lie}\, A)$ are coplanar.

Finally, we claim that $I \subset X^3$ dominates X via each projection: since I is irreducible and stable under permutations, we would otherwise have $I = C^3$ for a curve $C \subset X$, but then the image of I under the sum map would be $C + C + C \subset A$, which is not a point.

(2) Let P be the plane in $\mathbb{P}(\mathrm{Lie}\, A)$ spanned by L_x, L_y and L_z and $H \subset \mathrm{Gr}_2(\mathrm{Lie}\, A)$ the subvariety of points in $\mathrm{Gr}_2(\mathrm{Lie}\, A)$ corresponding to lines meeting P. The subvariety H is a hyperplane section of the Plücker embedding. The nondegeneracy of X implies $\gamma(X)$ meets any such hyperplane properly [33, Lemma II.12]; hence $\gamma^{-1}(H)$ is a divisor linearly equivalent to $\gamma^* \mathcal{L}$, which is isomorphic to the canonical bundle of X. By construction, we have

$$P \cap Y = L_x \cup L_y \cup L_z;$$

hence $\gamma^{-1}(H) = D_x \cup D_y \cup D_z$, which concludes the proof. □

Corollary 7.10. *For $U \subset X$ as above and $x, y \in U$, we have $D_x.D_y = 5$.*

Proof. By design, the divisors D_x form a flat family over U; thus the intersection number $n = D_x.D_y$ is independent of x and y. For $x, y, z \in U$ such that the lines L_x, L_y and L_z are pairwise distinct and span a plane, we thus have

$$9n = (D_x + D_y + D_z)^2 = c_1(X)^2 = 45,$$

because $D_x + D_y + D_z$ is a canonical divisor by Lemma 7.9. Therefore, $n = 5$. □

We can now prove that γ is birational onto its image: the divisors D_x are the pullback of Weil divisors on $\gamma(X)$ since the incidence relation defining them comes from $\mathrm{Gr}_2(\mathrm{Lie}\, A)$. The projection formula then implies that $\deg(\gamma)$ divides $D_x.D_y$ for $x, y \in U$. As $\deg(\gamma) \in \{1, 3\}$ and $D_x.D_y = 5$, it follows that $\deg(\gamma) = 1$. □

7.6. Smoothness: Proof of Proposition 7.6. It follows from Proposition 6.7 that the Hilbert polynomial of F with respect to the Plücker embedding is

$$\chi(F, \mathscr{L}^{\otimes i}) = 45\binom{i+1}{2} - 45i + 6,$$

where $\mathscr{L}$ is the line bundle on $\mathrm{Gr}_2(\mathrm{Lie}\, A)$ obtained as restriction via the Plücker embedding of the line bundle $\mathscr{O}(1)$ on $\mathbb{P}(\mathrm{Alt}^2 \mathrm{Lie}\, A)$. On the other hand, for all $i \geqslant 0$,

$$\chi(X, \gamma^*\mathscr{L}^{\otimes i}) = 45\binom{i+1}{2} - 45i + 6.$$

To see this, first notice that $\gamma^*\mathscr{L} \simeq \Omega_X^2$ by definition of the morphism γ; thus, by Serre's duality theorem, we have

$$\chi(X, \gamma^*\mathscr{L}) = \chi(X, \mathscr{O}_X) = 6.$$

On the other hand, the self-intersection of $\gamma^*\mathscr{L}$ is $c_1(X)^2 = 45$; hence the Hirzebruch–Riemann–Roch theorem implies the wanted identity. Now, consider the short exact sequence of $\mathscr{O}_F$-modules

$$0 \to \mathscr{O}_F \to \gamma_*\mathscr{O}_X \to Q \to 0.$$

Since the morphism γ is finite, the $\mathscr{O}_F$-module $\gamma_*\mathscr{O}_X$ is coherent; thus so is Q. Taking the tensor product with $\mathscr{L}^{\otimes i}$, we have the short exact sequence

$$0 \to \mathscr{L}^{\otimes i} \to \gamma_*\gamma^*\mathscr{L}^{\otimes i} \to Q \otimes \mathscr{L}^{\otimes i} \to 0$$

for all integers $i \geqslant 0$. The finiteness of the morphism γ also implies the vanishing of the higher direct images $\mathrm{R}^q\gamma_*\gamma^*\mathscr{L}^{\otimes i}$ for $q \geqslant 1$, which yields the identity

$$\chi(F, \gamma_*\gamma^*\mathscr{L}^{\otimes i}) = \chi(X, \gamma^*\mathscr{L}^{\otimes i}).$$

Since Euler characteristics are additive in short exact sequences, this implies that the Hilbert polynomial of Q vanishes altogether; hence $Q = 0$. The morphism γ being finite, this implies that $\gamma\colon X \to F$ is an isomorphism. By hypothesis, X is smooth; thus so is F. Since F has dimension 2, we can apply [1, Theorem 4.2] to derive that Y is smooth, too. □

8. How to rule out the Tannaka group E_7

In this section, which is largely independent of the rest of the paper, we explain how to rule out the occurrence of the Tannaka group E_7.

8.1. Main result. We will rule out E_7 via a general criterion for a subvariety to have a big Tannaka group if the alternating or symmetric convolution square has few summands. We go back to the notation introduced in Section 1.1, so let A be an abelian variety of dimension g over an algebraically closed field k of characteristic zero, and consider a smooth integral subvariety $X \subset A$ of dimension d. We again fix a fiber functor $\omega\colon \langle \delta_X \rangle \to \mathrm{Vect}(\mathbb{F})$ and denote the corresponding Tannaka group by

$$G_{X,\omega} \subset \mathrm{GL}(V), \quad \text{where } V := \omega(\delta_X).$$

The derived category of constructible sheaf complexes is a symmetric pseudoabelian tensor category with respect to the convolution product. So we have a decomposition

$$\delta_X * \delta_X = \mathrm{Sym}^2(\delta_X) \oplus \mathrm{Alt}^2(\delta_X),$$

where the involution of $\delta_X * \delta_X$ permuting the two factors acts as id on $\mathrm{Sym}^2(\delta_X)$ and $-$id on $\mathrm{Alt}^2(\delta_X)$. Define

$$T_+(\delta_X) := \begin{cases} \mathrm{Sym}^2(\delta_X) & \text{if } d \text{ is even,} \\ \mathrm{Alt}^2(\delta_X) & \text{if } d \text{ is odd.} \end{cases}$$

If $X \subset A$ is *symmetric up to translation* in the sense that $X = a - X$ for some point $a \in A(k)$, then the fiber of the sum morphism $\sigma\colon X \times X \to A$ over this point is the antidiagonal

$$\sigma^{-1}(a) = \{(x, a - x) \mid x \in X\}.$$

By proper base change and the decomposition theorem [3, Theorem 6.2.5], we then have a skyscraper direct summand $\delta_a \subset \delta_X * \delta_X$, and a closer look at the parity shows $\delta_a \subset T_+(\delta_X)$; see [20, Section 1.2], [26, Lemma 2.1]. When X is nondivisible, the point a is unique and we define

$$T_+(\delta_X) = S_+(\delta_X) \oplus \delta_a$$

for a unique complex $S_+(\delta_X)$. By the dimension formula (2.1), the skyscraper direct summand corresponds to a one-dimensional representation of the Tannaka group, and the projection onto it defines a bilinear form $\theta\colon V \otimes V \to \omega(\delta_a)$ which is symmetric if d is even and alternating if d is odd. Hence the derived subgroup $G^*_{X,\omega}$ of the connected component of $G_{X,\omega}$ is contained in $\mathrm{SO}(V, \theta)$ if d is even and in $\mathrm{Sp}(V, \theta)$ if d is odd. To have a uniform notation, when X is not symmetric up to translation, we set $S_+(\delta_X) := T_+(\delta_X)$.

Theorem 8.1. *If $X \subset A$ is nondivisible with ample normal bundle and $d < g/2$, then $S_+(\delta_X)$ is a perverse sheaf without negligible direct summands. If $S_+(\delta_X)$ is simple, then*

$$G^*_{X,\omega} = \begin{cases} \mathrm{SL}(V) & \text{if } X \text{ is not symmetric up to translation,} \\ \mathrm{SO}(V, \theta) & \text{if } X \text{ is symmetric up to translation and } d \text{ is even,} \\ \mathrm{Sp}(V, \theta) & \text{if } X \text{ is symmetric up to translation and } d \text{ is odd.} \end{cases}$$

Note that, in comparison to Larsen's alternative in [20, Lemma 3.7], we here only need an assumption on the symmetric or alternating square rather than on the full tensor square. The rest of this section will be devoted to the proof of the above criterion; see Section 8.5. But before coming to the proof, let us show how Theorem 8.1 rules out the exceptional Tannaka group E_7.

Corollary 8.2. *If $X \subset A$ has ample normal bundle and $d < g/2$, then $G^*_{X,\omega} \not\simeq E_7$.*

Proof. Suppose $G^*_{X,\omega} \simeq E_7$. If $X \subset A$ descends under an isogeny $f\colon A \to B$, then $f(X) \subset B$ is still smooth nondegenerate and irreducible of the same dimension with Tannaka group E_7 by [20, Remark 2.6], [24, Corollary 1.6(a)]. Hence, in what follows, we assume that X is nondivisible. Then, by [24, Corollary 1.10], the representation $V = \omega(\delta_X)$ of $G^*_{X,\omega}$ is minuscule and hence isomorphic to the 56-dimensional irreducible representation of E_7. By the highest weight theory of the exceptional group E_7, then $\mathrm{Alt}^2(V) \simeq W \oplus \mathbf{1}$, where W is irreducible and $\mathbf{1}$ is the trivial representation of dimension one. In particular, there is an alternating bilinear form on V which is invariant under the action of E_7. It follows from the parity discussion in [20, Section 1.2] that d is odd and, after a suitable translation, that X is symmetric. Then $\mathrm{Alt}^2(\delta_X) = S_+(\delta_X) \oplus \delta_0$ and $S_+(\delta_X)$ is perverse by the first part of Theorem 8.1 and corresponds to the representation $\omega(S_+(\delta_X)) \simeq W$. The latter is irreducible, so $S_+(\delta_X)$ is a simple perverse sheaf up to negligible direct summands. The second part of Theorem 8.1 then says that $G^*_{X,\omega}$ is the full symplectic group $\mathrm{Sp}(V, \theta)$, a contradiction. □

8.2. Topological preliminaries. In this section, we work over $k = \mathbb{C}$. For the proof of Theorem 8.1, we will detect perverse direct summands of a convolution product from the top cohomology of the fibers of the sum morphism, using the following well-known fact.

Lemma 8.3. *Let Z be a proper complex variety of dimension d, and let n be the number of its irreducible components of dimension d. Then $\dim_{\mathbb{C}} \mathrm{H}^{2d}(Z, \mathbb{C}) = n$.*

Proof. The intersection of distinct irreducible components has real codimension at least two; hence, via the Mayer–Vietoris sequence, one reduces to the case where Z is irreducible. The same argument allows us to pass to a resolution of singularities, so the claim follows. □

The above lemma is about the cohomology with constant coefficients, so we want a criterion to decide when the perverse intersection complex on a singular variety is the constant sheaf. A complex variety W is a *rational homology manifold* if

$$\mathrm{H}^i_{\{x\}}(W, \mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{for } i = 2\dim W, \\ 0 & \text{for } i \neq 2\dim W. \end{cases}$$

Any smooth variety is a rational homology manifold. More generally, we have the following.

Lemma 8.4. *For any rational homology manifold X and any $n \in \mathbb{N}$, the symmetric power $W = \mathrm{Sym}^n X$ is again a rational homology manifold. In particular, if X is irreducible, then*

$$\delta_W \simeq \mathbb{C}_W[m], \quad \textit{where } m = \dim W = n \dim X.$$

Proof. Any quotient of a rational homology manifold by the action of a finite group is a rational homology manifold [5, Proposition A.1 (iii)]. So $\mathrm{Sym}^n X = X^n/\mathfrak{S}_n$ is a rational homology manifold. But the intersection complex on any irreducible rational homology manifold is the constant sheaf [19, Proposition 8.2.21]. □

8.3. Kashiwara's estimate. In this section, we continue to work over $k = \mathbb{C}$. We will control negligible summands in convolution products in terms of characteristic varieties. As in [20, Definition 2.2], we attach to any subvariety $Y \subset A$ its projective conormal variety

$$\mathbb{A}_Y \subset \mathbb{P}(\Omega_A^1) = A \times \mathbb{P}_A,$$

where $\mathbb{P}_A = \mathbb{P}(\mathrm{Lie}(A)^\vee)$. The *characteristic variety* of a perverse sheaf $P \in \mathrm{Perv}(A)$ is defined as the support $\mathrm{Char}(P) = \mathrm{Supp}\,\mathrm{CC}(P)$ of the characteristic cycle $\mathrm{CC}(P)$, including possible negligible components. Its projectivization $\mathbb{P}\,\mathrm{Char}(P) \subset \mathbb{P}(\Omega_A^1)$ is a finite union of projective conormal varieties. For a sheaf complex $K \in \mathrm{D}_c^b(A, \mathbb{C})$, we denote by

$$\mathbb{P}\,\mathrm{Char}(K) = \bigcup_{i \in \mathbb{Z}} \mathbb{P}\,\mathrm{Char}({}^p\mathcal{H}^i(K))$$

the union of the characteristic varieties of all its perverse cohomology sheaves. We can then control the characteristic varieties of convolution products $K = P * P$ by the following incarnation of Kashiwara's estimate.

Lemma 8.5. *Let $P_1, P_2 \in \mathrm{Perv}(A)$. If the Gauss maps $\mathbb{P}\,\mathrm{Char}(P_i) \to \mathbb{P}_A$ are finite for both $i = 1, 2$, then so is the Gauss map $\mathbb{P}\,\mathrm{Char}(P_1 * P_2) \to \mathbb{P}_A$ and hence $\mathbb{P}\,\mathrm{Char}(P_1 * P_2)$ does not contain any negligible components.*

Proof. By definition, $P_1 * P_2 = R\sigma_*(P_1 \boxtimes P_2)$, where $\sigma\colon A \times A \to A$ denotes the sum morphism. Thus, by Kashiwara's upper estimate for the characteristic variety of proper direct images [21, Theorem 4.2 (b)], we have

$$\mathbb{P}\,\mathrm{Char}(P_1 * P_2) \subset \widetilde{\sigma}(\mathbb{A}_1 \times_{\mathbb{P}_A} \mathbb{A}_2),$$

where $\mathbb{A}_i = \mathbb{P}\,\mathrm{Char}(P_i)$ and $\widetilde{\sigma} = \sigma \times \mathrm{id}_{\mathbb{P}_A}$. Our finiteness assumption on the Gauss map implies that the projection $f\colon \mathbb{A}_1 \times_{\mathbb{P}_A} \mathbb{A}_2 \to \mathbb{P}_A$ is a finite morphism. The morphism f factors through the projection $g\colon \widetilde{\sigma}(\mathbb{A}_1 \times_{\mathbb{P}_A} \mathbb{A}_2) \to \mathbb{P}_A$. Thus g is finite and so is its restriction to the subvariety $\mathbb{P}\,\mathrm{Char}(P_1 * P_2) \subset \widetilde{\sigma}(\mathbb{A}_1 \times_{\mathbb{P}_A} \mathbb{A}_2)$. □

Corollary 8.6. *Let $P_1, P_2 \in \mathrm{Perv}(A)$. If the Gauss maps $\mathbb{P}\,\mathrm{Char}(P_i) \to \mathbb{P}_A$ are both finite, then for all $i \in \mathbb{Z}$ and any perverse direct summand $Q \subset {}^p\mathcal{H}^i(P_1 * P_2)$, the support $Y = \mathrm{Supp}(Q)$ is either of general type (in which case Q is not negligible), or $Y = A$.*

Proof. For $Q \subset {}^p\mathcal{H}^i(P_1 * P_2)$ with support Y, we have $\mathbb{A}_Y \subset \mathbb{P}\,\mathrm{Char}(P_1 * P_2)$. □

8.4. Decomposition of the convolution square. The main point in the proof of Theorem 8.1 will be to deduce from our assumption about the symmetric or alternating square the decomposition of the full tensor square. To explain this, we go back to the notation of Section 8.1. Let $T_-(\delta_X)$ denote the direct summand complementary to $T_+(\delta_X)$ in the decomposition of the convolution square $\delta_X * \delta_X$, so that

$$\delta_X * \delta_X = T_+(\delta_X) \oplus T_-(\delta_X).$$

We control the second direct summand in this decomposition as follows.

Proposition 8.7. *Suppose that $X \subset A$ has ample normal bundle and $d < g/2$.*

(1) *The convolution $\delta_X * \delta_X$ is a perverse sheaf without negligible summands.*

(2) *If $S_+(\delta_X)$ is simple, then*

(a) *the sum morphism $f\colon \mathrm{Sym}^2 X \to X + X$ is birational, and*

(b) *$T_-(\delta_X)$ is a simple perverse sheaf with support $X + X$.*

Proof. We can assume $k = \mathbb{C}$ and work in the analytic framework.

(1) By construction, $\delta_X * \delta_X$ is a sheaf complex with support $X + X$, which by our dimension assumption is a strict subvariety of A. Since for any smooth subvariety $X \subset A$ with nontrivial ample normal bundle the Gauss map $\mathbb{A}_X \to \mathbb{P}_A$ is finite, we see from Corollary 8.6 that the convolution $\delta_X * \delta_X$ cannot contain any negligible direct summand. The claim now follows from the general fact that the convolution of any two simple perverse sheaves is a direct sum of a semisimple perverse sheaf and a negligible complex, due to the decomposition theorem [3, Theorem 6.2.5] and generic vanishing [22, Lemma 4.3c].

(2) Part (1) implies that $S_+(\delta_X)$ is perverse. Now, the sum $X \times X \to X + X$ is the composite of the quotient morphism $\pi\colon X \times X \to \mathrm{Sym}^2 X$ with a unique map

$$f\colon \mathrm{Sym}^2 X \to X + X.$$

Write $\pi_*(\mathbb{C}_{X\times X}) \simeq L_+ \oplus L_-$, where L_+ are the invariants under the action of the deck transformation group of the double cover π and where L_- are the anti-invariants. Both L_+ and L_- are constructible sheaves of generic rank one on $\mathrm{Sym}^2 X$. In fact, by adjunction, one sees that $L_+ \simeq \mathbb{C}_{\mathrm{Sym}^2 X}$, and by Lemma 8.4, this sheaf is the perverse intersection complex on $\mathrm{Sym}^2 X$ up to a shift. The definition of the commutativity constraint for the convolution product in [40, Section 2.1] moreover shows

$$T_+(\delta_X) \simeq Rf_* L_+[2d] \quad \text{and} \quad T_-(\delta_X) \simeq Rf_* L_-[2d].$$

The decomposition theorem [3, Theorem 6.2.5] together with the fact that f is generically finite implies that $\delta_{X+X} \subset T_+(\delta_X)$. Since by assumption $S_+(\delta_X)$ is a simple perverse sheaf, we see that

$$S_+(\delta_X) = \delta_{X+X}.$$

Then $f_* L_+ \simeq \mathcal{H}^{-2d}(T_+(\delta_X))$ has generic rank one on its support $X + X$. Since by base change the generic rank of this direct image is the degree of the generically finite morphism

$$f\colon \mathrm{Sym}^2 X \to X + X,$$

it follows that f is birational. This implies that also $\mathcal{H}^{-2d}(T_-(\delta_X)) \simeq f_* L_-$ has generic rank one on its support $X + X$. The structure of perverse sheaves then forces

$$T_-(\delta_X) = Q \oplus R,$$

where Q is a simple perverse sheaf of generic rank one on its support $X + X$ and R is either zero or a semisimple perverse sheaf with strictly smaller support. Note that R cannot be a skyscraper sheaf: by adjunction [40, Corollary 1], any skyscraper summand of $\delta_X * \delta_X$ is supported at a point $a \in A(\mathbb{C})$ with $X = a - X$, but since X is nondivisible, there is at most one such skyscraper sheaf and it enters $T_+(\delta_X)$.

To show $R = 0$, we argue by contradiction. Suppose $R \neq 0$. The support $Y = \mathrm{Supp}(R)$ has dimension $e = \dim Y < 2d = \dim X + X$. For general $y \in Y$, we have

$$0 \neq \mathcal{H}^{-e}(R)_y \subset \mathcal{H}^{-e}(T_-(\delta_X))_y = R^{2d-e} f_*(L_-)_y \simeq \mathrm{H}^{2d-e}(F, L_{-|F})$$

for the fiber $F = f^{-1}(y)$. This nonvanishing forces $2 \dim F \geqslant 2d - e$. On the other hand,

$$\mathcal{H}^{2\dim F - 2d}(T_+(\delta_X))_y = R^{2\dim F} f_*(\mathbb{C}_{\mathrm{Sym}^2 X}) \simeq \mathrm{H}^{2\dim F}(F, \mathbb{C}) \neq 0$$

by Lemma 8.3. It follows that we have $2 \dim F = 2d - e$. Indeed, if $2 \dim F > 2d - e$, then the above nontrivial stalk cohomology would come from a non-perverse direct summand in $T_+(\delta_X)$, contradicting (1). The identity $2 \dim F = 2d - e$ implies the above nontrivial stalk cohomology comes from a perverse direct summand in $S_+(\delta_X)$ whose support Y is strictly smaller than $X + X$. This contradicts the simplicity of $S_+(\delta_X)$. □

The above statement is false if we exchange the roles of $S_+(\delta_X)$ and $T_-(\delta_X)$. Indeed, if X is the Fano surface of lines on a smooth cubic threefold embedded in $A = \mathrm{Alb}(X)$, then $T_-(\delta_X) = \mathrm{Alt}^2(V)$ is simple but $S_+(\delta_X) = \mathrm{Sym}^2 V$ is not.

8.5. Proof of Theorem 8.1. Part (1) of Proposition 8.7 says that $S_+(V)$ is a perverse sheaf without negligible direct summands, so it only remains to show that if $S_+(\delta_X)$ is a simple perverse sheaf, then

$$G^*_{X,\omega} = \begin{cases} \mathrm{SL}(V) & \text{if } X \text{ is not symmetric up to translation,} \\ \mathrm{SO}(V, \theta) & \text{if } X \text{ is symmetric up to translation and } d \text{ is even,} \\ \mathrm{Sp}(V, \theta) & \text{if } X \text{ is symmetric up to translation and } d \text{ is odd.} \end{cases}$$

But this follows from part (2) (b) of Proposition 8.7 and from the version of Larsen's alternative given in [20, Lemma 3.7]. For convenience, we recall the simple argument when the subvariety $X \subset A$ is symmetric up to translation, which is the only one needed for the application to E_7. The symmetry of X implies that the representation V is self-dual and we have an isomorphism $V \otimes V \simeq \mathrm{End}(V)$. Let $\mathfrak{h} \subset \mathrm{End}(V)$ be the Lie algebra of $\mathrm{SO}(V, \theta)$ if d is even and of $\mathrm{Sp}(V, \theta)$ if d is odd. Via the preceding isomorphism, we have

$$\mathfrak{h} \simeq \begin{cases} \mathrm{Alt}^2(V) & \text{if } d \text{ is even,} \\ \mathrm{Sym}^2(V) & \text{if } d \text{ is odd.} \end{cases}$$

The representation $\mathfrak{h}$ of $G_{X,\omega}$ therefore corresponds via Tannaka duality to the perverse sheaf $S_-(\delta_X)$, which is simple by part (2) (b) of Proposition 8.7. Thus $\mathfrak{h}$ is an irreducible representation of the group $G_{X,\omega}$. On the other hand, the adjoint representation $\mathfrak{g}$ of this group is a subrepresentation of $\mathfrak{h}$; thus $\mathfrak{g} = \mathfrak{h}$. □

A. Hodge number estimates

Let X be a smooth complex projective variety of dimension d. The arguments of Lazarsfeld–Popa [30, Theorem C (iii)] and Lombardi [31, Corollary 2.3] can be applied in a slightly more general context to get bounds on the Euler characteristics $\chi(X, \Omega_X^p)$ with the Albanese variety replaced by an arbitrary ambient abelian variety. For a linear subspace $V \subset \mathrm{H}^0(X, \Omega_X^1)$, put

$$m(V) := \min\{\operatorname{codim}(X, Z(\omega)) \mid 0 \neq \omega \in V\},$$

where $Z(\omega) \subset X$ denotes the zero locus of $\omega \in \mathrm{H}^0(X, \Omega_X^1)$ as in [31, Corollary 2.3]. We will be interested in the case $m(V) = d$, which is to say that every nonzero differential form $\omega \in V$ has at most isolated zeroes. In this case, we have the following result.

Theorem A.1. *If $V \subset \mathrm{H}^0(X, \Omega_X^1)$ satisfies $m(V) = d$ and $\dim V > d + 1$, then*

$$(-1)^{d-p}\chi(X, \Omega_X^p) \geqslant \begin{cases} \dim V - d + 1 & \text{for } p \in \{0, d\}, \\ 2 & \text{for } p \in \{1, d-1\}, \\ 1 & \text{for } 2 \leqslant p \leqslant d-2. \end{cases}$$

Proof. For $0 \leqslant p \leqslant d$ and any $v \in V \smallsetminus \{0\}$, the cup product with v gives rise to a complex

$$\mathbb{L}_{X,v}^p \colon [0 \longrightarrow \mathrm{H}^0(X, \Omega_X^p) \xrightarrow{v\smile} \mathrm{H}^1(X, \Omega_X^p) \xrightarrow{v\smile} \cdots \xrightarrow{v\smile} \mathrm{H}^d(X, \Omega_X^p) \longrightarrow 0].$$

For $m(V) = d$, this complex is exact in all degrees $i \neq d - p$ by [16, Proposition 3.4], Hodge symmetry and Serre duality; see [31, Proposition 2.1]. By varying the vector v, we can view these complexes as the fibers of a complex of vector bundles on the projective space $\mathbb{P} = \mathbb{P}(V)$, the Green–Lazarsfeld derivative complex

$$\mathbb{L}_{X,V}^p \colon [\cdots \to \mathcal{O}_{\mathbb{P}}(-d+i) \otimes \mathrm{H}^i(X, \Omega_X^p) \xrightarrow{\delta_i} \mathcal{O}_{\mathbb{P}}(-d+i+1) \otimes \mathrm{H}^{i+1}(X, \Omega_X^p) \to \cdots]$$

sitting in degrees $i = 0, 1, \ldots, d$. By the above, we have

(1) $\mathcal{H}^i(\mathbb{L}_{X,V}^p) = 0$ for all $i \neq d - p$, and

(2) the morphisms δ_i have constant rank for all i.

Property (2) implies that the cokernel $F = \operatorname{Coker}(\delta_{d-p-1})$ is locally free and then also that

$$E := \operatorname{Ker}\big(F \to \mathcal{O}_{\mathbb{P}}(-p+1) \otimes \mathrm{H}^{d-p+1}(X, \Omega_X^p)\big)$$

is locally free. Property (1) then shows that the rank of the vector bundle E is given by

$$\operatorname{rk}(E) = (-1)^{d-p}\chi(X, \Omega_X^p).$$

This shows $(-1)^{d-p}\chi(X, \Omega_X^p) \geqslant 1$, since necessarily $E \neq 0$: else the complex $\mathbb{L}_{X,V}^p$ would be exact in all degrees, which is impossible because any exact sequence of direct sums of line bundles on $\mathbb{P} = \mathbb{P}(V)$ with differentials of degree one has length $\geqslant \dim V + 1$ by [31, Lemma 2.2], and we assumed $\dim V > d$.

When $p = d - 1$, let $G := \mathrm{Ker}(\delta_1)$, so the complex $\mathbb{L}^p_{X,V}$ becomes an exact sequence

$$0 \to G \to \mathcal{O}_{\mathbb{P}}(-d+1) \otimes \mathrm{H}^1(X, \Omega_X^{d-1}) \to \cdots \to \mathcal{O}_{\mathbb{P}} \otimes \mathrm{H}^d(X, \Omega_X^{d-1}) \to 0.$$

The differential δ_0 factors through an injective map $h\colon \mathcal{O}_{\mathbb{P}}(-d) \otimes \mathrm{H}^0(X, \Omega_X^1) \hookrightarrow G$. The cokernel of h is locally free and it is nonzero by [31, Lemma 2.2]. If it were of rank 1, it would be of the form $\mathcal{O}_{\mathbb{P}}(j)$ for some integer j; hence we would have a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \otimes \mathrm{H}^0(X, \Omega_X^1) \to G \to \mathcal{O}_{\mathbb{P}}(j) \to 0.$$

The vanishing of $\mathrm{H}^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d+j))$ implies that the previous short exact sequence is split; hence G is a sum of line bundles on $\mathbb{P}$. By [31, Lemma 2.2], this is not possible; thus

$$-\chi(X, \Omega_X^{d-1}) = \mathrm{rk}\, G \geqslant 2.$$

It remains to prove the estimate $\chi(X, \Omega_X^d) \geqslant \dim V - d + 1$. For this, take $p = d$, and let $G := \mathrm{Ker}(\delta_0)$. We then have an exact sequence of vector bundles

$$0 \to G \to \mathcal{O}_{\mathbb{P}}(-d) \otimes \mathrm{H}^0(X, \Omega_X^d) \to \mathcal{O}_{\mathbb{P}}(-d+1) \otimes \mathrm{H}^1(X, \Omega_X^d) \to \cdots.$$

Taking the dual of the above sequence, one sees that

$$\mathrm{H}^i(\mathbb{P}, G^\vee(j)) = 0 \quad \text{for all } j \in \mathbb{Z} \text{ and all } i \in \{1, 2, \ldots, \dim V - d - 1\}.$$

If $\mathrm{rk}\, G \leqslant \dim V - d$, then by the Evans–Griffiths theorem [29, Example 7.3.10], it would follow that G is a direct sum of line bundles. This is seen to be impossible by applying [31, Lemma 2.2 (ii)] with $e = d$, $q = \dim V$ and $a = -j$. Therefore, we have $\mathrm{rk}\, G \geqslant \dim V - d + 1$ and the claim follows since, from the above exact sequence, $\mathrm{rk}\, G = \chi(X, \Omega_X^d)$. □

The above in particular applies when $X \subset A$ is a subvariety of an abelian variety and $V = \mathrm{H}^0(A, \Omega_A^1)$. For a smooth ample divisor $D \subset A$, the Euler characteristics can be computed exactly as follows:

$$(-1)^{g-1-i} \chi(D, \Omega_D^i) = n A(g, i),$$

where $n = D^g/g!$ and $A(g, i)$ is the Eulerian number; see [28, Section 3.2]. For subvarieties of codimension $\geqslant 2$, we obtain the following corollary.

Corollary A.2. *Let A be a complex abelian variety of dimension g, and let $X \subset A$ be a smooth subvariety of dimension $d \leqslant g - 2$ with ample normal bundle. Then we have*

$$(-1)^{d-p} \chi(X, \Omega_X^p) \geqslant \begin{cases} g - d + 1 & \text{for } p \in \{0, d\}, \\ 2 & \text{for } p \in \{1, d-1\}, \\ 1 & \text{for } 2 \leqslant p \leqslant d - 2. \end{cases}$$

Proof. Since $X \subset A$ has ample normal bundle, we know by [29, Proposition 6.3.10 (i)] that, for nonzero

$$\omega \in V := \mathrm{H}^0(A, \Omega_A^1) \subset \mathrm{H}^0(X, \Omega_X^1),$$

the zero locus $Z(\omega) \subset X$ is finite. Hence $m(V) = d$, and Theorem A.1 applies. □

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