Analytic curves in algebraic varieties over number fields

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To Yuri Ivanovich Manin

Summary. We establish algebraicity criteria for formal germs of curves in algebraic varieties over number fields and apply them to derive a rationality criterion for formal germs of functions on algebraic curves, which extends the classical rationality theorems of Borel-Dwork and Pólya-Bertrandias valid over the projective line to arbitrary algebraic curves over a number field. The formulation and the proof of these criteria involve some basic notions in Arakelov geometry, combined with complex and rigid analytic geometry (notably, potential theory over complex and p-adic curves). We also discuss geometric analogues, pertaining to the algebraic geometry of projective surfaces, of these arithmetic criteria.

Key words: Arakelov geometry, capacity theory, Borel-Dwork rationality criterion, algebricity of formal subschemes, rigid analytic geometry, slope inequality.

Subject Classifications: 14G40 (Primary); 14G22, 31A15, 14B20 (Secondary)

1 Introduction

The purpose of this article is to establish *algebraicity criteria* for *formal germs of curves* in algebraic varieties over number fields and to apply them to derive a *rationality criterion* for *formal germs of functions*, which extends the classical rationality theorems of Borel-Dwork ([6], [22]) and Pólya-Bertrandias ([43], [1, Chapter 5]; see also [16]), valid over the projective line, to arbitrary algebraic curves over a number field.

Our algebraicity criteria improve on the ones in [12] and [13], which themselves were inspired by the papers [19] and [20] of D. V. and G. V. Chudnovsky and by the subsequent works by André [2] and Graftieaux [26, 27]. As in [12] and [13], our results will be proved by means of a geometric version of "transcendence techniques", which avoids the traditional constructions of "auxiliary polynomials" and the explicit use of Siegel's Lemma, replacing them by a few basic concepts of Arakelov geometry. In the proofs, our main objects of interest will be some evaluation maps, defined on the spaces of global sections of powers of an ample line bundle on a projective variety by restricting these sections to formal subschemes or to subschemes of finite lengths. Arakelov geometry enters through the estimates satisfied by the heights of the evaluation maps, and the slopes and Arakelov degrees of the hermitian vector bundles defined by spaces of sections (see [17] and [14] for more details and references on this approach).

Our main motivation in investigating the algebraicity and rationality criteria presented in this article has been the desire to obtain theorems respecting the classical principle of number theory that "all places of number fields should appear on an equal footing" — which actually is not the case in "classical" Arakelov geometry and in its applications in [12]. A closely related aim has been to establish arithmetic theorems whose geometric counterparts (obtained through the analogy between number fields and function fields) have simple formulations and proofs. These concerns led us to two technical developments in this paper: the use of (rigid) analytic geometry over p-adic fields to define and estimate local invariants of formal curves over number fields[3], and the derivation of a rationality criterion from an algebraicity criterion by means of the Hodge index theorem on (algebraic or arithmetic) surfaces.

Let us describe the content of this article in more details.

In Section 2, we discuss geometric analogues of our arithmetic theorems. Actually, these are classical results in algebraic geometry, going back to Hartshorne [32] and Hironaka-Matsumura [35]. For instance, our algebraicity result admits as analogue the following fact. *Let X be a quasi-projective variety over a field k, and let Y be a smooth projective integral curve in X; let $\widehat{S}$ be a smooth formal germ of surface through Y* (that is, a smooth formal subscheme of dimension 2, containing Y, of the completion $\widehat{X}_Y$). *If the degree* $\deg_Y N_Y\widehat{S}$ *of the normal bundle to Y in $\widehat{S}$ is positive, then $\widehat{S}$ is algebraic.*

Our point is that, transposed to a geometric setting, the arguments leading to our algebraicity and rationality criteria in the arithmetic setting — which rely on the consideration of suitable evaluation maps, and on the Hodge index theorem — provide simple proofs of these non-trivial algebro-geometric results, in which the geometric punch-line of the arguments appears more clearly.

[3]Since the first version of this paper was written, the relevance of rigid analytic geometry *à la* Berkovich to develop a non-archimedean potential theory on p-adic curves, and consequently a "modern" version of Arakelov geometry of arithmetic surfaces satisfying the above principle of "equality of places", has been largely demonstrated by A. Thuillier in his thesis [51].

In Section 3, we introduce the notion of *A-analytic curve* in an algebraic variety X over a number field K. By definition, this will be a smooth formal curve $\widehat{C}$ through a rational point P in $X(K)$ — that is, a smooth formal subscheme of dimension 1 in the completion $\widehat{X}_P$ — which, firstly, is analytic at every place of K, finite or infinite. Namely, if v denotes any such place and K_v the corresponding completion of K, the formal curve $\widehat{C}_{K_v}$ in X_{K_v} deduced from $\widehat{C}$ by the extension of scalars $K \hookrightarrow K_v$ is the formal curve defined by a K_v-analytic curve in $X(K_v)$. Moreover the v-adic radius r_v (in $]0,1]$) of the open ball in $X(K_v)$ in which $\widehat{C}_{K_v}$ "analytically extends" is required to "stay close to 1 when v varies", in the sense that the series $\sum_v \log r_v^{-1}$ has to be convergent. The precise formulation of this condition relies on the notion of *size* of a smooth analytic germ in an algebraic variety over a p-adic field. This notion was introduced in [12, §3.1]; we review it in Section 3.A, adding some complements.

With the above notation, if $\mathscr{X}$ is a model of X over the ring of integers $\mathscr{O}_K$ of K, and if P extends to an integral point $\mathscr{P}$ in $\mathscr{X}(\mathscr{O}_K)$, then a formal curve $\widehat{C}$ through P is A-analytic if it is analytic at each archimedean place of K and extends to a smooth formal surface $\widehat{\mathscr{C}}$ in $\widehat{\mathscr{X}_{\mathscr{P}}}$. For a general formal curve $\widehat{C}$ that is analytic at archimedean places, being an A-analytic germ may be seen as a weakened form of the existence of such a smooth extension $\widehat{\mathscr{C}}$ of $\widehat{C}$ along $\mathscr{P}$. In this way, an A-analytic curve through the point P appears as an arithmetic counterpart of the smooth formal surface $\widehat{S}$ along the curve Y in the geometric algebraicity criterion above.

The tools needed to formulate the arithmetic counterpart of the positivity condition $\deg_Y N_Y\widehat{S} > 0$ are developed in Sections 4 and 5. We first show in Section 4 how, for any germ of analytic curve $\widehat{C}$ through a rational point P in some algebraic variety X over a local field K, one is led to introduce the so-called *canonical semi-norm* $\|\cdot\|^{\mathrm{can}}_{X,\widehat{C}}$ on the K-line $T_P\widehat{C}$ through the consideration of the metric properties of the evaluation maps involved in our geometric version of the method of auxiliary polynomials. This extends a definition introduced in [13] when $K = \mathbf{C}$. In Section 5, we discuss the construction of Green functions and capacities on rigid analytic curves over p-adic fields. We then extend the comparison of "canonical semi-norms" and "capacitary metrics" in [13, §3.4] to the non-archimedean setting.

In Section 6, we apply these notions to formulate and establish our algebraicity results. If $\widehat{C}$ is an A-analytic curve through a rational point P in an algebraic variety X over some number field K, then the K-line $T_P\widehat{C}$ may be equipped with a "K_v-adic semi-norm" for every place v by the above construction — namely, the semi-norm $\|.\|^{\mathrm{can}}_{X_{K_v},\widehat{C}_{K_v}}$ on

$$T_P\widehat{C}_{K_v} \simeq T_P\widehat{C} \otimes_K K_v.$$

The so-defined metrized K-line $\overline{T_P\widehat{C}}$ has a well defined Arakelov degree in $]-\infty,+\infty]$, and our main algebraicity criterion asserts that $\widehat{C}$ *is algebraic* *if the Arakelov degree* $\widehat{\deg}\,\overline{T_P\widehat{C}}$ *is positive.* Actually, the converse implication also holds: when $\widehat{C}$ is algebraic, the canonical semi-norms $\|.\|^{\mathrm{can}}_{X_{K_v},\widehat{C}_{K_v}}$ all vanish, and $\widehat{\deg}\,\overline{T_P\widehat{C}} = +\infty$.

Finally, in Section 7, we derive an extension of the classical theorems of Borel, Dwork, Pólya, and Bertrandias, which gives a criterion for the rationality of a formal germ of function φ on some algebraic curve Y over a number field. By considering the graph of φ — a formal curve $\widehat{C}$ in the surface $X := Y \times \mathbf{A}^1$ — we easily obtain the algebraicity of φ as a corollary of our previous algebraicity criterion. In this way, we are reduced to establishing a rationality criterion for an algebraic formal germ. Actually, rationality results for algebraic functions on the projective line have been investigated by Harbater [30], and used by Ihara [36] to study the fundamental group of some arithmetic surfaces. Ihara's results have been extended in [11] by using Arakelov geometry on arithmetic surfaces. Our rationality argument in Section 7, based on the Hodge index theorem on arithmetic surfaces of Faltings-Hriljac, is a variation on the proof of the Lefschetz theorem on arithmetic surfaces in *loc. cit.*.

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It would be difficult to acknowledge fairly the multifaceted influences of Yuri Ivanovich Manin on our work. We hope that this article will appear as a tribute, not only to his multiple contributions to algebraic geometry and number theory, but also to his global vision of mathematics, emphasizing geometric insights and analogies. The presentation of this vision in his 25th-Arbeitstagung report *New directions in geometry* [38] has been, since it was written, a source of wonder and inspiration to one of the authors, and we allowed ourselves to borrow the terminology "A-analytic" from the "A-geometry" programmatically discussed in [38]. It is an honour for us to dedicate this article to Yuri Ivanovich Manin.

2 Preliminary: the geometric case

The theorems we want to prove in this paper are analogues in arithmetic geometry of classical algebro-geometric results going back — at least in an implicit form — to Hartshorne, Hironaka, and Matsumura ([31], [34], [35]). Conversely, in this section we give short proofs of algebraic analogues of our main arithmetic theorems.

Proposition 2.1. *Let $\mathscr{X}$ be a quasi-projective scheme over a field k and let $\mathscr{P}$ be a projective connected subscheme of dimension 1 in $\mathscr{X}$. Let $\widehat{\mathscr{C}}$ be a formal subscheme of dimension 2 in $\widehat{\mathscr{X}_{\mathscr{P}}}$ admitting $\mathscr{P}$ as a scheme of definition. Assume that $\widehat{\mathscr{C}}$ is (formally) smooth over k, and that $\mathscr{P}$ has no embedded component (of dimension 0), or equivalently, that $\mathscr{P}$ defines a Cartier divisor in $\widehat{\mathscr{C}}$, and let $\mathscr{N}$ be the normal bundle of the immersion $\iota : \mathscr{P} \hookrightarrow \widehat{\mathscr{C}}$, that is, the invertible sheaf $\iota^* \mathscr{O}_{\widehat{\mathscr{C}}}(\mathscr{P})$ on $\mathscr{P}$.*

If the divisor $[\mathscr{P}]$ on the formal surface $\widehat{\mathscr{C}}$ is nef *and has* positive self-intersection*, then the formal surface $\widehat{\mathscr{C}}$ is algebraic, namely the Zariski-closure of $\widehat{\mathscr{C}}$ in $\mathscr{X}$ is an algebraic subvariety of dimension 2.*

Let $(\mathscr{P}_i)_{i\in I}$ be the family of irreducible components of $\mathscr{P}$, and $(n_i)_{i\in I}$ their multiplicities in $\mathscr{P}$. Recall that $[\mathscr{P}]$ is said to be nef on $\widehat{\mathscr{C}}$ when

$$[\mathscr{P}_i].[\mathscr{P}] := \deg_{\mathscr{P}_i} \mathscr{N} \geqslant 0 \text{ for any } i \in I,$$

and to have positive self-intersection if

$$[\mathscr{P}].[\mathscr{P}] := \sum_{i\in I} n_i. \deg_{\mathscr{P}_i} \mathscr{N} > 0,$$

or equivalently, when $[\mathscr{P}]$ is nef, if one the non-negative integers $\deg_{\mathscr{P}_i} \mathscr{N}$ is positive. Observe that these conditions are satisfied if $\mathscr{N}$ is ample on $\mathscr{P}$.

More general versions of the algebraicity criterion in Proposition 2.1 and of its proof below, without restriction on the dimensions of $\widehat{\mathscr{C}}$ and $\mathscr{P}$, can be found in [12, §3.3] [5], [13, Theorem 2.5] (see also [17, 18]). Besides it will be clear from the proof that, suitably reformulated, Proposition 2.1 still holds with the smoothness assumption on $\widehat{\mathscr{C}}$ omitted; we leave this to the interested reader.

Such algebraicity criteria may also be deduced from the works of Hironaka, Matsumura, and Hartshorne on the condition G_2 ([34], [35], [31]). We refer the reader to the monographs [33] and [3] for extensive discussions and references about related results concerning formal functions and projective algebraic varieties.

Note that Proposition 2.1 has consequences for the study of algebraic varieties over function fields. Let indeed S be a smooth, projective, and geometrically connected curve over a field k and let $K = k(S)$. Let $f \colon \mathscr{X} \to S$ be a surjective map of k-schemes and assume that $\mathscr{P}$ is the image of a section of f. Let $X = \mathscr{X}_K$, $P = \mathscr{P}_K$ and $\widehat{C} = \widehat{\mathscr{C}}_K$ be the generic fibers of $\mathscr{X}$, $\mathscr{P}$, and $\mathscr{C}$. Then P is a K-rational point of X and $\widehat{C}$ is a germ of curve in X at P. Observe that $\widehat{\mathscr{C}}$ is algebraic if and only if $\widehat{C}$ is algebraic. Consequently, in this situation, Proposition 2.1 appears as an algebraicity criterion for a formal germ of curves $\widehat{C}$ in X. In particular, it shows that such *a smooth formal curve $\widehat{C}$ in X is algebraic if it extends to a smooth formal scheme $\widehat{\mathscr{C}}$ through $\mathscr{P}$ in $\mathscr{X}$ such that the normal bundle of $\mathscr{P}$ in $\widehat{\mathscr{C}}$ has positive degree.*

Proof (of Proposition 2.1). We may assume that $\mathscr{X}$ is projective and that $\widehat{\mathscr{C}}$ is Zariski-dense in $\mathscr{X}$. We let $d = \dim \mathscr{X}$. One has obviously $d \geqslant 2$ and our goal is to prove the equality.

Let $\mathscr{O}(1)$ be any very ample line bundle on $\mathscr{X}$. The method of "auxiliary polynomials", borrowed from transcendence theory, suggests the introduction of the "evaluation maps"

$$\varphi_D \colon \Gamma(\mathscr{X}, \mathscr{O}(D)) \to \Gamma(\widehat{\mathscr{C}}, \mathscr{O}(D)), \quad s \mapsto s|_{\widehat{\mathscr{C}}},$$

for positive integers D.

Let us denote $E_D = \Gamma(\mathscr{X}, \mathscr{O}(D))$, and for any integer $i \geqslant 0$, let E_D^i be the set of all $s \in E_D$ such that $\varphi_D(s) = s|_{\widehat{\mathscr{C}}}$ vanishes at order at least i along $\mathscr{P}$, *i.e.* such that the restriction of $\varphi_D(s)$ to $i\mathscr{P}$ vanishes. Since $\widehat{\mathscr{C}}$ is Zariski-dense in $\mathscr{X}$, no non-zero section of $\mathscr{O}(D)$ has a restriction to $\widehat{\mathscr{C}}$ that vanishes at infinite order along $\mathscr{P}$, and we have

$$\bigcap_{i=0}^{\infty} E_D^i = 0.$$

Consequently,

$$\operatorname{rank} E_D = \sum_{i=0}^{\infty} \operatorname{rank}(E_D^i / E_D^{i+1}).$$

Moreover, there is a canonical injective map of k-vector spaces

$$E_D^i / E_D^{i+1} \hookrightarrow \Gamma(\mathscr{P}, \mathscr{O}(D) \otimes \mathscr{N}^{\vee \otimes i}),$$

which amounts to taking the ith jet along $\mathscr{P}$ — that is, the restriction to $(i+1)\mathscr{P}$ — of a section which vanishes at order at least i. Indeed the quotient sheaf

$$\big(\mathscr{O}(D) \otimes \mathscr{O}_{\widehat{\mathscr{C}}}(-i\mathscr{P})\big) \,/\, \big(\mathscr{O}(D) \otimes \mathscr{O}_{\widehat{\mathscr{C}}}(-(i+1)\mathscr{P})\big)$$

over $\widehat{\mathscr{C}}$ may be identified with $\mathscr{O}(D) \otimes \iota_* \mathscr{N}^{\vee \otimes i}$. Observe also that the dimension of the range of this injection satisfies an upper bound of the form

$$\dim \Gamma(\mathscr{P}, \mathscr{O}(D) \otimes \mathscr{N}^{\vee \otimes i}) \leqslant c(D + i),$$

valid for any non-negative integers D and i.

Assume that $E_D^i \neq 0$ and let $s \in E_D^i$ be any nonzero element. By assumption, $\varphi_D(s)$ vanishes at order i along $\mathscr{P}$, hence $\operatorname{div} \varphi_D(s) - i[\mathscr{P}]$ is an effective divisor on $\widehat{\mathscr{C}}$ and its intersection number with $[\mathscr{P}]$ is nonnegative, for $[\mathscr{P}]$ is nef. Consequently

$$\operatorname{div} \varphi_D(s) \cdot [\mathscr{P}] \geqslant i[\mathscr{P}] \cdot [\mathscr{P}].$$

Since

$$\operatorname{div} \varphi_D(s) \cdot [\mathscr{P}] = \deg_{\mathscr{P}}(\mathscr{O}(D)) = D \deg_{\mathscr{P}}(\mathscr{O}(1))$$

and $[\mathscr{P}] \cdot [\mathscr{P}] > 0$ by the assumption of positive self-intersection, this implies $i \leqslant aD$, where $a := \deg_{\mathscr{P}} \mathscr{O}(1)/[\mathscr{P}].[\mathscr{P}]$. Consequently E_D^i is reduced to 0 if $i > aD$.

Finally, we obtain:

$$\operatorname{rank} E_D = \sum_{i=0}^{\infty} \operatorname{rank}(E_D^i/E_D^{i+1}) = \sum_{i=0}^{\lfloor aD \rfloor} \operatorname{rank}(E_D^i/E_D^{i+1}) \leqslant \sum_{i=0}^{\lfloor aD \rfloor} c(D+i).$$

This proves that, when D goes to $+\infty$,

$$\operatorname{rank} E_D \ll D^2.$$

Besides

$$\operatorname{rank} E_D = \operatorname{rank} \Gamma(\mathscr{X}, \mathscr{O}(D)) \asymp D^d,$$

by Hilbert-Samuel's theorem. This establishes that the integer d, which is at least 2, actually equals 2. □

Proposition 2.2. *Let $f \colon S' \to S$ be a dominant morphism between two normal projective surfaces over a field k. Let $D \subset S$ and $D' \subset S'$ be effective divisors such that $f(D') = D$.*

Assume that $f|_{D'} \colon D' \to D$ is an isomorphism and that f induces an isomorphism $\widehat{f} \colon \widehat{S'_{D'}} \to \widehat{S_D}$ between formal completions. If moreover D is nef and $D \cdot D > 0$, then f is birational.

Recall that D is said to be nef if, for any effective divisor E on S the (rational) intersection number $D \cdot E$ is non-negative.

Proof. By hypothesis, f is étale in a neighborhood of D'. If $\deg(f) > 1$, one can therefore write $f^*D = D' + D''$, where D'' is a non zero effective Cartier divisor on S' which is disjoint from D'. Now, f^*D is a nef divisor on S' such that $f^*D \cdot f^*D = \deg(f) D \cdot D > 0$. As a classical consequence of the Hodge index theorem (see [24], [45], and also [11, Proposition 2.2]) the effective divisor f^*D is numerically connected, hence connected. This contradicts the decomposition $f^*D = D' \sqcup D''$. □

Proposition 2.3. *Let $\mathscr{S}$ be a smooth projective connected surface over a perfect field k. Let $\mathscr{P}$ be a smooth projective connected curve in $\mathscr{S}$. If the divisor $[\mathscr{P}]$ on $\mathscr{S}$ is big and nef, then any formal rational function along $\mathscr{P}$ is defined by a (unique) rational function on $\mathscr{S}$. In other words, one has an isomorphism of fields*

$$k(\mathscr{S}) \xrightarrow{\sim} \Gamma(\mathscr{P}, \operatorname{Frac} \mathscr{O}_{\widehat{\mathscr{S}_{\mathscr{P}}}}).$$

Proof. Let φ be any formal rational function along $\mathscr{P}$. We may introduce a sequence of blowing-ups of closed points $\nu \colon \mathscr{S}' \to \mathscr{S}$ such that $\varphi' = \nu^* \varphi$ has no point of indetermination and may be seen as a map (of formal k-schemes) $\widehat{\mathscr{S}'_{\mathscr{P}'}} \to \mathbf{P}^1_k$, where $\mathscr{P}' = \nu^* \mathscr{P}$.

Let us consider the graph $\mathrm{Gr}\,\varphi'$ of φ' in $\mathscr{S}' \times \mathbf{P}^1_k$. This is a formally smooth 2-dimensional formal scheme, admitting the graph of $\varphi'_{|\mathscr{P}'}\colon \mathscr{P}' \to \mathbf{P}^1_k$ as a scheme of definition, and the morphism φ' defines an isomorphism of formal schemes

$$\psi' := (\mathrm{Id}, \varphi)\colon \widehat{\mathscr{S}'}_{\mathscr{P}'} \to \mathrm{Gr}\,\varphi'.$$

Like the divisor $\mathscr{P}$ in $\mathscr{S}$, its inverse image $\mathscr{P}'$ in $\mathscr{S}'$ is nef and has positive self-intersection. Proposition 2.1 therefore implies that $\mathrm{Gr}\,\varphi'$ is algebraic in $\mathscr{S}' \times \mathbf{P}^1_k$. In other words, φ' is an algebraic function.

To establish its rationality, let us introduce the Zariski closure Γ of the graph of $\mathrm{Gr}\,\varphi'$ in $\mathscr{S}' \times \mathbf{P}^1_k$, the projections $\mathrm{pr}_1\colon \Gamma \to \mathscr{S}'$ and $\mathrm{pr}_2\colon \Gamma \to \mathbf{P}^1_k$, and the normalization $n\colon \tilde{\Gamma} \to \Gamma$ of Γ. Consider also the Cartier divisor $\mathscr{P}'_\Gamma$ (resp. $\mathscr{P}'_{\tilde{\Gamma}}$) defined as the inverse image $\mathrm{pr}_1^* \mathscr{P}'$ (resp. $n^*\mathscr{P}'_\Gamma$) of $\mathscr{P}'$ in Γ (resp. $\tilde{\Gamma}$). The morphisms n and pr_1 define morphism of formal completions:

$$\widehat{\tilde{\Gamma}}_{\mathscr{P}'_{\tilde{\Gamma}}} \xrightarrow{\widehat{n}} \widehat{\Gamma}_{\mathscr{P}'_\Gamma} \xrightarrow{\widehat{\mathrm{pr}_1}} \widehat{\mathscr{S}'}_{\mathscr{P}'}.$$

The morphism ψ' may be seen a section of $\widehat{\mathrm{pr}_1}$; by normality of $\widehat{\mathscr{S}'_{\mathscr{P}'}}$, it admits a factorization through $\widehat{n}$ of the form $\psi' = \widehat{n} \circ \tilde{\psi}$, for some uniquely determined morphism of k-formal schemes $\tilde{\psi}\colon \widehat{\mathscr{S}'}_{\mathscr{P}'} \to \widehat{\tilde{\Gamma}}_{\mathscr{P}'_{\tilde{\Gamma}}}$. This morphism $\tilde{\psi}$ is a section of $\widehat{\mathrm{pr}_1} \circ \widehat{n}$. Therefore the (scheme theoretic) image $\tilde{\psi}(\mathscr{P}')$ defines a (Cartier) divisor in $\tilde{\Gamma}$ such that

$$(f\colon S' \to S, D', D) = (\mathrm{pr}_1 \circ n\colon \tilde{\Gamma} \to \mathscr{S}', \tilde{\mathscr{P}}, \mathscr{P}')$$

satisfy the hypotheses of Proposition 2.2. Consequently the morphism $\mathrm{pr}_1 \circ n$ is birational. Therefore, pr_1 is birational too and φ' is the restriction of a rational function on $\mathscr{S}'$, namely $\mathrm{pr}_2 \circ \mathrm{pr}_1^{-1}$. This implies that φ is the restriction of a rational function on $\mathscr{S}$. The uniqueness of this rational function follows from the Zariski density of the formal neighborhood of $\mathscr{P}$ in $\mathscr{S}$. □

Remark 2.4. In the terminology of Hironaka and Matsumura [35], the last proposition asserts that $\mathscr{P}$ is G_3 in $\mathscr{S}$, and has been established by Hironaka in [34]. Hartshorne observes in [32, Proposition 4.3, and Remark p. 123] that Proposition 2.2 holds more generally under the assumption that D and D' are G_3 in $\mathscr{S}$ and $\mathscr{S}'$. Our approach to Propositions 2.2 and 2.3 follows an order opposite to the one in [34] and [32], and actually provides a simple proof of [32, Proposition 4.3].

3 *A*-analyticity of formal curves

3.A Size of smooth formal curves over *p*-adic fields

In this Section, we briefly recall some definitions and results from [12].

Let K be field equipped with some complete ultrametric absolute value $|.|$ and assume that its valuation ring R is a discrete valuation ring. Let also $\overline{K}$ be an algebraic closure of K. We shall still denote $|.|$ the non-archimedean absolute value on $\overline{K}$ that extends the absolute value $|.|$ on K.

For any positive real number r, we define the norm $\|g\|_r$ of a formal power series $g = \sum_{I \in \mathbf{N}^N} a_I X^I \in K[[X_1, \ldots, X_n]]$ by the formula

$$\|g\|_r = \sup_I |a_I| r^{|I|} \quad ;$$

it belongs to $\mathbf{R}_+ \cup \{\infty\}$. The power series g such that $\|g\|_r < \infty$ are precisely those that are convergent and bounded on the open N-ball of radius r in $\overline{K}^N$.

The group $G_{\mathrm{for},K} := \mathrm{Aut}(\widehat{\mathbf{A}^N_{K,0}})$ of automorphisms of the formal completion of $\mathbf{A}^N_K$ at 0 may be identified with the set of all N-tuples $f = (f_1, \ldots, f_n)$ of power series in $K[[X_1, \ldots, X_N]]$ such that $f(0) = 0$ and $Df(0) := \left(\frac{\partial f_j}{\partial X_i}(0)\right)$ belongs to $\mathrm{GL}_N(K)$. We consider its following subgroups:

- the subgroups G_{for} consisting of all elements $f \in G_{\mathrm{for},K}$ such that $Df(0) \in \mathrm{GL}_N(R)$;
- the subgroup $G_{\mathrm{an},K}$ consisting of those $f = (f_1, \ldots, f_N)$ in $G_{\mathrm{for},K}$ such that, for each j, f_j has a positive radius of convergence;
- $G_{\mathrm{an}} := G_{\mathrm{an},K} \cap G_{\mathrm{for}}$;
- for any positive real number r, the subgroup $G_{\mathrm{an},r}$ of G_{an} consisting of all N-tuples $f = (f_1, \ldots, f_N)$ such that $\|f_j\|_r \leqslant r$ for each j. This subgroup may be identified with the group of all analytic automorphisms, preserving the origin, of the open N-dimensional ball of radius r.

One has the inclusion $G_{\mathrm{an},r'} \subset G_{\mathrm{an},r}$ for any $r' > r > 0$, and the equalities

$$\bigcup_{r>0} G_{\mathrm{an},r} = G_{\mathrm{an}} \quad \text{and} \quad G_{\mathrm{an},1} = \mathrm{Aut}\left(\widehat{\mathbf{A}^N_{R,0}}\right).$$

It is straightforward that a formal subscheme $\widehat{V}$ of $\widehat{\mathbf{A}^N_{K,0}}$ is (formally) *smooth* of dimension d iff there exists $\varphi \in G_{\mathrm{for},K}$ such that $\varphi^* \widehat{V}$ is the formal subscheme $\widehat{\mathbf{A}^d_{K,0}} \times \{0\}$ of $\widehat{\mathbf{A}^N_{K,0}}$; when this holds, one can even find such a φ in G_{for}. Moreover such a smooth formal subscheme $\widehat{V}$ is K-*analytic* iff one can find φ as above in $G_{\mathrm{an},K}$, or equivalently in G_{an}.

Let $\mathscr{X}$ be a flat quasi-projective R-scheme, and $X = \mathscr{X} \otimes_R K$ its generic fibre. Let $\mathscr{P} \in \mathscr{X}(R)$ be an R-point, and let $P \in X(K)$ be its restriction to $\operatorname{Spec} K$. In [12, §3.1.1] we associated to any smooth formal scheme $\widehat{V}$ of dimension d in $\widehat{X}_P$, its *size* $S_{\mathscr{X}}(\widehat{V})$ *with respect to the model* $\mathscr{X}$ of X. It is a number in $[0, 1]$ whose definition and basic properties may be summarized in the following statement:

Theorem 3.1. *There is a unique way to attach a number $S_{\mathscr{X}}(\widehat{V})$ in* $[0,1]$ *to any such data $(\mathscr{X},\mathscr{P},\widehat{V})$ so that the following properties hold:*

a) if $\mathscr{X}\to\mathscr{X}'$ is an immersion, then $S_{\mathscr{X}'}(\widehat{V})=S_{\mathscr{X}}(\widehat{V})$ (invariance under immersions)*;*

b) for any two triples $(\mathscr{X},\mathscr{P},\widehat{V})$ and $(\mathscr{X}',\mathscr{P}',\widehat{V}')$ as above, if there exists an R-morphism $\varphi\colon\mathscr{X}\to\mathscr{X}'$ mapping $\mathscr{P}$ to $\mathscr{P}'$, étale along $\mathscr{P}$, and inducing an isomorphism $\widehat{V}\simeq\widehat{V}'$, then $S_{\mathscr{X}'}(\widehat{V}')=S_{\mathscr{X}}(\widehat{V})$ (invariance by étale localization)*;*

c) if $\mathscr{X}=\mathbf{A}^N_R$ is the affine space over R and $\mathscr{P}=(0,\dots,0)$, then $S_{\mathscr{X}}(\widehat{V})$ is the supremum in $[0,1]$ *of the real numbers $r\in(0,1]$ for which there exists $f\in G_{\mathrm{an},r}$ such that $f^*\widehat{V}=\widehat{\mathbf{A}^d_0}\times\{0\}$* (normalization).

As a straightforward consequence of these properties of the size, we obtain:

Proposition 3.2. *A smooth formal subscheme $\widehat{V}$ in $\widehat{X}_P$ is K-analytic if and only if its size $S_{\mathscr{X}}(\widehat{V})$ is a positive real number.*

Proposition 3.3. *Let $\mathscr{X}$, $\mathscr{P}$, and $\widehat{V}$ be as above and assume that there exists a smooth formal R-subscheme $\mathscr{V}\subset\widehat{\mathscr{X}_{\mathscr{P}}}$ such that $\widehat{V}=\mathscr{V}_K$. Then $S_{\mathscr{X}}(\widehat{V})=1$.*

The remainder of this section is devoted to further properties of the size.

Proposition 3.4. *The size is invariant under isometric extensions of valued fields (complete with respect to a discrete valuation).*

Proof. It suffices to check this assertion in the case of a smooth formal subscheme $\widehat{V}$ through the origin of the affine space $\mathbf{A}^N$. By its very definition, the size cannot decrease under extensions of the base field.

To show that it cannot increase either, let us fix an isomorphism of K-formal schemes

$$\xi=(\xi_1,\dots,\xi_N):\widehat{\mathbf{A}}^d_0\xrightarrow{\sim}\widehat{V}\hookrightarrow\widehat{\mathbf{A}}^N_0$$

given by N power series $\xi_i\in K[[T_1,\dots,T_d]]$ such that $\xi_1(0)=\dots=\xi_N(0)=0$. We then observe that, for any N-tuple $g=(g_1,\dots,g_N)$ of series in $K[[X_1,\dots,X_N]]$, the following two conditions are equivalent:

i) g belongs to $G_{\mathrm{for},K}$ and $(g^{-1})^*\widehat{V}=\widehat{\mathbf{A}^d_0}\times\{0\}$;

ii) $g_1(0)=\dots=g_N(0)=0$, $g_{d+1}(\xi_1,\dots,\xi_N)=\dots=g_N(\xi_1,\dots,\xi_N)=0$, and $\left(\frac{\partial g_i}{\partial X_j}(0)\right)$ belongs to $\mathrm{GL}_N(K)$.

Let K' be a valued field, satisfying the same condition as K, that contains K and whose absolute value restricts to the given one on K. Let R' be its valuation ring. We shall denote $G'_{\mathrm{for}},G'_{\mathrm{an}},G'_{\mathrm{an},r},\dots$ the analogues of $G_{\mathrm{for}},G_{\mathrm{an}},G_{\mathrm{an},r},\dots$ defined by replacing the valued field K by K'. Recall that there exists an “orthogonal projection” from K' to K, namely a K-linear map $\lambda\colon K'\to K$ such that $|\lambda(a)|\leqslant|a|$ for any $a\in K'$ and $\lambda(a)=a$ for any $a\in K$; see for instance [28, p. 58, Corollary (2.3)].

Let $\widehat{V}' = \widehat{V}_{K'}$ be the formal subscheme of $\widehat{\mathbf{A}^N_{K'}}$ deduced from $\widehat{V}$ by the extension of scalars $K \hookrightarrow K'$, and let r be an element in $]0, S_{\mathbf{A}^N_{R'}}(\widehat{V}')[$. By the very definition of the size, there exists some $g' = (g'_1, \ldots, g'_N)$ in $G'_{\mathrm{an},r}$ such that $(g'^{-1})^*\widehat{V} = \widehat{\mathbf{A}^d_0} \times \{0\}$. Since the tangent space at the origin of V' is defined over K, by composing g' with a suitable element in $GL_N(R')$, we may even find g' such that $Dg'(0)$ belongs to $GL_N(R)$. Then the series $g_i := \lambda \circ g'_i$, deduced from the series g'_i by applying the linear map λ to their coefficients, satisfy $g_i(0) = 0$, $(\partial g_i/\partial X_j)(0) = (\partial g'_i/\partial X_j)(0)$ and $\|g_i\|_r \leqslant \|g'_i\|_r$. Therefore $g := (g_1, \ldots, g_N)$ is an element of $G_{\mathrm{an},r}$. Moreover, from the equivalence of conditions (i) and (ii) above and its analogue with K' instead of K, we derive that g satisfies $(g^{-1})^*\widehat{V} = \widehat{\mathbf{A}^d_0} \times \{0\}$. This shows that $S_{\mathbf{A}^N_R}(\widehat{V}) \geqslant r$ and establishes the required inequality $S_{\mathbf{A}^N_R}(\widehat{V}) \geqslant S_{\mathbf{A}^N_{R'}}(\widehat{V})$. □

The next proposition relates sizes, radii of convergence, and Newton polygons.

Proposition 3.5. *Let $\varphi \in K[[X]]$ be a power series such that $\varphi(0) = 0$ and $\varphi'(0) \in R$, and let $\widehat{C}$ be its graph, namely the formal subscheme of $\widehat{\mathbf{A}}^2_0$ defined by the equation $x_2 = \varphi(x_1)$.*

1) The radius of convergence ρ of φ satisfies

$$\rho \geqslant S_{\mathbf{A}^2_R}(\widehat{C}).$$

2) Suppose that ρ is positive and that $\varphi'(0)$ is a unit in R. Then

$$S_{\mathbf{A}^2_R}(\widehat{C}) = \min(1, \exp \lambda_1),$$

where λ_1 denotes the first slope of the Newton polygon of the power series $\varphi(x)/x$.

Recall that, if $\varphi = \sum_{i \geqslant 1} c_i T^i$, under the hypothesis in 2), we have:

$$\lambda_1 := \inf_{i \geqslant 1} -\frac{\log|c_{i+1}|}{i} \leqslant \lim \inf_{i \to +\infty} -\frac{\log|c_{i+1}|}{i} = \log \rho.$$

Moreover $\exp \lambda_1$ is the supremum of the numbers $r \in]0, \rho[$ such that, for any t in $\overline{K}$ satisfying $|t| < r$, we have $|\varphi(t)| = |t|$.

Proof. Let r be a positive real number such that $r < S_{\mathbf{A}^2_0}(\widehat{C})$. By assumption, there are power series f_1 and $f_2 \in K[[X_1, X_2]]$ such that $f = (f_1, f_2)$ belongs to $G_{\mathrm{an},r}$ and such that $f^*\widehat{C} = \widehat{\mathbf{A}^1} \times \{0\}$. This last condition implies (actually is equivalent to) the identity in $K[[T]]$:

$$f_2(T, 0) = \varphi(f_1(T, 0)).$$

Let us write $f_1(T, 0) = \sum_{i \geqslant 1} a_i T^i$, $f_2(T, 0) = \sum_{i \geqslant 1} b_i T^i$, and $\varphi(X) = \sum_{i \geqslant 1} c_i X^i$.

One has $b_1 = c_1 a_1$ and $c_1 = \varphi'(0)$ belongs to R by hypothesis. Moreover, the first column of the matrix $Df(0)$ is $\binom{a_1}{b_1} = a_1 \binom{1}{c_1}$. Since $Df(0)$ belongs to $\mathrm{GL}_2(R)$ and c_1 to R, this imply that a_1 is a unit in R. Then, looking at the expansion of $f_1(T,0)$ (which satisfies $\|f_1(T,0)\|_r \leqslant r$), we see that $|f_1(t,0)| = |t|$ for any $t \in \overline{K}$ such that $|t| < r$. Consequently, if $g \in K[[T]]$ denotes the reciprocal power series of $f_1(T,0)$, then g converges on the open disc of radius r and satisfies $|g(t)| = |t|$ for any $t \in \overline{K}$ such that $|t| < r$.

The identity in $K[[T]]$

$$\varphi(T) = \varphi(f_1(g(T),0)) = f_2(g(T),0)$$

then shows that the radius of convergence of φ is at least r. This establishes 1).

Let us now assume that ρ is positive and that $\varphi'(0) (= c_1)$ is a unit of R. Then $b_1 = a_1 c_1$ also is a unit and similarly, we have $|f_2(t,0)| = |t|$ for any $t \in \overline{K}$ such that $|t| < r$. This implies that $|\varphi(t)| = |t|$ for any such t. This shows that $\exp \lambda_1 \geqslant S_{\mathbf{A}^2_R}(\widehat{C})$.

To complete the proof of 2), observe that the element f of G_{an} defined as $f(T_1,T_2) = (T_1 + T_2, \varphi(T_1))$ satisfies $f^*\widehat{C} = \widehat{\mathbf{A}^1} \times \{0\}$ and belongs to $G_{\mathrm{an},r}$ for any r in $]0, \min(1, \exp \lambda_1)[$. □

Observe that for any non zero $a \in R$, the series $\varphi(T) = T/(a - T)$ has radius of convergence $\rho = |a|$ while the size of its graph $\widehat{C}$ is 1 (observe that $f(T_1,T_2) := (aT_1 + T_2, T_1/(1-T_1))$ satisfies $f^*\widehat{C} = \widehat{\mathbf{A}^1} \times \{0\}$). Taking $|a| < 1$, this shows that the size of the graph of a power series φ can be larger than its radius of convergence when the assumption $\varphi'(0) \in R$ is omitted.

As an application of the second assertion in Proposition 3.5, we obtain that, when K is a p-adic field, the size of the graph of $\log(1+x)$ is equal to $|p|^{1/(p-1)}$. Considering this graph as the graph of the exponential power series with axes exchanged, this also follows from the first assertions of Proposition 3.5 and Proposition 3.6 below.

Finally, let us indicate that, by analyzing the construction *à la* Cauchy of local solutions of analytic ordinary differential equations, one may establish the following lower bounds on the size of a formal curve obtained by integrating an algebraic one-dimensional foliation over a p-adic field (*cf.* [13, Proposition 4.1]):

Proposition 3.6. *Assume that K is a field of characteristic 0, and that its residue field k has positive characteristic p. Assume also that $\mathscr{X}$ is smooth over R in a neighborhood of $\mathscr{P}$. Let $\mathscr{F} \subset T_{\mathscr{X}/R}$ be a rank 1 subbundle and let $\widehat{C}$ be the formal integral curve through P of the one-dimensional foliation $F = \mathscr{F}_K$. Then*

$$S_{\mathscr{X}}(\widehat{C}) \geqslant |p|^{1/(p-1)}.$$

If moreover K is absolutely unramified (that is, if the maximal ideal of R is pR) *and if the one-dimensional subbundle $\mathscr{F}_k \subset T_{\mathscr{X}_k}$ is closed under p-th power, then*

$$S_{\mathscr{X}}(\widehat{C}) \geqslant |p|^{1/p(p-1)}.$$

3.B A-analyticity of formal curves in algebraic varieties over number fields

Let K be a number field and let R denote its ring of integers. For any maximal ideal $\mathfrak{p}$ of R, let $|\cdot|_{\mathfrak{p}}$ denote the $\mathfrak{p}$-adic absolute value, normalized by the condition $|\pi|_{\mathfrak{p}} = (\#(R/\mathfrak{p}))^{-1}$ for any uniformizing element π at $\mathfrak{p}$. Let $K_{\mathfrak{p}}$ and $R_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic completions of K and R, and $\mathbf{F}_{\mathfrak{p}} := R/\mathfrak{p}$ the residue field of $\mathfrak{p}$.

In this Section, we consider a quasi-projective algebraic variety X over K, a rational point P in $X(K)$, and a smooth formal curve $\widehat{C}$ in $\widehat{X}_P$.

It is straightforward that, if N denotes a sufficiently divisible positive integer, there exists a model $\mathscr{X}$ of X, quasi-projective over $R[1/N]$, such that P extends to a point $\mathscr{P}$ in $\mathscr{X}(R[1/N])$. Then, for any maximal ideal $\mathfrak{p}$ not dividing N, the size $S_{\mathscr{X}_{R_{\mathfrak{p}}}}(\widehat{C}_{K_{\mathfrak{p}}})$ is a well-defined real number in $[0, 1]$.

Definition 3.7. *We will say that the formal curve $\widehat{C}$ in X is A-*analytic *if the following conditions are satisfied:*

(i) for any place v of K, the formal curve $\widehat{C}_{K_v}$ is K_v-analytic;
(ii) the infinite product $\prod_{\mathfrak{p}\nmid N} S_{\mathscr{X}_{R_{\mathfrak{p}}}}(\widehat{C}_{K_{\mathfrak{p}}})$ converges to a positive real number.

Condition (ii) asserts precisely that the series with non-negative terms

$$\sum_{\mathfrak{p}\nmid N} \log S_{\mathscr{X}_{R_{\mathfrak{p}}}}(\widehat{C}_{K_{\mathfrak{p}}})^{-1}$$

is convergent.

Observe that the above definition does not depend on the choices required to formulate it. Indeed, condition (i) does not involve any choice. Moreover, if condition (i) holds and if N' is any positive multiple of N, condition (ii) holds for $(N, \mathscr{X}, \mathscr{P})$ if and only if it holds for $(N', \mathscr{X}_{R[1/N']}, \mathscr{P}_{R[1/N']})$. Moreover, for any two such triples $(N_1, \mathscr{X}_1, \mathscr{P}_1)$ and $(N_2, \mathscr{X}_2, \mathscr{P}_2)$, there is a positive integer M, multiple of both N_1 and N_2, such that the models $(\mathscr{X}_1, \mathscr{P}_1)$ and $(\mathscr{X}_2, \mathscr{P}_2)$ of (X, P) become isomorphic over $R[1/M]$. This shows that, when (i) is satisfied, conditions (ii) for any two triples $(N, \mathscr{X}, \mathscr{P})$ are indeed equivalent.

It follows from the properties of the size recalled in Proposition 3.1 that A-analyticity is invariant under immersions and compatible to étale localization.

As a consequence of Proposition 3.2 and 3.3, we also have:

Proposition 3.8. *Let $\widehat{C}$ be a smooth formal curve which is K_v-analytic for any place v of K. Assume that $\widehat{C}$ extends to a smooth formal curve $\mathscr{C} \hookrightarrow \mathscr{X}$ over $R[1/N]$, for some $N \geqslant 1$. Then $\widehat{C}$ is A-analytic.*

Indeed, these conditions imply that the size of $\widehat{C}$ at almost every finite place of K is equal to 1, while being positive at every place.

As observed in substance by Eisenstein [23], any algebraic smooth formal curve satisfies the hypothesis of Proposition 3.8. Therefore:

Corollary 3.9. *If the smooth formal curve $\widehat{C}$ is algebraic, then it is A-analytic.*

The invariance of size under extensions of valued fields established in Proposition 3.4 easily implies that, *for any number field K' containing K, the smooth formal curve $\widehat{C}' := \widehat{C}_{K'}$ in $X_{K'}$ deduced from $\widehat{C}$ by the extension of scalars $K \hookrightarrow K'$ is A-analytic iff $\widehat{C}$ is A-analytic.*

Let $\varphi \in K[[X]]$ be any formal power series, and let $P := (0, \varphi(0))$. From the inequality in Proposition 3.5, 1), between the convergence radius of a power series and the size of its graph, it follows that *the A-analyticity of the graph $\widehat{C}$ of φ in $\widehat{\mathbf{A}^2_P}$ implies that the convergence radii R_v of φ at the places v of K satisfy the so-called Bombieri's condition*

$$\prod_v \min(1, R_v) > 0,$$

or equivalently

$$\sum_v \log^+ R_v^{-1} < +\infty.$$

However the converse does not hold, as can be seen by considering the power series $\varphi(X) = \log(1 + X)$, that satisfies that Bombieri's condition (since all the R_v equal 1), but is not A-analytic (its p-adic size is $|p|^{1/(p-1)}$ and the infinite series $\sum \frac{1}{p-1} \log p$ diverges).

Let us conclude this section by a brief discussion of the relevance of A-analyticity in the arithmetic theory of differential equations (we refer to [12, 17, 13] for more details).

Assume that X is smooth over K, that F is sub-vector bundle of rank one in the tangent bundle T_X (defined over K), and that $\widehat{C}$ is the formal leaf at P of the one-dimensional algebraic foliation on X defined by F. By a model of (X, F) over $R[1/N]$, we mean the data of a scheme $\mathscr{X}$ quasi-projective and smooth over $\operatorname{Spec} R$, of a coherent subsheaf $\mathscr{F}$ of $T_{\mathscr{X}/R}$, and of an isomorphism $X \simeq \mathscr{X} \otimes K$ inducing an isomorphism $F \simeq \mathscr{F} \otimes K$. Such models clearly exist if N is sufficiently divisible. Let us choose one of them $(\mathscr{X}, \mathscr{F})$. We say that the foliation F satisfies the *Grothendieck-Katz condition* if for almost every maximal ideal $\mathfrak{p} \subset R$, the subsheaf $\mathscr{F}_{\mathbf{F}_\mathfrak{p}}$ of $T_{\mathscr{X}_{\mathbf{F}_\mathfrak{p}}/\mathbf{F}_\mathfrak{p}}$ is closed under p-th powers, where p denotes the characteristic of $\mathbf{F}_\mathfrak{p}$. As above, this condition does not depend on the choice of the model $(\mathscr{X}, \mathscr{F})$.

Proposition 3.10. *With the above notation, if F satisfies the Grothendieck-Katz condition, then its formal integral curve $\widehat{C}$ through any rational point P in $X(K)$ is A-analytic.*

Proof. It follows from Cauchy's theory of analytic ordinary differential equations over local fields that the formal curve $\widehat{C}$ is K_v-analytic for any place v of K.

After possibly increasing N, we may assume that P extends to a section $\mathscr{P}$ in $\mathscr{X}(R[1/N])$. For any maximal ideal $\mathfrak{p} \subset R$ that is unramified over a prime number p, and such that $\mathscr{F}_{\mathbf{F}_\mathfrak{p}}$ is closed under p-th power, Proposition 3.6 show that the $\mathfrak{p}$-adic size of $\widehat{C}$ is at least $|p|^{1/p(p-1)}$. When F satisfies the Grothendieck-Katz condition, this inequality holds for almost all maximal ideals of R. Since the series over primes $\sum_p \frac{1}{p(p-1)} \log p$ converges, this implies the convergence of the series $\sum_{\mathfrak{p}\nmid N} \log S_{\mathscr{X}_{R_\mathfrak{p}}}(\widehat{C}_{K_\mathfrak{p}})^{-1}$ and consequently the A-analyticity of $\widehat{C}$. □

4 Analytic curves in algebraic varieties over local fields and canonical semi-norms

4.A Consistent sequences of norms

Let K be a local field, X a projective scheme over K, and L a line bundle over X.

We may consider the following natural constructions of sequences of norms on the spaces of sections $\Gamma(X, L^{\otimes n})$:

1) When $K = \mathbf{C}$ and X is reduced, we may choose an arbitrary continuous norm $\|.\|_L$ on the $\mathbf{C}$-analytic line bundle $L_{\rm an}$ defined by L on the compact and reduced complex analytic space $X(\mathbf{C})$. Then, for any integer n, the space of algebraic regular sections $\Gamma(X, L^{\otimes n})$ may be identified with a subspace of the space of continuous sections of $L_{\rm an}^{\otimes n}$ over $X(\mathbf{C})$. It may therefore be equipped with the restriction of the L^∞-norm, defined by:
$$\|s\|_{\mathrm{L}^\infty,n} := \sup_{x\in X(\mathbf{C})} \|s(x)\|_{L^{\otimes n}} \quad \text{for any } s \in \Gamma(X, L^{\otimes n}), \tag{4.1}$$
where $\|.\|_{L^{\otimes n}}$ denotes the continuous norm on $L_{\rm an}^{\otimes n}$ deduced from $\|.\|_L$ by taking the n-th tensor power.
This construction admits a variant where, instead of the sup-norms (4.1), one considers the L^p-norms defined by using some "Lebesgue measure" (*cf.* [12, 4.1.3], and [46, Théorème 3.10]).
2) When $K = \mathbf{R}$ and X is reduced, we may choose a continuous norm on $L_\mathbf{C}$ that is invariant under complex conjugation. The previous constructions define complex norms on the complex vector spaces
$$\Gamma(X, L^{\otimes n}) \otimes_\mathbf{R} \mathbf{C} \simeq \Gamma(X_\mathbf{C}, L_\mathbf{C}^{\otimes n})$$
which are invariant under complex conjugation, and by restriction, real norms on the real vector spaces $\Gamma(X, L^{\otimes n})$.
3) When K is a p-adic field, with ring of integers $\mathscr{O}$, we may choose a pair $(\mathscr{X}, \mathscr{L})$, where $\mathscr{X}$ is a projective flat model of X over $\mathscr{O}$, and $\mathscr{L}$ a line bundle over $\mathscr{X}$ extending L. Then, for any integer n, the $\mathscr{O}$-module

$\Gamma(\mathscr{X}, \mathscr{L}^{\otimes n})$ is free of finite rank and may be identified with an $\mathscr{O}$-lattice in the K-vector space $\Gamma(X, L^{\otimes n})$, and consequently defines a norm on the latter — namely, the norm $\|\cdot\|_n$ such that a section $s \in \Gamma(X, L^{\otimes n})$ satisfies $\|s\|_n \leqslant 1$ iff s extends to a section of $\mathscr{L}^{\otimes n}$ over $\mathscr{X}$.

4) A variant of Construction (1) can be used when K is a p-adic field and X is reduced. Let $\|\cdot\|$ be a metric on L (see Appendix A for basic definitions concerning metrics in the p-adic setting). For any integer n, the space $\Gamma(X, L^{\otimes n})$ admits a L^∞-norm, defined for any $s \in \Gamma(X, L^{\otimes n})$ by $\|s\|_{\mathrm{L}^\infty,n} := \sup_{x \in X(C)} \|s(x)\|$, where C denotes the completion of an algebraic closure of K. When the metric of L is defined by a model $\mathscr{L}$ of L on a *normal* projective model $\mathscr{X}$ of X on R, then this norm coincides with that defined by construction (3) (see, *e.g.*, [48, Proposition 1.2]).

For any given K, X, and L as above, we shall say that two sequences $(\|.\|_n)_{n\in\mathbf{N}}$ and $(\|.\|'_n)_{n\in\mathbf{N}}$ of norms on the finite dimensional K-vector spaces $(\Gamma(X, L^{\otimes n}))_{n\in\mathbf{N}}$ are *equivalent* when, for some positive constant C and any positive integer n,

$$C^{-n} \|.\|'_n \leqslant \|.\|_n \leqslant C^n \|.\|'_n .$$

One easily checks that, for any given K, X and L, the above constructions provide sequences of norms $(\|.\|_n)_{n\in\mathbf{N}}$ on the sequence of spaces $(\Gamma(X, L^{\otimes n}))_{n\in\mathbf{N}}$ that are all equivalent. In particular, their equivalence class does not depend on the auxiliary data (models, norms on L, ...) involved. (For the comparison of the L^2 and L^∞ norms in the archimedean case, see notably [46, Théorème 3.10].)

A sequence of norms on the spaces $\Gamma(X, L^{\otimes n})$ that is equivalent to one (or, equivalently, to any) of the sequences thus constructed will be called *consistent*. This notion immediately extends to sequences $(\|.\|_n)_{n\geqslant n_0}$ of norms on the spaces $\Gamma(X, L^{\otimes n})$, defined only for n large enough.

When the line bundle L is ample, consistent sequences of norms are also provided by additional constructions. Indeed we have:

Proposition 4.2. *Let K be a local field, X a projective scheme over K, and L an* ample *line bundle over X. Let moreover Y be a closed subscheme of X, and assume X and Y reduced when K is archimedean.*

For any consistent sequence of norms $(\|.\|_n)_{n\in\mathbf{N}}$ on $(\Gamma(X, L^{\otimes n}))_{n\in\mathbf{N}}$, the quotient norms $(\|.\|'_n)_{n\geqslant n_0}$ on the spaces $(\Gamma(Y, L^{\otimes n}_{|Y}))_{n\geqslant n_0}$, deduced from the norms $\|.\|_n$ by means of the restriction maps $\Gamma(X, L^{\otimes n}) \longrightarrow \Gamma(Y, L^{\otimes n}_{|Y})$ — which are surjective for $n \geqslant n_0$ large enough since L is ample — *constitute a consistent sequence.*

When K is archimedean, this is proved in [13, Appendix], by introducing a positive metric on L, as a consequence of Grauert's finiteness theorem for pseudo-convex domains applied to the unit disk bundle of $L^\vee$ (see also [46]).

When K is a p-adic field with ring of integers $\mathscr{O}$, Proposition 4.2 follows from the basic properties of ample line bundles over projective $\mathscr{O}$-schemes.

Let indeed $\mathscr{X}$ be a projective flat model of X over $\mathscr{O}$, $\mathscr{L}$ an ample line bundle on $\mathscr{X}$, $\mathscr{Y}$ the closure of Y in $\mathscr{X}$, and $\mathscr{I}_{\mathscr{Y}}$ the ideal sheaf of $\mathscr{Y}$. If the positive integer n is large enough, then the cohomology group $H^1(\mathscr{Y}, \mathscr{I}_{\mathscr{Y}} \cdot \mathscr{L}^{\otimes n})$ vanishes, and the restriction morphism $\Gamma(\mathscr{X}, \mathscr{L}^{\otimes n}) \to \Gamma(\mathscr{Y}, \mathscr{L}^{\otimes n}_{|\mathscr{Y}})$ is therefore surjective. Consequently, the norm on $\Gamma(Y, L^{\otimes n}_{|Y})$ attached to the lattice $\Gamma(\mathscr{Y}, \mathscr{L}^{\otimes n}_{|\mathscr{Y}})$ is the quotient of the norm on $\Gamma(X, L^{\otimes n})$ attached to $\Gamma(\mathscr{X}, \mathscr{L}^{\otimes n})$.

Let E be a finite dimensional vector space over the local field K, equipped with some norm, supposed to be euclidean or hermitian in the archimedean case. This norm induces similar norms on the tensor powers $E^{\otimes n}$, $n \in \mathbf{N}$, hence — by taking the quotient norms — on the symmetric powers $\mathrm{Sym}^n E$. If X is the projective space $\mathbf{P}(E) := \mathbf{Proj}\,\mathrm{Sym}^{\cdot}(E)$ and L the line bundle $\mathscr{O}(1)$ over $\mathbf{P}(E)$, then the canonical isomorphisms $\mathrm{Sym}^n E \simeq \Gamma(X, L^{\otimes n})$ allow one to see these norms as a sequence of norms on $(\Gamma(X, L^{\otimes n}))_{n \in \mathbf{N}}$. One easily checks that this sequence is consistent. (This is straightforward in the p-adic case. When K is archimedean, this follows for instance from [15, Lemma 4.3.6].)

For any closed subvariety Y in $\mathbf{P}(E)$ and any $n \in \mathbf{N}$, we may consider the commutative diagram of K-linear maps:

$$\begin{tikzcd}
\mathrm{Sym}^n E \arrow[r, "\sim"] & \mathrm{Sym}^n \Gamma(\mathbf{P}(E), \mathscr{O}(1)) \arrow[r, "\sim"] \arrow[d] & \Gamma(\mathbf{P}(E), \mathscr{O}(n)) \arrow[d, "\alpha_n"] \\
& \mathrm{Sym}^n \Gamma(Y, \mathscr{O}(1)) \arrow[r, "\beta_n"] & \Gamma(Y, \mathscr{O}(n))
\end{tikzcd}$$

where the vertical maps are the obvious restriction morphisms. The maps α_n, and consequently β_n, are surjective if n is large enough.

Together with Proposition 4.2, these observations yield the following corollary:

Corollary 4.3. *Let K, E and Y a closed subscheme of $\mathbf{P}(E)$ be as above. Assume that Y is reduced if K is archimedean. Let us choose a norm on E (resp. on $\Gamma(Y, \mathscr{O}(1))$) and let us equip* $\mathrm{Sym}^n E$ *(resp.* $\mathrm{Sym}^n \Gamma(Y, \mathscr{O}(1))$*) with the induced norm, for any $n \in \mathbf{N}$.*

Then the sequence of quotient norms on $\Gamma(Y, \mathscr{O}(n))$ defined for n large enough by means of the surjective morphisms $\alpha_n \colon \mathrm{Sym}^n E \to \Gamma(Y, \mathscr{O}(n))$ (resp. by means of $\beta_n \colon \mathrm{Sym}^n \Gamma(Y, \mathscr{O}(1)) \to \Gamma(Y, \mathscr{O}(n))$) is consistent.

4.B Canonical semi-norms

Let K be a local field. Let X be a *projective* variety over K, P a rational point in $X(K)$, and $\widehat{C}$ be a smooth K-*analytic* formal curve in $\widehat{X}_P$. To these data, we are going to attach a canonical semi-norm $\|\cdot\|^{\mathrm{can}}_{X,\widehat{C}}$ on the tangent line $T_P\widehat{C}$ of $\widehat{C}$ at P. It will be defined by considering an avatar of the evaluation map

$$E_D^i/E_D^{i+1} \hookrightarrow \Gamma(\mathscr{P}, \mathscr{O}(D) \otimes \mathscr{N}^{\vee \otimes i})$$

which played a prominent role in our proof of Proposition 2.1.

The construction of $\| \cdot \|_{X,\widehat{C}}^{\mathrm{can}}$ will require auxiliary data, of which it will eventually not depend.

Let us choose a line bundle L on X and a consistent sequence of norms on the K-vector spaces $E_D = \Gamma(X, L^{\otimes D})$, for $D \in \mathbf{N}$. Let us also fix norms $\|\cdot\|_0$ on the K-lines $T_P\widehat{C}$ and $L_{|P}$.

Let us denote by C_i the ith neighborhood of P in $\widehat{C}$. Thus we have $C_{-1} = \emptyset$, $C_0 = \{P\}$, and C_i is a K-scheme isomorphic to $\operatorname{Spec} K[t]/(t^{i+1})$; moreover, $\widehat{C} = \varinjlim C_i$. Let us denote by E_D^i the K-vector subspace of the $s \in E_D$ such that $s_{|C_{i-1}} = 0$. The restriction map $E_D \to \Gamma(C_i, L^{\otimes D})$ induces a linear map of finite dimensional K-vector spaces

$$\varphi_D^i \colon E_D^i \to \Gamma(C_i, \mathscr{I}_{C_{i-1}} \otimes L^{\otimes D}) \simeq (T_P^\vee \widehat{C})^{\otimes i} \otimes L_{|P}^{\otimes D}.$$

We may consider the $\|\varphi_D^i\|$ of this map, computed by using the chosen norms on E_D, $T_P\widehat{C}$, and $L_{|P}$, and the ones they induce by restiction, duality and tensor product on E_D^i and on $(T_P^\vee \widehat{C})^{\otimes i} \otimes L_{|P}^{\otimes D}$.

Let us now define $\rho(L)$ by the following formula:

$$\rho(L) = \limsup_{i/D \to +\infty} \frac{1}{i} \log\|\varphi_D^i\|.$$

The analyticity of $\widehat{C}$ implies that $\rho(L)$ belongs to $[-\infty, +\infty[$. Indeed, when K is $\mathbf{C}$ or $\mathbf{R}$, as observed in [13, §3.1], from Cauchy inequality we easily derive the existence of positive real numbers r and C such that

$$\|\varphi_D^i\| \leqslant C^{D+1} r^{-i}. \tag{4.4}$$

When K is ultrametric, we may actually bound $\rho(L)$ in terms of the size of $\widehat{C}$:

Lemma 4.5. *Assume that K is ultrametric and let R be its ring of integers. Let $\mathscr{X}$ be a projective flat R-model of X and let $\mathscr{P}\colon \operatorname{Spec} R \to \mathscr{X}$ the section extending P. Assume moreover that the metric of L is given by a line bundle $\mathscr{L}$ on $\mathscr{X}$ extending L and the consistent sequence of norms on (E_D) by the construction (3) in Section 4.A, and fix the norm $\|\cdot\|_0$ on $T_P\widehat{C}$ so that its unit ball is equal to $N_{\mathscr{P}}\mathscr{X} \cap T_P\widehat{C}$.*

Then, one has

$$\rho(L) \leqslant -\log S_{\mathscr{X},\mathscr{P}}(\widehat{C}).$$

Proof. Let r be an element of $]0, S_{\mathscr{X}}(\widehat{C})[$. We claim that, with the notation above, we have:

$$\|\varphi_i^D\| \leqslant r^{-i}.$$

This will establish that $\rho(L) = \limsup_{i/D \to +\infty} \frac{1}{i} \log\|\varphi_i^D\| \leqslant -\log r$, hence the required inequality by letting r go to $S_{\mathscr{X}}(\widehat{C})$.

To establish the above estimate on $\|\varphi_i^D\|$, let us choose an affine open neighbourhood U of $\mathscr{P}$ in $\mathscr{X}$ such that $\mathscr{L}_U$ admits a non-vanishing section l, and a closed embedding $i\colon U \hookrightarrow \mathbf{A}_R^N$ such that $i(\mathscr{P}) = (0, \ldots, 0)$. Let $\widehat{C}'$ denote the image of $\widehat{C}$ by the embedding of formal schemes $\widehat{i_{KP}} : \widehat{X}_P \hookrightarrow \widehat{\mathbf{A}_{K,0}^N}$. By the very definition of the size, we may find Φ in $G_{\mathrm{an},r}$ such that $\Phi^*\widehat{C}' = \widehat{\mathbf{A}_0^1} \times \{0\}^{N-1}$. Let s be an element of $\Gamma(\mathscr{X}, \mathscr{L}^{\otimes D})$. We may write $s_{|U} = i^*Q \cdot l^{\otimes D}$ for some Q in $R[X_1, \ldots, X_N]$. Then, Φ^*Q is given by a formal series $g = \sum b_I X^I$ which satisfies $\|g\|_r \leqslant 1$, or equivalently, $|b_I| r^{|I|} \leqslant 1$ for any multiindex I. If s belongs to E_D^i, with the chosen normalizations of norms, we have: $\|\varphi_i^D(s)\| = |b_{i,0,\ldots,0}| \leqslant r^{-i}$. □

The exponential of $\rho(L)$ is a well defined element in $[0, +\infty[$, and we may introduce the following

Definition 4.6. *The* canonical semi-norm *on $T_P\widehat{C}$ attached to $(X, \widehat{C}, L)$ is*

$$\|\cdot\|_{X,\widehat{C},L}^{\mathrm{can}} := e^{\rho(L)} \|\cdot\|_0.$$

Observe that, if $\widehat{C}$ is algebraic, then there exists a real number λ such that the filtration $(E_D^i)_{i\in\mathbf{N}}$ becomes stationary — or equivalently φ_D^i vanishes — for $i/D > \lambda$ (for instance, we may take the degree of the Zariski closure of $\widehat{C}$ for λ). Consequently, in this case, $\rho(L) = -\infty$ and the canonical semi-norm $\|\cdot\|_{X,\widehat{C},L}^{\mathrm{can}}$ vanishes.

The notation $\|\cdot\|_{X,\widehat{C},L}^{\mathrm{can}}$ for the canonical semi-norm — which makes reference to X, $\widehat{C}$, and L only — is justified by the first part in the next Proposition:

Proposition 4.7. *a) The semi-norm $\|\cdot\|_{X,\widehat{C},L}^{\mathrm{can}}$ is independent of the choices of norms on $T_P\widehat{C}$ and $L_{|P}$, and of the consistent sequence of norms on the spaces $E_D := \Gamma(X, L^{\otimes D})$.*

b) For any positive integer k, the semi-norm $\|\cdot\|_{X,\widehat{C},L}^{\mathrm{can}}$ is unchanged if L is replaced by $L^{\otimes k}$.

c) Let L_1 and L_2 be two line bundles and assume that $L_2 \otimes L_1^{-1}$ has a regular section σ over X that does not vanish at P. Then

$$\|\cdot\|_{X,\widehat{C},L_1}^{\mathrm{can}} \leqslant \|\cdot\|_{X,\widehat{C},L_2}^{\mathrm{can}}.$$

Proof. a) Let us denote with primes another family of norms on the spaces $T_P\widehat{C}$, $L_{|P}$, and E_D, and by $\rho'(L)$ and $(\|\cdot\|_{X,L,\widehat{C}}^{\mathrm{can}})'$ the attached "rho-invariant" and canonical semi-norm. There are positive real numbers a, b, c such that $\|t\|_0' = a\|t\|_0$ for any $t \in T_P\widehat{C}$, $\|s(P)\|' = b\|s(P)\|$ for any local section s of L at P, and

$$c^{-D}\|s\| \leqslant \|s\|' \leqslant c^D\|s\|$$

for any positive integer D and any global section $s \in E_D$. Consequently, for $(i,D) \in \mathbf{N}^2$ and $s \in E^i_D$,

$$\|\varphi^i_D(s)\|' = a^{-i}b^D\|\varphi^i_D(s)\| \leqslant a^{-i}b^D\|\varphi^i_D\|\|s\| \leqslant a^{-i}b^D\|\varphi^i_D\|c^D\|s\|',$$

hence

$$\|\varphi^i_D\|' \leqslant a^{-i}c^Db^D\|\varphi^i_D\|$$

and

$$\frac{1}{i}\log\|\varphi^i_D\|' \leqslant -\log a + \frac{D}{i}\log(bc) + \frac{1}{i}\log\|\varphi^i_D\|.$$

When i/D goes to infinity, this implies:

$$\rho'(L) \leqslant -\log a + \rho(L),$$

from which follows:

$$(\|\cdot\|^{\mathrm{can}}_{X,L,\widehat{C}})' \leqslant \|\cdot\|^{\mathrm{can}}_{X,L,\widehat{C}},$$

by definition of the canonical semi-norm. The opposite inequality also holds by symmetry, hence the desired equality.

b) To define $\rho(L)$ and $\rho(L^{\otimes k})$, let us use the same norm $\|\cdot\|_0$ on $T_P\widehat{C}$, and assume that the consistent sequence of norms chosen on $(\Gamma(X, L^{\otimes D})$ is defined by one of the constructions (1-4) in the above subsection 4.A, and finally that the one on $(\Gamma(X,(L^{\otimes k})^{\otimes D})) = (\Gamma(X, L^{\otimes kD}))$ is extracted from the one on $(\Gamma(X, L^{\otimes D})$.

Specifying the line bundle with a supplementary index, one has

$$\varphi^i_{D,L^{\otimes k}} = \varphi^i_{kD,L}.$$

The definition of an upper limit therefore implies that $\rho(L^k) \leqslant \rho(L)$.

To establish the opposite inequality, observe that, for any section s in $E^i_{D,L}$ and any positive integer k, the k-th tensor power $s^{\otimes k}$ belongs to $E^{ki}_{D,L^{\otimes k}}$ and

$$\varphi^{ki}_{D,L^{\otimes k}}(s^{\otimes k}) = (\varphi^i_{D,L}(s))^{\otimes k}.$$

Let ρ be any real number such that $\rho < \rho(L)$, and choose i, D, and $s \in E^i_{D,L}$ such that $\|\varphi^i_{D,L}(s)\| \geqslant e^{\rho i}\|s\|$. Then, for any positive integer k, we have

$$\|\varphi^{ki}_{D,L^{\otimes k}}(s^{\otimes k})\| = \|\varphi^i_{D,L}(s)\|^k \geqslant e^{\rho k i}\|s\|^k = e^{\rho k i}\|s^{\otimes k}\|,$$

so that $\|\varphi^{ki}_{D,L^{\otimes k}}\|^{1/ki} \geqslant e^{\rho}$. Consequently, $\rho(L^k) \geqslant \rho$.

c) Here again, we may use the same norm $\|\cdot\|_0$ on $T_P\widehat{C}$ to define $\rho(L_1)$ and $\rho(L_2)$, and assume that the consistent sequence of norms chosen on $(\Gamma(X, L_1^{\otimes D})$ and $(\Gamma(X, L_2^{\otimes D})$ are defined by one of the constructions (1-4) above.

If s is a global section of $L_1^{\otimes D}$, then $s \otimes \sigma^{\otimes D}$ is a global section of $L_2^{\otimes D}$; if s vanishes at order i along $\widehat{C}$, so does $s \otimes \sigma^{\otimes D}$ and

$$\varphi^i_{D,L_2}(s \otimes \sigma^{\otimes D}) = \varphi^i_{D,L_1}(s) \otimes \sigma(P)^{\otimes D}.$$

Consequently,

$$\|\varphi^i_{D,L_1}(s)\| \leqslant \|\varphi^i_{D,L_2}\|.\|s \otimes \sigma^{\otimes D}\|.\|\sigma(P)\|^{-D} \leqslant (\|\sigma(P)\|^{-1}\|\sigma\|)^D.\|\varphi^i_{D,L_2}\|.\|s\|,$$

and $\rho(L_1) \leqslant \rho(L_2)$, as was to be shown. □

Corollary 4.8. *The set of semi-norms on $T_P\widehat{C}$ described by $\|\cdot\|^{\mathrm{can}}_{X,\widehat{C},L}$ when L varies in the class of line bundles on X possesses a maximal element, namely the canonical semi-norm $\|\cdot\|^{\mathrm{can}}_{X,\widehat{C},L}$ attached to any ample line bundle L on X.*

We shall denote $\|\cdot\|^{\mathrm{can}}_{X,\widehat{C}}$ this maximal element. The formation of $\|\cdot\|^{\mathrm{can}}_{X,\widehat{C}}$ satisfies the following compatibility properties with respect to rational morphisms.

Proposition 4.9. *Let X' be another projective algebraic variety over K, and let $f\colon X \dashrightarrow X'$ be a rational map that is defined near P. Let $P' := f(P)$, and assume that f defines an (analytic, or equivalently, formal) isomorphism from $\widehat{C}$ onto a smooth K-analytic formal curve $\widehat{C'}$ in $\widehat{X'}_{P'}$.*

Then for any $v \in T_P\widehat{C}$,

$$\|Df(P)v\|_{X',f(\widehat{C})} \leqslant \|v\|_{X,\widehat{C}}.$$

If moreover f is an immersion in a neighborhood of P, then the equality holds.

When K is archimedean, this summarizes the results established in [13, Sections 3.2 and 3.3]. The arguments in *loc. cit.* may be immediately transposed to the ultrametric case, by using consistent norms as defined above instead of L^∞ norms on the spaces of sections E_D. We leave the details to the reader.

Observe finally that this Proposition allows us to define the canonical semi-norm $\|\cdot\|^{\mathrm{can}}_{X,\widehat{C}}$ when the algebraic variety X over K is only supposed to be quasi-projective. Indeed, if $\overline{X}$ denotes some projective variety containing X as an open subvariety, the semi-norm $\|\cdot\|^{\mathrm{can}}_{\overline{X},\widehat{C}}$ is independent of the choice of $\overline{X}$, and we let

$$\|\cdot\|^{\mathrm{can}}_{X,\widehat{C}} := \|\cdot\|^{\mathrm{can}}_{\overline{X},\widehat{C}}.$$

5 Capacitary metrics on p-adic curves

5.A Review of the complex case

Let M be a compact Riemann surface and let Ω be an open subset of M. We assume that the compact subset complementary to Ω in any connected component of M is not polar. Let D be an effective divisor on M whose support is contained in Ω. Potential theory on Riemann surfaces (see [11, 3.1.3–4]) shows the existence of a unique subharmonic function $g_{D,\Omega}$ on M satisfying the following assumptions:

1) $g_{D,\Omega}$ is harmonic on $\Omega \setminus |D|$;
2) the set of points $z \in M \setminus \Omega$ such that $g_{D,\Omega}(z) \neq 0$ is a polar subset of $\partial\Omega$;
3) for any open subset V of Ω and any holomorphic function f on V such that $\operatorname{div}(f) = D$, the function $g_{D,\Omega} - \log|f|^{-2}$ on $V \setminus |D|$ is the restriction of a harmonic function on V.

Moreover, $g_{D,\Omega}$ takes non-negative values, is locally integrable on M and defines a L^2_1-Green current for D in the sense of [11]. It is the so-called *equilibrium potential* attached to the divisor D in Ω.

If E is another effective divisor on M supported in Ω, one has $g_{D+E,\Omega} = g_{D,\Omega} + g_{E,\Omega}$. We can therefore extend by linearity the definition of the equilibrium potential $g_{D,\Omega}$ to arbitrary divisors D on M that are supported on Ω. Recall also that, if Ω_0 denotes the union of the connected components of Ω which meet $|D|$, then $g_{D,\Omega_0} = g_{D,\Omega}$ (*loc. cit.*, p. 258).

The function $g_{D,\Omega}$ allows one to define a generalized metric on the line bundle $\mathscr{O}_M(D)$, by the formula

$$\|\mathbf{1}_D\|^2(z) = \exp(-g_{D,\Omega}(z)),$$

where $\mathbf{1}_D$ denotes the canonical global section of $\mathscr{O}_M(D)$. We will call this metric the *capacitary metric*[4] on $\mathscr{O}_M(D)$ attached to Ω and denote by $\|f\|^{\mathrm{cap}}_{\Omega}$ the norm of a local section f of $\mathscr{O}_M(D)$.

5.B Equilibrium potential and capacity on p-adic curves

Let R be a complete discrete valuation ring, and let K be its field of fractions and k its residue field. Let X be a smooth projective curve over K and let U be an affinoid subspace of the associated rigid K-analytic curve X^{an}. We shall always require that U *meets every connected component of* X^{an} — this hypothesis is analogous to the non-polarity assumption in the complex case.

[4] Our terminology differs slightly from that in [11]. In the present article, the term *capacitary metric* will be used for two distinct notions: for the metrics on line bundles defined using equilibrium potentials just defined, and for some metrics on the tangent line to M at a point, see Subsection 5.C. In [11], it was used for the latter notion only.

We also let $\Omega = X^{\mathrm{an}} \setminus U$, which we view as a (non quasi-compact) rigid K-analytic curve; its affinoid subspaces are just affinoid subspaces of X^{an} disjoint of U — see Appendix B for a detailed proof that this endowes Ω with the structure of a rigid K-analytic space in the sense of Tate.

The aim of this subsection is to endow the line bundle $\mathscr{O}(D)$, where D is a divisor which does not meet U, with a metric (in the sense of Appendix A) canonically attached attached to Ω, in a way that parallels the construction over Riemann surfaces recalled in the previous subsection.

Related constructions of equilibrium potentials over p-adic curves have been developed by various authors, notably Rumely [49] and Thuillier [51] (see also [37]). Our approach will be self-contained, and formulated in the framework of classical rigid analytic geometry. Our main tool will be intersection theory on a model $\mathscr{X}$ of X over R. This point of view will allow us to combine potential theory on p-adic curves and Arakelov intersection theory on arithmetic surfaces in a straightforward way.

We want to indicate that, by using an adequate potential theory on analytic curves in the sense of Berkovich [4] such as the one developped by Thuillier [51], one could give a treatment of equilibrium potential on p-adic curves and their relations to canonical semi-norms that would more closely parallel the one in the complex case. For instance, in the Berkovich setting, the affinoid subspace U is a compact subset of the analytic curve attached to X, and Ω is an open subset. We leave the transposition and the extension of our results in the framework of Berkovich and Thuillier to the interested reader.

By Raynaud's general results on formal/rigid geometry, see for instance [8, 9], there exists a normal projective flat model $\mathscr{X}$ of X over R such that U is the set of rig points of X^{an} reducing to some open subset U of the special fibre X. We shall write $U =]\mathsf{U}[_{\mathscr{X}}$ and say that U is the tube of U in $\mathscr{X}$; similarly, we write $\Omega =]\mathscr{X} \setminus \mathsf{U}[_{\mathscr{X}}$. (We remove the index $\mathscr{X}$ from the notation when it is clear from the context.) The reduction map identifies the connected components of U with those of U, and the connected components of Ω with those of $\mathsf{X} \setminus \mathsf{U}$. Since we assumed that U meets every connected component of X, this shows that U meets every connected component of X.

Recall that to any two Weil divisors Z_1 and Z_2 on $\mathscr{X}$ such that $Z_{1,K}$ and $Z_{2,K}$ have disjoint supports is attached their the *intersection number* (Z_1, Z_2). It is a rational number, which depends linearly on Z_1 and Z_2. It may be defined *à la* Mumford (see [42, II.(b)]), and coincide with the degree over the residue field k of the intersection class $Z_1.Z_2$ in $\mathrm{CH}_0(\mathsf{X})$ when Z_1 or Z_2 is Cartier. Actually, when the residue field k is an algebraic extension of a finite field — for instance when K is a p-adic field, the case we are interested in in the sequel — any Weil divisor on $\mathscr{X}$ has a multiple which is Cartier (see [40, Théorème 2.8]), and this last property, together with their bilinearity, completely determines the intersection numbers.

The definition of intersection numbers immediately extends by bilinearity to pairs of Weil divisors with coefficients in $\mathbf{Q}$ (shortly, $\mathbf{Q}$-*divisors*) in $\mathscr{X}$ whose supports do not meet in X.

Proposition 5.1. *For any divisor D on X, there is a unique $\mathbf{Q}$-divisor $\mathscr{D}$ on $\mathscr{X}$ extending D and satisfying the following two conditions:*

1) *For any irreducible component v of codimension 1 of $\mathsf{X} \setminus \mathsf{U}$, $\mathscr{D} \cdot v = 0$.*
2) *The vertical components of $\mathscr{D}$ do not meet U.*

Moreover, the map $D \mapsto \mathscr{D}$ so defined is linear and sends effective divisors to effective divisors.

Proof. Let S denote the set of irreducible components of X and let $T \subset S$ be the subset consisting of components which do not meet U. Let $\mathscr{D}_0$ be the schematic closure of D in $\mathscr{X}$. Since U meets every connected component of X, T does not contain all of the irreducible components of some connected component of X, so that the restriction of the intersection pairing of $\mathrm{Div}_{\mathbf{Q}}(\mathscr{X})$ to the subspace generated by the components of X which belong to T is negative definite (see for instance [21, Corollaire 1.8] when $\mathscr{X}$ is regular; one reduces to this case by considering a resolution of $\mathscr{X}$, as in [42, II.(b)]). Therefore, there is a unique vertical divisor V, linear combination of components in T, such that $(\mathscr{D}_0 + V, s) = 0$ for any $s \in T$. (In the analogy with the theory of electric networks, the linear system one has to solve corresponds to that of a Dirichlet problem on a graph, with at least one electric source per connected component.) Set $\mathscr{D} = \mathscr{D}_0 + V$; it satisfies assumptions 1) and 2). The linearity of the map $D \mapsto \mathscr{D}$ follows immediately from the uniqueness of V.

Let us assume that D is effective and show that so is V. (In the graph theoretic language, this is a consequence of the maximum principle for the discrete Laplacian.) Denote by m_s the multiplicity of the component s in the special fibre of $\mathscr{X}$, so that $\sum_{s \in S} m_s s$ belongs to the kernel of the intersection pairing. Write $V = \sum_{s \in S} c_s s$, where $c_s = 0$ if $s \notin T$.

Let S' be the set of elements $s \in S$ where c_s/m_s achieves its minimal value. Then, for any element τ of $S' \cap T$,

$$\begin{aligned} 0 &= (c_\tau/m_\tau)\Big(\sum_{s \in S} m_s s, \tau\Big) = c_\tau(\tau,\tau) + \sum_{s \neq \tau} (c_\tau/m_\tau) m_s (s,\tau) \\ &\leqslant c_\tau(\tau,\tau) + \sum_{s \neq \tau} c_s (s,\tau) = \sum_{s \in S} (c_s s, \tau) \\ &\leqslant (\mathscr{D},\tau) - (\mathscr{D}_0,\tau) = -(\mathscr{D}_0,\tau). \end{aligned}$$

Since $\mathscr{D}_0$ is effective and horizontal, $(\mathscr{D}_0, \tau) \geqslant 0$, hence all previous inequalities are in fact equalities. In particular, $(\mathscr{D}_0, \tau) = 0$ and $c_s/m_s = c_\tau/m_\tau$ for any $s \in S$ such that $(s,\tau) \neq 0$.

Assume by contradiction that V is not effective, *i.e.*, that there is some s with c_s negative. Then S' is contained in T (for $c_s = 0$ is $s \notin T$) and the preceding argument implies that S' is a union of connected components of X. (In the graph theoretical analogue, all neighbours of a vertex in S' belong to S'.) This contradicts the assumption that U meets every connected component of X^{an} and concludes the proof that V is effective. $\square$

In order to describe the functoriality properties of the assignment $D \mapsto \mathscr{D}$ constructed in Proposition 5.1, we consider two smooth projective curves X and X' over K, some normal projective flat models $\mathscr{X}$ and $\mathscr{X}'$ over R of these curves, and $\pi\colon \mathscr{X}' \to \mathscr{X}$ an R-morphism such that the K-morphism $\pi_K\colon X' \to X$ is finite.

Recall that the direct image of 1-dimensional cycles defines a $\mathbf{Q}$-linear map between spaces of $\mathbf{Q}$-divisors:

$$\pi_*\colon\ \mathrm{Div}_{\mathbf{Q}}(\mathscr{X}') \longrightarrow \mathrm{Div}_{\mathbf{Q}}(\mathscr{X}),$$

and that the inverse image of Cartier divisors defines a $\mathbf{Q}$-linear map between spaces of $\mathbf{Q}$-Cartier divisors:

$$\pi^*\colon\ \mathrm{Div}^{\mathrm{Cartier}}_{\mathbf{Q}}(\mathscr{X}) \longrightarrow \mathrm{Div}^{\mathrm{Cartier}}_{\mathbf{Q}}(\mathscr{X}').$$

These two maps satisfy the following adjunction formula, valid for any Z in $\mathrm{Div}^{\mathrm{Cartier}}_{\mathbf{Q}}(\mathscr{X})$ and any Z' in $\mathrm{Div}_{\mathbf{Q}}(\mathscr{X}')$:

$$(\pi^* Z, Z') = (Z, \pi_* Z'). \tag{5.2}$$

When k is an algebraic extension of a finite field, as recalled above, $\mathbf{Q}$-divisors and $\mathbf{Q}$-Cartier divisors on $\mathscr{X}$ or $\mathscr{X}'$ coincide, and π^* may be seen as a linear map from $\mathrm{Div}_{\mathbf{Q}}(\mathscr{X})$ to $\mathrm{Div}_{\mathbf{Q}}(\mathscr{X}')$ adjoint to π_*.

In general, the map π^* above admits a unique extension to a $\mathbf{Q}$-linear map

$$\pi^*\colon\ \mathrm{Div}_{\mathbf{Q}}(\mathscr{X}) \longrightarrow \mathrm{Div}_{\mathbf{Q}}(\mathscr{X}'),$$

compatible with the pull-back of divisors on the generic fiber

$$\pi_K^*\colon\ \mathrm{Div}_{\mathbf{Q}}(X) \longrightarrow \mathrm{Div}_{\mathbf{Q}}(X'),$$

such that the adjunction formula (5.2) holds for any (Z, Z') in $\mathrm{Div}_{\mathbf{Q}}(\mathscr{X}) \times \mathrm{Div}_{\mathbf{Q}}(\mathscr{X}')$. The unicity of such a map map follows from the non-degeneracy properties of the intersection pairing, which show that if a divisor Z'_1 supported by the closed fiber X of $\mathscr{X}$ satisfies $Z'_1 \cdot Z'_2 = 0$ for every Z'_2 in $\mathrm{Div}_{\mathbf{Q}}(\mathscr{X}')$, then $Z'_1 = 0$. The existence of π^* is known when $\mathscr{X}'$ is regular (then $\mathrm{Div}_{\mathbf{Q}}(\mathscr{X})$ and $\mathrm{Div}^{\mathrm{Cartier}}_{\mathbf{Q}}(\mathscr{X})$ coincide), and when π is birational — *i.e.,* when π_K is an isomorphism — and $\mathscr{X}$ is regular, according to Mumford's construction in [42, II.(b)]. To deal with the general case, observe that there exist two projective flat regular curves $\tilde{\mathscr{X}}$ and $\tilde{\mathscr{X}}'$ equipped with birational R-morphisms $\nu\colon \tilde{\mathscr{X}} \to \mathscr{X}$ and $\nu'\colon \tilde{\mathscr{X}}' \to \mathscr{X}'$, and an R-morphism $\tilde{\pi}\colon \tilde{\mathscr{X}}' \to \tilde{\mathscr{X}}$ such that $\pi \circ \tilde{\nu} = \tilde{\pi} \circ \nu$. Then it is straightforward that $\pi^* := \tilde{\nu}_* \tilde{\pi}^* \nu^*$ satisfies the required properties.

Observe also that the assignment $\pi \mapsto \pi^*$ so defined is functorial, as follows easily from its definition.

Proposition 5.3. *Let* U *be a Zariski open subset of the special fibre* X *and let* $\mathsf{U}' = \pi^{-1}(\mathsf{U})$. *Assume that* $]\mathsf{U}[_{\mathscr{X}}$ *meets every connected component of* X^{an}*; then* $]\mathsf{U}'[_{\mathscr{X}'}$ *meets every connected component of* $(X')^{\mathrm{an}}$.

Let D *and* D' *be divisors on* X *and* X' *respectively, let* $\mathscr{D}$ *and* $\mathscr{D}'$ *be the extensions to* $\mathscr{X}$ *and* $\mathscr{X}'$*, relative to the open subsets* U *and* U' *respectively, given by Proposition 5.1.*

- *a) Assume that* $D' = \pi^* D$. *If* $|D|$ *does not meet* $]\mathsf{U}[$*, then* $|D'|$ *is disjoint from* $]\mathsf{U}'[$ *and* $\mathscr{D}' = \pi^*\mathscr{D}$.
- *b) Assume that* $D = \pi_* D'$. *If* $|D'|$ *does not meet* $]\mathsf{U}'[$*, then* $|D| \cap]\mathsf{U}[= \emptyset$ *and* $\mathscr{D} = \pi_*\mathscr{D}'$.

Proof. Let us denote by S the set of irreducible components of the closed fibre X of $\mathscr{X}$, and by T its subset of the components which do not meet U. Define similarly S' and T' to be the set of irreducible components of X' and its subset corresponding to the components that do not meet U'. Let also N denote the set of all irreducible components of X' which are contracted to a point by π.

By construction of π^*, the divisor $\pi^*(\mathscr{D})$ satisfies $(\pi^*(\mathscr{D}), n) = 0$ for any $n \in N$ and has no multiplicity along the components of N that are not contained in $\pi^{-1}(|\mathscr{D}|)$.

Since $\mathsf{U}' = \pi^{-1}(\mathsf{U})$, T' is the union of all components of X' that are mapped by π, either to a point outside U, or to a component in T.

a) Let $t' \in T'$. One has $(\pi^*\mathscr{D}, t') = (\mathscr{D}, \pi_* t') = 0$ since t' maps to a component in T, or to a point. Moreover, by the construction of π^*, the vertical components of $\pi^*\mathscr{D}$ are elements $s' \in S'$ such $\pi(s')$ meets the support of $\mathscr{D}$. By assumption, the Zariski closure of D in X is disjoint from U; in other words, the vertical components of $\pi^*\mathscr{D}$ all belong to T'. This shows that the divisor $\pi^*\mathscr{D}$ on $\mathscr{X}'$ satisfies the conditions of Proposition 5.1; since it extends $D' = \pi^* D$, one has $\pi^*\mathscr{D} = \mathscr{D}'$.

b) Let s be a vertical component appearing in $\pi_*(\mathscr{D}')$; necessarily, there is a vertical component s' of $\mathscr{D}'$ such that $s = \pi(s')$. This implies that $s' \in T'$, hence $s \in T$. For any $t \in T$, $\pi^*(t)$ is a linear combination of vertical components of X' contained in $\pi^{-1}(t)$. Consequently, they all belong to T' and one has $(\pi_*(\mathscr{D}'), t) = (\mathscr{D}', \pi^*(t)) = 0$. By uniqueness, $\pi_*(\mathscr{D}) = \mathscr{E}$. □

Corollary 5.4. *Let* X *be a projective smooth algebraic curve over* K*, let* U *be an affinoid subspace of* X^{an} *which meets any connected component of* X^{an}. *Let* D *be a divisor on* X *whose support is disjoint from* U.

Then the metrics on the line bundle $\mathscr{O}_X(D)$ *induced by the line bundle* $\mathscr{O}_{\mathscr{X}}(\mathscr{D})$ *defined by Prop 5.1 does not depend on the choice of the projective flat model* $\mathscr{X}$ *of* X *such that* U *is the tube of a Zariski open subset of the special fibre of* $\mathscr{X}$.

Proof. For $i = 1, 2$, let $(\mathscr{X}_i, \mathsf{U}_i)$ be a pair as above, consisting of a normal flat, projective model $\mathscr{X}_i$ of X over R and an open subset U_i of its special fibre X_i such that $]\mathsf{U}_i[_{\mathscr{X}_i} = U$. Let $\mathscr{D}_i$ denote the extension of D on $\mathscr{X}_i$ relative to U_i.

There exists a third model $(\mathscr{X}', \mathsf{U}')$ which admits maps $\pi_i\colon \mathscr{X}' \to \mathscr{X}_i$, for $i = 1, 2$, extending the identity on the generic fibre. Let $\mathscr{D}'$ denote the extension of D on $\mathscr{X}'$. For $i = 1, 2$, one has $\pi_i^{-1}(\mathsf{U}_i) = \mathsf{U}'$. By Proposition 5.3, one thus has the equalities $\pi^*\mathscr{D}_i = \mathscr{D}'$ hence the line bundles $\mathscr{O}_{\mathscr{X}'}(\mathscr{D}')$ on $\mathscr{X}'$ and $\mathscr{O}_{\mathscr{X}_i}(\mathscr{D}_i)$ on $\mathscr{X}$ induce the same metric on $\mathscr{O}_X(D)$. □

We shall call this metric the *capacitary metric* and denote as $\|f\|_\Omega^{\mathrm{cap}}$ the norm of a local section f of $\mathscr{O}_X(D)$ for this metric.

Proposition 5.5. *Let X be a projective smooth algebraic curve over K, let U be an affinoid subspace of X^{an} which meets any connected component of X^{an}. Let D be a divisor on X whose support is disjoint from U and let $\Omega = X^{\mathrm{an}} \setminus U$.*

If Ω' denotes the union of the connected components of Ω which meet $|D|$, then the capacitary metrics of $\mathscr{O}(D)$ relative to Ω and to Ω' coincide.

Proof. Let us fix a normal projective flat model $\mathscr{X}$ of X over R and a Zariski open subset U of its special fibre X such that $U =]\mathsf{U}[_{\mathscr{X}}$. Let $\mathsf{Z} = \mathsf{X} \setminus \mathsf{U}$ and let Z' denote the union of those connected components of Z which meet the specialization of $|D|$. Then $\Omega' =]\mathsf{Z}'[$ is the complementary subset to the affinoid $]\mathsf{U}'[$, where $\mathsf{U}' = \mathsf{X} \setminus \mathsf{Z}'$; in particular, $]\mathsf{U}'[$ meets every connected component of X^{an}.

Let $\mathscr{D}_0$ denote the horizontal divisor on $\mathscr{X}$ which extends D. The divisor $\mathscr{D}' := \mathscr{D}_{\Omega'}$ is the unique $\mathbf{Q}$-divisor of the form $\mathscr{D}_0 + V$ on $\mathscr{X}$ where V is a vertical divisor supported by Z' such that $(\mathscr{D}', t) = 0$ for any irreducible component of Z'. By the definition of Z', an irreducible component of Z which is not contained in Z' doesn't meet neither Z', nor $\mathscr{D}_0$. It follows that for any such component t, $(\mathscr{D}', t) = (\mathscr{D}_0, t) + (V, t) = 0$. By uniqueness, $\mathscr{D}'$ is the extension of D on $\mathscr{X}$ relative to U, so that $\mathscr{D}_{\Omega'} = \mathscr{D}_\Omega$. This implies the proposition. □

As an application of the capacitary metric, in the next proposition we establish a variant of a classical theorem by Fresnel and Matignon ([25, théorème 1]) asserting that affinoids of a curve can be defined by one equation. (While these authors make no hypothesis on the residue field of k, or on the complementary subset of the affinoid U, we are able to impose the polar divisor of f.) Using the terminology of Rumely [49, §4.2, p. 220], this proposition means that affinoid subsets of a curve are RL-domains ("rational lemniscates"), and that RL-domains with connected complement are PL-domains ("polynomial lemniscates"). It is thus essentially equivalent to Rumely's theorem [49, Theorem 4.2.12, p. 244] asserting that *island domains* coincide with PL-domains. Rumely's proof relies on his non-archimedean potential theory, which we replace here by Proposition 5.1.

This proposition will also be used to derive further properties of the capacitary metric.

Proposition 5.6. *Assume that the residue field k of K is algebraic over a finite field. Let (X, U, Ω) be as above, and let D be an effective divisor which* *does not meet* U *but meets every connected component of* Ω. *There is a rational function* $f \in K(X)$ *with polar divisor a multiple of* D *such that* $U = \{x \in X\,;\, |f(x)| \leqslant 1\}$.

Proof. Keep notations as in the proof of Proposition 5.1; in particular, S denotes the set of irreducible components of X. The closed subset $\mathsf{X} \setminus \mathsf{U}$ has only finitely many connected components, say $\mathsf{V}_1, \dots, \mathsf{V}_r$. Moreover, we may assume that for each i, V_i is the union of a family $T_i \subset S$ of components of X. For any i, the tube $]\mathsf{V}_i[$ in X^{an} consisting of the rig. points of X which reduce to points of V_i is a connected analytic subset of X^{an}, albeit not quasi-compact, and X^{an} is the disjoint union of $U =]\mathsf{U}[$ and of the $]\mathsf{V}_i[$. (See [47] for more details.) We let m_s denote the multiplicity of the component s in the special fibre, and $F = \sum_{s\in S} m_s s$.

Let $\mathscr{D} = \mathscr{D}_0 + V$ be the extension of D to a $\mathbf{Q}$-divisor of $\mathscr{X}$ given by Proposition 5.1, where $\mathscr{D}_0$ is horizontal and $V = \sum_{s\in S} c_s s$ is a vertical divisor supported by the special fibre X. One has $c_s = 0$ for $s \notin T$ and $c_s \geqslant 0$ if $s \in T$. For any $s \notin T$, we define $a_s = (V, s)$. This is a nonnegative rational number and we have

$$\sum_{s\in S\setminus T} a_s m_s = (V, F) - \sum_{t\in T} m_t (V, t) = \sum_{t\in T} m_t(\mathscr{D}_0, t) = \sum_{s\in S} m_s(\mathscr{D}_0, s)$$

since D does not meet U, hence

$$\sum_{s\in S\setminus T} a_s m_s = (\mathscr{D}_0, F) = \deg(D). \tag{5.7}$$

For any $s \in S \setminus T$, let us fix a point x_s of X which is contained on the component s as well as on the smooth locus of $\mathscr{X}$. Using either a theorem of Rumely [49, Theorem 1.3.1, p. 48], or van der Put's description of the Picard group of any one-dimensional K-affinoid, *cf.* [44, Proposition 3.1],[5] there is a rational function $f_s \in K(X)$ with polar divisor a multiple of D and of which all zeroes specialize to x_s. We may write its divisor as a sum

$$\operatorname{div}(f_s) = -n_s\mathscr{D} + E_s + W_s,$$

where n_s is a positive integer, E_s is a horizontal effective divisor having no common component with $\mathscr{D}$, and W_s is a vertical divisor. Since E_s is the closure of the divisor of zeroes of f_s, it only meets the component labelled s. One thus has $(E_s, s') = 0$ for $s' \in S \setminus \{s\}$, while

$$(E_s, s) = \frac{1}{m_s}(E_s, m_s s) = \frac{1}{m_s}(E_s, F) = \frac{n_s}{m_s}\deg D.$$

Let $t \in T$. One has $(\operatorname{div}(f_s), t) = 0$, hence

[5] The proofs in both references are similar and rely on the Abel-Jacobi map, together with the fact that K is the union of its locally compact subfields.

$$(W_s,t)=n_s(\mathscr{D},t)-(E_s,t)=0.$$

Similarly, if $s'\in S\setminus T$, $(\operatorname{div}(f_s),s')=0$ and

$$\begin{aligned}(W_s,s')&=n_s(\mathscr{D},s')-(E_s,s')\\&=n_s(\mathscr{D}_0,s')+n_s(V,s')-(E_s,s')\\&=0+n_sa_{s'}-(E_s,s').\end{aligned}$$

If $s'\neq s$, it follows that

$$(W_s,s')=n_sa_{s'},$$

while

$$(W_s,s)=n_sa_s-\frac{n_s}{m_s}\deg(D).$$

We now define a vertical divisor

$$W=\sum_{s\notin T}\frac{a_sm_s}{n_s}W_s.$$

For any $t\in T$, $(W,t)=0$. Moreover, for any $s'\in S\setminus T$,

$$\begin{aligned}(W,s')&=\sum_{s\notin T}\frac{a_sm_s}{n_s}(W_s,s')\\&=\sum_{s\notin T}a_sm_sa_{s'}-\frac{a_{s'}}{n_{s'}}n_{s'}\deg(D)\\&=a_{s'}\left(\sum_{s\notin T}a_sm_s\right)-a_{s'}\deg(D),\end{aligned}$$

hence $(W,s')=0$ by (5.7). Therefore, the vertical $\mathbf{Q}$-divisor W is a multiple of the special fibre and there is $\lambda\in\mathbf{Q}$ such that $W=\lambda F$. Finally,

$$\sum_{s\notin T}\frac{a_sm_s}{n_s}\operatorname{div}(f_s)-\lambda F=-\deg(D)\mathscr{D}+\sum_{s\notin T}\frac{a_sm_s}{n_s}E_s$$

is a principal $\mathbf{Q}$-divisor. It follows that there are positive integers μ and λ_s, for $s\notin T$, such that

$$\mathscr{P}=\sum_{s\notin T}\lambda_sE_s-\mu\mathscr{D}$$

is the divisor of a rational function $f\in K(X)$.

By construction, the polar divisor of f on X is a multiple of D. Moreover, the reduction of any $x\notin U$ belongs to a component labelled by T at which the multiplicity of $\mathscr{P}$ is positive. Consequently, $|f(x)|>1$. On the contrary, if $x\in U$, it reduces to a component outside T and $|f(x)|\leqslant 1$. More precisely, $|f(x)|<1$ if and only if x reduces to one of the points x_s, $s\notin T$. □

The definition of an algebraic metric now implies the following explicit description of the capacitary metric.

Corollary 5.8. *Let (X, U, Ω) be as above let D be any divisor which does not meet U, and let f be a rational function defining U, as in preceding proposition, and whose polar divisor is equal to mD, for some positive integer m. Then, the capacitary metric on $\mathscr{O}_X(D)$ can be computed as*

$$-\log\|1_{\mathsf{D}}\|^{\mathrm{cap}}_{\Omega}(x) = \frac{1}{m}\log^+|f(x)| = \max(0, \log|f(x)|^{1/m}).$$

Proposition 5.9. *Let (X, U, Ω) and (X', U', Ω') be as above and let $\varphi\colon \Omega' \to \Omega$ be any rigid analytic isomorphism. Let D' be any divisor in X' whose support does not meet U' and let $D = \varphi(D')$.*

Then, for any $x \in \Omega'$,

$$\|1_{D'}\|^{\mathrm{cap}}_{\Omega'}(x) = \|1_D\|^{\mathrm{cap}}_{\Omega}(\varphi(x)).$$

Proof. By linearity, we may assume that D is effective. Let $f \in K(X)$ and $f' \in K(X')$ be rational functions as in Proposition 5.6. Let m and m' be positive integers such that the polar divisor of f and f' are mD and $m'D'$ respectively. The function $f\circ\varphi$ is a meromorphic function on Ω' whose divisor is mD'. Consequently, the meromorphic function on Ω'

$$g = (f \circ \varphi)^{m'}/(f')^m$$

is in fact invertible. We have to prove that $|g|(x) = 1$ for any $x \in \Omega'$.

Let (ε_n) be any decreasing sequence of elements of $\sqrt{|K^*|}$ converging to 1. The sets $V'_n = \{x \in X' \,;\, |f'(x)| \geqslant \varepsilon_n\}$ are affinoid subspaces of Ω' and exhaust it. By the maximum principle (see Proposition B.1 below), one has

$$\sup_{x\in V'_n}|g(x)| = \sup_{|f'(x)|=\varepsilon_n}|g(x)| \leqslant 1/(\varepsilon_n)^m \leqslant 1.$$

Consequently, $\sup_{x\in\Omega'}|g(x)| \leqslant 1$. The opposite inequality is shown similarly by considering the isomorphism $\varphi^{-1}\colon \Omega \to \Omega'$. This proves the proposition. □

5.C Capacitary norms on tangent spaces

Definition 5.10. *Let (X, U, Ω) be as above and let $P \in X(K)$ be a rational point lying in Ω. Let us endow the line bundle $\mathscr{O}_X(P)$ with its capacitary metric relative to Ω. The capacitary norm $\|\cdot\|^{\mathrm{cap}}_{P,\Omega}$ on the K-line T_PX is then defined as the restriction of $(\mathscr{O}_X(P), \|\cdot\|^{\mathrm{cap}}_{\Omega})$ to the point P, composed with the adjunction isomorphism $\mathscr{O}_X(P)|_P \simeq T_PX$.*

Example 5.11. Let us fix a normal projective flat model $\mathscr{X}$, let $\mathscr{P}$ be the divisor extending P, meeting the special fibre X in a smooth point P. Let $\mathsf{U} = \mathsf{X} \setminus \{\mathsf{P}\}$ and define $U =]\mathsf{U}[$, $\Omega =]\mathsf{P}[$. In other words, Ω is the set of rig-points of X^{an} which have the same reduction P as P. Then $\Omega =]\mathsf{P}[$ is isomorphic to an open unit ball, the divisor $\mathscr{P}$ is simply the image of the section which extends the point P, and the capacitary metric on $T_P S$ is simply the metric induced by the integral model.

Example 5.12 (Comparison with other definitions). Let us show that how this norm fits with Rumely's definition in [49] of the capacity of U with respect to the point P. Let f be a rational function on X, without pole except P, such that $U = \{x \in X\,;\, |f(x)| \leqslant 1\}$. Let m be the order of f at P and let us define $c_P \in K^*$ so that $f(x) = c_P t(x)^{-m} + \dots$ around P, where t is a fixed local parameter at P. By definition of the adjunction map, the local section $\frac{1}{t} 1_P$ of $\mathscr{O}_X(P)$ maps to the tangent vector $\frac{\partial}{\partial t}$. Consequently,

$$\|\frac{\partial}{\partial t}\|^{\mathrm{cap}}_{P,\Omega} = \|\frac{1}{t} 1_P\|(P) = \lim_{x\to P} |t(x)|^{-1} \min(1, |f(x)|^{-1/m}) = |c_P|^{-1/m}. \quad (5.13)$$

As an example, and to make explicit the relation of our rationality criterion below with the classical theorem of Borel-Dwork later on, let us consider the classical case in which $X = \mathbf{P}^1$ (containing the affine line with t coordinate), and U is the affinoid subspace of $\mathbf{P}^1$ defined by the inequality $|t| \geqslant r$ (to which we add the point at infinity), where $r \in \sqrt{|K^*|}$. Let us note $\Omega = \complement U$ and choose for the point $P \in \Omega$ the point with coordinate $t = 0$. Let m be a positive integer and $a \in K^*$ such that $r^m = |a|$; let $f = a/t^m$; this is a rational function on $\mathbf{P}^1$ with a single pole at P and U is defined by the inequality $|f| \leqslant 1$. It follows that

$$\|\frac{\partial}{\partial t}\|^{\mathrm{cap}}_{P,\Omega} = |a|^{-1/m} = 1/r.$$

Similarly, assume that U is an affinoid subset of $\mathbf{P}^1$ which does not contain the point $P = \infty$. Then U is bounded and $\|t^2 \frac{\partial}{\partial t}\|_{P,\Omega}$. is nothing but its transfinite diameter in the sense of Fekete. (See [1], the equivalence of both notions follows from [49, Theorem 4.1.19, p. 204]; see also [49, Theorem 3.1.18, p. 151] for its archimedean counterpart.)

Remarks 5.14. a) Let (X, U) be as above, let $P \in X(K)$ be a rational point such that $P \notin U$. Let $\Omega = X^{\mathrm{an}} \setminus U$ and define Ω_0 to be the connected component of Ω which contains P. It follows from Proposition 5.5 that the norms $\|\cdot\|^{\mathrm{cap}}_{P,\Omega_0}$ and $\|\cdot\|^{\mathrm{cap}}_{P,\Omega}$ on $T_P X$ coincide.

b) Let U' be another affinoid subspace of X^{an} such that $U' \subset U$; the complementary subset Ω' to U' satisfies $\Omega \subset \Omega'$. If moreover Ω and Ω' are connected, then for any $P \in \Omega$ and any vector $v \in T_P X$, one has

$$\|v\|^{\mathrm{cap}}_{P,\Omega'} \leqslant \|v\|^{\mathrm{cap}}_{P,\Omega}.$$

Indeed, since Ω and Ω' are connected and contain P, Proposition 5.6 implies that there exist rational functions f and f' on X, without pole except P, such that the affinoids U and U' are defined by the inequalities $|f| \leqslant 1$ and $|f'| \leqslant 1$ respectively. Replacing f and f' by some positive powers, we may also assume that $\operatorname{ord}_P(f) = \operatorname{ord}_P(f')$; let us denote it by $-d$. Let t be a local parameter at P; it is enough to prove the desired inequality for $v = \frac{\partial}{\partial t}$.

We may expand f and f' around P as Laurent series in $t - t(P)$, writing

$$f = \frac{c}{(t-t(P))^d} + \dots, \qquad f' = \frac{c'}{(t-t(P))^d} + \dots \quad .$$

The rational function $g = f/f'$ on X defines a holomorphic function on the affinoid subspace defined by the inequality $\{|f'| \geqslant 1\}$, since the poles at P at the numerator and at the denominator cancel each other; moreover, $g(P) = c/c'$. Using twice the maximum principle (Proposition B.1), we have

$$\begin{aligned} |g(P)| &\leqslant \sup_{|f'(x)|\geqslant 1} |g(x)| = \sup_{|f'(x)|=1} |g(x)| = \sup_{|f'(x)|=1} |f(x)| \\ &= \sup_{|f'(x)|\leqslant 1} |f(x)| \leqslant 1 \end{aligned}$$

since $\Omega \subset \Omega'$. This implies that $|g(P)| \leqslant 1$, so that $|c| \leqslant |c'|$. Therefore,

$$\|\frac{\partial}{\partial t}\|^{\mathrm{cap}}_{P,\Omega'} = |c'|^{-1/d} \leqslant |c|^{-1/d} = \|\frac{\partial}{\partial t}\|^{\mathrm{cap}}_{P,\Omega},$$

as was to be shown.

5.D Canonical semi-norms and capacities

Let K be a local field.

In the case K is archimedean, we assume moreover that $K = \mathbf{C}$; let M be a connected Riemann surface, Ω be an open subset in M, relatively compact. In the case K is ultrametric, let M be a smooth projective curve over K, let U be an affinoid in M^{an}, let us denote $\Omega = M^{\mathrm{an}} \setminus U$.

In both cases, let O be a point in Ω.

We endow the K-line $T_O M$ with its capacitary semi-norm, as defined by the first author in [13] when $K = \mathbf{C}$, or in the previous section in the p-adic case.

Let X be a projective variety over K, let $P \in X(K)$ be a rational point and let $\widehat{C}$ be a smooth formal curve in $\widehat{X}_P$. Assume that $\widehat{C}$ is K-analytic and let $\varphi\colon \Omega \to X^{\mathrm{an}}$ be an analytic map such that $\varphi(O) = P$ which maps the germ of Ω at O to $\widehat{C}$. (Consequently, if $D\varphi(O) \neq 0$, then φ defines an analytic isomorphism from the formal germ of Ω at O to $\widehat{C}$.) We endow $T_P X$ with its canonical semi-norm $\|\cdot\|^{\mathrm{can}}_{X,\widehat{C}}$.

Proposition 5.15. *For any $v \in T_O\Omega$, one has*

$$\|D\varphi(O)(v)\|^{\mathrm{can}}_{X,\widehat{C}} \leqslant \|v\|^{\mathrm{cap}}_{P,\Omega}.$$

Proof. The case $K = \mathbf{C}$ is treated in [13, Proposition 3.6]. It therefore remains to treat the ultrametric case.

In view of remark 5.14, *a*), we may assume that Ω is connected. By Proposition 5.6, there exists a rational function $f \in K(M)$ without pole except O such that $U = \{x \in M\,;\, |f(x)| \leqslant 1\}$. Let $m > 0$ denote the order of the pole of f at the point O. For any real number $r > 1$ belonging to $\sqrt{|K|^*}$, let us denote by U_r and ∂U_r the affinoids $\{|f(x)| \geqslant r\}$ and $\{|f(x)| = r\}$ in M. One has $\bigcup_{r>1} U_r = \Omega$. We shall denote by φ_r the restriction of φ to the affinoid U_r. Let us also fix a local parameter t at O and let us define $c_P = \lim_{x\to O} t(x)^m f(x)$. One has $\|\frac{\partial}{\partial t}\|^{\mathrm{cap}}_{O,\Omega} = |c_P|^{-1/m}$.

Let L be an ample line bundle on X For the proof of the proposition, we may assume that $D\varphi(O)$ is non zero; then φ is a formal isomorphism and we may consider the formal parameter $\tau = t\circ\varphi^{-1}$ on $\widehat{C}$ at P. We have $dt = \varphi^* d\tau$, hence $D\varphi(O)(\frac{\partial}{\partial t}) = \frac{\partial}{\partial \tau}$. Let us also fix a norm $\|\cdot\|_0$ on the K-line $T_P\widehat{C}$, and let us still denote by $\|\cdot\|_0$ the associated norm on on its dual $T^\vee_P\widehat{C}$.

Let us choose a real number $r > 1$ such that $r \in \sqrt{|K^*|}$, fixed for the moment. Since the residue field of K is finite, the line bundle $\varphi_r^* L$ on U_r is torsion (see [44, Proposition 3.1]); we may therefore consider a positive integer n and a nonvanishing section ε of $\varphi_r^* L^{\otimes n}$. For any integer D and any section $s \in \Gamma(X, L^{\otimes nD})$, let us write $\varphi_r^* s = \sigma\varepsilon^{\otimes nD}$, where σ is an analytic function on U_r. Since we assumed that $D\varphi(O) \neq 0$, the condition that s vanishes at order i along $\widehat{C}$ means exactly that σ vanishes at order i at O. Consequently, the i-th jet of $\varphi_r^* s$ at O is given by

$$\mathrm{j}^i_O(\varphi_r^* s) = (\sigma t^{-i})(O)\varepsilon^{nD}(O) \otimes d\tau^{\otimes i}.$$

Writing $(\sigma t^{-i})^m = (\sigma^m f^i)(f t^m)^{-i}$, it follows that

$$\|\mathrm{j}^i_O(\varphi_r^* s)\|^m = |\sigma^m f^i|(O)|c_P|^{-i}\|\varepsilon(O)\|^{nmD}\|d\tau\|_0^{im}.$$

Notice that $\sigma^m f^i$ is an analytic function on U_r. By the maximum principle (Proposition B.1),

$$|\sigma^m f^i|(O) \leqslant \sup_{U_r}|\sigma^m f^i| = \sup_{x\in\partial U_r}|\sigma^m f^i(x)| = \|\sigma\|^m_{\partial U_r} r^i.$$

Consequently,

$$\begin{aligned}\|\mathrm{j}^i_O(s)\| &\leqslant \|\sigma\|_{\partial U_r}|c_P|^{-i/m} r^i \|\varepsilon(O)\|^{nD}\|d\tau\|_0^i \\ &\leqslant \|s\|_{\partial U_r}|c_P|^{-i/m} r^i \left(\frac{\|\varepsilon(O)\|}{\inf_{x\in\partial U_r}\|\varepsilon(x)\|}\right)^{nD}\|d\tau\|_0^i.\end{aligned}$$

With the notations of Section 4.B, it follows that the norm of the evaluation morphism

$$\varphi^i_{nD}\colon E^i_{nD} \to L^{\otimes nD}_{|P} \otimes (T^\vee_P \widehat{C})^{\otimes i}$$

satisfies the inequality

$$\|\varphi^i_{nD}\|^{1/i} \leqslant r^{1/m} |c_P|^{-1/m} \left(\|\varepsilon(O)\| / \inf_{\partial U_r} \|\varepsilon\| \right)^{nD/i} \|d\tau\|_0,$$

hence

$$= \limsup_{i/D\to\infty} \frac{1}{i} \log\|\varphi^i_{nD}\| \leqslant \frac{1}{m} \log \frac{r}{|c_P|} = \log\|d\tau\|_0.$$

Using the notations introduced for defining the canonical semi-norm, we thus have $\rho(L) = \rho(L^{\otimes n}) \leqslant \log\|d\tau\|_0$ and

$$\begin{aligned} \|D\varphi(O)(\frac{\partial}{\partial t})\|^{\mathrm{can}}_{X,\widehat{C},P} &= \|\frac{\partial}{\partial \tau}\|^{\mathrm{can}}_{X,\widehat{C},P} = e^{\rho(L)} \|\frac{\partial}{\partial \tau}\|_0 \\ &\leqslant \left(\frac{r}{|c_P|}\right)^{1/m} = r^{1/m} \|\frac{\partial}{\partial t}\|^{\mathrm{cap}}_{\Omega,P}. \end{aligned}$$

Letting r go to 1, we obtain the desired inequality. □

5.E Global capacities

Let K be a number field, let R denote the ring of integers in K. Let X be a projective smooth algebraic curve over K. For any ultrametric place v of R, let us denote by $\mathbf{F}_v$ the residue field of R at v, by K_v the completion of K at v, and by X_v the rigid K_v-analytic variety attached to X_{K_v}. For any archimedean place v of X, corresponding to an embedding $\sigma\colon K \hookrightarrow \mathbf{C}$, we let X_v be the compact Riemann surface $X_\sigma(\mathbf{C})$. When v is real, by an open subset of X_v, we shall mean an open subset of $X_\sigma(\mathbf{C})$ invariant under complex conjugation.

Our goal in this Section is to show how capacitary metrics at all places fit within the framework of the Arakelov intersection theory (with L^2_1-regularity) introduced in [11]. Let us briefly recall here the main notations and properties of this arithmetic intersection theory, referring to this article for more details.

For any normal projective flat model $\mathscr{X}$ of X over R, the Arakelov Chow group $\widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$ consists of equivalence classes of pairs $(\mathscr{D}, g) \in \widehat{\mathrm{Z}}^1_{\mathbf{R}}(\mathscr{X})$ where $\mathscr{D}$ is a $\mathbf{R}$-divisor on $\mathscr{X}$ and g is a Green current with L^2_1-regularity on $\mathscr{X}(\mathbf{C})$ for the real divisor $\mathscr{D}_K$, stable under complex conjugation. For any class α of an Arakelov divisor $(\mathscr{D}, g)$, we shall denote, as usual, $\omega(\alpha) = dd^c g + \delta_{\mathscr{D}_{\mathbf{C}}}$.

Arithmetic intersection theory endowes the space $\widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$ with a symmetric $\mathbf{R}$-valued bilinear form. Any morphism $\pi\colon \mathscr{X}' \to \mathscr{X}$ between normal projective flat models of curves X' and X induces morphisms of abelian groups $\pi_*\colon \widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X}') \to \widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$ and $\pi^*\colon \widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X}) \to \widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X}')$. For any classes

α and $\beta \in \widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$, $\gamma \in \widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X}')$, one has $\pi^*\alpha \cdot \pi^*\beta = \alpha \cdot \beta$ and a projection formula $\pi_*(\pi^*\alpha \cdot \gamma) = \deg(\pi)\alpha \cdot \pi_*(\gamma)$, when π has constant generic degree $\deg(\pi)$.

Any class $\alpha \in \widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$, defines a *height* function h_α which is a linear function on the subspace of $Z^1_{\mathbf{R}}(\mathscr{X})$ consisting of real 1-cycles Z on $\mathscr{X}$ such that $\omega(\alpha)$ is locally L^∞ on a neighbourhood of $|Z|(\mathbf{C})$. If D is a real divisor on X such that $\omega(\alpha)$ is locally L^∞ in a neighbourhood of $|D|(\mathbf{C})$, we shall still denote $h_\alpha(D)$ the height of the unique horizontal 1-cycle on $\mathscr{X}$ which extends D. Moreover, for any effective divisor D on X such that $\omega(\alpha)$ is locally L^∞ in a neighbourhood of $|\pi_*(D)|(\mathbf{C})$, then $\omega(\pi^*\alpha)$ is locally L^∞ in a neighbourhood of $|D|(\mathbf{C})$ and one has the equality $h_{\pi^*\alpha}(D) = h_\alpha(\pi_*(D))$.

Definition 5.16. *Let D be a divisor on X. For each place v of K, let Ω_v be an open subset of X_v (stable under complex conjugation if v is archimedean). One says that the collection (Ω_v) is an* adelic tube adapted to D *if the following conditions are satisfied:*

1) *for any ultrametric place v, the complement of Ω_v in any connected component of X_v is a nonempty affinoid subset ;*
2) *for any archimedean place v, the complement of Ω_v in any connected component of X_v is non-polar;*
3) *there exists an effective reduced divisor E containing $|D|$, a finite set of places F of K, a normal projective flat model $\mathscr{X}$ of X over R such that for any ultrametric place v of K such that $v \notin F$, $\Omega_v =]\overline{E}[_v$ is the tube in X_v around the specialization of E in the special fibre $\mathscr{X}_{\mathbf{F}_v}$.*

Let $\Omega = (\Omega_v)$ be a family, where, for each place v of K, Ω_v is an open subset of the analytic curve X_v satisfying the conditions (1) and (2). Let D be a divisor on X whose support is contained in Ω_v for any place v of K. By the considerations of this section, the line bundle $\mathscr{O}_X(D)$ is then endowed, for each place v of K, of a v-adic metric $\|\cdot\|^{\mathrm{cap}}_{\Omega_v}$. If Ω is an adelic tube adapted to D, then, for almost all places of K, this metric is in fact induced by the horizontal extension of the divisor D in an adequate model $\mathscr{X}$ of X. Actually, one has the following proposition:

Proposition 5.17. *Assume that Ω is an adelic tube adapted to $|D|$. There is a normal, flat, projective model $\mathscr{X}$ of X over R and a (unique) Arakelov $\mathbf{Q}$-divisor extending D, inducing at any place v of K the v-adic capacitary metric on $\mathscr{O}_X(D)$.*

Such an arithmetic surface $\mathscr{X}$ will be said *adapted to Ω*. Then, the Arakelov $\mathbf{Q}$-divisor on $\mathscr{X}$ whose existence is asserted by the proposition will be denoted $\widehat{D}_\Omega$. Observe moreover that the current $\omega(\widehat{D}_\Omega)$ is locall L^∞ on Ω since it vanishes there. Consequently, the height $h_{\widehat{D}_\Omega}(E)$ is defined when E is any 0-cycle on X which is supported by Ω.

Proof. It has already been recalled that archimedean Green functions defined by potential theory have the required L^2_1-regularity. It thus remains to show that the metrics at finite places can be defined using a single model $(\mathscr{X},\mathscr{D})$ of (X,D) over R.

Lemma 5.18. *There exists a normal, flat projective model $\mathscr{X}$ of X over R, and, for any ultrametric place v of K, a Zariski closed subset Z_v of the special fibre $\mathsf{X}_{\mathbf{F}_v}$ at v such that $\Omega_v =]\mathsf{Z}_v[$. We may moreover assume that for almost all ultrametric places v of K, $\mathsf{Z}_v = \mathscr{E} \cap \mathsf{X}_{\mathbf{F}_v}$, where $\mathscr{E}$ is an effective reduced horizontal divisor on $\mathscr{X}$.*

Proof. Let $\mathscr{X}_1$ be a projective flat model of X over R, E an effective reduced divisor on $\mathscr{X}$, and F be a finite set of places satisfying the condition 3) of the definition of an adelic tube. Up to enlarging F, we may assume that the fibre product $\mathscr{X}_1 \otimes_R R_1$ is normal, where R_1 denotes the subring of K obtained from R by localizing outside places in F.

By Raynaud's formal/rigid geometry comparison theorem, there is, for each finite place $v \in F$, a normal projective and flat model $\mathscr{X}_v$ of X over the completion $\widehat{R_v}$, a Zariski closed subset Z_v the special fibre of $\mathscr{X}_v$, such that $\Omega_v =]\mathsf{Z}_v[$.

By a general descent theorem of Moret-Bailly ([41, Theorem 1.1]; see also [10, 6.2, Lemma D]), there exists a projective and flat R-scheme $\mathscr{X}$ which coincides with $\mathscr{X}_1$ over $\operatorname{Spec} R_1$ and such that its completion at any finite place $v \in F$ is isomorphic to $\mathscr{X}_v$. By faithfully flat descent, such a scheme is normal (see [39, 21.E, Corollary]).

For any ultrametric place v over $\operatorname{Spec} R_1$, we just let Z_v be the specialization of E in $\mathscr{X}_{\mathbf{F}_v} = (\mathscr{X}_0)_{\mathbf{F}_v}$; one has $\Omega_v =]\mathsf{Z}_v[$ by assumption since v does not belong to the finite set F of excluded places. For any ultrametric place $v \in F$, Z_v identifies with a Zariski closed subset of the special fibre $\mathscr{X}_{\mathbf{F}_v}$ and its tube is equal to Ω_v by construction. This concludes the proof of the lemma. □

Fix such a model $\mathscr{X}$ and let $\mathscr{D}_0$ be the Zariski closure of D in $\mathscr{X}$. For any ultrametric place v of F, let V_v be the unique divisor on the special fibre $\mathscr{X}_{\mathbf{F}_v}$ such that $\mathscr{D}_0 + V_v$ satisfies the assumptions of Proposition 5.1. One has $V_v = 0$ for any ultrametric place v such that Z_v has no component of dimension 1, hence for all but finitely places v. We thus may consider the $\mathbf{Q}$-divisor $\mathscr{D} = \mathscr{D}_0 + \sum_v V_v$ on $\mathscr{X}$ and observe that it induces the capacitary metric at all ultrametric places. □

Proposition 5.19. *Let D be a divisor on X and let Ω be an adelic tube adapted to $|D|$. One has the equality*

$$\widehat{D}_\Omega \cdot \widehat{D}_\Omega = h_{\widehat{D}_\Omega}(D).$$

Proof. Let us consider a model $\mathscr{X}$ of X and an Arakelov $\mathbf{Q}$-divisor $\mathscr{D}$ on $\mathscr{X}$ defining the capacitary metric $\|\cdot\|^{\mathrm{cap}}_{\Omega_v}$ at all ultrametric places v of K.

Let $\mathscr{D}_0$ denote the Zariski closure of D in $\mathscr{X}$. For any ultrametric place v of K, and let V_v be the vertical part of $\mathscr{D}$ lying above v so that $\mathscr{D} = \mathscr{D}_0 + \sum_v V_v$. By [11, Corollary 5.4], one has

$$\widehat{D}_\Omega \cdot \widehat{D}_\Omega = h_{\widehat{D}_\Omega}(\mathscr{D}).$$

By the definition of the capacitary metric at ultrametric places, the geometric intersection number of $\mathscr{D}$ with any vertical component of $\mathscr{D}$ is zero. Consequently,

$$\widehat{D}_\Omega \cdot \widehat{D}_\Omega = h_{\widehat{D}_\Omega}(\mathscr{D}_0) + \sum_v h_{\widehat{D}_\Omega}(V_v) = h_{\widehat{D}_\Omega}(\mathscr{D}_0),$$

as was to be shown. □

Corollary 5.20. *Let $P \in X(K)$ be a rational point of X and let Ω be an adelic tube adapted to P. One has*

$$\widehat{P}_\Omega \cdot \widehat{P}_\Omega = \widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}}).$$

6 An algebraicity criterion for *A*-analytic curves

Let K be a number field, R its ring of integers, X a quasi-projective algebraic variety over K and let P a point in $X(K)$. Let $\widehat{C} \hookrightarrow \widehat{X}_P$ be a smooth formal curve that is A-analytic.

For any place v of K, the formal curve $\widehat{C}$ is K_v-analytic, and we may equip the K-line $T_P\widehat{C}$ with the canonical v-adic semi-norm $\|\cdot\|_v^{\mathrm{can}} = \|\cdot\|_{X,\widehat{C},P,v}^{\mathrm{can}}$ constructed in Section 4.B. We claim that, equipped with these semi-norms, $T_P\widehat{C}$ defines a semi-normed K-line $(T_P\widehat{C}, \|\cdot\|^{\mathrm{can}})$ with a well-defined Arakelov degree in $]-\infty, +\infty]$, in the sense of [13, 4.2]. Recall that it means that, for any (or equivalently, for some) non-zero element in $T_P\widehat{C}$, the series $\sum_v \log^+\|t\|_v^{\mathrm{can}}$ is convergent. To see this, consider a quasi-projective flat R-scheme $\mathscr{X}$ with generic fibre X, together with a section $\mathscr{P}\colon \operatorname{Spec} R \to \mathscr{X}$ which extends P. According to Lemma 4.5 (applied to projective compactifications of X and $\mathscr{X}$, and an ample line bundle $\mathscr{L}$), the inequality

$$\log\|t\|_v^{\mathrm{can}} \leqslant -\log S_{\mathscr{X},v}(\widehat{C}),$$

holds for almost all finite places v, where $S_{\mathscr{X},v}$ denotes the size of $\widehat{C}$ with respect to the R_v model $\mathscr{X} \otimes R_v$. Since by definition of A-analyticity the series with non-negative terms $\sum_v \log S_{\mathscr{X},v}(\widehat{C})^{-1}$ has a finite sum, this establishes the required convergence.

The Arakelov degree of $(T_P\widehat{C}, \|\cdot\|^{\mathrm{can}})$ is defined as the sum:

$$\widehat{\deg}(T_P\widehat{C}, \|\cdot\|^{\mathrm{can}}) := \sum_v (-\log\|t\|_v^{\mathrm{can}}).$$

It is a well defined element in $]-\infty, +\infty]$, independent of the choice of t by the product formula (we follow the usual convention $-\log 0 = +\infty$.)

The following criterion extends Theorem 4.2 of [13], where instead of canonical semi-norms, larger norms constructed by means of the sizes were used at finite places.

Theorem 6.1. *Let $\widehat{C}$ be, as above, an A-analytic curve through a rational point P in some algebraic variety X over K.*

If $\widehat{\deg}(T_P\widehat{C}, \|\cdot\|^{\mathrm{can}}) > 0$, then $\widehat{C}$ is algebraic.

Proof. We keep the above notation, and we assume, as we may, X (resp. $\mathscr{X}$) to be projective over K (resp. over R). We choose an ample line bundle $\mathscr{L}$ over $\mathscr{X}$ and we let $L := \mathscr{L}_K$.

We let $\mathscr{E}_D := \Gamma(\mathscr{X}, \mathscr{L}^{\otimes D})$ and, for any embedding $\sigma : K \hookrightarrow \mathbf{C}$, we choose a consistent sequence of hermitian norms $(\|\cdot\|_{D,\sigma})$ on the $\mathbf{C}$-vector spaces $\mathscr{E}_{D,\sigma} \simeq \Gamma(X_\sigma, L_\sigma^{\otimes D})$, in a way compatible with complex conjugation. Using these norms, we define hermitian vector bundles $\overline{\mathscr{E}}_D := (\mathscr{E}_D, (\|\cdot\|_{D,\sigma})_\sigma)$ over $\operatorname{Spec} R$.

We also choose an hermitian structure on $\mathscr{P}^*\mathscr{L}$, and we denote $\overline{\mathscr{P}^*\mathscr{L}}$ the so-defined hermitian line bundle over $\operatorname{Spec} R$. Finally, we equip $T_P\widehat{C}$ with the R-structure defined by $N_{\mathscr{P}}\mathscr{X} \cap T_P\widehat{C}$ and with an arbitrary hermitian structure, and in this way we define an hermitian line bundle $\overline{T}_0$ over $\operatorname{Spec} R$ such that $(T_0)_K = T_P\widehat{C}$.

We define the K-vector spaces $E_D := \mathscr{E}_{D,K} \simeq \Gamma(X, L^{\otimes D})$, their subspaces E_D^i, and the evaluation maps

$$\varphi_D^i \colon E_D^i \to (T_P^\vee\widehat{C})^{\otimes i} \otimes L_{|P}^{\otimes D}.$$

as in the "local" situation considered in Section 4.B. According to the basic algebraicity criteria in [13, 2.2], to prove that $\widehat{C}$ is algebraic, it suffices to prove that the ratio

$$\frac{\sum_{i\geqslant 0} (i/D)\operatorname{rank}(E_D^i/E_D^{i+1})}{\sum_{i\geqslant 0} \operatorname{rank}(E_D^i/E_D^{i+1})} \tag{6.2}$$

stays bounded when D goes to $+\infty$.

For any place v of K, the morphism φ_D^i has a v-adic norm, defined by means of the integral and hermitian structures introduced above. If $\varphi_D^i \neq 0$, the height of φ_D^i is the real number defined as the (finite) sum:

$$h(\varphi_D^i) = \sum_v \log\|\varphi_D^i\|_v.$$

When φ_D^i vanishes, we define $h(\varphi_D^i) = -\infty$; observe that, in this case, $E_D^{i+1} = E_D^i$.

As established in the proof of Lemma 4.5 above (see also [12, Lemma 3.3]), the following inequality holds for any finite place v, and any two non-negative integers i and D:

$$\log\|\varphi_D^i\|_v \leqslant -i\log S_{\mathscr{X},v}(\widehat{C}). \tag{6.3}$$

Since $\widehat{C}$ is A-analytic, the upper bounds (4.4) and (6.3) show the existence of some positive real number c such that

$$h(\varphi_D^i) \leqslant c(i+D). \tag{6.4}$$

For any place v of K, we let

$$\rho_v(L) = \limsup_{i/D \to \infty} \frac{1}{i} \log \|\varphi_D^i\|_v.$$

This is an element in $[-\infty, +\infty[$, which, according to (6.3), satisfies:

$$\rho_v(L) \leqslant -\log S_{\mathscr{X},v}(\widehat{C})$$

for any finite place v. Moreover, by its very definition, the Arakelov degree of $(T_P\widehat{C}, \|\cdot\|^{\mathrm{can}})$ is given by:

$$\begin{aligned}\widehat{\deg}(T_P\widehat{C}, \|\cdot\|^{\mathrm{can}}) &= \sum_v (-\rho_v(L)) + \widehat{\deg}\,\overline{T}_0 \\ &= \sum_{v \text{ finite}} \left(-\rho_v(L) - \log S_{\mathscr{X},v}(\widehat{C})\right) \\ &\qquad + \sum_{v \text{ finite}} \log S_{\mathscr{X},v}(\widehat{C}) + \sum_{v|\infty} (-\rho_v(L)) + \widehat{\deg}\,\overline{T}_0.\end{aligned}$$

In the last expression, the terms of the first sum belong to $[0, +\infty]$ — and the sum itself is therefore well-defined in $[0, +\infty]$ — and the second sum is convergent by A-analyticity of $\widehat{C}$.

Observe also that, since the sums

$$\sum_{v \text{ finite}} \left(-\frac{1}{i} \log \|\varphi_D^i\|_v + \log S_{\mathscr{X},v}(\widehat{C})\right)$$

have non-negative terms, we get, as a special instance of Fatou's Lemma:

$$\begin{aligned}&\sum_{v \text{ finite}} \liminf_{i/D \to \infty} \left(-\frac{1}{i} \log \|\varphi_D^i\|_v + \log S_{\mathscr{X},v}(\widehat{C})\right) \\ &\qquad\qquad \leqslant \liminf_{i/D \to \infty} \sum_{v \text{ finite}} \left(-\frac{1}{i} \log \|\varphi_D^i\|_v + \log S_{\mathscr{X},v}(\widehat{C})\right).\end{aligned}$$

Consequently

$$\limsup_{i/D \to \infty} \frac{1}{i} h(\varphi_D^i) \leqslant \sum_v \rho_v(L),$$

and

$$\widehat{\deg}(T_P\widehat{C}, \|\cdot\|^{\mathrm{can}}) \leqslant -\limsup_{i/D \to \infty} \frac{1}{i} h(\varphi_D^i) + \widehat{\deg}\,\overline{T}_0. \tag{6.5}$$

When $\widehat{\deg}(T_P\widehat{C}, \|\cdot\|^{\mathrm{can}})$ is positive, the inequality (6.5) implies the existence of positive real numbers ε and λ such that, for any two positive integers i and D,

$$\widehat{\deg}\,\overline{T}_0 - \frac{1}{i}h(\varphi_D^i) \geqslant \varepsilon \quad \text{if } i \geqslant \lambda D. \tag{6.6}$$

Let $\mathscr{E}_D^i := \mathscr{E}_D \cap E_D^i$ and let $\overline{\mathscr{E}_D^i/\mathscr{E}_D^{i+1}}$ be the hermitian vector bundle on $\operatorname{Spec} R$ defined by the quotient $\mathscr{E}_D^i/\mathscr{E}_D^{i+1}$ equipped with the hermitian structure induced by the one of $\overline{\mathscr{E}}_D$. The evaluation map φ_D^i induces an injection $E_D^i/E_D^{i+1} \hookrightarrow (T_P^\vee \widehat{C})^{\otimes i} \otimes L_{|P}^{\otimes D}$. Actually, either $\varphi_D^i = 0$ and then $E_D^i = E_D^{i+1}$, or $\varphi_D^i \neq 0$, and this inclusion is an isomorphism of K-lines. In either case, we have:

$$\widehat{\deg}\,\overline{\mathscr{E}_D^i/\mathscr{E}_D^{i+1}} = \operatorname{rank}(E_D^i/E_D^{i+1})\left(\widehat{\deg}(\overline{\mathscr{P}^*\mathscr{L}}^{\otimes D} \otimes \overline{T}_0^{\vee\otimes i}) + h(\varphi_D^i)\right).$$

Indeed, if $\varphi_D^i = 0$, both sides vanish (we follow the usual convention $0\cdot(-\infty) = 0$). If $\varphi_D^i \neq 0$, the equality is a straightforward consequence of the definitions of the Arakelov degree of an hermitian line bundle over $\operatorname{Spec} R$ and of the heights $h(\varphi_D^i)$.

The above equality may also be written:

$$\widehat{\deg}\,\overline{\mathscr{E}_D^i/\mathscr{E}_D^{i+1}} = \operatorname{rank}(E_D^i/E_D^{i+1})\left(D\,\widehat{\deg}\,\overline{\mathscr{P}^*\mathscr{L}} - i\,\widehat{\deg}\,\overline{T}_0 + h(\varphi_D^i)\right). \tag{6.7}$$

Moreover, by [12, Proposition 4.4], there is a constant c', such that for any $D \geqslant 0$ and any saturated submodule $\mathscr{F}$ of $\mathscr{E}_D$,

$$\widehat{\deg}\,\overline{\mathscr{E}_D/\mathscr{F}} \geqslant -c' D \operatorname{rank}(\mathscr{E}_D/\mathscr{F}).$$

(This is an easy consequence of the fact that the K-algebra $\bigoplus_{D\geqslant 0} \mathscr{E}_{D,K}$ is finitely generated.) Applied to $\mathscr{F} := \bigcap_{i\geqslant 0} \mathscr{E}_D^i$, this estimate becomes:

$$\sum_{i\geqslant 0} \widehat{\deg}\,\overline{\mathscr{E}_D^i/\mathscr{E}_D^{i+1}} \geqslant -c' D \sum_{i\geqslant 0} \operatorname{rank}(E_D^i/E_D^{i+1}). \tag{6.8}$$

Using (6.7) and (6.8), we derive the inequality:

$$\begin{aligned} -(c' + \widehat{\deg}\,\overline{\mathscr{P}^*\mathscr{L}}) D \sum_{i\geqslant 0} \operatorname{rank}(E_D^i/E_D^{i+1}) \\ \leqslant \sum_{i\geqslant 0} \operatorname{rank}(E_D^i/E_D^{i+1})(-i\,\widehat{\deg}\,\overline{T}_0 + h(\varphi_D^i)). \end{aligned} \tag{6.9}$$

Finally, using (6.9), (6.4), and (6.6), we obtain

$$\begin{aligned} \sum_{i<\lambda D} \operatorname{rank}(E_D^i/E_D^{i+1})\left(\frac{i}{D}\widehat{\deg}\,\overline{T}_0 - c\frac{i+D}{D}\right) + \sum_{i\geqslant\lambda D} \operatorname{rank}(E_D^i/E_D^{i+1})\varepsilon\frac{i}{D} \\ \leqslant (c' + \widehat{\deg}\,\overline{\mathscr{P}^*\mathscr{L}}) \sum_{i\geqslant 0} \operatorname{rank}(E_D^i/E_D^{i+1}). \end{aligned}$$

This implies that the ratio (6.2) is bounded by

$$\lambda + \frac{1}{\varepsilon}\left(c' + \widehat{\deg}\,\overline{\mathscr{P}^*\mathscr{L}} + c + \lambda \max(0, c - \widehat{\deg}\,\overline{T}_0)\right),$$

and completes the proof. □

7 Rationality criteria

7.A Numerical equivalence and numerical effectivity on arithmetic surfaces

The following results are variations on a classical theme in Arakelov geometry of arithmetic surfaces. The first theorem characterizes numerically trivial Arakelov divisors with real coefficients. It is used in the next proposition to describe effective Arakelov divisors whose sum is numerically effective. We allow ourselves to use freely the notations of [11].

Theorem 7.1 (Compare [11, Theorem 5.5]). *Let $\mathscr{X}$ be a normal flat projective scheme over the ring of integers of a number field K whose generic fibre is a smooth and geomerically connected curve. Let (D,g) be any element in $\widehat{\mathrm{Z}}^1_{\mathbf{R}}(\mathscr{X})$ which is numerically trivial. Then there exist an integer n, real numbers λ_i and rational functions $f_i \in K(\mathscr{X})^*$, for $1 \leqslant i \leqslant n$, and a family $(c_\sigma)_{\sigma\colon K\hookrightarrow\mathbf{C}}$ of real numbers such that $c_{\overline{\sigma}} = c_\sigma$, $\sum c_\sigma = 0$, and $(D,g) = (0,(c_\sigma)) + \sum_{i=1}^n \lambda_i \widehat{\mathrm{div}}(f_i)$.*

Proof. There are real numbers λ_i and Arakelov divisors $(D_i, g_i) \in \widehat{\mathrm{Z}}^1(\mathscr{X})$ such that $(D,g) = \sum \lambda_i (D_i, g_i)$. We may assume that the λ_i are linearly independent over $\mathbf{Q}$. By assumption, the degree of D on any vertical component of $\mathscr{X}$ is zero; the linear independence of the λ_i implies that the same holds for any D_i. Let us then denote by g_i' any Green current for D_i such that $\omega(D_i, g_i') = 0$. One has

$$0 = \omega(D,g) = \sum \lambda_i \omega(D_i, g_i) = \sum \lambda_i \omega(D_i, g_i'),$$

so that the difference $g - \sum \lambda_i g_i'$ is harmonic, and therefore constant on any connected component of $\mathscr{X}(\mathbf{C})$. By adding a locally constant function to some g_i', we may assume that $g = \sum \lambda_i g_i'$. Then, $(D,g) = \sum \lambda_i (D_i, g_i')$. This shows that we may assume that one has $\omega(D_i, g_i) = 0$ for any i. By Faltings-Hriljac's formula, the Néron-Tate quadratic form on $\mathrm{Pic}^0(\mathscr{X}_K) \otimes \mathbf{R}$ takes the value 0 on the class of the real divisor $\sum \lambda_i (D_i)_K$. Since this quadratic form is positive definite (see [50, 3.8, p. 42]), this class is zero. Using that the λ_i are linearly independent over $\mathbf{Q}$, we deduce that the class of each divisor $(D_i)_K$ in $\mathrm{Pic}^0(\mathscr{X}_K)$ is torsion. Since D_i has degree zero on any vertical component of $\mathscr{X}$ and the Picard group of the ring of integers of K is finite, the class in $\mathrm{Pic}(\mathscr{X})$ of the divisor D_i is torsion too. Let then choose positive integers n_i and rational functions f_i on $\mathscr{X}$ such that $\mathrm{div}(f_i) = n_i D_i$. The Arakelov divisors $\widehat{\mathrm{div}}(f_i) - n_i(D_i, g_i)$ are of the form $(0, c_i)$, where $c_i = (c_{i,\sigma})_{\sigma\colon K\hookrightarrow\mathbf{C}}$ is a family of real numbers such that $c_{i,\overline{\sigma}} = c_{i,\sigma}$ and $\sum_\sigma c_{i,\sigma} = 0$. Then, letting $c_\sigma = \sum_i (\lambda_i/n_i) c_{i,\sigma}$, one has

$$(D,g) = (0,(c_\sigma)) + \sum \frac{\lambda_i}{n_i} \widehat{\mathrm{div}}(f_i)$$

as requested. □

Let $f_1, \dots, f_n$ be meromorphic functions on some Rieman surface M, and $\lambda_1, \dots, \lambda_n$ real numbers, and let $f \in \mathbf{C}(M)^* \otimes_{\mathbf{Z}} \mathbf{R}$ be defined as $f = \sum_{i=1}^n f_i \otimes \lambda_i$. We shall denote $|f|$ the real function on M given by $\prod |f_i|^{\lambda_i}$, and by $\operatorname{div} f$ the $\mathbf{R}$-divisor $\sum \lambda_i \operatorname{div}(f_i)$; they don't depend on the decomposition of f as a sum of tensors. One has $\mathrm{dd}^c \log|f|^{-2} + \delta_{\operatorname{div}(f)} = 0$.

We shall say that a pair (D, g) formed of a divisor D on M and of a Green current g with L^2_1 regularity for D is *effective*[6] if the divisor D is effective, and if the Green current g of degree 0 for D may be represented by a nonnegative summable function (see [11, Definition 6.1]).

Similarly, we say that an Arakelov divisor $(D, g) \in \widehat{\mathrm{Z}}^1_{\mathbf{R}}(\mathscr{X})$ on the arithmetic surface $\mathscr{X}$ is effective if D is effective on $\mathscr{X}$ and if $(D_{\mathbf{C}}, g)$ is effective on $\mathscr{X}(\mathbf{C})$.

We say that an Arakelov divisor, or the class α of an Arakelov divisor, is *numerically effective* (or shortly, *nef*) if $[(D, g)] \cdot \alpha \geqslant 0$ for any effective Arakelov divisor $(D, g) \in \widehat{\mathrm{Z}}^1_{\mathbf{R}}(\mathscr{X})$ (according to [11, Lemma 6.6], it is sufficient to consider Arakelov divisors (D, g) with $\mathscr{C}^\infty$-regularity). If (D, g) is an effective and numerically effective Arakelov divisor, then the current $\omega(g) := \mathrm{dd}^c g + \delta_D$ is a positive measure (see [11, proof of Proposition 6.9]).

Proposition 7.2. *Let $\mathscr{X}$ be a normal, flat projective scheme over the ring of integers of a number field K whose generic fibre is a smooth geometrically connected algebraic curve.*

Let (D, g) and (E, h) be non-zero elements of $\widehat{\mathrm{Z}}^1_{\mathbf{R}}(\mathscr{X})$; let α and β denote their classes in $\widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$. Let us assume that the following conditions are satisfied:

1) *the Arakelov divisors (D, g) and (E, h) are effective;*
2) *the supports of D and E do not meet and $\int_{\mathscr{X}(\mathbf{C})} g * h = 0$.*

If the class $\alpha + \beta$ is numerically effective, then there exist a positive real number λ, an element $f \in K(\mathscr{X})^* \otimes_{\mathbf{Z}} \mathbf{R}$ and a family $(c_\sigma)_{\sigma\colon K \hookrightarrow \mathbf{C}}$ of real numbers which is invariant by conjugation and satisfies $\sum_\sigma c_\sigma = 0$, such that for any embedding $\sigma\colon K \hookrightarrow \mathbf{C}$,

$$g_\sigma = (c_\sigma + \log|f|^{-2})^+, \quad \textit{and} \quad h_\sigma = \lambda(c_\sigma + \log|f|^{-2})^-,$$

where, for any real valued function φ, we denoted $\varphi^+ = \max(0, \varphi)$ and $\varphi^- = \max(0, -\varphi)$, so that $\varphi^+ - \varphi^- = \varphi$.

Moreover, $\alpha^2 = \alpha\beta = \beta^2 = 0$.

Proof. Since (D, g) and (E, h) are effective and non-zero, the classes α and β are not equal to zero [11, Proposition 6.10]. Moreover, the assumptions of the proposition imply that

$$\alpha \cdot \beta = \deg \pi_*(D, E) + \frac{1}{2} \int_{\mathscr{X}(\mathbf{C})} g * h = 0.$$

[6] in the terminology of [11], *nonnegative*

Since $\alpha+\beta$ is numerically effective, it follows from Lemma 6.11 of [11] (which in turn is an application of the Hodge index theorem in Arakelov geometry) that there exists $\lambda \in \mathbf{R}_+^*$ such that $\beta = \lambda\alpha$ in $\widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$. In particular, α and β are nef, and $\alpha^2 = \beta^2 = \alpha\cdot\beta = 0$.

Replacing (E,h) by $(\lambda E, \lambda h)$, we may assume that $\lambda = 1$. Then, $(D - E, g-h)$ belongs to the kernel of the canonical map $\rho\colon \widehat{\mathrm{Z}}^1_{\mathbf{R}}(\mathscr{X}) \to \widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$, so is numerically trivial. By Theorem 7.1, there exist real numbers λ_i, rational functions $f_i \in K(\mathscr{X})^*$ and a family $c = (c_\sigma)_{\sigma\colon K\hookrightarrow \mathbf{C}}$ of real numbers, invariant by conjugation, such that $\sum_\sigma c_\sigma = 0$ and $(D-E, g-h) = (0,c) + \sum \lambda_i \widehat{\mathrm{div}}(f_i)$ in $\widehat{\mathrm{Z}}^1_{\mathbf{R}}(\mathscr{X})$. Let us denote by f the element $\sum f_i \otimes \lambda_i$ of $K(\mathscr{X})^* \otimes_{\mathbf{Z}} \mathbf{R}$. The proposition now follows by applying Lemma 7.3 below to the connected Riemann surface $\mathscr{X}_\sigma(\mathbf{C})$, the pairs (D, g_σ), (E, h_σ) and the "meromorphic function" $e^{-2c_\sigma} f_{\mathscr{X}_\sigma(\mathbf{C})}$, for each embedding $\sigma\colon K \hookrightarrow \mathbf{C}$. □

Lemma 7.3. *Let M be a compact connected Riemann surface, let D and D' two nonzero $\mathbf{R}$-divisors on M, and let g and g' be two Green functions with L^2_1 regularity for D and D'. We make the following assumptions: $|D| \cap |D'| = \emptyset$, the pairs (D,g) and (D',g') are effective, the currents $\omega(g) = \mathrm{dd}^c g + \delta_D$ and $\omega(g') = \mathrm{dd}^c g' + \delta_{D'}$ are positive measures, $\int_M g * g' = 0$. If there exists an element $f \in \mathbf{C}(M)^* \otimes \mathbf{R}$ such that $g - g' = \log|f|^{-2}$, then $g = \max(0, \log|f|^{-2})$ and $g' = \max(0, \log|f|^2)$.*

Proof. First observe that

$$\omega(g) - \omega(g') = \mathrm{dd}^c(g-g') + \delta_D - \delta_{D'} = \mathrm{dd}^c \log|f|^{-2} + \delta_D - \delta_{D'} = \delta_{D-D'-\mathrm{div}(f)},$$

by the Poincaré-Lelong formula. By assumption, the current $\omega(g) - \omega(g')$ belongs to the Sobolev space L^2_{-1}; it is therefore non-atomic (see [11, Appendix, A.3.1]), so that $D - D' = \mathrm{div}(f)$ and $\omega(g) = \omega(g')$.

Observe also that $g_{|M\setminus|D|}$ (resp. $g'_{|M\setminus|D|'}$) is a subharmonic current. In the sequel, we denote by g (resp. g') the unique subharmonic function on $M \subset |D|$ (resp. on $M' \subset |D|'$) which represents this current.

Let F be the set of points $x \in M$ where $|f(x)| = 1$ and let $\Omega = M \setminus F$ be its complementary subset. The functions $h = \max(0, \log|f|^{-2})$ and $h' = \max(0, \log|f|^2)$ are continuous Green functions with L^2_1 regularity for D and D' respectively. The currents $\mathrm{dd}^c h + \delta_D$, $\mathrm{dd}^c h' + \delta_{D'}$ are equal to a common positive measure, which we denote by ν. Since h (resp. h') is harmonic on $M \setminus (|D| \cup F)$ (resp. on $M \setminus (|D|' \cup F)$), this measure is supported by F.

Let S be the support of the positive measure $\omega(g)$. It follows from [11, Remark 6.5] that g and g' vanish $\omega(g)$-almost everywhere on M. Consequently, the equality $\log|f|^{-2} = g - g' = 0$ holds $\omega(g)$-almost everywhere; in particular, $S \subset F$.

Let us pose $u = h - g = h' - g'$; this is a current with L^2_1 regularity on M and $\mathrm{dd}^c u = \mathrm{dd}^c h - \mathrm{dd}^c g = \nu - \omega(g)$. In particular, $\mathrm{dd}^c(u|_\Omega) = 0$: u is harmonic on Ω. Since g is nonnegative, one has $u \leqslant 0$ on $F = \complement\Omega$. By the maximum principle, this implies that $u \leqslant 0$ on Ω (*cf.* [11, Theorem A.6.1]; observe that u is finely continuous on M).

Finally, one has

$$0 = \int_M g * g' = \int_M h * h' - \int_M u\nu - \int_M u\omega(g) \geqslant \int_M h * h'.$$

By [11, Corollary 6.4], this last term is nonnegative, so that all terms of the formula vanish. In particular, $\int u\nu = 0$, hence $u = 0$ (ν-a.e.). Using again that u is harmonic on Ω, it follows that its Dirichlet norm vanishes, and finally that $u \equiv 0$. □

Remark 7.4. The Green currents g and h appearing in the conclusion of Proposition 7.2 are very particular. Assume for example that the Arakelov divisors $\widehat{D}$ and $\widehat{E}$ are defined using capacity theory at the place σ, with respect to an open subset Ω_σ of X_σ. Then, g_σ and h_σ vanish nearly everywhere on $\complement\Omega_\sigma$. In other words, $\complement\Omega_\sigma$ is contained in the set of $x \in X_\sigma$ such that $|f(x)|^2 = \exp(-c_\sigma)$, which is a real semi-algebraic curve in X_σ, viewed as a real algebraic surface. In particular, it contradicts any of the following hypothesis on Ω_σ, respectively denoted $(4.2)_{\mathscr{X},\Omega_\sigma}$ and $(4.3)_{\mathscr{X},\Omega_\sigma}$ in [11]:

1) the interior of $\mathscr{X}_\sigma(\mathbf{C}) \setminus \Omega_\sigma$ is not empty;
2) there exists an open subset U of $\mathscr{X}_\sigma(\mathbf{C}) \setminus |D|(\mathbf{C})$ not contained in Ω such that any harmonic function on U which vanishes nearly everywhere on $U \setminus \Omega$ vanishes on U.

7.B Rationality criteria for algebraic and analytic functions on curves over number fields

Let K be a number field and X be a smooth projective geometrically connected curve over K. For any place v of K, we denote by X_v the associated rigid analytic curve over K_v if v is ultrametric, resp. the corresponding Riemann surface $X_\sigma(\mathbf{C})$ if v is induced by an embedding of K in $\mathbf{C}$.

Let D be an effective divisor in X and $\Omega = (\Omega_v)_v$ an adelic tube adapted to $|D|$. We choose a normal projective flat model of X over the ring of integers $\mathscr{O}_K$ of K, say $\mathscr{X}$, and an Arakelov $\mathbf{Q}$-divisor $\widehat{D}_\Omega$ on $\mathscr{X}$ inducing the capacitary metrics $\|\cdot\|^{\mathrm{cap}}_{\Omega_v}$ at all places v of K. In particular, we assume that for any ultrametric place v, Ω_v is the tube $]\mathsf{Z}_v[$ around a closed Zariski subset Z_v of its special fibre $\mathscr{X}_{\mathbf{F}_v}$, and $\mathsf{Z}_v = \overline{D} \cap \mathscr{X}_{\mathbf{F}_v}$ for almost all places v.

Our first statement in this section is the following arithmetic analogue of Proposition 2.2.

Proposition 7.5. *Let X' be another geometrically connected smooth projective curve over K and $f\colon X' \to X$ be a non constant morphism. Let D' be an effective divisor in X'. We make the following assumptions:*

1) *by restriction, f defines an isomorphism from the subscheme D' of X' to the subscheme D of X and is étale in a neighbourhood of $|D'|$;*

2) *for any place v of K, the morphism f admits an analytic section $\varphi_v \colon \Omega_v \to X'_v$ defined over Ω_v whose formal germ is equal to $\widehat{f_{D}}^{-1}_{K_v}$;*
3) *the class of the Arakelov $\mathbf{Q}$-divisor $\widehat{D}_\Omega$ is numerically effective.*

Assume moreover

4′) *either that $\widehat{D}_\Omega \cdot \widehat{D}_\Omega > 0$;*
4″) *or that there is an archimedean place v such that the complementary subset to Ω_v in X_v is not contained in a real semi-algebraic curve of X_v.*

Then f is an isomorphism.

Proof. Let us denote by E the divisor f^*D on X'; we will prove that $E = D'$. Observe that, according to assumption 1), this divisor may be written

$$E := f^*D = D' + R,$$

where R denotes an effective or zero divisor on X, whose support is disjoint from the one of D'.

Let $\mathscr{X}'$ denote the normalization of $\mathscr{X}$ in the function field of X' and let us still denote by f the natural map from $\mathscr{X}'$ to $\mathscr{X}$ which extends f. Then $\mathscr{X}'$ is a normal projective flat model of X' over $\mathscr{O}_K$. For any place v of K, let Ω'_v denote the preimage $f^{-1}(\Omega_v)$ of Ω_v by f. The complementary subset of Ω_v is a nonempty affinoid subspace of X'_v if v is ultrametric, and a non-polar compact subset of X'_v if v is archimedean. Moreover, for almost all ultrametric places v, Ω'_v is the tube around the specialization in $\mathscr{X}'_{\mathbf{F}_v}$ of $f^{-1}(D)$. In particular, the collection $\Omega' = (\Omega'_v)$ is an adelic tube adapted to $|E|$.

We thus may assume that the capacitary metrics on $\mathscr{O}_{X'}(D')$ and $\mathscr{O}_{X'}(E)$ relative to the open subsets Ω'_v are induced by Arakelov $\mathbf{Q}$-divisors on $\mathscr{X}'$. Let us denote them by $\widehat{D'}_{\Omega'}$ and $\widehat{E}_{\Omega'}$ respectively.

Since X and X' are normal, and the associated rigid analytic spaces as well, the image $\varphi_v(\Omega_v)$ of Ω_v by the analytic section φ_v is a closed and open subset Ω^1_v of Ω'_v containing $|D'|$, and the collection $\Omega^1 = (\Omega^1_v)$ is an adelic tube adapted to $|D'|$. Consequently, by Proposition 5.5, one has $\widehat{D'}_{\Omega'} = \widehat{D'}_{\Omega^1}$. Similarly, denoting $\Omega^2_v = \Omega'_v \setminus \Omega^1_v$, the collection $\Omega^2 = (\Omega^2_v)$ is an adelic tube adapted to $|R|$ and $\widehat{R}_{\Omega'} = \widehat{R}_{\Omega^2}$. One has $\widehat{E}_{\Omega'} = f^*\widehat{D}_\Omega = \widehat{D'}_{\Omega^1} + \widehat{R}_{\Omega^2}$. Since $\Omega^1_v \cap \Omega^2_v = \emptyset$ for any place v, Lemma 7.6 below implies that $[\widehat{R}_{\Omega^2}] \cdot [\widehat{D'}_{\Omega^1}] = 0$.

Since $\widehat{D}$ is non-zero and its class is numerically effective, the class in $\widehat{\mathrm{CH}}^1_{\mathbf{Q}}(\mathscr{X}')$ of the Arakelov divisor $f^*\widehat{D} = \widehat{D} + \widehat{R}$ is numerically effective too. Proposition 7.2 and Remark 7.4 show that, when either of the hypotheses (4′) or (4″) is satisfied, necessarily $\widehat{R}_{\Omega^2} = 0$. In particular, $R = 0$ and $E = D'$. It follows that f has degree one, hence is an isomorphism. □

Lemma 7.6. *Let X be a geometrically connected smooth projective curve over a number field K, let D_1 and D_2 be divisors on X and let Ω_1 and Ω_2 be adelic* *tubes adapted to $|D_1|$ and $|D_2|$. Let us consider a normal projective and flat model $\mathscr{X}$ of X over the ring of integers of K as well as Arakelov divisors $\widehat{D_1}_{\Omega_1}$ and $\widehat{D_2}_{\Omega_2}$ inducing the capacitary metrics on $\mathscr{O}_X(D_1)$ and $\mathscr{O}_X(D_2)$ relative to the adelic tubes Ω_1 and Ω_2.*

If $\Omega_{1,v} \cap \Omega_{2,v} = \emptyset$ for any place v of K, then

$$\widehat{D_1}_{\Omega_1} \cdot \widehat{D_2}_{\Omega_2} = 0.$$

Proof. Observe that D_1 and D_2 have no common component, since any point P common to D_1 and D_2 would belong to $\Omega_{1,v} \cap \Omega_{2,v}$.

Let $\mathscr{X}$ be a normal projective flat model of X adapted to Ω_1 and Ω_2, so that the classes $\widehat{D_i}_{\Omega_i}$ live in $\widehat{\mathrm{CH}}^1_{\mathbf{R}}(\mathscr{X})$. Namely $\widehat{D_i}_{\Omega_i} = (\mathscr{D}_i, g_i)$, where $\mathscr{D}_i$ is the $\mathbf{Q}$-divisor on $\mathscr{X}$ extending D_i defined by Proposition 5.1 and $g_i = (g_{D_i,\Omega_{i,v}})$ is the family of capacitary Green currents at archimedean places. The vertical components of $\mathscr{D}_1$ and $\mathscr{D}_2$ lying over any finite place v are distinct the one from the other, since $\Omega_{1,v} \cap \Omega_{2,v} = \emptyset$. Consequently, the geometric part of the Arakelov intersection product is zero. In view of [11, Lemma 5.1], the contribution of any archimedean place v is zero too, since $\Omega_{1,v}$ and $\Omega_{2,v}$ are disjoint. This concludes the proof. □

The following proposition makes more explicit the numerical effectivity hypothesis in Proposition 7.5.

Proposition 7.7. *Let X, Ω, D, $\mathscr{X}$, $\widehat{D}_\Omega$ be as in the beginning of this Subsection.*

a) *If D is effective, then The Arakelov divisor $\widehat{D}_\Omega$ on $\mathscr{X}$, attached to the effective divisor D and to the adelic tube Ω, is effective.*
b) *Write $D = \sum_i n_i P_i$, for some closed points P_i of X and positive integers n_i. Then $\widehat{D}_\Omega$ is numerically effective if and only if $h_{\widehat{D}_\Omega}(P_i) \geqslant 0$ for each i.*
c) *If D is a rational point P, then $\widehat{D}_\Omega$ is numerically effective (resp. $\widehat{D}_\Omega . \widehat{D}_\Omega > 0$) if and only if the Arakelov degree $\widehat{\deg}(T_P X, \|\cdot\|^{\mathrm{cap}}_\Omega)$ is nonnegative (resp. positive).*

Proof. *a*) Let us assume that D is an effective divisor. For each archimedean place v of K, the capacitary Green function g_{D,Ω_v} is therefore nonnegative ([11, 3.1.4]). Moreover, we have proved in Proposition 5.1 that the $\mathbf{Q}$-divisor $\mathscr{D}$ in $Z^1_{\mathbf{Q}}(\mathscr{X})$ is effective. These two facts together imply that $\widehat{D}_\Omega$ is an effective Arakelov divisor.

b) For any archimedean place v, the definition of the archimedean capacitary Green currents involved in $\widehat{D}_\Omega$ implies that $\omega(\widehat{D}_{\Omega_v})$ is a positive measure on X_v, zero near $|D|$ [11, Theorem 3.1, (iii)]. By [11, Proposition 6.9], in order to $\widehat{D}_\Omega$ being numerically effective, it is necessary and sufficient that $h_{\widehat{D}_\Omega}(E) \geqslant 0$ for any irreducible component E of $\mathscr{D}$. This holds by construction if E is a vertical component of $\mathscr{X}$: according to the conditions of Proposition 5.1, on has $\mathscr{D} \cdot V = 0$ for any vertical component V of the support of $\mathscr{D}$; for any other vertical component V, one has $\mathscr{D} \cdot V \geqslant 0$ because the divisor $\widehat{D}_\Omega$ is effective. Consequently, $\widehat{D}_\Omega$ is nef if and only if $h_{\widehat{D}_\Omega}(P_i) \geqslant 0$ for all i.

c) This follows from b) and from the equality (Corollary 5.20)

$$h_{\widehat{D}_\Omega}(P) = \widehat{D}_\Omega.\widehat{D}_\Omega = \widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}}).$$

□

Theorem 7.8. *Let X be a geometrically connected smooth projective curve over K, P is a rational point in $X(K)$, and $\Omega := (\Omega_v)$ an adelic tube adapted to P.*

Let $\varphi \in \widehat{\mathscr{O}_{X,P}}$ be any formal function around P satisfying the following assumptions:

1) *for any $v \in F$, φ extends to an analytic meromorphic function on Ω_v;*
2) *φ is algebraic over $\mathscr{O}_{X,P}$;*
3) $\widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}}) \geqslant 0$.

If equality holds in the last inequality, assume moreover that there is an archimedean place v of F such that $X_v \setminus \Omega_v$ is not contained in a real semi-algebraic curve of X_v.

Then φ is the formal germ at P of a rational function in $K(X)$.

Proof. Let X' be the normalization of X in the field extension of $K(X)$ generated by φ. This is a geometrically connected smooth projective curve over K, which may be identified with the normalization of the Zariski closure Z in $X \times \mathbf{P}^1_K$ of the graph of φ. It is endowed with a finite morphism $f \colon X' \to X$, namely the composite morphism $X' \to Z \overset{\mathrm{pr}_1}{\to} X$. Moreover the formal function φ may be identified with the composition of the formal section σ of f at P that lifts the formal section (Id_X, φ) of $Z \overset{\mathrm{pr}_1}{\to} X$ and of the rational function $\tilde{\varphi}$ in the local ring $\mathscr{O}_{X',\sigma(P)}$ defined as the composition $X' \to Z \overset{\mathrm{pr}_2}{\to} \mathbf{P}^1_K$.

To show that φ is the germ at P of a rational function, we want to show that f is an isomorphism.

For any place v, Ω_v is a smooth analytic curve in X_v, and σ extends to an analytic section $\sigma_v \colon \Omega_v \to X'_v$ of f. Indeed, according to 1), the formal morphism (Id_X, φ) extends to an analytic section of $Z \overset{\mathrm{pr}_1}{\to} X$ over Ω_v, which in turn lifts to an analytic section of f by normality.

By Corollary 5.20, the Arakelov $\mathbf{Q}$-divisor $\widehat{P}_\Omega$ attached to the point P and the adelic tube Ω is nef. When $\widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}})$ is positive, Proposition 7.5 implies that f is an isomorphism, hence φ is the formal germ to a rational function on X. This still holds when $\widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}}) = 0$, thanks to the supplementary assumption at archimedean places in that case. □

As an example, this theorem applies when X is the projective line, P is the origin and when, for each place v in F, Ω_v is the disk of center 0 and of radius $R_v \in \sqrt{|K_v^*|}$ in the affine line. Then (Ω_v) is an adelic tube adapted to P iff almost every R_v equals 1, and $\widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}})$ is non-negative iff $\prod_v R_v \geqslant 1$. In this special case, Theorem 7.8 becomes Harbater's rationality criterion ([30, Proposition 2.1]).

Actually Harbater's result is stated without the assumption $R_v \in \sqrt{|K_v^*|}$ on the non-archimedean radii. The reader will easily check that his rationality criterion may be derived in full generality from Theorem 7.8, by shrinking the disks Ω_v for v non archimedean, and replacing them by larger simply connected domains for v archimedean.

When $\widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}}) = 0$, some hypothesis on the sets $X_v \setminus \Omega_v$ is really necessary for a rationality criterion to hold. As an example, let us consider the Taylor series of the algebraic function $\varphi(x) = 1/\sqrt{1-4x} - 1$, viewed as a formal function aroung the origin of the projective line $\mathbf{P}^1_{\mathbf{Q}}$. As shown by the explicit expansion

$$\frac{1}{\sqrt{1-4x}} - 1 = \sum_{n=1}^{\infty} (-4)^n \binom{-1/2}{n} x^n = \sum_{n=1}^{\infty} \binom{2n}{n} x^n,$$

the coefficients of this series are rational integers. Moreover, the complementary subset Ω of the real interval $[1/4, \infty]$ in $\mathbf{P}^1(\mathbf{C})$ is a simply connected open Riemann surface on which the algebraic function has no ramification. Consequently, there is a meromorphic function φ_∞ on Ω such that $\varphi_\infty(x) = (1-4x)^{-1/2} - 1$ around 0. One has $\mathrm{cap}_0(\Omega) = 1$, hence $\widehat{\deg}(T_0\mathbf{P}^1, \|\cdot\|_\Omega^{\mathrm{cap}})) = 0$. However, φ is obviously not a rational function.

By combining the algebraicity criterion of Theorem 6.1 and the previous corollary, we deduce the following result, a generalization to curves of any genus of Borel-Dwork's criterion.

Theorem 7.9. *Let X be a geometrically connected smooth projective curve over K, P is a rational point in $X(K)$, and $\Omega := (\Omega_v)$ an adelic tube adapted to P.*

Let $\varphi \in \widehat{\mathscr{O}_{X,P}}$ be any formal function around P satisfying the following assumptions:

1) *for any $v \in F$, φ extends to an analytic meromorphic function on Ω_v;*
2) *the formal graph of φ in $\widehat{X \times \mathbf{A}^1}_{(P,\varphi(P))}$ is A-analytic.*

If moreover $\widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}}) > 0$, then φ is the formal germ at P of a rational function on X (in other words, φ belongs to $\mathscr{O}_{X,P}$).

Proof. In view of Corollary 7.8, it suffices to prove prove that φ is algebraic. Let $V = X \times \mathbf{P}^1$ and let $\widehat{C} \subset \widehat{V}_{(P,\varphi(P))}$ be the formal graph of φ. We need to prove that $\widehat{C}$ is algebraic. Indeed, since at each place v of K, the canonical v-adic semi-norms on $T_P\widehat{C}$ is smaller than the capacitary one, $\widehat{\deg}(T_P X, \|\cdot\|_{X,\widehat{C}}^{\mathrm{can}}) \geqslant \widehat{\deg}(T_P X, \|\cdot\|_\Omega^{\mathrm{cap}}) > 0$. By Theorem 6.1, $\widehat{C}$ is then algebraic, and φ is algebraic over $K(X)$. $\square$

Observe that, when condition 1) is satisfied in Theorem 7.9, the A-analyticity condition 2) is implied by the following one:

2′) *there exist a positive integer N and a smooth model $\mathscr{X}$ of X over* $\operatorname{Spec}\mathscr{O}_K[1/N]$ *such that P extends to an integral point $\mathscr{P}$ in $\mathscr{X}(\mathscr{O}_K[1/N])$, and φ extends to a regular formal function on the formal completion $\widehat{\mathscr{X}_{\mathscr{P}}}$.*

This follows from Proposition 3.8, since then the formal graph of φ extends to a smooth formal curve in $\mathscr{X} \times \mathbf{A}^1$ over $\operatorname{Spec}\mathscr{O}_K[1/N]$.

Example 7.10. Theorem 7.9 may be applied when X is $\mathbf{P}^1_K$, P is the origin 0 in $\mathbf{A}^1(K) \hookrightarrow \mathbf{P}^1(K)$, and when, for each place v, $\Omega_v \subset F_v$ is a disk of center 0 and of positive radius R_v in the affine line, provided these radii are almost all equal to 1 and satisfy $\prod R_v > 1$. In this case, the rationality of any φ in $\widehat{\mathscr{O}_{X,P}} \simeq K[[X]]$ under the assumptions 1) and 2′) is precisely Borel-Dwork's rationality criterion ([6, 22]).

More generally, the expression of capacitary norms in terms of transfinite diameters and a straightforward approximation argument[7] allows one to recover the criterion of Pólya-Bertrandias ([43, 1]) from our Theorem 7.9 with $X = \mathbf{P}^1_K$.

Appendix

A Metrics on line bundles

Let K be a field which is complete with respect to the topology defined by a discrete absolute value $|\cdot|$ on K. Let R be its valuation ring and π be an uniformizing element of R. We denote by $v = \log|\cdot|/\log|\pi|$ the corresponding normalized valuation on K.

Let X be an algebraic variety over K and let L be a line bundle on X. In this appendix, we precise basic facts concerning the definition of a metric on the fibres of L.

Let $\overline{K}$ be an algebraic closure of K; endow it with the unique absolute value which extends the given one on K. It might not be complete, however its completion, denoted C, is a complete field containing $\overline{K}$ as a dense subset on which the absolute value extends uniquely, endowing it with the structure of a complete valued field.

A metric on the fibres of L is the data, for any $x \in X(C)$, of a norm $\|\cdot\|$ on the one-dimensional C-vector space $L(x)$. Namely, $\|\cdot\|$ is a map $L(x) \to \mathbf{R}_+$ satisfying the following properties:

- $\|s_1 + s_2\| \leqslant \max(\|s_1\|, \|s_2\|)$ for all s_1, $s_2 \in L(x)$;
- $\|as\| = |a|\|s\|$ for all $a \in C$ and $s \in L(x)$;

[7] using the fact that bounded subsets of $\mathbf{C}_p$ are contained in affinoids (actually, lemniscates) with arbitrarily close transfinite diameters.

- $\|s\| = 0$ implies $s = 0$.

We also assume that these norms are stable under the natural action of the Galois group $\mathrm{Gal}(C/K)$, namely that for any $x \in X(C)$, $s \in L(x)$ and $\sigma \in \mathrm{Gal}(C/K)$, $\|\sigma(s)\| = \|s\|$.

We say a metric is continuous if for any open subset $U \subset X$ (for the Zariski topology) and any section $s \in \Gamma(U, L)$, the function $x \mapsto \|s(x)\|$ on $U(C)$ is continuous. This definition corresponds to that classical notion of a Weil function attached to a Cartier divisor on X and will be sufficient for our purposes; a better one would be to impose that this function extends to a continuous function on the analytic space attached to U by Berkovich [4]; see *e.g.* [29] for this point of view.

Assume that X is projective and let $\mathscr{X}$ be any projective and flat R-scheme with generic fibre X, together with a line bundle $\mathscr{L}$ on $\mathscr{X}$ extending L. Let $x \in X(C)$; if C^0 denotes the valuation ring of C, there is a unique morphism $\varepsilon_x \colon \operatorname{Spec} C^0 \to \mathscr{X}$, by which the generic point of $\operatorname{Spec} C^0$ maps to x. Then, $\varepsilon_x^*\mathscr{L}$ is a sub-C^0-module of $L(x)$. For any section $s \in L(x)$, there exists $a \in C^0$ such that $as \in \varepsilon_x^*\mathscr{L}$. Define, for any $s \in L(x)$,

$$\|s\| = \inf\{|a|^{-1}\ ,\ as \in \varepsilon_x^*\mathscr{L}, \quad a \in C \setminus \{0\}\}.$$

This is a continuous metric on the fibres of L, which we call an algebraic metric.

Algebraic metrics are in fact the only metrics that we use in this article, where the language of metrics is just a convenient way of comparing various extensions of X and L over R. In that respect, we make the following two remarks:

1) Let Y be another projective algebraic variety over K and let $f : Y \to X$ be a morphism. Let $(L, \|\cdot\|_L)$ be a metrized line bundle on X. Then, the line bundle f^*L on Y admits a metric $\|\cdot\|_{f^*L}$, defined by the formula $\|f^*s(y)\|_{f^*L} = \|s(f(y))\|_L$, where $y \in Y(C)$ and s is a section of L in a neighbourhood of $f(y)$. Assume that the metric of L is algebraic, defined by a model $(\mathscr{X}, \mathscr{L})$. Let $\mathscr{Y}$ be any projective flat model of Y over R such that f extends to a morphism $\varphi\colon \mathscr{Y} \to \mathscr{X}$. Then, the metric $\|\cdot\|_{f^*L}$ is algebraic, defined by the pair $(\mathscr{Y}, \varphi^*\mathscr{L})$.

2) Let $\mathscr{X}$ be a projective and flat model of X on R and let $\mathscr{L}$ and $\mathscr{L}'$ be two line bundles on $\mathscr{X}$ which induce the same (algebraic) metric on L. If $\mathscr{X}$ is *normal* then the identity map $\mathscr{L}_K = \mathscr{L}'_K$ on the generic fibre extends uniquely to an isomorphism $\mathscr{L} \simeq \mathscr{L}'$.

B Background on rigid analytic geometry

The results of this appendix are basic facts of rigid analytic geometry: the first one is a version of the maximum principle, while the second proposition states that the complementary subsets to an affinoid subspace in a rigid analytic space has a canonical structure of a rigid space. They are well known to specialists but, having been unable to find a convenient reference, we decided to write them here.

Let K be a field, endowed with a ultrametric absolute value for which it is complete.

Proposition B.1. *Let C be a smooth projective connected curve over K, let $f \in K(C)$ be a non constant rational function and let X denote the Weierstrass domain $C(f) = \{x \in X\,;\, |f(x)| \leqslant 1\}$ in X. Then, any affinoid function g on X is bounded; moreover, there exists $x \in U$ such that*

$$|g(x)| = \sup_X |g| \quad \textit{and} \quad |f(x)| = 1.$$

The fact that g is bounded and attains it maximum is the classical maximum principle; we just want to assure that the maximum is attained on the "boundary" of U.

Proof. The analytic map $f\colon C^{\mathrm{an}} \to (\mathbf{P}^1)^{\mathrm{an}}$ induced by f is finite hence restricts to a finite map $f_X\colon X \to \mathbf{B}$ of rigid analytic spaces, where $\mathbf{B} = \operatorname{Sp} K\langle t\rangle$ is the unit ball. It corresponds to f_X a morphism of affinoid algebras $K\langle t\rangle \hookrightarrow \mathscr{O}(X)$ which makes $\mathscr{O}(X)$ a $K\langle t\rangle$-module of finite type. Let $g \in \mathscr{O}(X)$ be an analytic function. Then g is integral over $K\langle t\rangle$, hence there is a smallest positive integer n, as well as analytic functions $a_i \in K\langle t\rangle$, for $1 \leqslant i \leqslant n$, such that

$$g(x)^n + a_1(f(x))g(x)^{n-1} + \cdots + a_n(f(x)) = 0$$

for any $x \in X$. Then, (see [7, p. 239, Proposition 6.2.2/4])

$$\sup_{x\in X} |g(x)| = \max_{1\leqslant i\leqslant n} |a_i(t)|^{1/i}.$$

The usual proof of the maximum principle on $\mathbf{B}$ shows that there is for each integer $i \in \{1,\dots,n\}$ a point $t_i \in \mathbf{B}$ satisfying $|t_i| = 1$ and $|a_i(t_i)| = \|a_i\|$. (After having reduced to the case where $\|a_i\| = 1$, it suffices to lift any nonzero element of the residue field at which the reduced polynomial $\overline{t_i}$ does not vanish.) Consequently, there is therefore a point $t \in \mathbf{B}$ such that $|t| = 1$ and

$$\max_i |a_i(t)|^{1/i} = \max_i \|a_i\|^{1/i}.$$

Applying Proposition 3.2.1/2, p. 129, of [7] to the polynomial $Y^n + a_1(t)Y^{n-1} + \cdots + a_n(t)$, there is a point $y \in \mathbf{P}^1$ and $|y| = \max_i \|a_i\|^{1/i}$. Since the morphism $K\langle t\rangle[g] \subset \mathscr{O}(X)$ is integral, there is a point $x \in X$ such that $f(x) = t$ and $g(x) = y$. For such a point, one has $|f(x)| = 1$ and $|g|(x) = \|g\|$. $\square$

Proposition B.2. *Let X be a rigid analytic variety over K and let $A \subset X$ be the union of finitely many affinoid subsets.*

Then $X \setminus A$, endowed with the induced G-topology, is a rigid analytic variety.

Proof. By [7, p. 357, Proposition 9.3.1/5], and the remark which follows that proposition, it suffices to prove that $X \setminus A$ is an admissible open subset.

Let (X_i) be an admissible affinoid covering of X; then, for each i, $A_i = A \cap X_i$ is a finite union of affinoid subsets of X_i. Assume that the Proposition holds when X is affinoid; then, each $X_i \setminus A_i$ is an admissible open subset of X_i, hence of X. Then $X \setminus A = \bigcup_i (X_i \setminus A_i)$ is an admissible open subset of X, by the property (G_1) satisfied by the G-topology of rigid analytic varieties.

We thus may assume that X is an affinoid variety. By Gerritzen-Grauert's theorem ([7, p. 309, Corollary 7.3.5/3]), A is a finite union of rational subdomains $(A_i)_{1 \leqslant i \leqslant m}$ in X. For each i, let us consider affinoid functions $(f_{i,1}, \dots, f_{i,n_i}, g_i)$ on X generating the unit ideal such that

$$\begin{aligned} A_i &= X\left(\frac{f_{i,1}}{g_i}, \dots, \frac{f_{i,n_i}}{g_i}\right) \\ &= \left\{ x \in X \,;\, |f_{i,1}(x)| \leqslant |g_i(x)|, \dots, |f_{i,n_i}(x)| \leqslant |g_i(x)| \right\}. \end{aligned}$$

We have

$$X \setminus A = \bigcap_{i=1}^{m} (X \setminus A_i) = \bigcap_{i=1}^{m} \bigcup_{j=1}^{n_i} \left\{ x \in X \,;\, |f_{i,j}(x)| > |g_i(x)| \right\}.$$

Since any finite intersection of admissible open subsets is itself admissible open, it suffices to treat the case where $m = 1$, *i.e.*, when A is a rational subdomain $X(f_1, \dots, f_n; g)$ of X, which we now assume.

By assumption, $f_1, \dots, f_n, g$ have no common zero. By the maximum principle [7, p. 307, Lemma 7.3.4/7], there is $\delta \in \sqrt{|K^*|}$ such that for any $x \in X$,

$$\max(|f_1(x)|, \dots, |f_n(x)|, |g(x)|) \geqslant \delta.$$

For any $\alpha \in \sqrt{|K^*|}$ with $\alpha > 1$, and any $j \in \{1, \dots, n\}$, define

$$X_{j,\alpha} = X\left(\delta \frac{1}{f_j}, \alpha^{-1} \frac{g}{f_j}\right) = \left\{ x \in X \,;\, \delta \leqslant |f_j(x)|, \quad \alpha |g(x)| \leqslant |f_j(x)| \right\}.$$

This is a rational domain in X. For any $x \in X_{j,\alpha}$, one has $f_j(x) \neq 0$, and $|g(x)| < |f_j(x)|$, hence $x \in X \setminus A$. Conversely, if $x \in X \setminus A$, there exists $j \in \{1, \dots, n\}$ such that $\max(|f_1(x)|, \dots, |f_n(x)|, |g(x)|) = |f_j(x)| > |g(x)|$; it follows that there is $\alpha \in \sqrt{|K^*|}$, $\alpha > 1$, such that $x \in X_{j,\alpha}$. This shows that the affinoid domains $X_{j,\alpha}$ of X, for $1 \leqslant j \leqslant n$ and $\alpha \in \sqrt{|K^*|}$, $\alpha > 1$, form a covering of $X \setminus A$. Let us show that this covering is admissible. Let Y be an affinoid space and let $\varphi \colon Y \to X$ be an affinoid map such that $\varphi(Y) \subset X \setminus A$. By [7, p. 342, Proposition 9.1.4/2], we need to show that the covering $(\varphi^{-1}(X_{j,\alpha}))_{j,\alpha}$ of Y has a (finite) affinoid covering which refines it. For that, it is sufficient to prove that there are real numbers $\alpha_1, \dots, \alpha_n$ in $\sqrt{|K^*|}$, greater than 1, such that $\varphi(Y) \subset \bigcup_{j=1}^{n} X_{j,\alpha_j}$.

For $j \in \{1, \ldots, n\}$, define an affinoid subspace Y_j of Y by

$$Y_j = \Big\{ y \in Y\,;\ |f_i(\varphi(y))| \leqslant |f_j(\varphi(y))| \text{ for } 1 \leqslant i \leqslant n \Big\}.$$

One has $Y = \bigcup_{j=1}^n Y_j$. Fix some $j \in \{1, \ldots, n\}$. Since $\varphi(Y_j) \subset X \setminus A$, $|g(x)| < |f_j(x)|$ on Y_j. It follows that $f_j \circ \varphi$ does not vanish on Y_j, hence $g \circ \varphi / f_j \circ \varphi$ is an affinoid function on Y_j such that

$$\left| \frac{g \circ \varphi}{f_j \circ \varphi}(y) \right| < 1$$

for any $y \in Y_j$. By the maximum principle, there is $\alpha_j \in \sqrt{|K^*|}$ such that $\alpha_j > 1$ and $\left| \frac{g \circ \varphi}{f_j \circ \varphi} \right| < \frac{1}{\alpha_j}$ on Y_j. One then has $\varphi(Y) \subset \bigcup_{j=1}^n X_{j,\alpha_j}$, which concludes the proof of the proposition. □

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