Germs of analytic varieties in algebraic varieties: canonical metrics and arithmetic algebraization theorems.

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Contents

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Conventions The following notation and terminology are used throughout this paper.

The open (resp. closed) disk of center a and radius r in $\mathbb{C}$ is will be denoted $D(a, r)$ (resp. $\overline{D}(a, r)$).

The rank of a vector bundle E (resp. of a linear map φ) will be denoted $\operatorname{rk} E$ (resp. $\operatorname{rk} \varphi$).

By an *algebraic variety* over some field k, we mean an integral scheme of finite type over k. Integral subschemes of such an algebraic variety X over k will be called *algebraic subvarieties* of X.

On a complex analytic manifold, we write as usual $d = \partial + \overline{\partial}$ and we let

$$d^c := (i/4\pi)(\overline{\partial} - \partial);$$

consequently:

$$dd^c = (i/2\pi)\partial\overline{\partial}.$$

1 Introduction

Consider a number field K, a quasi-projective variety X over K, a point P in $X(K)$, and a germ $\hat{V}$ of formal subvariety of X through P, namely, a smooth formal subscheme of the formal completion $\hat{X}_P$ of the K-scheme X at the closed point P. We shall say that such a formal scheme is *algebraic* when it is a branch (*i.e.*a component of the formal completion at P) of an algebraic subvariety Y of X containing P (see section 2.1, *infra*, for a more complete discussion of the concept of algebraic formal germ).

Various questions in arithmetic geometry may be rephrased in terms of the algebraicity of such formal germs $\hat{V}$: one would like to know natural *arithmetic conditions* on $\hat{V}$ implying its algebraicity.

The main examples we have in mind are the following ones:

A.Formal series. Let $f \in K[[t_1, \ldots, t_N]]$ be a formal series in N variables which has a positive radius of convergence at every place of K, finite or infinite. In other words, for any non-zero prime ideal $\mathbf{p}$ in $\mathcal{O}_K$ (resp. for any field embedding $\sigma : K \hookrightarrow \mathbb{C}$), the series f seen as an element of $K_{\mathbf{p}}[[t_1, \ldots, t_N]]$ (resp. of $\mathbb{C}[[t_1, \ldots, t_N]]$) by means of the embedding

$K \hookrightarrow K_{\mathbf{p}}$ of K in its $\mathbf{p}$-adic completion $K_{\mathbf{p}}$ (resp. by means of $\sigma : K \hookrightarrow \mathbb{C}$) has a positive $\mathbf{p}$-adic (resp. complex) radius of convergence.

Then the graph of f defines a smooth formal germ of dimension N,

$$\hat{V} := \mathrm{Graph}(f)$$

in $\mathbb{A}_K^{N+1}$ through $(0, f(0))$—formally, it is defined by the principal ideal generated by $z - f(t_1, \ldots, t_N)$ in $K[[t_1, \ldots, t_N, z - f(0)]]$)—and the algebraicity of $\mathrm{Gr}(f)$ is equivalent to the algebraicity of f over the subfield $K(t_1, \ldots, t_N)$ of the field of fraction of $K[[t_1, \ldots, t_N]]$ (or to the fact that f belongs to the integral closure of the local ring $\mathcal{O}_{\mathbb{A}^N,0}$ in its completion $\hat{\mathcal{O}}_{\mathbb{A}^N,0} \simeq K[[t_1, \ldots, t_N]]$).

B.Formal subgroups of algebraic groups. Assume that X is a K-algebraic group G and $P = e$, the unit element of $G(K)$, and let $\mathbf{h}$ be a Lie subalgebra (over K) of $\mathbf{g} := \mathrm{Lie}\, G$. We may consider the formal Lie subgroup $\hat{V} := \widehat{\mathrm{Exp}}\, \mathbf{h}$ of the formal group $\hat{G}_e$ over K attached to G, namely the smooth formal subgroup of $\hat{G}_e$ which admits $\mathbf{h}$ as Lie algebra[1]. Then the formal germ $\hat{V}$ is algebraic iff $\mathbf{h}$ is an algebraic Lie algebra, *i.e.*, is the Lie algebra of some algebraic K-subgroup H of G.

For instance, if G is the product $G_1 \times G_2$ of two K-algebraic groups G_1 and G_2 with Lie algebras $\mathbf{g}_1$ and $\mathbf{g}_2$, a K-Lie algebra isomorphism $\varphi : \mathbf{g}_1 \to \mathbf{g}_2$ is the differential of a K-isogeny from G_1 to G_2 iff the formal germ $\hat{V} := \widehat{\mathrm{Exp}}\, \mathbf{h}$ is algebraic, where $\mathbf{h}$ denotes the Lie subalgebra of $\mathbf{g} = \mathbf{g}_1 \oplus \mathbf{g}_2$ defined by the graph of φ.

C.Ordinary differential equations. Consider an algebraic ordinary differential equation over a number field K, and define $\hat{V}$ as its formal solution for some initial conditions defined over K.

For instance, if $Q = (Q_1, \ldots, Q_n)$ is an element in $K(X, Y_1, \ldots, Y_n)^n$ and y_0 a point in K^n such that $(0, y_0)$ does not lie on the polar divisor of any component Q_i of Q, we may consider the formal solution f in $K[[t]]^n$ of the differential equation

$$f'(t) = Q(t, f(t)) \tag{1.1}$$

satisfying the initial condition

$$f(0) = y_0. \tag{1.2}$$

This solution f is an "algebraic function" iff the graph $\hat{V}$ of f in $\mathbb{A}^{N+1}_{(0,y_0)}$ is algebraic.

More generally, we may consider a smooth variety X over K, a point P in $X(K)$ and a sub-vector bundle F of rank one of the tangent bundle $T_{X/K}$, and consider the smooth formal germ of curve $\hat{V}$ defined by "integrating" the line bundle F. Formally, it is defined as the unique smooth formal germ of curve in X through P such that, if $i : \hat{V} \hookrightarrow X$ denotes the inclusion morphism, the differential Di, which *a priori* is an element of $\Gamma(\hat{V}, i^* T_X)$, indeed belongs to $\Gamma(\hat{V}, i^* F)$. We recover the previous situation by defining X as the complement

[1]It may be constructed as follows: if $\widehat{\mathrm{Exp}}$ denotes the "formal exponential map" of G—that is, the isomorphism of K-formal schemes from the completion at 0 of the K-affine space defined by $\mathbf{g}$ onto $\hat{G}_e$ defined by the Campbell-Hausdorff series—then $\widehat{\mathrm{Exp}}\, \mathbf{h}$ is the image by $\widehat{\mathrm{Exp}}$ of the formal completion at 0 of the K-affine subspace $\mathbf{h}$ of $\mathbf{g}$.

of the polar divisors of the Q_i's in $\mathbb{A}^{n+1}$, P as $(0, y_0)$, and F as the line bundle generated by the vector field

$$\frac{\partial}{\partial X} - \sum_{i=1}^{n} Q_i \frac{\partial}{\partial Y_i}.$$

(Actually, both constructions **B** and **C** are special cases of the construction of formal germs of leaves of algebraic foliations over number fields; *cf.* [ESBT99] and [Bos01].)

In the three situations **A**, **B**, and **C** above, the formal germ $\hat{V}$ satisfies the following analyticity conditions:

For any non-zero prime ideal $\mathbf{p}$ *in* $\mathcal{O}_K$ *(resp. for any field embedding* $\sigma : K \hookrightarrow \mathbb{C}$*), the formal germ* $\hat{V}_{K_{\mathbf{p}}}$ *(resp.* V_σ*) in the formal completion at* P *of* $X_{K_{\mathbf{p}}}$ *(resp.* X_σ*) deduced from* $\hat{V}$ *by the base field extension* $K \hookrightarrow K_{\mathbf{p}}$ *(resp.* $\sigma : K \hookrightarrow \mathbb{C}$*) is analytic.* Namely, it is the formal germ attached to some (necessarily smooth) germ of $K_{\mathbf{p}}$-analytic (resp. $\mathbb{C}$-analytic) subvariety through P of the $K_{\mathbf{p}}$-analytic (resp. $\mathbb{C}$-analytic space) $X(K_{\mathbf{p}})$ (resp. $X(\mathbb{C})$).

This is tautological in case **A**; in case **B** and **C**, this follows from the well known analyticity properties of the Campbell-Hausdorff series and from the classical Cauchy's theorem and its p-adic versions (see for instance [Ser92], section V.4, and [DGS94], Appendix III).

These analyticity conditions are easily seen to be necessary for the algebraicity of $\hat{V}$. Actually, the latter imposes much stronger conditions. For instance, as early as 1852, Eisenstein discovered the following fact, now known as Eisenstein's theorem: *if a formal series* $\sum_{k=0}^{+\infty} a_k t^k$ *in* $\mathbb{Q}[[t]]$ *is algebraic, then there exists integers* $A, B \geq 1$ *such that* $AB^k a_k \in \mathbb{Z}$ *for every* $k \in \mathbb{N}$. Concerning solutions of differential equations considered in **B** above, it was pointed out by Grothendieck and Katz around 1970 ([Kat72]) that, if the differential system defined by a line bundle F in the tangent bundle T_X of a smooth variety X over a number field K is algebraically integrable, then the following arithmetic condition—which we shall call *condition GK*—necessarily holds:

For almost every non-zero prime ideal $\mathbf{p}$ *in* $\mathcal{O}_K$*, the sub-line bundle* $F_{\mathbb{F}_{\mathbf{p}}}$ *of* $T_{X_{\mathbb{F}_{\mathbf{p}}}}$ *on the variety* $X_{\mathbb{F}_{\mathbf{p}}}$ *obtained by reduction modulo* $\mathbf{p}$ *from some smooth model* $\mathcal{X}$ *of* X *over some open dense subscheme* S *of* $\operatorname{Spec} \mathcal{O}_K$ *and from a line bundle* $\mathcal{F} \hookrightarrow T_{\mathcal{X}/S}$ *extending* F *is closed under the p-th power map* (where p denotes the characteristic of the residue field $\mathbb{F}_{\mathbf{p}} := \mathcal{O}_K/\mathbf{p}$).

Actually, Grothendieck and Katz were considering *linear* differential systems only; the case of general differential systems explicitly appears in [Miy87], [SB92] and [ESBT99].

In this paper, we are interested in *sufficient conditions* implying the algebraicity of $\hat{V}$ in the context of examples **A**, **B**, and **C** above. The investigation of such conditions has a long and rich history, about which we shall give only a few indications.

The first result concerning sufficient conditions for algebraicity appears to be a theorem established by E. Borel in 1892 asserting that, *if a formal series* $f \in \mathbb{Z}[[t]]$ *is the Taylor expansion at 0 of some fonction meromorphic on a disk* $D(0; R)$ *of radius* $R > 1$*, then* f *is the expansion of a rational function.*

Concerning linear differential equations, Grothendieck and Katz conjectured that condition GK is indeed a sufficient condition for algebraic integrability. This conjecture is formulated in the seminal paper [Kat72] of Katz, where he proves it in the significant special case of linear differential systems "of geometric origin" (see also [Kat82], [Kat96], [And99] and [And02] for more recent developments in this direction).

Besides, in their famous works [Ser68] and [Fal83], Serre and Faltings obtained deep results concerning isogenies between elliptic curves and abelian varieties, which may be used to handle non-trivial cases of the algebraicity problem in the situation **B** (see for instance [ESBT99], sections 3-5).

Finally, around 1984, D.V. and G.V. Chudnovsky discovered how to apply "transcendence techniques" to establish algebraicity statements in the situations A, B, and C ([CC85a] and [CC85b]). Their work was subsequently extended by André ([And89], [And99] and [And02]), Graftieaux ([Gra01a] and [Gra01b]), and the author ([Bos01]). We refer the reader to [CL02] for a synthetic view of these results.

Bernard Dwork himself played a distinguished role in contributing to various aspects of the algebraicity problem in the situations **A** and **C**. It is barely necessary to recall that, in his famous rationality proof ([Dwo60]), he established a generalized version of Borel's rationality criterion discussed above—the *Borel-Dwork criterion.* Let us also mention its investigations of Eisenstein's theorem ([DR79], [DvdP92]) and his papers ([BD79], [Dwo81], [Dwo99]) devoted to the "arithmetic theory of differential equations". The latter also constitutes one of the main themes of the beautiful book [DGS94] by B. Dwork, G. Gerotto and F. Sullivan.

This paper is devoted to some algebraicity criterions, implying the algebraicity of formal germs of curves over number fields in the situations **A**, **B**, and **C** considered above. These criterions, which are refined versions of the main results of [Bos01] in the special case of germs of formal curves, are expressed in terms of *positivity properties*—defined in terms of its Arakelov degree—of the tangent line $T_P\hat{V}$ equipped with some natural p-adic and archimedean semi-norms. As our previous results in [Bos01], they are established by a geometric version of "transcendence techniques", which avoids the traditional constructions of "auxiliary polynomials" but is based instead on some geometric version of these, namely the study of *evaluation maps* on the spaces of global sections of ample line bundles on a projective variety, defined by restricting them to formal subschemes or to subschemes of finite lengths.

Dealing with formal germs of curves only—instead of formal germs of arbitrary dimension as in [Bos01]—allows various technical simplifications and leads to an algebraization theorem (Theorem 4.2, *infra*) whose statement and proof are particularly simple. However, Theorem 4.2 admits higher dimensional generalizations on which we plan to return in the future.

This paper is organized as follows.

In section 2, we discuss the notion of algebraicity of formal germs in algebraic varieties, and we provide an introduction to the use of auxiliary polynomials, in the geometric guise of evaluation maps, by showing how simply they lead to non-trivial algebraicity results in some purely geometric situations. In particular, we establish an algebraicity criterion for formal germs over functions fields, which we use to investigate the positivity properties of the Lie algebras of group schemes over a field of characteristic zero.

Section 3 is of a more analytic nature: we assume that $\hat{V}$ is a germ of analytic curve in a complex algebraic variety X, and we explain how the consideration of the metric properties of the evaluation maps involved in the method of auxiliary polynomials leads to the construction of some remarkable semi-norm on the complex line $T_P\hat{V}$. We also study some "naturality" and "functoriality" properties of this semi-norm, and we establish some upper-bound on it in terms of potential theoretic invariants.

In section 4, we present an algebraicity theorem concerning formal germs of curves in algebraic varieties over number fields, which may be seen as an arithmetic counterpart of the criterion over function fields discussed in section 2. This criterion involves the canonical complex semi-norms investigated in section 3. Actually, it may be used to formulate a conjecture about *complex* linear algebraic differential systems, whose solution would provide a proof of the conjecture of Grothendieck-Katz asserting that condition GK is a sufficient condition of algebraic integrability for algebraic linear differential systems over number fields.

2 Algebraicity of smooth formal germs in algebraic varieties and auxiliary polynomials

2.1 Algebraic formal germs

Let X be a variety over a field K, P a point of $X(K)$, $\hat{X}_P$ the formal completion of X at P, and $\hat{V} \hookrightarrow \hat{X}_P$ a smooth formal subscheme. For any non-negative integer i, we shall denote V_i the i-th infinitesimal neighborhood of P in $\hat{V}$. Thus,

$$V_0 = \{P\} \subset V_1 \subset V_2 \subset \cdots$$

and

$$\hat{V} = \lim_{\rightarrow} V_i.$$

It will be convenient to let:

$$V_{-1} = \emptyset.$$

We may consider the Zariski closure of $\hat{V}$ in X, namely, the smallest closed subscheme Z of X which contain V_i for every $i \geq 0$, or equivalently, such that $\hat{Z}_P$ contain $\hat{V}$. Observe that it is a subvariety (*i.e.*an integral subscheme) of X containing P. The ideal in $\mathcal{O}_{X,P}$ defining its germ at P is the intersection of $\mathcal{O}_{X,P}$ and the ideal in its completion $\hat{\mathcal{O}}_{X,P} = \mathcal{O}_{\hat{X}_P}$ that defines $\hat{V}$. Since $\hat{Z}_P$ contains $\hat{V}$, the dimension of Z greater or equal to the dimension of $\hat{V}$.

The following proposition is an easy application of the basic properties of dimension and normalization:

Proposition 2.1 *The following three conditions are equivalent:*

(i) There exists an algebraic variety Y over K, a point 0 of $Y(K)$ and a K-morphism which maps 0 to P and such that the induced morphism on formal completions

$$\hat{f}_0 : \hat{Y}_0 \longrightarrow \hat{X}_P$$

factorizes through $\hat{V} \hookrightarrow \hat{X}_P$ and defines a formal isomorphism from $\hat{Y}_0$ to $\hat{V}$.

(ii) There exists a closed subvariety Z of X such that P belongs to $Z(K)$ and $\hat{V}$ is a branch of Z through P.

(iii) The dimension of the Zariski closure Z of $\hat{V}$ in X coincides with the dimension of the formal scheme $\hat{V}$.

We shall say that *the formal germ $\hat{V}$ is algebraic* when the above conditions are satisfied.

2.2 Evaluation maps and an algebraicity criterion

Let us keep the notation of the preceding paragraph. Let us moreover assume that X *is projective* and consider an ample line bundle L on X.

Let us introduce the following K-vector spaces and K-linear maps:

$$E_D := \Gamma(X, L^{\otimes D}),$$

$$\begin{array}{rccc} \eta_D : & E_D & \longrightarrow & \Gamma(\hat{V}, L^{\otimes D}) \\ & s & \longmapsto & s_{|\hat{V}}, \end{array}$$

$$\begin{array}{rccc} \eta_D^i : & E_D & \longrightarrow & \Gamma(V_i, L^{\otimes D}) \\ & s & \longmapsto & s_{|V_i}, \end{array}$$

and

$$E_D^i := \{s \in E_D \mid s_{V_{i-1}} = 0\} = \ker \eta_D^{i-1}.$$

Observe that we have a canonical isomorphism

$$\Gamma(\hat{V}, L^{\otimes D}) \simeq \lim_{\substack{\leftarrow \\ i}} \Gamma(V_i, L^{\otimes D}),$$

by means of which the map η_D gets identified with

$$\lim_{\substack{\leftarrow \\ i}} \eta_D^i.$$

The subspaces E_D^i define a decreasing filtration of the finite dimensional K-vector space E_D:

$$\ldots \subset E_D^{i+1} \subset E_D^i \subset \ldots \subset E_D^1 \subset E_D^0 = E_D.$$

Moreover the very definition of Z as the Zariski closure of $\hat{V}$ shows that, if $\mathcal{I}_Z$ denotes the ideal sheaf in $\mathcal{O}_X$ defining Z, we have

$$\bigcap_{i \geq 0} E_D^i = \ker \eta_D = \Gamma(X, \mathcal{I}_Z . L^{\otimes D}). \tag{2.1}$$

Finally, if $T_{\hat{V}}$ denotes the tangent space of $\hat{V}$, then, for any non-negative integer i, the kernel of the restriction map from $\Gamma(V_i, L^{\otimes D})$ to $\Gamma(V_{i-1}, L^{\otimes D})$ may be identified with $S^i\check{T}_{\hat{V}} \otimes L_P^D$, and the restriction of the evaluation map η_D^i to E_D^i defines a K-linear map:

$$\gamma_D^i : E_D^i \longrightarrow S^i\check{T}_{\hat{V}} \otimes L_P^{\otimes D}.$$

Roughly speaking, it is the map which sends a section of $L^{\otimes D}$ vanishing up to order $i-1$ at P along $\hat{V}$ to the i-th "Taylor coefficient" of its restriction to $\hat{V}$. By construction,

$$\ker \gamma_D^i = E_D^{i+1}. \tag{2.2}$$

Proposition 2.2 *The following two conditions are equivalent:*

(i) The formal germ $\hat{V}$ is algebraic.

(ii) There exists $c > 0$ such that, for any $(D, i) \in \mathbb{N}_{>0} \times \mathbb{N}$ satisfying $i/D > c$, the map γ_D^i vanishes.

Condition (ii) may be also expressed by saying that, for every positive integer D the filtration $(E_D^i)_{i\geq 0}$ becomes stationary—or equivalently that η_D vanishes on E_D^i—when $i > cD$.

The direct implication (i) $\Rightarrow$ (ii) will be a consequence of the following lemma, which belongs to the basic theory of ample line bundles (see for instance [Laz01], Chapter 5, notably Proposition 5.1.9).

Lemma 2.3 *Let M be a projective variety of dimension d over a field K, H an ample line bundle over M, and 0 a point in $M(k)$. Let $\varepsilon(H, 0)$ denote the Seshadri constant of H at 0 and*

$$\deg_H M := c_1(H)^d \cap [M]$$

be the degree of M with respect to H. Then, for any positive integer D and any regular section s of $H^{\otimes D}$ over M which does not vanishes identically, the order of vanishing $\mathrm{mult}_0 s$ of s at 0 satisfies the following upper bound:

$$\mathrm{mult}_0 s \leq \frac{\deg_H M}{\varepsilon(H, 0)^{d-1}} D. \tag{2.3}$$

Recall that $\varepsilon(H, 0)$ is the positive real number defined as follows: let

$$\nu : \tilde{M} \longrightarrow M$$

be the blow-up of 0 in M and let $E := \nu^{-1}(0)$ be its exceptional divisor; then $\sigma_0(H)$ is the supremum of the rational numbers q such that the $\mathbb{Q}$-line bundle $\nu^* H \otimes \mathcal{O}(-qE)$ is ample.

To prove (2.3), one observes that the Cartier divisor on $\tilde{M}$

$$\nu^* \mathrm{div}\, s - \mathrm{mult}_0 s.E$$

is effective; therefore, for any q as above, the intersection number

$$c_1(\nu^* H \otimes \mathcal{O}(-qE))^{d-1} \cap (\nu^* \mathrm{div}\, s - \mathrm{mult}_0 s.E)$$

is non-negative. Since this intersection number is easily seen to be

$$D.\deg_H M - \mathrm{mult}_0 s.q^{d-1},$$

we get (2.3) by letting q go to $\varepsilon(H,0)$.

Proof of Proposition 2.2: To prove the implication (i) $\Rightarrow$ (ii), let us assume that $\hat{V}$ is algebraic and let us consider the normalization $n : Z_n \to Z$ of the Zariski closure Z of $\hat{V}$ in X. Like Z, it is a projective variety of dimension $d := \dim \hat{V}$. Indeed, the line bundle n^*L on Z_n is ample and, since n is birational,

$$\deg_{n^*L} Z_n = \deg_L Z.$$

Let $0 \in Z_n(K)$ be the preimage of P by n corresponding to the branch $\hat{V}$ of $\hat{Z}_P$. In other words, the completion of n at 0 induces a formal isomorphism:

$$\hat{n}_0 : \widehat{Z_{n0}} \longrightarrow \hat{V}.$$

Let s be an element of E^i_D. Pulling back s by n, we get a regular section n^*s of $n^*L^{\otimes N}$ over Z_n which vanishes at order at least i at the point 0. Lemma (2.3) shows that n^*s vanishes on Z_n if

$$i > \frac{\deg_L Z}{\varepsilon(n^*L,0)^{d-1}} D.$$

This proves that any $s \in E^i_D$ vanishes on $\hat{V}$ when $i > cD$, where

$$c := \frac{\deg_L Z}{\varepsilon(n^*L,0)^{d-1}}.$$

Conversely, let assume that condition (ii) holds, and let d still denote $\dim \hat{V}$. Then, for any $(D,i) \in \mathbb{N}^2$, the quotient vector space

$$E^i_D / E^{i+1}_D = E^i_D / \ker \gamma^i_D \simeq \mathrm{im}\, \gamma^i_D$$

vanishes if $i > cD$ and its rank is always at most

$$\mathrm{rk}\,(S^i \check{T}_{\hat{V}} \otimes L^D_P) = \binom{d+i-1}{i}.$$

This implies that

$$\mathrm{rk}\,(E_D / \bigcap_{i \geq 0} E^i_D) = \sum_{i \geq 0} \mathrm{rk}\,(E^i_D / E^{i+1}_D) \leq \sum_{i=0}^{[cD]} \binom{d+i-1}{i}.$$

Moreover the last sum is equivalent to $\frac{c^d}{d!} D^d$ when D goes to infinity.

Besides, according to (2.1):

$$E_D / \bigcap_{i \geq 0} E^i_D = \Gamma(X, L^{\otimes D}) / \Gamma(X, \mathcal{I}_Z . L^{\otimes D}).$$

For D large enough, this space may be identified with $\Gamma(Z, L^{\otimes D})$ and its rank is equivalent to $\frac{\deg_L Z}{(\dim Z)!} D^{\dim Z}$ when D goes to infinity.

This shows that $\dim Z$ is less or equal—hence equal—to d and that $\deg_L Z \leq c^d$.

$\square_{\text{Proposition 2.2}}$

The implication (ii) $\Rightarrow$ (i) in Proposition 2.2 asserts that, when $\hat{V}$ is not algebraic, there exists non-vanishing maps γ_D^i with arbitrary large values of the ratio i/D. Actually, it is possible to establish a strengthened version of this implication, which will turn out to be useful in the sequel:

Lemma 2.4 *If $\hat{V}$ is not algebraic, then*

$$\lim_{D \to +\infty} \frac{\sum_{i \geq 0} (i/D) \mathrm{rk}\,(E_D^i / E_D^{i+1})}{\sum_{i \geq 0} \mathrm{rk}\,(E_D^i / E_D^{i+1})} = +\infty. \tag{2.4}$$

Observe that, if (2.4) holds, then, for any $\lambda > 0$,

$$\lim_{D \to +\infty} \frac{\sum_{i \geq \lambda D} (i/D) \mathrm{rk}\,(E_D^i / E_D^{i+1})}{\sum_{i \geq 0} \mathrm{rk}\,(E_D^i / E_D^{i+1})} = +\infty. \tag{2.5}$$

Indeed,

$$\frac{\sum_{i < \lambda D} (i/D) \mathrm{rk}\,(E_D^i / E_D^{i+1})}{\sum_{i \geq 0} \mathrm{rk}\,(E_D^i / E_D^{i+1})} \leq \lambda.$$

Proof of Lemma 2.4: As observed in the previous proof, we have:

$$\sum_{i \geq 0} \mathrm{rk}\,(E_D^i / E_D^{i+1}) = \mathrm{rk}\,(E_D / \bigcap_{i \geq 0} E_D^i)$$

is equal to $\mathrm{rk}\,(\Gamma(Z, L^{\otimes D}))$ when D is large enough, and therefore grows like $D^{\dim Z}$ when D goes to infinity.

Moreover, for any $\lambda \geq 0$ and any $D \in \mathbb{N}$,

$$\sum_{i \geq 0} \frac{i}{D} \mathrm{rk}\,(E_D^i / E_D^{i+1}) \geq \lambda \sum_{i \geq \lambda D} \mathrm{rk}\,(E_D^i / E_D^{i+1}) = \lambda \mathrm{rk}\,(E_D^{\lceil \lambda D \rceil} / \bigcap_{i \geq 0} E_D^i), \tag{2.6}$$

where $\lceil \lambda D \rceil$ denotes the smallest integer $\geq \lambda D$. To derive a lower bound on this quantity, observe that

$$\mathrm{rk}\,(E_D^{\lceil \lambda D \rceil} / \bigcap_{i \geq 0} E_D^i) = \mathrm{rk}\,(E_D / \bigcap_{i \geq 0} E_D^i) - \mathrm{rk}\,(E_D / E_D^{\lceil \lambda D \rceil})$$

and that

$$\mathrm{rk}\,(E_D / E_D^{\lceil \lambda D \rceil}) = \mathrm{rk}\,(\mathrm{im}\,\gamma_D^{\lceil \lambda D \rceil - 1})$$

is bounded from above by the length $\lg(V_{\lceil \lambda D \rceil - 1})$ of $V_{\lceil \lambda D \rceil - 1}$. This shows that

$$\frac{\sum_{i \geq \lambda D} (i/D) \mathrm{rk}\,(E_D^i / E_D^{i+1})}{\sum_{i \geq 0} \mathrm{rk}\,(E_D^i / E_D^{i+1})} \geq \lambda \left(1 - \frac{\lg(V_{\lceil \lambda D \rceil - 1})}{\mathrm{rk}\,(\Gamma(Z, L^{\otimes D}))} \right).$$

Finally, if $\hat{V}$ is not algebraic, then $\dim Z > d := \dim \hat{V}$ and, when D goes to infinity,

$$\lg(V_{\lceil \lambda D \rceil - 1}) = \binom{\lceil \lambda D \rceil + d - 1}{d} = O(D^d) = o(D^{\dim Z}) = o(\mathrm{rk}\,(\Gamma(Z, L^{\otimes D})),$$

and therefore

$$\liminf_{D \to +\infty} \frac{\sum_{i \geq 0} (i/D) \mathrm{rk}\,(E^i_D / E^{i+1}_D)}{\sum_{i \geq 0} \mathrm{rk}\,(E^i_D / E^{i+1}_D)} \geq \lambda.$$

As λ is arbitrary, this completes the proof.

$\square_{\text{Lemma 2.4}}$

2.3 An algebraicity criterion for smooth formal germs in varieties over function fields

Let C be a smooth projective connected curve over some field k, and let $K := k(C)$ be its function field. Consider X a variety over K, P a point in $X(K)$, and $\hat{V} \subset \hat{X}_P$ a smooth formal germ of subvariety through P of X.

In this section, we discuss a criterion for the algebraicity of $\hat{V}$, involving a model $\mathcal{X}$ of X over C and the positivity properties of the thickenings of the closure $\mathcal{P}$ of P in $\mathcal{X}$ attached to $\hat{V}$. This algebraicity criterion will appear as a geometric model for the arithmetic algebraicity criterion presented in section 4.3 below. Moreover, its proof demonstrates how simply the use of "auxiliary polynomials" leads to non-trivial results, even in a purely geometric framework (see for instance Theorem 2.6 *infra*). The reader is referred to [BM01] and to [Bos01], section 3.3, for related geometric results and discussions of their relations with the classical works of Andreotti on pseudo-concavity, and of Hartshorne-Hironaka-Matsumura on the G2 condition.

After possibly shrinking X, we may assume that it is quasi-projective and choose a quasi-projective model[2] $\pi : \mathcal{X} \longrightarrow C$ such that P extends to a section $\mathcal{P}$ of π.

As in the preceding section, we denote V_i the i-th infinitesimal neighbourhood of P in X. We may consider the subschemes $\mathcal{V}_i$ of $\mathcal{X}$ defined as the closures of these subschemes V_i of $\mathcal{X}_K$. For any $i \in \mathbb{N}$, the support of $\mathcal{V}_i$ is exactly the image of the section $\mathcal{P}$. In particular, the subschemes $\mathcal{V}_i$ are finite over C. Moreover, their ideal sheaves $\mathcal{I}_{\mathcal{V}_i}$ satisfy the relations

$$\mathcal{I}_{\mathcal{V}_i}.\mathcal{I}_{\mathcal{V}_j} \subset \mathcal{I}_{\mathcal{V}_{i+j+1}}, \quad \text{for any } (i,j) \in \mathbb{N}^2; \tag{2.7}$$

indeed, the restriction to the generic fiber of any local section of $\mathcal{I}_{\mathcal{V}_i}.\mathcal{I}_{\mathcal{V}_j}$ is a section of $\mathcal{I}_{V_i}.\mathcal{I}_{V_j} = \mathcal{I}_{V_{i+j+1}}$. In particular

$$\mathcal{I}_{\mathcal{V}_0}.\mathcal{I}_{\mathcal{V}_i} \subset \mathcal{I}_{\mathcal{V}_{i+1}},$$

and the coherent sheaf $\mathcal{I}_{\mathcal{V}_i}/\mathcal{I}_{\mathcal{V}_{i+1}}$ may be identified with a coherent sheaf on $\mathcal{V}_0$, or equivalently, thanks to the isomorphism $\mathcal{P} : C \xrightarrow{\sim} \mathcal{V}_0$, with a coherent sheaf

$$\mathcal{J}_{i+1} := \pi_* \mathcal{I}_{\mathcal{V}_i}/\mathcal{I}_{\mathcal{V}_{i+1}}$$

[2] namely, a quasi-projective k-variety $\mathcal{X}$, equipped with a flat k-morphism $\pi : \mathcal{X} \to C$ and an isomorphism of its generic fiber $\mathcal{X}_K$ with X.

over C. Actually, the sheaves $\mathcal{J}_{i+1}$ are easily checked to be torsion free, and therefore may be identified with the sheaves of sections of some vector bundles J_{i+1} over C.

Recall that, if E is a vector bundle of positive rank on C, its *slope* is defined as the quotient

$$\mu(E) := \frac{\deg E}{\operatorname{rk} E},$$

and its *maximal slope* $\mu_{\max}(E)$ is the maximum of the slopes $\mu(F)$ of sub-vector bundles of positive rank in E. Observe that, if L is any line bundle on C,

$$\mu_{\max}(E \otimes L) = \mu_{\max}(E) + \deg L.$$

Moreover, if E_1 and E_2 are vector bundles over C, with E_2 of positive rank, and if there exists some (generically) injective morphism of vector bundles

$$\varphi : E_1 \longrightarrow E_2,$$

then the following slope inequality holds:

$$\deg (E_1) \leq \operatorname{rk} (E_1) \mu_{\max}(E_2). \tag{2.8}$$

We are now in position to formulate our algebraicity criterion:

Theorem 2.5 *With the notation above, if*

$$\limsup_{j \to +\infty} \frac{1}{j} \mu_{\max}(J_j) < 0, \tag{2.9}$$

then $\hat{V}$ is algebraic.

Observe that, if $\hat{V}$ extends to a formal subscheme $\hat{\mathcal{V}}$ of $\hat{\mathcal{X}}_{\mathcal{P}}$ that is smooth over C, then for any $j \in \mathbb{N}$, we have

$$J_j \simeq S^j(N_{\mathcal{P}}\hat{\mathcal{V}}^{\check{}}),$$

and the numerical condition (2.9) is equivalent to the ampleness[3] of the vector bundle $\mathcal{P}^* N_{\mathcal{P}} \hat{\mathcal{V}}$ over C. In general, we still have natural maps of vector bundles over C

$$S^j J_1 \longrightarrow J_j,$$

which are isomorphisms at the generic point $\operatorname{Spec} K$ of C. However, they are not always isomorphisms over C, and in general condition (2.9) is stronger than the ampleness of $\mathcal{P}^* N_{\mathcal{P}} \hat{\mathcal{V}}$.

Proof of Theorem 2.5: One easily checks that one may find a projective compactification of $\mathcal{X}$ to which the morphism π extends. Therefore, we may assume that $\mathcal{X}$ is indeed projective, and choose an ample line bundle $\mathcal{L}$ on it. Let L be its restriction $\mathcal{L}_K$ to X, and let E_D, E_D^i, η_D^i and γ_D^i be as in the previous section 2.2.

[3] See for instance [Laz01], part II, and its references for the basic results of the theory of ample vector bundles; see also [Bar71] in the positive characteristic case.

By replacing X by the Zariski closure Z of $\hat{V}$ in X and $\mathcal{X}$ by the closure $\mathcal{Z}$ of Z in $\mathcal{X}$ (which leaves the subschemes $\mathcal{V}_i$ and the sheaves $\mathcal{J}_i$ unchanged), we may also assume that $\hat{V}$ is Zariski dense in X, and therefore that, for any integer D, E^i_D is the zero subspace for i large enough.

We are going to show that, when condition (2.9) is satisfied, the "average value" of i/D, namely

$$\frac{\sum_{i\geq 0}(i/D)\mathrm{rk}\,(E^i_D/E^{i+1}_D)}{\sum_{i\geq 0}\mathrm{rk}\,(E^i_D/E^{i+1}_D)} = \frac{\sum_{i\geq 0}(i/D)\mathrm{rk}\,(E^i_D/E^{i+1}_D)}{\mathrm{rk}\,E_D}, \tag{2.10}$$

stays bounded when D goes to infinity. According to Lemma 2.4, this will prove that $\hat{V}$ is algebraic.

To achieve this, let us consider the direct images $\mathcal{E}_D := \pi_*\mathcal{L}^{\otimes D}$ and $\pi_{|\mathcal{V}_i\,*}\mathcal{L}^{\otimes D}$. These are torsion free coherent sheaves, or equivalently vector bundles, on C, which at the generic point $\mathrm{Spec}\,K$ of C coincide with the K-vector spaces E_D and $\Gamma(V_i, L^{\otimes D})$. Moreover, the restriction map $\eta^i_D : E_D \longrightarrow \Gamma(V_i, L^{\otimes D})$ extends to a morphism of vector bundles:

$$\begin{array}{cccc} \overline{\eta}^i_D : & \mathcal{E}_D & \longrightarrow & \pi_{|\mathcal{V}_i\,*}\mathcal{L}^{\otimes D} \\ & s & \longmapsto & s_{|\mathcal{V}_i}. \end{array}$$

The filtration $(E^i_D)_{i\geq 0}$ of E_D also extends to the filtration of $\mathcal{E}_D$ by the sub-vector bundles $\mathcal{E}^i_D := \ker \overline{\eta}^{i-1}_D$. Finally, the kernel of the restriction map from $\pi_{|\mathcal{V}_i\,*}\mathcal{L}^{\otimes D}$ to $\pi_{|\mathcal{V}_{i-1}\,*}\mathcal{L}^{\otimes D}$ may be identified with $\mathcal{J}_i \otimes \mathcal{P}^*\mathcal{L}^{\otimes D}$ and the restriction of the evaluation map $\overline{\eta}^i_D$ to $\mathcal{E}^i_D$ defines a morphism of vector bundles

$$\overline{\gamma}^i_D : \mathcal{E}^i_D \longrightarrow \mathcal{J}_i \otimes \mathcal{P}^*\mathcal{L}^{\otimes D},$$

which coincides with γ^i_D at the generic point of C. The kernel of $\overline{\gamma}^i_D$ is $\mathcal{E}^{i+1}_D$ and therefore $\overline{\gamma}^i_D$ factorizes through a (generically) injective morphism of vector bundles:

$$\tilde{\gamma}^i_D : \mathcal{E}^i_D/\mathcal{E}^{i+1}_D \longrightarrow \mathcal{J}_i \otimes \mathcal{P}^*\mathcal{L}^{\otimes D}.$$

Since $\mathcal{L}$ is ample, for D large enough, the sheaf $\mathcal{E}_D$ is generated by its global sections, and consequently:

$$\deg \mathcal{E}_D \geq 0. \tag{2.11}$$

Moreover, as $\mathcal{E}^i_D = \{0\}$ when $i >> 0$, we may write:

$$\deg \mathcal{E}_D = \sum_{i\geq 0} \deg\,(\mathcal{E}^i_D/\mathcal{E}^{i+1}_D). \tag{2.12}$$

Combined with the slope inequality 2.8 applied to the morphisms $\tilde{\gamma}^i_D$, the relations (2.11) and (2.12) and the identity

$$\mu_{\max}(\mathcal{J}_i \otimes \mathcal{P}^*\mathcal{L}^{\otimes D}) = \mu_{\max}(\mathcal{J}_i) + D\deg\,(\mathcal{P}^*\mathcal{L})$$

show that:

$$D\,\mathrm{rk}\,E_D \deg \mathcal{P}^*\mathcal{L} + \sum_{i\geq 0} \mathrm{rk}\,(E^i_D/E^{i+1}_D)\mu_{\max}(\mathcal{J}_i) \geq 0. \tag{2.13}$$

If we now assume that condition (2.9) is satisfied, then there exists $i_0 \in \mathbb{N}$ and $c > 0$ such that, for any integer $i \geq i_0$,

$$\mu_{\max}(\mathcal{J}_i) \leq -c\,i.$$

Therefore, from (2.13), we deduce that

$$D \operatorname{rk} E_D \deg \mathcal{P}^*\mathcal{L} + \sum_{0 \leq i < i_0} \operatorname{rk}(E_D^i/E_D^{i+1})(\mu_{\max}(\mathcal{J}_i) + ci) - c\sum_{i \geq 0} i \operatorname{rk}(E_D^i/E_D^{i+1}) \geq 0.$$

Since the second sum is bounded by

$$\sum_{0 \leq i < i_0} \operatorname{rk}(S^i \check{T}_{\hat{V}})(\mu_{\max}(\mathcal{J}_i) + ci),$$

which does not depend on D, this establishes that the ratio (2.10) is indeed bounded.

$\square_{\text{Theorem 2.5}}$

2.4 Application: positivity properties of Lie algebras of group schemes

In spite of the simplicity of its proof, the algebraization criterion in Theorem 2.5 has significant applications, for instance to algebraic foliations, as demonstrated by the recent work of Bogomolov and McQuillan [BM01]. In this section, we briefly describe how easily it leads to some basic positivity properties of Lie algebras of group schemes over a field of characteristic 0.

Let C be a smooth projective connected curve over a field k, and let $\pi : \mathcal{G} \to C$ be a smooth quasi-projective[4] group scheme over C. Let us denote by ε its zero-section, and by $\operatorname{Lie}\mathcal{G} := \varepsilon^* T_\pi$ its Lie algebra. It is a vector bundle over C, equipped with a $\mathcal{O}_C$-bilinear Lie bracket:

$$\begin{array}{rccc} [.,.]: & \operatorname{Lie}\mathcal{G} \otimes_{\mathcal{O}_C} \operatorname{Lie}\mathcal{G} & \longrightarrow & \operatorname{Lie}\mathcal{G} \\ & v_1 \otimes v_2 & \longmapsto & [v_1, v_2]. \end{array}$$

Observe that, by restricting to the generic point $\operatorname{Spec} K$ of C, one defines a bijection between the set of sub-vector bundles of $\operatorname{Lie}\mathcal{G}$ and the set of K-vector subspaces of $(\operatorname{Lie}\mathcal{G})_K$. Moreover, $(\operatorname{Lie}\mathcal{G})_K$ may be identified with the K-Lie algebra $\operatorname{Lie} G$ of the K-algebraic group $G := \mathcal{G}_K$, and a sub-vector bundle F of $\operatorname{Lie}\mathcal{G}$ is a bundle of Lie subalgebras (in other words, its sheaf of sections is closed under the above Lie bracket) iff F_K is a Lie sub-algebra of $\operatorname{Lie} G$.

Theorem 2.6 *Assume that k is a field of characteristic zero, and consider a Lie subalgebra F_K of $\operatorname{Lie} G$. If the corresponding sub-vector bundle F of $\operatorname{Lie}\mathcal{G}$ is ample, then F_K is an algebraic Lie subalgebra of $\operatorname{Lie} G$. Actually, it is the Lie algebra of a connected linear unipotent[5] K-algebraic subgroup H of G.*

[4] Actually, according to Raynaud [Ray70], Corollaire VI 2.6 et Théorème VIII 2, any smooth group scheme (of finite type) over a smooth curve is quasi-projective. Besides, a classical theorem of Cartier shows that, over a field of characteristic zero, any flat group scheme of finite type is smooth.

[5] Recall that, over a perfect field K, a (smooth) K-algebraic group G is called unipotent if, for some n, it is isomorphic to a closed subgroup of the algebraic subgroup of the subgroup $U(n)_K$ of $GL(n)_K$ of matrices

Since an abelian variety over a field of characteristic zero contains no non-trivial unipotent subgroup, this implies in particular:

Corollary 2.7 *Assume that k is a field of characteristic zero. If G is an abelian variety over K, then the vector bundle $\Omega^1_{\mathcal{G}/C} := (\mathrm{Lie}\,\mathcal{G})^\vee$ is semi-positive* (in other words, any quotient of $\Omega^1_{\mathcal{G}/C}$ has non-negative degree).

This corollary is indeed a classical result of Griffiths, which he established by transcendental techniques. (When $k = \mathbb{C}$, and $\mathcal{G}$ is an abelian scheme, this follows from the computation of the curvature of Hodge bundles; *cf.* [Gri70], Theorem 5.2. When $\mathcal{G}$ is semi-abelian, the singularities on the metric on $\Omega^1_{\mathcal{G}/C}$, seen as a Hodge bundle, at points of bad reduction of $\mathcal{G}$ are logarithmic, and consequently, the positivity properties of its curvature outside the divisor of bad reduction still allows to derive the semi-positivity of $\Omega^1_{\mathcal{G}/C}$. The general case follows by semi-stable reduction.) It seems plausible that Theorem 2.6 can be recovered by combining structure theorems on algebraic groups, the rigidity properties of reductive Lie groups, and Corollary 2.7. (The author confesses that he did not check it in detail.)

Observe that Corollary 2.7, and *a fortiori* Theorem 2.6, do not hold in general over a base field k of characteristic $p > 0$. As demonstrated by Moret-Bailly in [MB81], Proposition 3.1, counter-examples may be obtained from two supersingular elliptic curves E_1 and E_2 by considering the abelian surface $\mathcal{G}$ over C obtained by quotienting the abelian surface $\mathcal{A} := (E_1 \times E_2)_C$ over C by a subgroup scheme of $(\alpha_p \oplus \alpha_p)_C$ (which lies in $\mathcal{A}_{[p]}$) that is locally isomorphic to $\alpha_{p,C}$, but not constant.

Proof of Theorem 2.6: To prove the first assertion, we need to show that, if F is ample, then the germ of formal subvariety $\hat{V} := \hat{\mathrm{Exp}}_G F_K$ of $\hat{G}_e$ is algebraic. This is a straightforward consequence of Theorem 2.5 and of the subsequent observation, since $\hat{V}$ extends to a formal subscheme of the completion $\hat{\mathcal{G}}_\varepsilon$ of $\mathcal{G}$ along its zero section ε which is smooth over C, and the normal bundle to ε in $\hat{\mathcal{V}}$, pulled back to C by ε, may be identified with F. Indeed, we may consider the relative formal exponential map $\hat{\mathrm{Exp}}_{\mathcal{G}/C}$. It maps the completion of the total space $\mathbb{V}(\mathrm{Lie}\,\mathcal{G})$ of the vector bundle $\mathrm{Lie}\,\mathcal{G}$ along its zero section isomorphically onto the completion $\hat{\mathcal{G}}_\varepsilon$ of $\mathcal{G}$ along its unit section ε. Consequently, the image under $\hat{\mathrm{Exp}}_{\mathcal{G}/C}$ of the completion along its zero section of the total space $\mathbb{V}(\check{F})$ of the subbundle F of $\mathrm{Lie}\,\mathcal{G}$ provides the required extension.

For any vector bundle E over C, we shall denote $\mathbb{E}$ the commutative algebraic group over C defined by the total space $\mathbb{V}(\check{E})$ of this vector bundle equipped with the group law deduced from the additive structure of E.

We now turn to the special case of Theorem 2.6 when F_K is an *abelian* Lie subalgebra of $\mathrm{Lie}\,\mathcal{G}$, namely:

Lemma 2.8 *With the notation of Theorem 2.6, if the vector bundle is ample and if F_K is an abelian Lie subalgebra of* $\mathrm{Lie}\,G$ (*i.e.*, if $[.,.]$ vanishes on $F \otimes F$), *then there exists an*

$(g_{ij})_{1 \leq i,j \leq n}$ such that $g_{ii} = 1$ and $g_{ij} = 0$ if $i < j$. A connected K-algebraic group is unipotent iff there exists a composition series $G = G_0 \supset G_1 \supset \cdots \supset G_s = \{e\}$ by connected K-algebraic subgroups such that the quotients G_i/G_{i+1} are isomorphic to additive groups ${\mathbb{G}_a}^r$ over K. See for instance [Bor91], I.4 and V.15 for proofs and references.

embedding of algebraic group

$$j : \mathbb{F}_K \longrightarrow G,$$

the differential of which along ε *coincides with the inclusion map* $i_K : F_K \hookrightarrow \mathrm{Lie}\, G$.

Observe that Lemma 2.8 already implies Corollary 2.7.

Proof of Lemma 2.8: Consider the group scheme $\mathbb{F} \times_C \mathcal{G}$ over C. Its Lie algebra $F \oplus \mathrm{Lie}\, \mathcal{G}$ contains the graph of i as a Lie subalgebra, which moreover is ample. Therefore the graph of i_K is the Lie algebra of a connected algebraic subgroup H of $\mathbb{F}_K \times G$, which is easily shown to be the graph of a group embedding $j : \mathbb{F}_K \to G$.

$\square_{\text{Lemma 2.8}}$

In the sequel of the proof, we freely use the basic properties of the Harder-Narasimhan filtration and of the slopes of vector bundles on a curve over a field of characteristic zero.

To prove the second assertion in Theorem 2.6, we may assume that the vector bundle $\mathrm{Lie}\, \mathcal{G}$ has positive rank and consider its Harder-Narasimhan filtration:

$$E_0 = \{0\} \subset E_1 \subset \ldots \subset E_N = \mathrm{Lie}\, \mathcal{G}.$$

By definition, the quotient bundles E_i/E_{i-1}, $1 \leq i \leq N$, are semi-stable and their slopes

$$\mu_i := \mu(E_i/E_{i-1})$$

satisfy

$$\mu_1 > \ldots > \mu_N.$$

We also define

$$\begin{aligned} i_+ \; &:= \; \max\{i \in \{1, \ldots, n\} \mid \mu_i > 0\} \quad && \text{if } \mu_1 > 0 \\ &:= \; 0 && \text{if } \mu_1 = 0, \end{aligned}$$

and

$$E_+ := E_{i_+}.$$

Any sub-vector bundle of $\mathrm{Lie}\, \mathcal{G}$ which is ample is contained in E_+. Consequently, to complete the proof of Theorem 2.6, it is sufficient to show that $E_{+,K}$ *is the Lie algebra of a unipotent algebraic subgroup of* G. This will follow from the following Lemma, inspired by a similar observation by Shepherd-Barron ([SB92], Lemma 9.1.3.1):

Lemma 2.9 *(i) For any* $i \in \{1, \ldots, N\}$ *such that* $\mu_i \geq 0$, $E_{i,K}$ *is a Lie subalgebra of* $\mathrm{Lie}\, G$. *Moreover, for any element* $j \in \{1, \ldots, i\}$, $E_{j,K}$ *is a Lie ideal in* $E_{i,K}$.

(ii) For any $i \in \{1, \ldots, N\}$ *such that* $\mu_i > 0$, *the quotient Lie algebra* $E_{i,K}/E_{i-1,K}$ *is abelian.*

Indeed, combined with the first assertion of Theorem 2.6, Lemma 2.9 (i) shows that the K-vector spaces $E_{1,K} \subset \ldots \subset E_{i_+,K}$ are the Lie algebras of connected algebraic subgroups $H_1 \subset \ldots \subset H_{i_+}$ in G, and that $H_1, \ldots, H_{i_+-1}$ are normal subgroups of H_{i_+}. Moreover, Lemma 2.9 (ii) and Lemma 2.8 show that the algebraic groups H_1, H_2/H_1,$\ldots$,H_{i_+}/H_{i_+-1} are additive groups.

Proof of Lemma 2.9: Observe that, for any $i \in \{1, \dots, N\}$, the maximal slope of E/E_{i-1} is μ_i. Moreover, for any $(i,j) \in \{1, \dots, N\}^2$, $E_i/E_{i-1} \otimes E_j/E_{j-1}$ is semi-stable of slope $\mu_i + \mu_j$, and consequently the minimal slope of $E_i \otimes E_j$ is $\mu_i + \mu_j$.

For any $i \in \{1, \dots, N-1\}$, we may consider the following morphism of vector bundles over C:

$$\alpha_i : E_i \otimes E_i \hookrightarrow E \otimes E \xrightarrow{[.,.]} E \twoheadrightarrow E/E_i.$$

If $\mu_i \geq 0$, the minimal slope $2\mu_i$ of its source $E_i \otimes E_i$ is larger than the maximal slope μ_{i+1} of E/E_i, and therefore α_i is the zero morphism. This shows that, if $\mu_i \geq 0$, then $E_{i,K}$ is a Lie subalgebra of $\mathrm{Lie}\, G$.

The other assertions of Lemma 2.9 are similarly established by considering the morphisms:

$$\beta_{ij} : E_i \otimes E_j \hookrightarrow E \otimes E \xrightarrow{[.,.]} E \twoheadrightarrow E/E_j.$$

and

$$\gamma_i : E_i \otimes E_i \xrightarrow{[.,.]} E_i \twoheadrightarrow E_i/E_{i-1}.$$

$\square_{\text{Lemma 2.9 and Theorem 2.6}}$

3 The canonical semi-norm attached to a germ of analytic curve in a complex algebraic variety

3.1 The basic construction

Consider a complex algebraic variety X, a point P in X, and a germ C of smooth analytic curve through P in X. In this section, we describe a construction which attaches—in a canonical way—a semi-norm $\|.\|_{(X,P,C)}$ on the tangent line $T_P C$ to any such data (X, P, C). This construction focuses on the metric behavior of the evaluation maps η^i_D and γ^i_D already considered in the proof of the algebraicity criterion Proposition 2.2, and turns out to play a key role in the arithmetic algebraization theorem, Theorem 4.2 *infra*.

For a while, let us assume that X is complete and consider a line bundle L on X. Let us also choose a norm $\|.\|_0$ on the complex line $T_P C$ and a continuous hermitian metric $\|.\|$ on L. Then, for any non-negative integer D, we may consider the D-th tensor power of this hermitian metric on $L^{\otimes D}$ and the L^∞-norm $\|.\|_{L^\infty}$ it induces on the finite dimensional complex vector space

$$E_D := \Gamma(X, L^{\otimes D}).$$

For any non-negative integer i, we may also consider the norm $\|.\|_{i,D}$ on the complex line

$$\check{T}_P C^{\otimes i} \otimes L_P^{\otimes D}$$

deduced by duality and tensor product from the norm $\|.\|_0$ on $T_P C$ and the norm $\|.\|$ on L_P.

By applying the construction of “auxiliary functions” in section 2.2 to the formal germ of curve $\hat{V}$ through P defined by the germ of analytic curve C, we define subspaces

$$E_D^i := \{s \in E^D \mid s_{|C} \text{ has a zero of order } \geq i \text{ at } P\},$$

and evaluation maps

$$\gamma_D^i : E_D^i \longrightarrow S^i \check{T}_P C \otimes L_P^{\otimes D} \simeq \check{T}_P C^{\otimes i} \otimes L_P^{\otimes D}$$

which send a section of $L^{\otimes D}$ vanishing up to order $i-1$ at P along C to the i-th “Taylor coefficient” of its restriction to C. Finally we may consider the operator norm

$$\|\gamma_D^i\| := \max_{s \in E_D^i, \|s\|_{L^\infty} \leq 1} \|\gamma_D^i(s)\|_{i,D}$$

of γ_D^i with respect to the norms on E_D^i and $\check{T}_P C^{\otimes i} \otimes L_P^{\otimes D}$ considered above.

A straightforward application of Cauchy's inequalities establishes the existence of positive real numbers r and C such that, for any non-negative integers i and D,

$$\|\gamma_D^i\| \leq r^{-i} C^D. \tag{3.1}$$

Equivalently, if we let $a := \log r^{-1}$ and $b := \log C$, we have:

$$\log \|\gamma_D^i\| \leq ai + bD, \tag{3.2}$$

and consequently the upper limit

$$\rho(X, P, C, L) := \limsup_{\frac{i}{D} \to +\infty} \frac{1}{i} \log \|\gamma_D^i\| (= \lim_{x \to +\infty} \sup_{\frac{i}{D} \geq x} \frac{1}{i} \log \|\gamma_D^i\|) \tag{3.3}$$

belongs to $[-\infty, +\infty[$. Moreover one easily checks that it does not depend on the choice of the metric $\|.\|$ on L and that, if $\|.\|_0$ is replaced by $e^\lambda \|.\|_0$, then $\rho(X, P, C, L)$ is replaced by $\rho(X, P, C, L) - \lambda$. This shows that

$$\|\cdot\|_{(X,P,C,L)} := e^{\rho(X,P,C,L)} \|.\|_0 \tag{3.4}$$

is a semi-norm on the complex line $T_P C$ independent of the choices of the auxiliary metrics $\|.\|_0$ and $\|.\|$. It vanishes iff $\rho(X, P, C, L) = -\infty$.

The following properties of the semi-norm $\|.\|_{(X,P,C,L)}$ are simple consequences of its definition:

Lemma 3.1 *1) For any two line bundles L_1 and L_2 over X such that there exists a regular section of $\check{L}_1 \otimes L_2$ which does not vanish at P, we have:*

$$\|\cdot\|_{(X,P,C,L_1)} \leq \|\cdot\|_{(X,P,C,L_2)}. \tag{3.5}$$

2) For any line bundle L over X and any positive integer k,

$$\|\cdot\|_{(X,P,C,L^{\otimes k})} = \|\cdot\|_{(X,P,C,L)}. \tag{3.6}$$

Lemma 3.1 shows that, when X is projective, the set of semi-norms $\|.\|_{(X,P,C,L)}$ on $T_P C$ obtained by varying the line bundle L possesses one greatest element, namely the semi-norm $\|.\|_{(X,P,C,L)}$ where L is any ample line bundle on X. This greatest semi-norm will be called the *canonical semi-norm* on $T_P C$ and denoted $\|.\|_{(X,P,C)}$.

3.2 Birational invariance of the canonical semi-norm

It turns out that the construction of the canonical semi-norm may be extended to the situation where X is an arbitrary complex variety (not necessarily projective) and that it satisfies remarkable "functorial" properties. This will be a consequence of the following proposition.

Proposition 3.2 *Let $f : X' \to X$ be a morphism of complete complex algebraic varieties, and let P' be a point in X' and C' a germ of smooth complex analytic curve through P' in X' such that the restriction of f to C' defines an analytic isomorphism from C' onto a germ of smooth complex analytic curve C through $P := f(P')$*[6].

1) For any line bundle L on X, the isomorphism of complex lines

$$Df_{|C}(P') = Df(P')_{|T_{P'}C'} : T_{P'}C' \longrightarrow T_P C$$

satisfies, for any $v \in T_{P'}C'$:

$$\|Df(P')v\|_{(X,P,C,L)} \leq \|v\|_{(X',P',C',f^*L)}. \tag{3.7}$$

2) Moreover, equality holds in (3.7) if one of the following conditions holds:
i) the canonical morphism of sheaves

$$\mathcal{O}_X \longrightarrow f_*\mathcal{O}_{X'}$$

is an isomorphism;
ii) the line bundle L is ample and the canonical morphism of sheaves

$$\mathcal{O}_X \longrightarrow f_*\mathcal{O}_{X'}$$

is an isomorphism on some open neighborhood of P in X.

Observe that condition i) in 2) holds for instance when f is dominant (or equivalently surjective) with geometrically connected generic fiber and X is normal.

Proof : For any non-negative integer i, we shall denote C_i (resp. C'_i) the i-th infinitesimal neighbourhood of p (resp. P') in C (resp. C'). Let us choose a continuous hermitian metric on L and let us endow f^*L with this metric pulled back by f. Let us also choose some norms on the complex lines T_PC and $T_{P'}C'$ such that $Df_{|C}(P') : T_{P'}C' \longrightarrow T_PC$ is an isometry.

The inequality (3.7) will be obtained by examining the following commutative diagrams:

$$\begin{array}{ccc} E_D := \Gamma(X, L^{\otimes D}) & \xrightarrow{\varphi_D} & E'_D := \Gamma(X', f^*L^{\otimes D}) \\ \downarrow \eta^i_D & & \downarrow {\eta'}^i_D \\ \Gamma(C_i, L^{\otimes D}) & \xrightarrow{\sim} & \Gamma(C'_i, f^*L^{\otimes D}) \end{array}$$

[6] This condition is satisfied iff the tangent space T_PC is not contained in the kernel of the differential $Df(P)$ (which is a linear map between Zariski tangent spaces, from $T_{P'}X'$ to T_PX).

where the horizontal maps are defined by pulling back sections of $L^{\otimes D}$ by f, and where η_D^i and ${\eta'}_D^i$ denotes the restriction maps.

Indeed, these diagrams induce the following ones:

$$\begin{array}{ccc} E_D^i := \ker \eta_D^{i-1} & \xrightarrow{\varphi_D} & {E'}_D^i := \ker \eta_D^{i-1} \\ \Big\downarrow \gamma_D^i & & \Big\downarrow {\gamma'}_D^i \\ \check{T}_P C^{\otimes i} \otimes L_P^{\otimes D} & \xrightarrow{I_D^i} & \check{T}_{P'} C'^{\otimes i} \otimes f^* L_{P'}^{\otimes D}. \end{array} \tag{3.8}$$

As in the construction of $\|.\|_{(X,P,C,L)}$ described in the previous section, the metrics introduced above may be used to define the norms of γ_D^i and of ${\gamma'}_D^i$. Moreover, in the commutative diagram (3.8), the map φ_D decreases the L^∞-norms, while the map

$$I_D^i : \check{T}_P C^{\otimes i} \otimes L_P^{\otimes D} \longrightarrow \check{T}_{P'} C'^{\otimes i} \otimes f^* L_{P'}^{\otimes D} \simeq \check{T}_{P'} C'^{\otimes i} \otimes L_P^{\otimes D}$$

may be identified with ${}^t Df_{|C}(P')^{\otimes i} \otimes Id_{L_P^{\otimes D}}$, which is an isometry. This shows that

$$\|\gamma_D^i\| \leq \|{\gamma'}_D^i\|. \tag{3.9}$$

Using the definition of $\|.\|_{(X,P,C,L)}$ and $\|.\|_{(X',P',C',f^*L)}$ (see (3.3) and (3.4)), this yields (3.7).

To prove that equality holds in (3.7) when condition i) or ii) holds, first observe that these conditions imply that f is surjective, and therefore that the maps φ_D preserve the L^∞-norms.

Moreover, when condition i) is satisfied, the linear maps φ_D define isomorphisms

$$\varphi_D : E_D^i \longrightarrow {E'}_D^i,$$

and therefore equality holds in (3.9), hence in (3.7).

To prove the equality in case ii), one may consider the Stein factorization of f and write it as the composition of a morphism satisfying i) and of a finite morphism. Thus one is reduced to handle the case where L is ample and where f is a finite morphism that defines an isomorphism between open neighbourhoods of P in X and of P' in X'. In this case, as L and f^*L are ample, the metrics which appear in (3.7) are the canonical metrics $\|.\|_{(X,P,C)}$ and $\|.\|_{(X',P',C')}$, and we may replace L by any ample line bundle on X.

In particular, if $\mathcal{I}$ denotes the coherent ideal sheaf in $\mathcal{O}_X$ defined as the annihilator of the cokernel of the canonical morphism $\mathcal{O}_X \longrightarrow f_*\mathcal{O}_{X'}$, we may assume that there exists a section s_0 in $\Gamma(X, \mathcal{I}.L)$ which does not vanishes at P. Then for any non-negative integers i and D, and any $s' \in {E'}_D^i$, the product $f^* s_0 \otimes s'$ may be written $\varphi_D(s)$ with $s \in E_{D+1}^i$ and its image $s_0(P) \otimes {\gamma'}_D^i(s')$ by ${\gamma'}_{D+1}^i$ coincides with $I_{D+1}^i(s_0(P) \otimes \gamma_D^i(s))$. This shows that

$$\|{\gamma'}_D^i\| \leq \frac{\|s_0\|_{L^\infty(X)}}{\|s_0(P)\|} \|\gamma_{D+1}^i\|.$$

These estimates lead to the inequality opposite to (3.7).

$\Box_{\text{Proposition 3.2}}$

Corollary 3.3 *Let X and X' be two projective complex varieties and $i : U' \to U$ an isomorphism between Zariski open subsets U and U' of X and X' respectively. If P' is a point of X and C a germ of smooth analytic curve through P in X, and if $P := f(P')$ and $C := f(C')$, then the isomorphism*

$$Di(P') : T_{P'}C' \longrightarrow T_P C$$

satisfies:

$$\|Di(P')v\|_{(X,P,C)} = \|v\|_{(X',P',C')}, \text{ for any } v \in T_{P'}C'. \tag{3.10}$$

Proof : By considering the closure of the graph of i in $X \times X'$ and its projections to X and X', we see that to prove (3.10) we may assume that i is the restriction of some (birational) morphism $i : X' \to X$. Let us also choose ample line bundles L and L' on X and X' respectively.

By the equality case (ii) in Proposition 3.2, we have:

$$\|Di(P')v\|_{(X,P,C)} = \|Di(P')v\|_{(X,P,C,L)} = \|v\|_{(X',P',C',i^*L)}. \tag{3.11}$$

Besides, if k is a large enough positive integer, the line bundle $\check{L}' \otimes i^* L^{\otimes k}$ admits a regular section on X' which does not vanish at P'. (Indeed, $\Gamma(X', \check{L}' \otimes i^* L^{\otimes k})$ may be identified with $\Gamma(X, i_* \check{L}' \otimes L^{\otimes k})$ and $i_* \check{L}' \otimes L^{\otimes k}$ is generated by its global sections for $k >> 0$.) Therefore, by applying Lemma 3.1, 1) and 2), we get:

$$\|\cdot\|_{(X',P',C',L')} \leq \|\cdot\|_{(X',P',C',i^*L^{\otimes k})} = \|\cdot\|_{(X',P',C',i^*L)}.$$

This shows that

$$\|\cdot\|_{(X',P',C',i^*L)} = \|\cdot\|_{(X',P',C')}. \tag{3.12}$$

Finally, (3.10) follows from (3.11) and (3.12).

$\square_{\text{Corollary 3.3}}$

Let us now assume that the variety X is arbitrary, and consider some quasi-projective open neighbourhood U of P in X and some projective variety $\overline{U}$ containing U as an open subvariety. Corollary 3.3 shows that the canonical metric $\|\cdot\|_{(\overline{U},P,C)}$ on $T_P C$ is independent of the choices of U and $\overline{U}$, and we shall extend the previous definition of the canonical metric by letting:

$$\|\cdot\|_{(X,P,C)} := \|\cdot\|_{(\overline{U},P,C)}.$$

3.3 Functorial properties of the canonical semi-norm

We may now generalize the "functoriality properties" established in Proposition 3.2 and Corollary 3.3 when the ambient varieties are projective. Indeed, from these properties and the definition of the canonical semi-norm, it is straightforward to deduce assertions 1) and 2-i) in the following proposition:

Proposition 3.4 *Let X (resp. X') a complex algebraic variety, P (resp. P') a point in X (resp. X'), and C (resp. C') a germ of smooth analytic curve through P (resp. P') in X*

(resp. X'). Let also $f : X' \to X$ be a morphism of complex algebraic varieties such that $f(P') = P$ and $f_{|C'}$ is an analytic isomorphism from C' to C.

1) The isomorphism of complex lines

$$Df_{|C}(P') : T_{P'}C' \longrightarrow T_P C$$

satisfies, for any $v \in T_{P'}C'$:

$$\|Df(P')v\|_{(X,P,C)} \leq \|v\|_{(X',P',C')}. \tag{3.13}$$

2) Moreover, equality holds in (3.13) if one of the following conditions holds:
i) the morphism f defines an isomorphism from some open neighborhood of P' in X' onto some open neighborhood of P in X;
ii) the morphism f is an embedding.

To prove that equality holds in (3.13) when f is an embedding, we may assume that X and Y are projective. Then it is a consequence of the following proposition of independent interest, a stronger form of which is established in Appendix A at the end of this article.

Proposition 3.5 *Let X be a complex projective variety, Y a closed subvariety of X, L an ample line bundle over X, and $\|.\|$ an arbitrary continuous metric on L. There exists $C \in \mathbb{R}_+^*$ satisfying the following condition: for any positive large enough integer D and any $s \in \Gamma(Y, L^{\otimes D})$, there exists $\tilde{s} \in \Gamma(X, L^{\otimes D})$ such that*

$$\tilde{s}_{|Y} = s$$

and

$$\|\tilde{s}\|_{L^\infty} \leq C^D \|s\|_{L^\infty}.$$

3.4 Canonical semi-norm and capacity

Observe that, *if a germ C of smooth analytic curve* through a point P in a complex algebraic variety *is algebraic, then the canonical semi-norm $\|.\|_{(X,P,C)}$ on $T_P C$ vanishes.* Indeed the direct implication in the algebraicity criterion Proposition 2.2 shows that, if we assume—as we can—the variety X projective and if we denote by L an ample line bundle on X, then the evaluation maps considered in 3.1

$$\gamma_D^i : E_D^i \longrightarrow S^i \check{T}_P C \otimes L_P^{\otimes D} \simeq \check{T}_P C^{\otimes i} \otimes L_P^{\otimes D}$$

vanish if i/D is large enough; accordingly,

$$\rho(X, P, C, L) := \limsup_{\frac{i}{D} \to +\infty} \frac{1}{i} \log \|\gamma_D^i\| = -\infty.$$

In this section, we derive an upper bound on the canonical metric $\|.\|_{(X,P,C)}$ in terms of classical potential theoretic invariants of a Riemann surface "extending" C, which implies its vanishing when C is algebraic as a very special case. As will be clear in the proof, this lower bound is a geometric version of the classical Schwarz lemma, which plays a prominent role in transcendance and Diophantine approximation proofs.

We defer examples of analytic germs with non-trivial canonical semi-norms to section 4.5 *infra.*

3.4.1 Green functions and Riemann surfaces

Let us briefly recall some basic facts and introduce some notation concerning Green functions on Riemann surfaces. We refer the reader to the monographs [Tsu59],[Rum89], and [Ran95] and to [Bos99], 3.1 and Appendix, for proofs and additional information.

Let M be a connected Riemann surface and O a point of M. Consider a relatively compact domain Ω in M containing O with a non-empty and regular enough boundary $\partial\Omega$. Precisely, we assume that Ω has only regular boundary points in the sense of potential theory. (This condition is satisfied for instance if the non-empty compact set $\partial\Omega$ is locally connected without isolated points. Actually, it would be enough for the sequel to consider the case where Ω is the interior of some compact submanifold of codimension 0 with C^∞ boundary.) Then we may consider the *Green function*, or *equilibrium potential*, of P in Ω. It is the unique continuous function $g_{P,\Omega}$ on $M \setminus \{O\}$ satisfying the following three conditions:

EP1. It vanishes identically on $M \setminus \Omega$;

EP2. It is harmonic on $\Omega \setminus \{O\}$;

EP3. It possesses a logarithmic singularity at O; namely, if z denotes a local holomorphic coordinates on some open neighborhood U of P, we have:

$$g_{P,\Omega} = \log |z - z(O)|^{-1} + h \text{ on } U \setminus \{O\},$$

where h is an harmonic function on U.

The Green function $g_{O,\Omega}$ represents the electric field of a unit charge placed at the point O in the two-dimensional world modeled by M, when Ω (resp. $M \setminus \Omega$) is made of an insulating material (resp. of a conducting material wired to the earth). It is positive on $\Omega \setminus \{O\}$, and conditions **EP2** and **EP3** may be expressed as the equality of currents:

$$dd^c g_{O,\Omega} = -\frac{1}{2}\delta_O \text{ on } \Omega. \tag{3.14}$$

The value $h(P)$ of the function h in condition **EP3** may be interpreted as the *capacity* of $M \setminus \Omega$ with respect to P. Of course, this value depends on the choice of the local coordinate z. Intrinsically, we may define a "capacitary norm" $\|.\|^{cap}_{P,\Omega}$ on the complex line $T_O M = \mathbb{C}\frac{\partial}{\partial z}_{|P}$ by the equation:

$$\|\frac{\partial}{\partial z}_{|P}\|^{cap}_{P,\Omega} := e^{-h(P)} = \lim_{Q\to O} \frac{e^{-g_{O,\Omega}(Q)}}{|z(Q) - z(O)|}. \tag{3.15}$$

Let us now assume that M is not compact and consider an increasing sequence $(\Omega_n)_{n\in\mathbb{N}}$ of relatively compact domains of M containing O, with "regular" boundaries, such that

$M = \bigcup_{n\in\mathbb{N}} \Omega_n$. Then the sequence of Green functions $(g_{O,\Omega_n})_{n\in\mathbb{N}}$ is non-decreasing, and consequently the sequence of norms $(\|.\|^{cap}_{O,\Omega_n})_{n\in\mathbb{N}}$ is non-increasing.

Their limit behavior turns out to depend on the "type" of the Riemann surface M in the sense of the classical works of Myrberg-Nevanlinna-Ahlfors (see for instance [Ahl52] and [AS60], chapter IV). Recall that a connected Riemann surface S is said to be "parabolic" in the sense of Myrberg, or equivalently, to have "null boundary" in the sense of R. Nevanlinna, or to belong to the class O_G, when any negative subharmonic function on S is constant. This arises for instance when S is a complex (smooth connected) algebraic curve. Otherwise, S is said to be "hyperbolic", or to have "positive boundary". Using this terminology, the following alternative holds:

(1) If the Riemann surface M is hyperbolic, then the pointwise limit $g_{O,M}$ of $(g_{O,\Omega_n})_{n\in\mathbb{N}}$ is everywhere finite on $M\setminus\{O\}$. Moreover it is a positive harmonic function on $M\setminus\{O\}$, with a logarithmic singularity at O—indeed, $g_{O,M}$ is minimal amongst the functions satisfying these conditions, and, by definition, is the Green function of O in M. We may also define a capacitary norm $\|.\|^{cap}_{O,M}$ on $T_O M$ by the equality

$$\|\frac{\partial}{\partial z_{|P}}\|^{cap}_{O,\Omega} := \lim_{Q\to O} \frac{e^{-g_{O,M}(Q)}}{|z(Q)-z(O)|}. \tag{3.16}$$

This norm coincides with the limit $\lim_{n\to\infty} \|.\|^{cap}_{O,\Omega_n}$.

(2) If the Riemann surface M is "parabolic", then the point-wise limit of $(g_{O,\Omega_n})_{n\in\mathbb{N}}$ is everywhere $+\infty$ and

$$\lim_{n\to\infty} \|.\|^{cap}_{O,\Omega_n} = 0.$$

Then we let:

$$\|.\|^{cap}_{O,M} = 0.$$

To sum up, the "capacitary semi-norm" $\|.\|^{cap}_{O,M}$ on $T_O M$ always coincide with the limit $\lim_{n\to\infty} \|.\|^{cap}_{O,\Omega_n}$, and vanishes iff M is parabolic.

It is natural to extend the preceding discussion to the situation where M is compact (hence parabolic) by letting $\|.\|^{cap}_{O,M} = 0$ in that case also.

Observe finally that, if F is any closed polar subset of M (*e.g.*, a closed discrete subset) not containing O, then the semi-norms $\|.\|^{cap}_{O,M}$ and $\|.\|^{cap}_{O,M\setminus F}$ coincides. Indeed, the Riemann surfaces M and $M \setminus F$ have the same type, and, when they are hyperbolic, $g_{O,M\setminus F}$ is the restriction of $g_{O,M}$ to $M \setminus F$.

3.4.2 An upper bound on canonical semi-norms

As before, we consider a complex algebraic variety X, a point P in X, and a germ C of smooth analytic curve through P in X.

Let also M be a connected Riemann surface, O a point in M and

$$f : M \longrightarrow X$$

an analytic map which sends O to P and maps the germ of M at O to the germ C. (Thus f defines an analytic isomorphism from the germ of M at O onto the germ C, unless $Df(O)$ vanishes.)

Proposition 3.6 *For any v in $T_O M$, we have:*

$$\|Df(O)v\|_{(X,P,C)} \leq \|v\|^{cap}_{O,M}. \tag{3.17}$$

In particular, this proves:

Corollary 3.7 *If f maps the germ of M at O isomorphically onto the germ C at P and if the Riemann surface M is parabolic, then the canonical semi-norm $\|.\|_{(X,P,C)}$ vanishes.*

Applied to the normalization of the Zariski closure of C, this corollary shows again that the canonical semi-norm $\|.\|_{(X,P,C)}$ vanishes when the germ C is algebraic.

Observe that the capacitary norm at the origin on the open disk $D(0,1)$ is the "standard norm":

$$\|\frac{\partial}{\partial z}_{|0}\|^{cap}_{0,D(0,1)} = 1.$$

Indeed, the disk $D(0,1)$ is hyperbolic and

$$g_{0,D(0,1)}(z) = \log|z|^{-1} \quad \text{for any } z \in \mathbb{C}.$$

Therefore the special case of Proposition 3.6 where $(M,O) = (D(0,1),0)$ reads:

Corollary 3.8 *For any analytic map*

$$f : D(0,1) \longrightarrow X$$

which sends 0 to P and maps the germ of $\mathbb{C}$ at 0 to the germ C, we have:

$$\|Df(0)(\frac{\partial}{\partial z})\|_{(X,P,C)} \leq 1. \tag{3.18}$$

In more geometric terms, this estimate asserts that the canonical semi-norm $\|.\|_{(X,P,C)}$ on $T_P C$ is bounded from above by the Poincaré metric at P on any Riemann surface which"extends C and maps to X".

Proof of Proposition 3.6:. To establish (3.17), we may assume that M is not compact (by deleting one point if necessary) and then it is enough to prove that, for any relatively compact domain Ω in M with regular boundary containing O, the following inequality holds for any v in $T_O M$:

$$\|Df(O)v\|_{(X,P,C)} \leq \|v\|^{cap}_{O,\Omega}. \tag{3.19}$$

Clearly, we may also assume that $Df(O)$ is not zero (hence an isomorphism) and that X is projective.

To derive (3.19), we choose an ample line bundle L on X, a C^∞ hermitian metric $\|.\|$ on L, an holomorphic coordinate z on some open neighborhood of O in M that vanishes at O, and we define a norm $\|.\|_0$ on $T_P C$ by letting

$$\|Df(O)(\frac{\partial}{\partial z}_{|O})\|_0 = 1.$$

Finally, we choose a real valued C^∞ function ψ, defined on some open neighborhood of $\overline{\Omega}$ in M such that

$$\psi(O) = 0 \quad \text{and} \quad dd^c\psi \geq f^*c_1(\overline{L}) \quad \text{on } \overline{\Omega}.$$

(If h_1 and h_2 are two holomorphic functions vanishing at O defined on some open neighborhood of $\overline{\Omega}$ with disjoint ramification divisors, then we can take

$$\psi := C(|h_1|^2 + |h_2|^2)$$

for any large enough C in $\mathbb{R}_+^*$.)

Using these data, we may define E_D^i, γ_D^i, $\|\gamma_D^i\|$ and $\rho(X, P, C, L)$ as in section 3.1, and the inequality (3.19) may be rewritten as:

$$\rho(X, P, C, L) \leq \|\frac{\partial}{\partial z_{|O}}\|_{O,\Omega}^{cap}. \tag{3.20}$$

To prove (3.20), observe that, for any section s in E_D^i, we have:

$$\|\gamma_D^i(s)\| = \lim_{Q\to O} \frac{\|s \circ f(Q)\|}{|z(Q)|^i}.$$

Therefore, provided s does not vanish identically on $f(M)$, $\log \|\gamma_D^i(s)\|$ is the value at O of

$$\log \|s \circ f\| - i \log |z|,$$

which defines a locally integrable continuous function with values in $[-\infty, +\infty[$ on a neighborhood of O in M. This is also the value at O of the function

$$\log \|s \circ f\| + i(g_{O,\Omega} + \log \|\frac{\partial}{\partial z_{|O}}\|_{O,\Omega}^{cap}) + \frac{D}{2}\psi \tag{3.21}$$

from M to $[-\infty, +\infty[$, which indeed is subharmonic on Ω. This follows from the equality of currents on M

$$dd^c \log \|s \circ f\|^2 = \delta_{f^*\operatorname{div} s} - f^*c_1(\overline{L}),$$

from which we derive:

$$dd^c(\log \|s \circ f\|^2 + D\psi) \geq i\delta_O = -2i dd^c g_{O,\Omega} \quad \text{on } \Omega.$$

By the the maximum principle, $\log \|\gamma_D^i(s)\|$ is therefore not greater than the supremum of (3.21) on $\partial\Omega$. Since $g_{P,\Omega}$ vanishes on $\partial\Omega$, we finally get:

$$\log \|\gamma_D^i(s)\| \leq \log \|s\|_{L^\infty} + i \log \|\frac{\partial}{\partial z_{|O}}\|_{O,\Omega}^{cap} + \frac{D}{2} \max_{\partial\Omega} \psi.$$

This shows that

$$\frac{1}{i} \log \|\gamma_D^i\| \leq \log \|\frac{\partial}{\partial z_{|O}}\|_{O,\Omega}^{cap} + \frac{D}{2i} \max_{\partial\Omega} \psi$$

and yields (3.19).

$\square_{\text{Proposition 3.6}}$

4 Algebraicity criteria for smooth formal germs of subvarieties in algebraic varieties over number fields

In this section, we discuss some algebraization theorems concerning formal germs of subvarieties in algebraic varieties over number fields, which involve the canonical semi-norm studied in the previous paragraphs. These theorems are improvements of the main result of [Bos01] applied to formal germs of curves.

When dealing with number fields and p-adic fields, we will use the following notation and terminology.

If K is a number field, its ring of integers will be denoted $\mathcal{O}_K$ and the set of its finite places (or, equivalently, the set of non-zero prime ideals of $\mathcal{O}_K$, or of closed points of $\operatorname{Spec} \mathcal{O}_K$) will be denoted $V_f(K)$. For any $\mathbf{p}$ in $V_f(K)$, we let $\mathbf{F}_{\mathbf{p}}$ be the finite field $\mathcal{O}_K/\mathbf{p}$, $N\mathbf{p} := |\mathbf{F}_{\mathbf{p}}|$ the norm of $\mathbf{p}$, $K_{\mathbf{p}}$ (resp. $\mathcal{O}_{\mathbf{p}}$) the $\mathbf{p}$-adic completion of K (resp. of $\mathcal{O}_K$), and $|\ |_{\mathbf{p}}$ the $\mathbf{p}$-adic absolute value on $K_{\mathbf{p}}$ normalized in such a way that, for any uniformizing element ϖ in $\mathcal{O}_{\mathbf{p}}$, we have:

$$|\varpi|_{\mathbf{p}} = N\mathbf{p}^{-1};$$

equivalently, if p denotes the residue characteristic of $\mathbf{p}$ and e the absolute ramification index of $K_{\mathbf{p}}$, then:

$$|p|_{\mathbf{p}} = N\mathbf{p}^{-e} = p^{-[K_{\mathbf{p}}:\mathbb{Q}_p]}.$$

If Λ is an $\mathcal{O}_{\mathbf{p}}$-lattice in some finite dimensional $K_{\mathbf{p}}$-vector space E, the $\mathbf{p}$-adic norm $\|\ \|$ on E attached to Λ is defined by the equality

$$\|\sum_{i=1}^{n} x_i e_i\| := \max_{1\leq i\leq n} |x_i|_{\mathbf{p}},$$

for any $\mathcal{O}_{\mathbf{p}}$-basis $(e_1, \ldots, e_n)$ of Λ and any $(x_1, \ldots, x_n) \in K^n$.

4.1 Sizes of formal subschemes over p-adic fields

We now recall some constructions from [Bos01], to which we refer for details and proofs.

Let k be a p-adic field (*i.e.*, a finite extension of $\mathbb{Q}_p$), $\mathcal{O}$ its subring of integers (*i.e.*, the integral closure of $\mathbb{Z}_p$ in k), $|\ | : k \to \mathbb{R}_+$ its absolute value, and $\mathbb{F}$ its residue field. (Actually we might assume more generally that k is any field equipped with a complete non-Archimedean absolute value $|\ | : k \to \mathbb{R}_+$ and let $\mathcal{O} := \{t \in k \mid |t| \leq 1\}$ be its valuation ring.)

4.1.1 Groups of formal and analytic automorphisms

If $g := \sum_{I\in\mathbb{N}^d} a_I X^I$ is a formal power series in $k[[X_1, \cdots, X_d]]$ and if $r \in \mathbb{R}_+^*$, we define

$$\|g\|_r := \sup_I |a_I| r^{|I|} \in \mathbb{R}_+ \cup \{+\infty\}.$$

The "norm" $\|g\|_r$ is finite iff the series g is convergent and bounded on the open d-dimensional ball of radius r.

The group Aut $\hat{\mathbf{A}}^d_k$ of automorphisms of $\hat{\mathbf{A}}^d_k$, the formal completion at the origin of the d-dimensional affine space over k, may be identified with the space of d-tuples $f = (f_i)_{1\leq i\leq d}$ of formal series $f_i \in k[[x_1, \cdots, x_d]]$ such that $f(0) = 0$ and $Df(0) := \left(\frac{\partial f_i}{\partial x_j}(0)\right)_{1\leq i,j\leq d}$ belongs to $GL_n(k)$. We shall consider the following subgroups of Aut $\hat{\mathbf{A}}^d_k$:

- the subgroup G_{for} formed by the formal automorphisms f such that $Df(0)$ belongs to $GL_n(\mathcal{O})$;
- the subgroup G_ω formed by the elements $f := (f_i)_{1\leq i\leq d}$ of G_{for} such that the series f_i have positive radii of convergence;
- for any $r \in \mathbb{R}^*_+$, the subgroup $G_\omega(r)$ of G_ω formed by the elements $f := (f_i)_{1\leq i\leq d}$ of G_{for} such that the series $(f_i)_{1\leq i\leq d}$ satisfy the bounds $\|f_i\|_r \leq r$. This group may be seen as the group of analytic automorphisms, preserving the origin, of the open d-dimensional ball of radius r. Moreover,

$$r' > r > 0 \Rightarrow G_\omega(r') \subset G_\omega(r) \quad \text{and} \quad \bigcup_{r>0} G_\omega(r) = G_\omega.$$

4.1.2 The size $R(\hat{V})$ of a formal germ $\hat{V}$

The filtration $(G_\omega(r))_{r>0}$ of the group G_ω will now be used to attach a number $R(\hat{V})$ in $[0, 1]$ to any smooth formal germ $\hat{V}$ in an algebraic variety over k, which will provide some quantitative measure of its analyticity.

Let $\hat{V}$ be a formal subscheme of $\hat{\mathbf{A}}^d_k$. For any φ in Aut $\hat{\mathbf{A}}^d_k$, we may consider its inverse image $\varphi^*(\hat{V})$, which is again a formal subscheme of $\hat{\mathbf{A}}^d_k$. Moreover, the following conditions are equivalent:

1. $\hat{V}$ is a smooth formal scheme of dimension v.

2. There exists φ in Aut $\hat{\mathbf{A}}^d_k$ such that $\varphi^*(\hat{V})$ is the formal subscheme $\hat{\mathbf{A}}^v_k \times \{0\}$ of $\hat{\mathbf{A}}^d_k$.

3. There exists φ in G_{for} such that $\varphi^*(\hat{V})$ is the formal subscheme $\hat{\mathbf{A}}^v_k \times \{0\}$ of $\hat{\mathbf{A}}^d_k$.

Similarly, the following two conditions are equivalent:

1. $\hat{V}$ is the formal scheme attached to some germ at 0 of smooth analytic subspace of dimension v of the d-dimensional affine space over k.

2. There exists φ in G_ω such that $\varphi^*(\hat{V})$ is the formal subscheme $\hat{\mathbf{A}}^v_k \times \{0\}$ of $\hat{\mathbf{A}}^d_k$.

When they are satisfied, we shall say that the formal germ $\hat{V}$ is *analytic and smooth.*

These observations lead to define the *size of a smooth formal subscheme* $\hat{V}$ of dimension v of $\hat{\mathbf{A}}^d_k$ as the supremum $R(\hat{V})$ in $[0, 1]$ of the real numbers $r \in]0, 1]$ for which there exists φ in $G_\omega(r)$ such that $\varphi^*(\hat{V})$ is the formal subscheme $\hat{\mathbf{A}}^v_k \times \{0\}$ of $\hat{\mathbf{A}}^d_k$. It is positive iff $\hat{V}$ is analytic.

More generally, if $\mathcal{X}$ is an $\mathcal{O}$-scheme of finite type equipped with a section $P \in \mathcal{X}(\mathcal{O})$ and if $\hat{V}$ is a smooth formal subscheme of the formal completion $\hat{X}_{P_K}$ of $X := \mathcal{X}_K$ at P_K, then the *size* $R_\mathcal{X}(\hat{V})$ *of* $\hat{V}$ *with respect to the model* $\mathcal{X}$ of X may be defined as the size of $i(\hat{V})$, where $i : U \hookrightarrow \mathbf{A}^d_\mathcal{O}$ is an embedding of some open neighbourhood U in $\mathcal{X}$ of the section P into an affine space of large enough dimension d, which moreover maps P to the origin $0 \in \mathbf{A}^d_{\mathcal{O}}(\mathcal{O})$. This definition is independent of the choices of U, d, and i, and extends the previous one.

When $\hat{V}$ is a smooth germ of analytic curve, we shall define a p-adic norm on the tangent line $T_P\hat{V}$ by letting:

$$\|.\|_{(\mathcal{X},\mathcal{P},\hat{V})} := R_{\mathcal{X}}(\hat{V})^{-1}\|.\|_0,$$

where $\|.\|_0$ denotes the p-adic norm on $T_P\hat{V}$ which makes the differential $Di(P) : T_P\hat{V} \to T_0\mathbf{A}^d_k \simeq k^d$ isometric when k^d is equipped with the "standard" p-adic norm, the unit ball of which is $\mathcal{O}^d$.

Observe that, if $\hat{V}$ extends to a formal subscheme $\hat{\mathcal{V}}$ of the formal completion of $\mathcal{X}$ along $\mathcal{P}$ which is smooth along $\mathcal{P}$, then $R_{\mathcal{X}}(\hat{V}) = 1$. If moreover $\hat{V}$ is a formal germ of curve, the norm $\|.\|_{(\mathcal{X},\mathcal{P},\hat{V})}$ on $T_P\hat{V}$ is therefore the p-adic norm attached to its $\mathcal{O}$-lattice defined by the normal bundle of $\mathcal{P}$ in $\hat{\mathcal{V}}$.

4.1.3 Sizes of solutions of algebraic differential equations

It is possible to establish lower bounds on the sizes of formal germs of solutions of algebraic ordinary differential equations. These play a key role in the application of our arithmetic algebraization criterion to the solutions of algebraic differential equations over number fields (see [Bos01], 2.2 and 3.4.3, and *infra*, 4.6.

Proposition 4.1 *Let $\mathcal{X}$ be a smooth scheme over* Spec $\mathcal{O}$*, $\mathcal{P}$ a section in $\mathcal{X}(\mathcal{O})$, and $\mathcal{F}$ a sub-vector bundle of rank 1 in $T_{\mathcal{X}/\mathcal{O}}$. Let $X := \mathcal{X}_k$, $P := \mathcal{P}_k$, and $F := \mathcal{F}_k$, and let $\hat{V}$ be the formal germ of curve in $\hat{X}_P$ defined by integration of the (involutive) line bundle F in T_X.*

1) The size $R(\hat{V})$ of $\hat{V}$ with respect to $\mathcal{X}$ satisfies the lower bound:

$$R(\hat{V}) \geq |\pi| := |p|^{\frac{1}{p-1}}. \tag{4.1}$$

2) If moreover k is absolutely unramified and if the reduction $\mathcal{F}_{\mathbb{F}} \hookrightarrow T_{\mathcal{X}_{\mathbb{F}}}$ of $\mathcal{F}$ to the closed fiber $\mathcal{X}_{\mathbb{F}}$ of $\mathcal{X}$ is closed under p-th power, then

$$R(\hat{V}) \geq |p|^{\frac{1}{p(p-1)}}. \tag{4.2}$$

This is proved in [Bos01], Proposition 3.9, with the exponent $3/p^2$ instead of $1/p(p-1)$ in (4.2); however, a closer inspection of the proof shows that indeed it holds with the exponent $1/p(p-1)$.

Observe that the lower bound (4.1) is basically optimal, as demonstrated by the differential system

$$\mathcal{X} := \mathbb{A}^2_{\mathcal{O}}, \;\; \mathcal{P} := (0,0), \;\; \text{and } \mathcal{F} := \left(\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial y}\right)\mathcal{O}_{\mathbb{A}^2}.$$

Indeed, then $\hat{V}$ is the formal germ

$$\mathrm{Graph}(x \mapsto \exp x - 1),$$

the size of which is not larger than the radius of convergence $|\pi|$ of the exponential series.

Observe also that, after exchanging the two coordinates, $\hat{V}$ may also be seen as the graph of the series

$$\log(1+t) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} t^n,$$

whose radius of convergence is 1. This shows that the size $R(\mathrm{Graph}(\phi))$ of the graph of some formal series ϕ may be strictly smaller than its radius of convergence.

4.2 Normed and semi-normed lines over number fields

We define a *normed line*

$$\overline{L} := (L_K, (\|.\|_{\mathbf{p}}), (\|.\|_\sigma))$$

over a number field K as the data of a rank one K-vector space L_K, of a family $(\|.\|_{\mathbf{p}})$ of $\mathbf{p}$-adic norms on the $K_{\mathbf{p}}$-lines $L_K \otimes_K K_{\mathbf{p}}$ indexed by the non-zero prime ideals $\mathbf{p}$ of $\mathcal{O}_K$, and of a family $(\|.\|_\sigma)$ of hermitian norms on the complex lines $L_K \otimes_{K,\sigma} \mathbb{C}$, indexed by the fields embeddings $\sigma : K \hookrightarrow \mathbb{C}$. Moreover the family $(\|.\|_\sigma)$ is required to be stable under complex conjugation. (The data of these families of norms is equivalent to the data of a family $(\|.\|_v)_v$, indexed by the set of all places v of K, of v-adic norms on the rank one vector spaces $L_v := L_K \otimes_K K_v$ over the v-adic completions K_v of K.)

If $\overline{L}$ and $\overline{M}$ are normed lines over K, then we will denote by $\check{\overline{L}}$ (resp. by $\overline{L} \otimes \overline{M}$) the normed line over K defines by the K-line $\check{L_K} := Hom_K(L_K, K)$ (resp. the K-line $L_K \otimes_K M_K$) equipped with the $\mathbf{p}$-adic and hermitian norms deduced by duality (resp. by tensor product) from the ones defining $\overline{L}$ (resp. $\overline{L}$ and $\overline{M}$).

We shall say that a normed K-line is *summable* if, for some (or equivalently, for any), non-zero element l of L_K, the family of real numbers $(\log \|l\|_{\mathbf{p}})_{\mathbf{p}}$ is summable. Then we may define its *Arakelov degree* as the real number

$$\widehat{\deg}\, \overline{L} := \sum_{\mathbf{p}} \log \|l\|_{\mathbf{p}}^{-1} + \sum_{\sigma} \log \|l\|_\sigma^{-1}. \tag{4.3}$$

Indeed, by the product formula, the right-hand side of (4.3) does not depend on the choice of l.

If $\overline{L}$ and $\overline{M}$ are summable normed lines over K, then the normed K-lines $\check{\overline{L}}$ and $\overline{L} \otimes \overline{M}$ also are summable. Moreover, as a straightforward consequence of the definition of the Arakelov degree, we have:

$$\widehat{\deg}\, \check{\overline{L}} = -\widehat{\deg}\, \overline{L} \tag{4.4}$$

and

$$\widehat{\deg}\, \overline{L} \otimes \overline{M} = \widehat{\deg}\, \overline{L} + \widehat{\deg}\, \overline{M}. \tag{4.5}$$

Observe that hermitian line bundles over $\operatorname{Spec} \mathcal{O}_K$, as usually defined in Arakelov geometry (see for instance [Bos01], 4.1.1) provide examples of normed lines over K. Namely, if $\overline{\mathcal{L}} = (\mathcal{L}, (\|.\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}}))$ is such an hermitian line bundle—so $\mathcal{L}$ is a projective $\mathcal{O}_K$-module of rank 1, and $(\|.\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$ is a family, invariant under complex conjugation, of norms on the complex lines $\mathcal{L}_\sigma := \mathcal{L} \otimes_{\sigma:\mathcal{O}_K \to \mathbb{C}} \mathbb{C}$—the corresponding normed K-line is $\mathcal{L}_K$ equipped with the p-adic norms defined by the $\mathcal{O}_{\mathbf{p}}$-lattices $\mathcal{L} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathbf{p}}$ in $\mathcal{L} \otimes_{\mathcal{O}_K} K_{\mathbf{p}} \simeq L \otimes_K K_{\mathbf{p}}$ and with the hermitian norms $(\|.\|_\sigma)$. The normed lines so-defined are summable, and their Arakelov degree, as defined by (4.3), coincide with the usual Arakelov degree of hermitian line bundles.

It is convenient to extend the definitions of normed lines and Arakelov degree as follows: we shall define a *semi-normed K-line* $\overline{L}$ as a rank one K-vector space L_K equipped with families of *semi-norms* $(\|.\|_{\mathbf{p}})$ and $(\|.\|_\sigma)$, where the latter is assumed to be stable under complex conjugation. (In other words, we allow some of the $\|.\|_{\mathbf{p}}$ or $\|.\|_\sigma$ to vanish.) We shall say that *the Arakelov degree* of a semi-normed K-line $\overline{L}$ *is defined* if, for some (or equivalently, for any), non-zero element l of L_K, the family of real numbers $(\log^+ \|l\|_{\mathbf{p}})_{\mathbf{p}}$ is summable. Then we may again define its Arakelov degree by means of (4.3), where we follow the usual convention

$$\log 0^{-1} = +\infty.$$

It is a well defined element of $]-\infty, +\infty]$. The definition of the tensor product of normed K-lines immediately extends to semi-normed K-lines. Moreover, if two semi-normed K-lines have well defined Arakelov degrees, then their tensor product also and the additivity relation (4.5) still holds. (Observe however that the duality relation (4.4) makes sense only for summable normed K-lines.)

4.3 An arithmetic algebraization theorem

We are now in position to state an arithmetic analogue of the algebraization criterion of section 2.3, which concerns germs of formal curves in algebraic varieties over number fields:

Theorem 4.2 *Let X be a quasi-projective variety over a number field K, P a point in $X(K)$ and $\hat{V}$ a germ of smooth formal curve in X through P that is analytic at every place*[7].

Let $\mathcal{X}$ be a model of X, quasi-projective over $\operatorname{Spec} \mathcal{O}_K$*, such that P extends to a section $\mathcal{P}$ in $\mathcal{X}(\mathcal{O}_K)$, and let $\bar{t}$ be the semi-normed K-line defined by the tangent line $T_P\hat{V}$ equipped with the* $\mathbf{p}$*-adic norms $\|.\|_{(\mathcal{X}_{\mathcal{O}_{\mathbf{p}}}, \mathcal{P}_{\mathcal{O}_{\mathbf{p}}}, \hat{V}_{\mathcal{O}_{\mathbf{p}}})}$ and the hermitian semi-norms $\|.\|_{(X_\sigma, P_\sigma, \hat{V}_\sigma)}$.*

If the Arakelov degree of $\bar{t}$ is well-defined and if

$$\widehat{\deg}\, \bar{t} > 0, \tag{4.6}$$

then the formal germ $\hat{V}$ is algebraic.

Observe that, conversely, if $\hat{V}$ is any algebraic smooth formal germ through a rational point in an algebraic variety over a number field, then it is analytic at every place, almost all its $\mathbf{p}$-adic sizes are equal to 1 [8], and its complex canonical semi-norms vanish. In particular, the Arakelov degree of $\bar{t}$ is well defined, and assumes the value $+\infty$.

[7]Recall that this means that $\hat{V}$ is a one-dimensional smooth formal subscheme of $\hat{X}_P$ such that, for any non-zero prime ideal $\mathbf{p}$ in $\mathcal{O}_K$ (resp. any field embedding $\sigma : K \hookrightarrow \mathbb{C}$), the smooth formal curve $\hat{V}_{K_{\mathbf{p}}}$ (resp. $\hat{V}_\sigma$) in $X_{K_{\mathbf{p}}}$ (resp. X_σ) is indeed $K_{\mathbf{p}}$-analytic (resp. $\mathbb{C}$-analytic).

[8]Actually, for any model $\mathcal{X}$ of X over $\mathcal{O}_K$, there is a non-empty subscheme $\operatorname{Spec} \mathcal{O}_K[1/N]$ and a section $\mathcal{P} \in \mathcal{X}(\mathcal{O}_K[1/N])$ such $\hat{V}$ extends to a formal subscheme of $\mathcal{X}$ along $\mathcal{P}$ that is smooth over $\mathcal{O}_K[1/N]$.

Corollary 4.3 *Let $\hat{V}$ be a smooth formal germ of curve through a rational point P in an algebraic variety X over a number field K, analytic at every place.*

Let us denote by $(R_{\mathbf{p}})$ the family of $\mathbf{p}$-adic sizes of $\hat{V}$, defined with respect to some model $\mathcal{U}$ over $\mathcal{O}_K$ of an open neighborhood U of P in X such that P extends to an integral point $\mathcal{P} \in \mathcal{U}(\mathcal{O}_K)$, and suppose that the following conditions are satisfied:

1) the product

$$\prod_{\mathbf{p} \in \operatorname{Spec} \mathcal{O}_{\mathbf{K}} \setminus \{(\mathbf{0})\}} R_{\mathbf{p}},$$

which is a well-defined number in $[0, 1]$, is positive;

2) for at least one embedding $\sigma : K \hookrightarrow \mathbb{C}$, the canonical semi-norm $\|.\|_{(X_\sigma, P_\sigma, \hat{V}_\sigma)}$ vanishes.

Then the formal germ $\hat{V}$ is algebraic.

Observe that condition 1) is actually independent of the choice of U and $\mathcal{U}$. Moreover, condition 2) is satisfied if , for some embedding σ, there exists a parabolic Riemann surface M, a point O in M and an analytic map $f : M \longrightarrow X_\sigma(\mathbb{C})$ which defines an isomorphism from the formal germ of M at O to $\hat{V}_\sigma$. In this way, we recover the main result of [Bos01] (Theorem 3.4) for one-dimensional formal germs as a special instance of Theorem 4.2.

Corollary 4.3 is a straightforward consequence of Theorem 4.2: after possibly shrinking U and changing $\mathcal{U}$, we may assume that $\mathcal{U}$ is quasi-projective over $\mathcal{O}_K$ and apply Theorem 4.2 to $\mathcal{X} = \mathcal{U}$; indeed, condition 1) shows that $\widehat{\deg}\, \bar{t}$ is well defined, and condition 2) that its value is $+\infty$.

Theorem 4.2 and Corollary 4.3 are in the same spirit as the algebraization theorems for formal germs of D.V. and G.V. Chudnovsky ([CC85a], Section 5, and [CC85b], Theorem 1.2) and André ([And89], Chapter VIII, especially Theorem 1.2, [And99], Theorem 2.3.1, and [And02], Theorem 5.4.3), which however are technically somewhat different.

4.4 Proof of the algebraization theorem

The proof of Theorem 4.2 is similar to the proof of the algebraization criterion over function fields, Theorem 2.5. It constitutes a refined variant of the proof of the main result (Theorem 3.4) in [Bos01], and, like the latter, it relies on some simple inequalities relating slopes of hermitian vector bundles and heights of linear maps, for which we refer to [Bos01], 4.1.

In the sequel, we freely use the basic definitions and results concerning hermitian vector bundles, slopes and height of linear maps which are recalled in *loc. cit.*

4.4.1 Auxiliary hermitian vector bundles and linear maps

Observe that $\mathcal{X}$ may be imbedded, as a scheme over $\operatorname{Spec} \mathcal{O}_K$, into some projective space $\mathbb{P}^N_{\mathcal{O}_K}$. By replacing $\mathcal{X}$ by its closure in $\mathbb{P}^N_{\mathcal{O}_K}$, we may assume that it is projective. We may also assume that $\hat{V}$ is Zariski dense in X, by replacing $\mathcal{X}$ by the closure in $\mathbb{P}^N_{\mathcal{O}_K}$ of the Zariski closure Z of $\hat{V}$ in X considered in section 2.1. Observe that these reductions leave unchanged the (semi-)norms defining the generalized hermitian line bundle $\bar{t}$. (For the

p-adic norms, this follows from the independence of the size of a formal germ with respect to the imbedding i used to define it; for the archimedean canonical semi-norms, this follows from Proposition 3.4. Actually, we could avoid to rely on this non-trivial Proposition by not assuming that $\hat{V}$ is Zariski dense. This would only require more complicated notation and minor modifications in the proof below.)

Let us also choose the following additional data:

– an hermitian line bundle $\overline{\mathcal{L}} := (\mathcal{L}, \|.\|_{\overline{\mathcal{L}}})$ on $\mathcal{X}$ such that $L := \mathcal{L}_K$ is ample on $X := \mathcal{X}_K$;

– a positive Lebesgue measure μ on $X(\mathbb{C})$, invariant under complex conjugation (see [Bos01], 4.1.3);

– a family $(\|.\|_{0,\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}$, invariant under complex conjugation, of norms on the complex lines $(T_P\hat{V}_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}$.

Using these data, we may define:

– for any positive integer D, the direct image $\mathcal{E}_D := \pi_*\mathcal{L}^{\otimes D}$ of $\mathcal{L}^{\otimes D}$ by the structural morphism $\pi : \mathcal{X} \to \operatorname{Spec}\mathcal{O}_K$. (In other words, $\mathcal{E}_D$ is the locally free coherent sheaf on $\operatorname{Spec}\mathcal{O}_K$ associated to the $\mathcal{O}_K$-module $\Gamma(\mathcal{X}, \mathcal{L}^{\otimes D})$.)

– the L^2-norms $(\|.\|_{L^2,\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}$ on the finite dimensional complex vector spaces

$$\mathcal{E}_{D,\sigma} \simeq \Gamma(X_\sigma(\mathbb{C}), L_\sigma^{\otimes D})$$

associated to the measure $\mu_{X_\sigma(\mathbb{C})}$ and the D-th tensor power of the given metric $\|.\|_{\overline{\mathcal{L}}}$ on L_σ. By endowing $\mathcal{E}_D$ with these hermitian norm, we obtain an hermitian vector bundle $\overline{\mathcal{E}}_D$.

– a normed K-line $\bar{t}_0$, associated to an hermitian line bundle over $\operatorname{Spec}\mathcal{O}_K$, by endowing the K-line $T_P\hat{V}$ with the archimedean norms $(\|.\|_{0,\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}$ and with its naive $\mathcal{O}_K$-structure[9] defined from the model $\mathcal{X}$ of X.

Theorem 4.2 will be established by applying the algebraicity criterion involving evaluation maps established in section 2.2 (see Proposition 2.2 and Lemma 2.4), and we define E_D, E_D^i, η_D^i, and γ_D^i as in this section. Observe that $E_D := \Gamma(X, L^{\otimes D})$ may be identified with $\mathcal{E}_{D,K}$. Moreover, since $\hat{V}$ is Zariski dense in X, for any given D, the evaluation map

$$\eta_D^i : E_D := \Gamma(X, L^{\otimes D}) \longrightarrow \Gamma(V_i, L^{\otimes D})$$

is injective—and therefore E_D^{i+1} vanishes—provided i is large enough. In particular,

$$\sum_{i\geq 0} \operatorname{rk}(E_D^i/E_D^{i+1}) = \operatorname{rk} E_D.$$

For any $\mathbf{p} \in \operatorname{Spec}\mathcal{O}_K \setminus \{0\}$, the size of the formal germ $\hat{V}_{K_\mathbf{p}}$ with respect to the model $\mathcal{X}_{\mathcal{O}_\mathbf{p}}$ will be denoted $R_\mathbf{p}$. Since the Arakelov degree of $\bar{t}$ is well defined, the series with

[9] In other words, for any $\mathbf{p} \in \operatorname{Spec}\mathcal{O}_K \setminus \{0\}$, the $\mathbf{p}$-adic norm on $T_P\hat{V} \otimes_K K_\mathbf{p}$ defining $\bar{t}_0$ is the norm $\|.\|_0$ considered at the end of section 4.1.2. Equivalently, the $\mathcal{O}_K$-submodule of $\check{T}_P\hat{V}$ defining the integral structure of the dual hermitian vector bundle $\check{\bar{t}}_0$ is given by the image of the composite map

$$\mathcal{P}^*\Omega^1_{\mathcal{X}/\mathcal{O}_K} \to (\mathcal{P}^*\Omega^1_{\mathcal{X}/\mathcal{O}_K})_K \simeq \Omega^1_{X/K,P} \to \check{T}_P\hat{V}.$$

positive terms

$$\sum_{\mathbf{p}\in \operatorname{Spec}\mathcal{O}_K\setminus\{0\}} \log R_{\mathbf{p}}^{-1}$$

has a finite sum.

By definition, the canonical semi-norm on $T_P\hat{V}_\sigma$ is given by

$$\|.\|_{(X_\sigma,P_\sigma,\hat{V}_\sigma)} = \exp\left(\limsup_{i/D\to+\infty} \frac{1}{i}\log\|\gamma_D^i\|_\sigma\right)\|.\|_{0,\sigma},$$

where $\|\gamma_D^i\|_\sigma$ denotes the operator norm of

$$\gamma^i_{D,\sigma} : E^i_{D,\sigma} \longrightarrow \check{T}_P\hat{V}_\sigma^{\otimes i}\otimes L_{P_\sigma}^{\otimes D}$$

when the source space is equipped with the L^∞-norm and the range space with the norm deduced by tensor product from the norms $\|.\|_{0,\sigma}$ on $T_P\hat{V}_\sigma$ and $\|.\|_{\overline{\mathcal{L}}}$ on L_{P_σ}. As a matter of fact, we could—and, in the sequel, we shall—use the L^2-metric on $E^i_{D,\sigma}$ (namely, the restriction of the one on $E_{D,\sigma} = \mathcal{E}_{D,\sigma}$ considered above), and still define the same canonical semi-norm. Indeed, the logarithm of the ratio of the L^∞ and L^2 norms on $E_{D,\sigma}$ is $O(D)$ when D goes to infinity (see for instance [Bos01], 4.1.3).

The very definitions of the normed lines $\bar{t}$ and $\bar{t}_0$ and of their Arakelov degree show that the latter satisfy the following relation:

$$\widehat{\deg}\,\bar{t} = \widehat{\deg}\,\bar{t}_0 + \sum_{\mathbf{p}\in \operatorname{Spec}\mathcal{O}_K\setminus\{0\}} \log R_{\mathbf{p}} - \sum_{\sigma:K\hookrightarrow\mathbb{C}} \limsup_{i/D\to+\infty}\frac{1}{i}\log\|\gamma_D^i\|_\sigma. \tag{4.7}$$

Consequently, as $\widehat{\deg}\,\bar{t}$ belongs to $]0,+\infty]$, there exists positive real numbers λ and d such that, for any $(D,i)\in\mathbb{N}_{>0}\times\mathbb{N}$ satisfying $i>\lambda D$,

$$\widehat{\deg}\,\bar{t}_0 + \sum_{\mathbf{p}\in \operatorname{Spec}\mathcal{O}_K\setminus\{0\}} \log R_{\mathbf{p}} - \sum_{\sigma:K\hookrightarrow\mathbb{C}} \frac{1}{i}\log\|\gamma_D^i\|_\sigma \geq d. \tag{4.8}$$

4.4.2 Application of the slope inequalities

We are going to show that the ratio

$$\frac{\sum_{i\geq\lambda D}(i/D)\operatorname{rk}(E_D^i/E_D^{i+1})}{\operatorname{rk}E_D} \tag{4.9}$$

stays bounded when D goes to infinity. According to Lemma 2.4 and (2.5), this will prove that $\hat{V}$ is algebraic. As in [Bos01], our main tool will be the *slope inequalities* applied to the evaluation morphisms

$$\eta_D^n : E_D := \mathcal{E}_{D,K} \longrightarrow \Gamma(V_n, L^{\otimes D}).$$

Specifically, if n is so large that η_D^n is injective, the slope inequalities of *loc. cit.*, Proposition 4.6, applied to the hermitian vector bundle $\overline{\mathcal{E}}_D$, the linear map η_D^n, and the filtration of $\Gamma(V_n, L^{\otimes D})$ by the order of vanishing read as the following estimates (compare [Bos01], (4.18)):

$$\widehat{\mu}(\overline{\mathcal{E}}_D) \leq \frac{1}{\operatorname{rk} E_D} \sum_{i \geq 0} \operatorname{rk}\left((E_D^i/E_D^{i+1}\right) \left[\widehat{\deg}\,(\check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^*\overline{\mathcal{L}}^{\otimes D}) + [K:\mathbb{Q}]h(\overline{\mathcal{E}}_D^i, \check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^*\overline{\mathcal{L}}^{\otimes D}, \gamma_D^i)\right]. \tag{4.10}$$

The left hand side of (4.10) is the slope of $\overline{\mathcal{E}}_D$:

$$\widehat{\mu}(\overline{\mathcal{E}}_D) := \frac{\widehat{\deg}\,(\overline{\mathcal{E}}_D)}{\operatorname{rk} E_D}.$$

Recall also that $h(\overline{\mathcal{E}}_D, \check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^*\overline{\mathcal{L}}^{\otimes D}, \gamma_D^i)$ denotes the height of the linear map γ_D^i. By definition, it is given by the sum of the "local norms" of γ_D^i:

$$[K:\mathbb{Q}]h(\overline{\mathcal{E}}_D^i, \check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^*\overline{\mathcal{L}}^{\otimes D}, \gamma_D^i) = \sum_{\mathbf{p} \in \operatorname{Spec} \mathcal{O}_K \setminus \{0\}} \log \|\gamma_D^i\|_{\mathbf{p}} + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|\gamma_D^i\|_\sigma,$$

where the archimedean norm $\|\gamma_D^i\|_\sigma$ has the same meaning as above, and where the $\mathbf{p}$-adic norm $\|\gamma_D^i\|_{\mathbf{p}}$ is defined as the operator norm of

$$\gamma_{D,K_{\mathbf{p}}}^i : E_{D,K_{\mathbf{p}}}^i \longrightarrow \check{T}_P \hat{V}_{K_{\mathbf{p}}}^{\otimes i} \otimes L_{P,K_{\mathbf{p}}}^{\otimes D},$$

defined by using the $\mathbf{p}$-adic norm on $E_{D,K_{\mathbf{p}}}^i$ (resp. on $\check{T}_P \hat{V}_{K_{\mathbf{p}}}^{\otimes i} \otimes L_{P,K_{\mathbf{p}}}^{\otimes D}$) defined by the lattice $\mathcal{E}_{D,\mathcal{O}_{\mathbf{p}}}^i$ (resp. by $\mathcal{P}^*(\check{t}_0^{\otimes i} \otimes \mathcal{L}^{\otimes D})_{\mathcal{O}_{\mathbf{p}}}$).

As shown in [Bos01], Proposition 4.4, the left hand side of (4.10) satisfies the following lower bound, where c denotes some positive constant, and D any natural integer:

$$\widehat{\mu}(\overline{\mathcal{E}}_D) \geq -cD. \tag{4.11}$$

To derive an upper bound on the right hand side of (4.10), first observe that

$$\widehat{\deg}\,(\check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^*\overline{\mathcal{L}}^{\otimes D}) = -i\,\widehat{\deg}\,\overline{t}_0 + D\,\widehat{\deg}\,\overline{\mathcal{L}}. \tag{4.12}$$

To estimate the height of γ_D^i, recall that, from the definition of the $\mathbf{p}$-adic sizes $R_{\mathbf{p}}$, it follows that, for any $\mathbf{p} \in \operatorname{Spec} \mathcal{O}_K \setminus \{0\}$,

$$\|\gamma_D^i\|_{\mathbf{p}} \leq R_{\mathbf{p}}^{-1}$$

(see [Bos01], Lemma 3.3 and 4.9). Consequently,

$$[K:\mathbb{Q}]\, h(\overline{\mathcal{E}}_D^i, \check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^*\overline{\mathcal{L}}^{\otimes D}, \gamma_D^i) \leq i \sum_{\mathbf{p} \in \operatorname{Spec} \mathcal{O}_K \setminus \{0\}} \log R_{\mathbf{p}}^{-1} + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|\gamma_D^i\|_\sigma \tag{4.13}$$

Moreover, the archimedean norms $\|\gamma_D^i\|_\sigma$ satisfy the Cauchy type estimates (3.1) and (3.2). Therefore, there exist constants α and β, such that, for any non-negative inetegers D and i:

$$\sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|\gamma_D^i\|_\sigma \leq \alpha i + \beta D. \tag{4.14}$$

From (4.13) and (4.14), we already derive the existence of some constant $c(\lambda)$ such that, for any natural integer D,

$$\frac{1}{\operatorname{rk} E_D} \sum_{0 \leq i \leq \lambda D} \operatorname{rk}((E_D^i/E_D^{i+1}) \left[-i \,\widehat{\deg}\, \overline{t}_0 + [K:\mathbb{Q}] h(\overline{\mathcal{E}}_D^i, \check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^* \overline{\mathcal{L}}^{\otimes D}, \gamma_D^i)\right] \leq c(\lambda) D. \tag{4.15}$$

The slope inequality (4.10), combined with the lower bound (4.11) on its left hand side and with (4.12) and (4.15), leads to the estimate:

$$-cD \leq D \,\widehat{\deg}\, \mathcal{P}^* \overline{\mathcal{L}} + c(\lambda) D + \frac{1}{\operatorname{rk} E_D} \sum_{i > \lambda D} \operatorname{rk}((E_D^i/E_D^{i+1}) \left[-i \,\widehat{\deg}\, \overline{t}_0 + [K:\mathbb{Q}] h(\overline{\mathcal{E}}_D^i, \check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^* \overline{\mathcal{L}}^{\otimes D}, \gamma_D^i)\right]. \tag{4.16}$$

Moreover, (4.8) and (4.13) show that, if $i > \lambda D$, then

$$-i \,\widehat{\deg}\, \overline{t}_0 + [K:\mathbb{Q}] h(\overline{\mathcal{E}}_D, \check{\overline{t}}_0^{\otimes i} \otimes \mathcal{P}^* \overline{\mathcal{L}}^{\otimes D}, \gamma_D^i) \leq -id.$$

Together with (4.16), this leads to the upper bound

$$\frac{\sum_{i \geq \lambda D} (i/D) \operatorname{rk}(E_D^i/E_D^{i+1})}{\operatorname{rk} E_D} \leq \frac{c + c(\lambda) + \widehat{\deg}\, \mathcal{P}^* \overline{\mathcal{L}}}{d}$$

and concludes the proof.

4.5 Analytic germs with positive canonical semi-norms

In this section, we apply our algebraization theorem to investigate the canonical semi-norm associated to a germ of smooth analytic curve in the affine plane $\mathbb{A}^2(\mathbb{C})$.

We may restrict to analytic germs C through the origin $(0,0)$ in $\mathbb{A}^2(\mathbb{C})$, the restriction to which of the first projection

$$\begin{array}{rcl} \mathbb{A}^2 & \longrightarrow & \mathbb{A}^1 \\ (z_1, z_2) & \longmapsto & z_1 \end{array}$$

is étale. These germs are exactly the germs of the form

$$C_\varphi := \operatorname{Graph}(\varphi),$$

where

$$\varphi(z) = \sum_{n=1}^{+\infty} a_n z^n$$

is a complex formal series with positive radius of convergence. For any such germ, we let

$$v_\varphi := \frac{\partial}{\partial z_1} + \varphi'(0) \frac{\partial}{\partial z_2}.$$

It a basis vector of the complex line $T_{(0,0)} C_\varphi$.

Observe that, according to Corollary 3.8, for any such series φ of radius of convergence at least 1, we have:

$$\|v_\varphi\|_{(\mathbb{A}^2,(0,0),C_\varphi)} \leq 1.$$

Moreover, as observed in section 3.4,

$$\|v_\varphi\|_{(\mathbb{A}^2,(0,0),C_\varphi)} = 0$$

when C_φ, or equivalently φ, is algebraic.

Besides, if the coefficients a_n of the series φ are integers, then the formal germ $\hat{V}$ through the origin in $\mathbb{A}^2_{\mathbb{Q}}$ defined as the graph of φ seen as a formal series is analytic at every place. Actually it is straightforward to check that, for any prime number p, the p-adic size of this germ, computed with respect to the model $\mathbb{A}^2_{\mathbb{Z}_p}$ is 1 and that, with the notation of Theorem 4.2 applied with $K = \mathbb{Q}$, $X = \mathbb{A}^2_Q$, and $\mathcal{X} = \mathbb{A}^2_{\mathbb{Z}}$, we have:

$$\widehat{\deg}\, \bar{t} = -\log \|v_\varphi\|_{(\mathbb{A}^2,(0,0),C_\varphi)}.$$

According to Theorem 4.2, the germ C_φ is therefore algebraic if

$$\|v_\varphi\|_{(\mathbb{A}^2,(0,0),C_\varphi)} < 1.$$

These observations establish the following proposition:

Proposition 4.4 *If $\varphi(z) = \sum_{n=1}^{+\infty} a_n z^n$ is an element, vanishing at* 0*, of the ring $\mathcal{R}$ of formal series with integer coefficients whose complex radius of convergence is ≥ 1, then either (i) φ is algebraic and $\|v_\varphi\|_{(\mathbb{A}^2,(0,0),C_\varphi)} = 0$, or (ii) φ is not algebraic and $\|v_\varphi\|_{(\mathbb{A}^2,(0,0),C_\varphi)} = 1$.*

It is not difficult to prove that, in case (i), the series φ is actually the expansion of a function in $\mathbb{Q}(z)$[10]. We shall not use this fact in the sequel.

Observe that the set of algebraic elements of $\mathcal{R}$, vanishing at 0, is infinite countable (indeed the Zariski-closure in $\mathbb{A}^2_{\mathbb{C}}$ of a germ C_φ with $\varphi \in \mathbb{Q}[[z]]$ is defined over $\mathbb{Q}$). Therefore the set of series of type (ii) in Proposition 4.4 constitute a set with the power of the continuum. Explicit elements of this set are provided by lacunary series such as

$$\varphi(z) := \sum_{k=0}^{+\infty} z^{2^k},$$

or, more generally, by the series

$$\varphi_{\mathbf{n}}(z) := \sum_{k=0}^{+\infty} z^{n_k},$$

[10]This a special case of Proposition 2.1 and Corollary 2.2 in [Har88], which may be established as follows. For any holomorphic function φ over the open unit disk $D(0,1)$ that is algebraic over $\mathbb{C}[t]$, there is a non-zero polynomial Q in $\mathbb{C}[t]$ such that $Q\varphi$ is integral over $\mathbb{C}[t]$, and therefore extends to a continuous function on the closed disk $\overline{D}(0,1)$. In particular the coefficients of the Taylor expansion of $Q\varphi$ at 0 converge to 0. If moreover f belongs to $\mathcal{R}$, then Q may be chosen in $\mathbb{Z}[t]$, and consequently these coefficients belong to $\mathbb{Z}$, and only a finite number of them does not vanish.

where $\mathbf{n} = (\mathbf{n_k})_{\mathbf{k}\in\mathbb{N}}$ is a sequence of positive integers such that

$$\inf_{k\in\mathbb{N}} \frac{n_{k+1}}{n_k} > 1. \tag{4.17}$$

Indeed, according to a classical theorem of Hadamard, the holomorphic functions on the unit disc $D(0,1)$ defined by such series admit the full circle $\partial D(0,1)$ as natural boundary, and therefore cannot be algebraic.

Observe also that, for any polynomial P in $\mathbb{C}[z]$, vanishing at 0, the automorphism

$$\begin{array}{cccc} T_P: & \mathbb{A}^2_{\mathbb{C}} & \longrightarrow & \mathbb{A}^2_{\mathbb{C}} \\ & (z_1, z_2) & \longmapsto & (z_1, z_2 + P(z_1)) \end{array}$$

of $\mathbb{A}^2_{\mathbb{C}}$ transforms the germ C_φ into the germ $C_{P+\varphi}$, and its differential $DT_P(0,0)$ maps v_φ to $v_{P+\varphi}$. In particular,

$$\|v_{P+\varphi}\|_{(\mathbb{A}^2,(0,0),C_{P+\varphi})} = \|v_\varphi\|_{(\mathbb{A}^2,(0,0),C_\varphi)}.$$

In particular, for any $P \in \mathbb{C}[z]$ and any non-algebraic element of $\mathcal{R}$ vanishing at 0,

$$\|v_{P+\varphi}\|_{(\mathbb{A}^2,(0,0),C_{P+\varphi})} = 1.$$

This construction shows in particular that, amongst the series φ holomorphic on the unit disk, the ones such that $\|v_\varphi\|_{(\mathbb{A}^2,(0,0),C_\varphi)} = 1$ are dense in the topology of uniform convergence on compact subsets of $D(0,1)$.

4.6 Application to differential equations

We finally discuss how our algebraicity criterion Theorem 4.2 may be applied to ordinary differential equations.

As in the situation C described in the introduction, we consider a smooth variety X over a number field K, a point P in $X(K)$ and a sub-vector bundle F of rank 1 of the tangent bundle $T_{X/K}$, and we are interested in the algebraicity of the formal germ of integral curve $\hat{V}$ through P.

The conjecture of Grothendieck-Katz has been generalized to possibly non-linear differential systems by Ekedahl, Shepherd-Barron, and Taylor ([ESBT99]) as the following question:

With the notation above, does the condition GK—which asserts that almost all the reductions of F modulo a prime ideal $\mathbf{p}$ of $\mathcal{O}_K$ are closed under the p-th power map—*imply the algebraicity of* $\hat{V}$ *?*

Observe that the formal germ $\hat{V}$ is analytic at every place and that, when moreover the condition GK is satisfied, we may apply the lower bound (4.2) to $\hat{V}_{K_\mathbf{p}}$ for almost every non-zero prime ideal $\mathbf{p}$ in $\mathcal{O}_K$. Therefore, under this assumption, the sizes $R_\mathbf{p}$ defined as in Corollary 4.3 satisfy the following lower bounds:

$$R_\mathbf{p} > 0 \text{ for every } \mathbf{p} \in \operatorname{Spec} \mathcal{O}_K \setminus \{0\}$$

and

$$R_{\mathbf{p}} \geq p^{-\frac{[K_{\mathbf{p}}:\mathbb{Q}_p]}{p(p-1)}} \text{ for almost every } \mathbf{p} \in \operatorname{Spec} \mathcal{O}_K \setminus \{0\}.$$

(As usual, p denotes the residue characteristic of $\mathbf{p}$.) In particular, the product

$$\prod_{\mathbf{p} \in \operatorname{Spec} \mathcal{O}_{\mathbf{K}} \setminus \{(\mathbf{0})\}} R_{\mathbf{p}}$$

is positive. Together with Corollary 4.3, this establishes the following:

Proposition 4.5 *If a sub-line bundle F of the tangent bundle T_X of smooth variety X over a number field K satisfies the condition GK, then its formal germ of integral curve through a point P in $X(K)$ is algebraic if (and only if), for at least one embedding $\sigma : K \hookrightarrow \mathbb{C}$, the canonical semi-norm $\|.\|_{(X_\sigma, P_\sigma, \hat{V}_\sigma)}$ vanishes.*

Consequently, the conjecture of Grothendieck-Katz and its non-linear generalization leads us to wonder wether *the canonical semi-norm attached to a germ of integral curve of a complex algebraic differential equation always vanishes.*

It seems quite sensible to expect that this is true for *linear* differential equations. According to Proposition 4.5, this would establish the original conjecture of Grothendieck-Katz.

A Appendix: extensions of sections of large powers of ample line bundles

A.1 Recall that a continuous metric $\|.\|$ on a line bundle L over an analytic space X is called *positive* if, for any trivializing section s of L over an open subset U of X, the function $\log \|s\|^{-1}$ is strongly plurisubharmonic on U.

In this Appendix, we prove the following sharp version of Proposition 3.5, concerning line bundles equipped with positive metrics :

Theorem A.1 *Let X be a complex projective variety, Y a closed subvariety of X, L an ample line bundle over X, and $\|.\|$ a positive metric on L. There exist an integer $D_0 \geq 0$ and, for any $\varepsilon > 0$, a positive real number C_ε satisfying the following condition: for any integer $D \geq D_0$ and any $s \in \Gamma(Y, L^{\otimes D})$, there exists $\tilde{s} \in \Gamma(X, L^{\otimes D})$ such that*

$$\tilde{s}_{|Y} = s$$

and

$$\|\tilde{s}\|_{L^\infty(X)} \leq C_\varepsilon e^{\varepsilon D} \|s\|_{L^\infty(Y)}. \tag{A.1}$$

Since the validity of Proposition 3.5 does not depend on the choice of the metric on L, and since any ample line bundle on a projective variety admits a positive metric, Theorem A.1 implies Proposition 3.5.

Observe that, besides the proof of Proposition 3.4, Theorem A.1 also possesses applications to Arakelov geometry, in the study of heights of cycles and subschemes (*cf.* [Zha95], [Ran02]). Actually, similar results have been established in the literature by means of L^2 estimates *à la* Hörmander. However, they are often less precise, and require some smoothness hypothesis on X and Y (see for instance [Man93], [Zha95] Theorem 2.2, [Dem00], [Ran02] section 3.1.1). The proof that we present in this Appendix is based instead on the classical finiteness results of Grauert on strictly pseudo-convex domains (in the spirit of the proof of Satz 2 in [Gra62], p. 343) and the Banach open mapping theorem, and allows us to handle singular varieties as well.

A.2 Specifically, we shall use the following theorem of Grauert, which he established in course of his famous solution of the Levi problem ([Gra58], Proposition 4, p. 466; in this paper, Grauert considers only analytic manifolds, however, as observed in [Gra62], p. 344, the proof immediately extends to analytic spaces):

Theorem A.2 *Let M be a reduced complex analytic space and Ω a relatively compact open subset of M, with strictly pseudo-convex boundary. For any coherent analytic sheaf $\mathcal{F}$ on Ω, the cohomology group $H^1(\Omega; \mathcal{F})$ is finite dimensional.*

Besides, we shall use the following version of the open mapping theorem (see for instance [Bou81], I.28 exercice 4), and I.19 Corollaire 3):

Theorem A.3 *Let E and F be two Fréchet spaces and $u : E \longrightarrow F$ a continuous linear map.*

If $\operatorname{coker} u := F/u(E)$ *is finite dimensional, then* $u(E)$ *is closed in* F *and the map* $u : E \longrightarrow u(E)$ *is open. In particular, for any continuous semi-norm* p *on* E*, there exists a continuous semi-norm* q *on* F *satisfying the following condition: for any* y *in* $u(E)$*, there exists* x *in* E *such that*

$$u(x) = y$$

and

$$p(x) \leq q(y).$$

A.3 In the sequel, the algebra of analytic functions on some complex analytic space M will be denoted $\mathcal{O}^{\mathrm{an}}(M)$.

Let X, Y, L, and $\| \ \|$ be as in the statement of Theorem A.1, and let $\| \ \|$ also denote the metric on $\check{L}$ dual to the metric $\| \ \|$ on L. We may consider the total spaces $\mathbb{V}(X, L)$ and $\mathbb{V}(Y, L)$ of the line bundle $\check{L}$ over X and Y, and, for any $r \in \mathbb{R}_+^*$, the disk bundles

$$D(X, r) \hookrightarrow \mathbb{V}(X, L)(\mathbb{C})$$

and

$$D(Y, r) \hookrightarrow \mathbb{V}(Y, L)(\mathbb{C}),$$

formed by the elements v in the fibers of $\check{L}$ such that $\|v\| < r$. These are relatively compact open subsets of the analytic spaces $\mathbb{V}(X, L)(\mathbb{C})$ and $\mathbb{V}(Y, L)(\mathbb{C})$, and their boundary is strongly pseudo-convex, as a consequence of the positivity of the metric $\| \ \|$ on L.

We shall also denote by $D(X)$ (resp. $D(Y)$) the unit disk bundle $D(X, 1)$ (resp. $D(Y, 1)$). Observe that the closed embedding of complex algebraic varieties

$$i : \mathbb{V}(Y, L) \hookrightarrow \mathbb{V}(X, L)$$

restricts to a closed embedding of analytic spaces

$$j : D(Y) \hookrightarrow D(X).$$

For any $r \in]0, 1[$, we denote $\| \ \|_{X,r}$ (resp. $\| \ \|_{Y,r}$) the norm $\| \ \|_{L^\infty(D(X,r))}$ (resp. the norm $\| \ \|_{L^\infty(D(Y,r))}$) on $\mathcal{O}^{\mathrm{an}}(D(X))$ (resp. on $\mathcal{O}^{\mathrm{an}}(D(Y))$). The family of norms $(\| \ \|_{X,r})_{r\in]0,1[}$ (resp. $(\| \ \|_{Y,r})_{r\in]0,1[}$) defines the natural Fréchet space structure on $\mathcal{O}^{\mathrm{an}}(D(X))$ (resp. on $\mathcal{O}^{\mathrm{an}}(D(Y))$).

The spaces $\mathbb{V}(X, L)$ and $\mathbb{V}(Y, L)$ are equipped with natural $\mathbb{G}_m$-actions, defined by the action of homotheties on fibers of $\check{L}$, and the imbedding i is $\mathbb{G}_m$-equivariant. These actions restrict to analytic actions of

$$U(1) := \{u \in \mathbb{C} \mid |u| = 1\}$$

on $D(X)$ and $D(Y)$, and, for any integer k, we shall define $\mathcal{O}^{\mathrm{an}}(D(X))_k$ as the subspace of $\mathcal{O}^{\mathrm{an}}(D(X))$ consisting of the analytic functions f on $D(X)$ such that, for any $u \in U(1)$ and any $z \in D(X)$,

$$f(uz) = u^k f(z).$$

One defines a projection

$$p_{X,k} : \mathcal{O}^{\mathrm{an}}(D(X)) \longrightarrow \mathcal{O}^{\mathrm{an}}(D(X))_k$$

by letting

$$p_{X,k}(f)(z) := \int_0^1 e^{-2\pi i k t} f(e^{2\pi i t} z) dt.$$

It is continuous; indeed, for any $r \in]0,1[$ and any $f \in \mathcal{O}^{\mathrm{an}}(D(X))$,

$$\|p_{X,k}(f)\|_{X,r} \leq \|f\|_{X,r}. \tag{A.2}$$

Observe also that $\mathcal{O}^{\mathrm{an}}(D(X))_k$ may be identified with the vector space $\Gamma(X, L^{\otimes k})$ of algebraic regular—or equivalently, of analytic—sections of $L^{\otimes k}$, by means of the map which sends $s \in \Gamma(X, L^{\otimes k})$ to the analytic function f on $D(X)$ defined by

$$f(z) := \langle s(\pi(z)), z^{\otimes k} \rangle \quad \text{for any } z \in D(X).$$

(Observe that $s(\pi(z))$ belongs to the complex line $L^{\otimes k}_{\pi(z)}$, and $z^{\otimes k}$ to the dual line $\check{L}^{\otimes k}$.) Moreover, with the above notation, the norms of f and s are related by:

$$\|f\|_{X,r} := r^k \|s\|_{L^\infty(X)}. \tag{A.3}$$

Similarly, we may define a subspace $\mathcal{O}^{\mathrm{an}}(D(Y))_k$ of $\mathcal{O}^{\mathrm{an}}(D(Y))$, and a projection

$$p_{Y,k} : \mathcal{O}^{\mathrm{an}}(D(Y)) \longrightarrow \mathcal{O}^{\mathrm{an}}(D(Y))_k.$$

The subspace $\mathcal{O}^{\mathrm{an}}(D(Y))_k$ may be identified with $\Gamma(Y, L^{\otimes k})$, and (A.3) still holds (with Y instead of X). Moreover, for any $f \in \mathcal{O}^{\mathrm{an}}(D(X))$,

$$p_{Y,k}(f_{|D(Y)}) = p_{X,k}(f)_{|D(Y)}. \tag{A.4}$$

A.4 Consider the ideal sheaf $\mathcal{I}_{\mathbb{V}(Y,L)}$ of $\mathbb{V}(Y,L)$ in $\mathbb{V}(X,L)$ and the associated short exact sequence of sheaves of $\mathcal{O}_{\mathbb{V}(X,L)}$-modules:

$$0 \longrightarrow \mathcal{I}_{\mathbb{V}(Y,L)} \longrightarrow \mathcal{O}_{\mathbb{V}(X,L)} \longrightarrow i_* \mathcal{O}_{\mathbb{V}(Y,L)} \longrightarrow 0.$$

This sequence induces a short exact sequence of analytic coherent sheaves on $D(X)$:

$$0 \longrightarrow \mathcal{I}^{\mathrm{an}}_{D(Y)} \longrightarrow \mathcal{O}^{\mathrm{an}}_{D(X)} \longrightarrow j_* \mathcal{O}^{\mathrm{an}}_{D(Y)} \longrightarrow 0,$$

and, consequently, by taking the cohomology on $D(X)$, an exact sequence of complex vector spaces

$$\mathcal{O}^{\mathrm{an}}(D(X)) \xrightarrow{\rho} \mathcal{O}^{\mathrm{an}}(D(Y)) \longrightarrow H^1(D(X); \mathcal{I}^{\mathrm{an}}_{D(Y)}),$$

where ρ denotes the restriction map from functions on $D(X)$ to functions on $D(Y)$.

According to Theorem A.2, the cohomology group $H^1(D(X); \mathcal{I}^{\mathrm{an}}_{D(Y)})$ is finite dimensional. Therefore, by Theorem A.3, $\rho(\mathcal{O}^{\mathrm{an}}(D(X)))$ is a closed subspace of $\mathcal{O}^{\mathrm{an}}(D(Y))$, and, for any positive ε, there exists C_ε and $r(\varepsilon) \in]0,1[$ such that, for any $f \in \rho(\mathcal{O}^{\mathrm{an}}(D(Y)))$, there exists $\tilde{f} \in \mathcal{O}^{\mathrm{an}}(D(X))$ mapped to f by ρ such that

$$\|\tilde{f}\|_{X,e^{-\varepsilon}} \leq C_\varepsilon \|f\|_{Y,r(\varepsilon)}. \tag{A.5}$$

The restriction morphism ρ is clearly equivariant with respect to the $U(1)$-action on $\mathcal{O}^{\mathrm{an}}(D(X))$ and $\mathcal{O}^{\mathrm{an}}(D(Y))$. Therefore its cokernel—which is a finite dimensional separated locally convex complex vector space—is naturally endowed with a continuous action of $U(1)$, and consequently, may be decomposed as a finite direct sum

$$\operatorname{coker} \rho = \bigoplus_{k \in I} (\operatorname{coker} \rho)_k,$$

where $(\operatorname{coker} \rho)_k$ denotes the subspace of $\operatorname{coker} \rho$ on which $U(1)$ acts by the character $(u \mapsto u^k)$.

Let D_0 be any non-negative integer larger that all the integers in I. Then, for any integer $D \geq D_0$ and any $s \in \Gamma(Y, L^{\otimes D})$, the class in $\operatorname{coker} \rho$ of the function $f \in \mathcal{O}^{\mathrm{an}}(D(Y))_D$ associated to s vanishes, and therefore f may be written $\rho(\tilde{f})$, where $\tilde{f}$ is an element of $\mathcal{O}^{\mathrm{an}}(D(X))$ satisfying (A.5). Moreover, (A.4) and (A.2) show that, by replacing $\tilde{f}$ by $p_{X,D}(\tilde{f})$, we may also assume that $\tilde{f}$ belongs to $\mathcal{O}^{\mathrm{an}}(D(X))_D$. Then the corresponding section $\tilde{s}$ in $\Gamma(X, L^{\otimes D})$ satisfies

$$\tilde{s}_{|Y} = s$$

and, according to (A.5) and (A.3),

$$e^{-\varepsilon D} \|\tilde{s}\|_{L^\infty(X)} \leq C_\varepsilon r(\varepsilon)^D \|s\|_{L^\infty(Y)}.$$

Since $r(\varepsilon) < 1$, this establishes the required estimate (A.1).

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