Evaluation maps, slopes, and algebraicity criteria

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Abstract. We discuss criteria for the algebraicity of a formal subscheme $\hat{V}$ in the completion $\hat{X}_P$ at some rational point P of an algebraic variety X over some field K. In particular we consider the case where K is a function field or a number field, and we discuss applications concerning the algebraicity of leaves of algebraic foliations, algebraic groups, absolute Tate cycles, and the rationality of germs of formal functions on a curve over a number field.

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1. Introduction

This article presents a survey of the *arithmetic algebraicity criteria* and their applications that have been developed in [12], [13], and in the subsequent joint work with Chambert-Loir [14].

The proofs of these criteria rely on what might be called the *method of slopes*, that is inspired by the classical techniques of auxiliary polynomials in Diophantine approximation, but is formulated in a geometric framework. In this approach, when investigating a projective algebraic variety X equipped with some ample line bundle L defined over a number field K, and some zero-dimensional subschemes Σ_i of X, the basic objects of interest are the *evaluation maps*

$$\eta_{D,i} : \Gamma(X, L^{\otimes D}) \longrightarrow \Gamma(\Sigma_i, L^{\otimes D}),$$

which map global sections of $L^{\otimes D}$ to their restrictions to Σ_i. Typically, when X is the compactification of an algebraic group, the Σ_i's may be some sets of multiples of some rational points, or some thickenings of such subsets. In the situation we shall deal with in this paper, the Σ_i's will be the successive infinitesimal neighbourhoods of some point P of $X(K)$ in a formal subscheme $\hat{V}$ of the formal completion $\hat{X}_P$.

The geometry of the Σ_i's in X turns out to be reflected by the injectivity properties of these evaluation maps — this is the contents of the so-called zero

lemmas when X is some compactified algebraic groups, and, in the setting of this paper, of the algebraicity criterion in Proposition 2.1 below. This geometry is finally related to the arithmetic properties of the data X, L, and Σ_i through the *slopes inequalities* satisfied by the K-linear maps $\eta_{D,i}$. Indeed, after the choice of auxiliary data (such as integral models for X, L, and the Σ_i's, and hermitian metrics on $X(\mathbb{C})$ and $L_{\mathbb{C}}$), the source and range of $\eta_{D,i}$ appear as the underlying K-vector spaces attached to some hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$. To them, elementary Arakelov geometry attaches arithmetic invariants, such as the *height* of $\eta_{D,i}$ and the *Arakelov degree* and the *slopes* of these hermitian vector bundles. The slope inequality for $\eta_{D,i}$ asserts that, when for instance $\eta_{D,i}$ is injective, the maximal slope of its source is bounded from above by the sum of its height and of the maximal slope of its range.

The approach to proving Diophantine statements by the consideration of evaluation maps and of the associated slope estimates has been introduced in a Bourbaki report [10], devoted to the work by Masser and Wüstholz on periods and minimal abelian subvarieties of abelian varieties over number fields [35]. The flexibility of this new geometric approach allows one to combine the original arguments in [35], phrased in terms of classical theta functions, with the "modern" theory of abelian schemes (notably various basic results due to Néron, Mumford, and Moret-Bailly), and to establish variants of the original work of Masser and Wüstholz where constants occuring in various estimates are explicitely bounded.

These effective versions of the "period theorem" of Masser and Wüstholz and of the consequent "isogeny estimates" have been recently improved by Viada ([43], [42]), by means of the same techniques. The combination of the method of slopes and of the modern theory of abelian schemes has also been used by Gaudron ([26]) to derive effective estimates on linear forms in logarithms on abelian varieties.

The results in the present article have been inspired by the work of D. and G. Chudnovsky [19], [18], and by the generalization of the results in [19] to abelian varieties by Graftieaux in [27] and [28], who also used the above combination of techniques. However, in the formulation and the proofs of the results discussed below, the flexibility of the method of slopes has not been exploited to derive Diophantine statements on abelian varieties involving *explicit* estimates, but instead to establish results valid in some *general geometric* setting by means of relatively non-technical arguments, at the expense of explicitness.

Another illustration of the flexibility of the method of slopes, in a spirit similar to this paper, is provided by the recent work by Gasbarri [25], who used this method to derive generalizations of transcendance theorems *à la* Schneider-Lang-Bombieri in general geometric situations and to clarify their relations with higher dimensional Nevanlinna theory.

In this article, we shall focus on these geometric aspects, instead of going into the details of the arguments of Arakelov geometry involved in proofs. For those, we refer to the original papers [12], [13], and [14] and to Chambert-Loir's Bourbaki report [16], which also discusses the link between slopes inequalities and more traditional techniques, such as Siegel's lemma and the interpolation determinants of Laurent [33].

To emphasize the geometric content of our approach, we shall first explain how the methods of auxiliary polynomials, in the guise of the study of the maps $\eta_{D,i}$ above, provides a simple algebraicity criterion for formal germs in an arbitrary projective variety over a field K (section 2). Then, assuming that K is the function field $k(C)$ of some projective curve over some field k, we shall derive some geometric analogues of the results presented in the later sections. The proofs of these analogues will demonstrate how inequalities between slopes — of vector bundles over the curve C in this geometric setting — may be used to establish that, under suitable positivity conditions, the hypotheses of our previous algebraicity criterion are fullfilled (section 3). Albeit technically simpler in the function field case, these arguments will give some insight into the proof of the arithmetic algebraization criteria stated in sections 6 and 7.

Besides, we shall illustrate these arithmetic criteria by applications to the algebraicity of leaves of *algebraic foliation*. Here, by an algebraic foliation over some base field K, we mean a smooth algebraic variety X over K, equipped with some sub-vector bundle F of the tangent bundle T_X, that is involutive (*i.e.*, whose sheaf of sections is closed under Lie bracket). When K is a field of characteristic $p > 0$, the sheaf of sections of T_X is equipped with the operation of p-th power, and it makes sense to require the sheaf of section of F to be closed under this operation. When K is a field of characteristic zero, for any point P in $X(K)$, one may consider the formal leaf of the foliation (X, F) through P, namely the unique smooth formal subscheme $\hat{V}$ of dimension $\operatorname{rk} F$ in the completion $\hat{X}_p$ whose formal tangent bundle coincides with the restriction of F.

When K is a number field with ring of integers $\mathcal{O}_K$, we may introduce some smooth model $\mathcal{X}$ of X over an open subscheme S of $\operatorname{Spec} \mathcal{O}_K$ such that F extends to a sub-vector bundle $\mathcal{F}$ of $T_{\mathcal{X}/S}$, and consider the following condition, which we shall call the *Grothendieck-Katz condition*:

For almost every maximal ideal $\mathbf{p}$ *in* $\operatorname{Spec} \mathcal{O}_K$, *of residue characteristic* p, *the involutive subbundle* $\mathcal{F}_{\mathbb{F}_\mathbf{p}}$ *of* $T_{\mathcal{X}_{\mathbb{F}_\mathbf{p}}}$ *is stable under* p*-th power.*

It is easily seen to be satisfied when the foliation (X, F) is algebraically integrable[1]. The *generalized conjecture of Grothendieck-Katz* asserts that the converse holds. It was initially formulated for linear differential systems by Katz [31] who attributes it to Grothendieck. The general formulation of the conjecture is due to Ekedahl, Shepherd-Barron, and Taylor [24]. Its investigation has been one of the main motivations behind the algebraicity results presented in this survey (see in particular, Section 6 and Theorems 6.1 and 6.2 below) which may also be considered as extensions of the earlier works of Chudnovsky ([18]) and André ([2], Chapter VIII, and [4], Section 5) on the original conjecture of Grothendieck-Katz.

For lack of space, we do not attempt to give any complete historical account of the origins of the algebraization techniques discussed below. Let us however indicate that these techniques — based on the consideration of maps sending global sections of ample line bundles to their restrictions to thickened points — goes back at least to the paper of Poincaré [37], where he presented an overview of his main

[1]This means, by definition, that for any field extension Ω of K, the formal leaf through any point of $X(\Omega)$ of the algebraic foliation (X_Ω, F_Ω) is itself algebraic.

results concerning abelian functions (see especially its Section II). Besides, the original proof of Chow's theorem relies on a criterion somewhat in the spirit of Proposition 2.1 below (see [17], Theorem IV). One might also refer to the article of Siegel [40] for historical comments on the proofs of algebraization statements in the framework of analytic geometry during the "pre-GAGA" era.

Algebraicity criteria in a geometric setting involving positivity conditions in the spirit of Theorem 3.1 below may also be deduced from classical results by Andreotti and Grauert on fields of meromorphic functions ([6], [5]) and by Hironaka, Matsumura, and Hartshorne on fields of formal meromorphic functions ([30], [29]); see [12], Section 3.3, and [9].

We refer to the monograph [7] for additional references concerning these techniques of formal geometry and their applications to extension and connectedness problems in projective geometry over a field. Many of these applications may be expected to have arithmetic counterparts which would extend the results in this article.

Conventions. The following notation and terminology are used throughout the article.

By an *algebraic scheme* over some field k, we mean a separated scheme of finite type over k. Integral subschemes of such an algebraic scheme X over k will be called *algebraic subvarieties* of X.

Let G be an algebraic group over a field K of characteristic 0. Its Lie algebra $\mathrm{Lie}\, G$ is the fiber at the unit element $e \in G(K)$ of the tangent bundle T_G, and may be identified with the K-vector space of the left-invariant regular sections of T_G over G. The Lie bracket on $\mathrm{Lie}\, G$ is, by definition, the restriction of the Lie bracket on vector fields in $\Gamma(G, T_G)$. A Lie subalgebra $\mathbf{h}$, defined over K, of $\mathrm{Lie}\, G$ is called an *algebraic Lie subalgebra* when it is the Lie algebra of some algebraic subgroup H in G. When this holds, the subgroup H may be supposed connected, and then is unique and defined over any field of definition of G and $\mathbf{h}$.

If K is a number field, its ring of integers will be denoted $\mathcal{O}_K$. For any non-zero prime ideal $\mathbf{p}$ of $\mathcal{O}_K$, we let $N\mathbf{p} := |\mathcal{O}_K/\mathbf{p}|$ its norm, $K_\mathbf{p}$ (resp. $\mathcal{O}_\mathbf{p}$) the $\mathbf{p}$-adic completion of K (resp. of $\mathcal{O}_K$), and $|\ |_\mathbf{p}$ the $\mathbf{p}$-adic absolute value on $K_\mathbf{p}$ normalized in such a way that, for any uniformizing element ϖ of $\mathcal{O}_\mathbf{p}$, $|\varpi|_\mathbf{p} = N\mathbf{p}^{-1}$. We shall also denote K_v the completion of K at some place v (possibly archimedean).

2. Algebraic formal germs and auxiliary polynomials

Let X be an algebraic scheme over a field K, P a point of $X(K)$, $\hat{X}_P$ the formal completion of X at P, and $\hat{V} \hookrightarrow \hat{X}_P$ a smooth formal subscheme. Such a $\hat{V}$ will also be called a *smooth formal germ of subvariety* through P in X. For any non-negative integer i, we shall denote V_i the i-th infinitesimal neighborhood of P in $\hat{V}$. Thus,

$$V_0 = \{P\} \subset V_1 \subset V_2 \subset \cdots$$

and

$$\hat{V} = \varinjlim V_i.$$

We may consider the Zariski closure of $\hat{V}$ in X, namely, the smallest closed subscheme Z of X such that $\hat{Z}_P$ contain $\hat{V}$. Observe that it is a subvariety of X containing P: the ideal in $\mathcal{O}_{X,P}$ defining its germ at P is the intersection of $\mathcal{O}_{X,P}$ and of the ideal in its completion $\hat{\mathcal{O}}_{X,P} = \mathcal{O}_{\hat{X}_P}$ that defines $\hat{V}$, hence is prime. Moreover, since $\hat{Z}_P$ contains $\hat{V}$, the dimension of Z is at least equal to the dimension of $\hat{V}$.

The formal germ $\hat{V}$ is called *algebraic* when these two dimensions are equal. Indeed, using the basic properties of dimension and normalization, one easily checks that this condition is equivalent to each of the following ones, which we could have been used as alternative definitions:

(i) There exists a closed subvariety Z of X such that P belongs to $Z(K)$ and $\hat{V}$ is a branch of Z through P (*i.e.*, a component of the completion $\hat{Z}_P$).

(ii) There exist an integral algebraic scheme Y over K, a point 0 of $Y(K)$ and a K-morphism $f : Y \to X$ which maps 0 to P, such that the induced morphism on formal completions

$$\hat{f}_0 : \hat{Y}_0 \longrightarrow \hat{X}_P$$

factorizes through $\hat{V} \hookrightarrow \hat{X}_P$ and defines a formal isomorphism from $\hat{Y}_0$ to $\hat{V}$.

Let us moreover assume that X *is projective* over K. We may choose an ample line bundle L on X, and introduce the following K-vector spaces and K-linear maps, for any non-negative integers D and i:

$$E_D := \Gamma(X, L^{\otimes D}),$$

$$\begin{array}{rccc} \eta_D : & E_D & \longrightarrow & \Gamma(\hat{V}, L^{\otimes D}) \\ & s & \longmapsto & s_{|\hat{V}}, \end{array}$$

$$\begin{array}{rccc} \eta_D^i : & E_D & \longrightarrow & \Gamma(V_i, L^{\otimes D}) \\ & s & \longmapsto & s_{|V_i}, \end{array}$$

and[2]

$$E_D^i := \{s \in E_D \mid s_{V_{i-1}} = 0\} = \ker \eta_D^{i-1}.$$

Observe that there is a canonical isomorphism $\Gamma(\hat{V}, L^{\otimes D}) \simeq \varprojlim_i \Gamma(V_i, L^{\otimes D})$, by means of which the map η_D gets identified with $\varprojlim_i \eta_D^i$.

The subspaces E_D^i define a decreasing filtration of E_D:

$$E_D = E_D^0 \supset E_D^1 \supset \ldots \supset E_D^i \supset E_D^{i+1} \supset \ldots$$

Since the K-vector space E_D is finite dimensional, this filtration is stationary, and the very definition of Z as the Zariski closure of $\hat{V}$ shows that, if $\mathcal{I}_Z$ denotes its ideal sheaf in $\mathcal{O}_X$, we have

$$\bigcap_{i \geq 0} E_D^i = \ker \eta_D = \Gamma(X, \mathcal{I}_Z.L^{\otimes D}). \tag{2.1}$$

[2]In this definition, when $i = 0$, we let $V_{-1} = \emptyset$ and $\eta_D^{-1} = 0$.

Finally, if $T_{\hat{V}}$ denotes the tangent space of $\hat{V}$, then, for any non-negative integer i, the kernel of the restriction map from $\Gamma(V_i, L^{\otimes D})$ to $\Gamma(V_{i-1}, L^{\otimes D})$ may be identified with $S^i\check{T}_{\hat{V}} \otimes L_P^D$, and the restriction of the evaluation map η_D^i to E_D^i defines a K-linear map:

$$\gamma_D^i : E_D^i \longrightarrow S^i\check{T}_{\hat{V}} \otimes L_P^{\otimes D}.$$

Roughly speaking, it is the map which sends a section of $L^{\otimes D}$ vanishing up to order i at P along $\hat{V}$ to the $(i+1)$-th "Taylor coefficient" of its restriction to $\hat{V}$. By construction,

$$\ker \gamma_D^i = E_D^{i+1}. \tag{2.2}$$

The following proposition shows how the algebraicity of $\hat{V}$ and the asymptotic behaviour of the ranks of the subquotients E_D^i/E_D^{i+1} are related. It may be considered as a geometric version of the techniques of "auxiliary polynomials" used in the theory of Diophantine approximation: in our setting, the elements of the space $E_D := \Gamma(X, L^{\otimes D})$ play the role of auxiliary polynomials of degree D.

Proposition 2.1. *The following three conditions are equivalent:*

(i) the formal germ $\hat{V}$ is algebraic;

(ii) there exists $c > 0$ such that, for any $(D, i) \in \mathbb{N}^2$ satisfying $i > cD$, the map γ_D^i vanishes;

(iii) the ratio

$$\frac{\sum_{i\geq 0}(i/D)\mathrm{rk}\,(E_D^i/E_D^{i+1})}{\sum_{i\geq 0}\mathrm{rk}\,(E_D^i/E_D^{i+1})} \tag{2.3}$$

does not *admit the limit $+\infty$ when D goes to infinity.*

Condition (ii) may be also expressed by saying that, for every positive integer D the filtration $(E_D^i)_{i\geq 0}$ becomes stationary — or equivalently that η_D vanishes on E_D^i — when $i > cD$.

The implication (i) $\Rightarrow$ (ii) is a straightforward consequence of the basic theory of ample line bundles and their Seshadri constants (see for instance [34], Chapter 5, notably Proposition 5.1.9). The implication (ii) $\Rightarrow$ (iii) is clear. We sketch the proof of (ii) $\Rightarrow$ (i) below. The one of the implication (iii) $\Rightarrow$ (i) — which constitutes the algebraicity criterion we shall use in the sequel — is similar, but slightly more elaborate; see [13], Section 2.2, for more details.

Let us assume that condition (ii) holds, and let d denote the dimension of $\hat{V}$, which we may assume positive Then, for any non-negative integers D and i, the quotient vector space $E_D^i/E_D^{i+1} = E_D^i/\ker\gamma_D^i \simeq \mathrm{im}\,\gamma_D^i$ has rank at most $\mathrm{rk}\,(S^i\check{T}_{\hat{V}} \otimes L_P^D) = \binom{d+i-1}{i}$ and vanishes if $i > cD$. This implies that

$$\mathrm{rk}\,(E_D/\bigcap_{i\geq 0} E_D^i) = \sum_{i\geq 0}\mathrm{rk}\,(E_D^i/E_D^{i+1}) \leq \sum_{i=0}^{[cD]}\binom{d+i-1}{i}.$$

Moreover the last sum is equivalent to $\frac{c^d}{d!}D^d$ when D goes to infinity.

Besides, according to (2.1),

$$E_D/\bigcap_{i\geq 0} E_D^i = \Gamma(X, L^{\otimes D})/\Gamma(X, \mathcal{I}_Z.L^{\otimes D}).$$

For D large enough, this space may be identified with $\Gamma(Z, L^{\otimes D})$ and its rank is equivalent to $\frac{\deg_L Z}{(\dim Z)!} D^{\dim Z}$ when D goes to infinity.

This shows that $\dim Z$ is at most d, and therefore is equal to d. This establishes condition (i), and completes the proof of (ii) $\Rightarrow$ (i).

3. An algebraicity criterion for smooth formal germs in varieties over function fields

Let C be a smooth projective, geometrically connected curve over some field k and let $K := k(C)$ be the associated function field.

Let X be a quasi-projective K-scheme, P a point in $X(K)$, and $\hat{V} \subset \hat{X}_P$ a smooth formal germ of subvariety through P in X.

After possibly shrinking X, we may assume that it is quasi-projective and choose a quasi-projective model[3] $\pi : \mathcal{X} \longrightarrow C$ such that P extends to a section $\mathcal{P}$ of π.

Recall that, if E is a vector bundle of positive rank on C, its *slope* is defined as the quotient of its degree by its rank

$$\mu(E) := \frac{\deg E}{\operatorname{rk} E},$$

and its *maximal slope* $\mu_{\max}(E)$ is the maximum of the slopes $\mu(F)$ of sub-vector bundles of positive rank in E. Observe that, if L is any line bundle on C,

$$\mu_{\max}(E \otimes L) = \mu_{\max}(E) + \deg L.$$

Moreover, if E_1 and E_2 are vector bundles over C with E_2 of positive rank, and if there exists some (generically) injective morphism of vector bundles $\varphi : E_1 \to E_2$, then the following slope inequality holds:

$$\deg E_1 \leq \operatorname{rk} E_1.\mu_{\max}(E_2). \tag{3.1}$$

Finally, recall that a vector bundle E over C is ample iff it has positive rank and there exists $c > 0$ such that, for any non-negative integer i,

$$\mu_{\max}(S^i \check{E}) \leq -c.i.$$

[3] namely, a quasi-projective k-variety $\mathcal{X}$, equipped with a flat k-morphism $\pi : \mathcal{X} \to C$ and an isomorphism of its generic fiber $\mathcal{X}_K$ with X.

Theorem 3.1 ([13], Theorem 2.5). *With the above notation, assume that the following two conditions are satisfied:*
(i) the formal subscheme $\hat{V}$ in $\hat{X}_P$ extends to a formal subscheme $\hat{\mathcal{V}}$ of $\hat{\mathcal{X}}_{\mathcal{P}}$ that is smooth over C;
(ii) the normal bundle $N_{\mathcal{P}}\hat{\mathcal{V}}$ of $\hat{\mathcal{V}}$ along $\mathcal{P}$ is ample.
Then $\hat{V}$ is algebraic.

This algebraicity criterion may be seen as a "geometric model", concerning functions fields, of the arithmetic algebraicity criterion in Theorem 6.1 below, devoted to formal germs in varieties over number fields.

Our proof of Theorem 3.1 will rely on the implication (iii) $\Rightarrow$ (i) in Proposition 2.1. Indeed there exists a projective compactification of $\mathcal{X}$ to which the morphism π extends. Therefore we may assume that $\mathcal{X}$ is projective, and choose some ample line bundle $\mathcal{L}$ on $\mathcal{X}$. Let $L := \mathcal{L}_K$ be its restriction to X, and let E_D, E_D^i, η_D^i, and γ_D^i be as in the previous section. We are going to show that, when conditions (i) and (ii) in Theorem 3.1 are satisfied, the ratio (2.3) stays bounded when D goes to infinity.

To achieve this, let us consider the direct images $\mathcal{E}_D := \pi_*\mathcal{L}^{\otimes D}$ and $\pi_{|\mathcal{V}_i *}\mathcal{L}^{\otimes D}$, where $\mathcal{V}_i$ denotes the i-th infinitesimal neighbourhood of $\mathcal{P}$ in $\mathcal{V}$. These are torsion free coherent sheaves — or equivalently vector bundles — on C, which at the generic point $\operatorname{Spec} K$ of C coincide with the K-vector spaces E_D and $\Gamma(V_i, L^{\otimes D})$. Moreover, every restriction map $\eta_D^i : E_D \longrightarrow \Gamma(V_i, L^{\otimes D})$ extends to a morphism of vector bundles:

$$\begin{array}{rccc} \overline{\eta}_D^i : & \mathcal{E}_D & \longrightarrow & \pi_{|\mathcal{V}_i *}\mathcal{L}^{\otimes D} \\ & s & \longmapsto & s_{|\mathcal{V}_i}. \end{array}$$

The filtration $(E_D^i)_{i\geq 0}$ of E_D also extends to the filtration of $\mathcal{E}_D$ by the sub-vector bundles $\mathcal{E}_D^i := \ker \overline{\eta}_D^{i-1}$. Finally, the kernel of the restriction map from $\pi_{|\mathcal{V}_i *}\mathcal{L}^{\otimes D}$ to $\pi_{|\mathcal{V}_{i-1} *}\mathcal{L}^{\otimes D}$ may be identified with $S^i(\check{N}_{\mathcal{P}}\hat{\mathcal{V}}) \otimes \mathcal{P}^*\mathcal{L}^{\otimes D}$ and the restriction of the evaluation map $\overline{\eta}_D^i$ to $\mathcal{E}_D^i$ defines a morphism of vector bundles $\overline{\gamma}_D^i : \mathcal{E}_D^i \to S^i(\check{N}_{\mathcal{P}}\hat{\mathcal{V}}) \otimes \mathcal{P}^*\mathcal{L}^{\otimes D}$, which coincides with γ_D^i at the generic point of C. The kernel of $\overline{\gamma}_D^i$ is $\mathcal{E}_D^{i+1}$ and therefore $\overline{\gamma}_D^i$ factorizes through a (generically) injective morphism of vector bundles $\tilde{\gamma}_D^i : \mathcal{E}_D^i/\mathcal{E}_D^{i+1} \to S^i(\check{N}_{\mathcal{P}}\hat{\mathcal{V}}) \otimes \mathcal{P}^*\mathcal{L}^{\otimes D}$.

The ampleness of $N_{\mathcal{P}}\hat{\mathcal{V}}$ and the slope inequality (3.1) applied to the morphisms $\tilde{\gamma}_D^i$ now show that, for some $c > 0$ independent of i and D, we have:

$$\deg\left(\mathcal{E}_D^i/\mathcal{E}_D^{i+1}\right) \leq \operatorname{rk}\left(E_D^i/E_D^{i+1}\right)(-c.i + D.\deg \mathcal{P}^*\mathcal{L}).$$

Besides, since $\mathcal{L}$ is ample, the sheaf $\mathcal{E}_D$ is generated by its global sections for D large enough, and consequently:

$$\deg\left(\mathcal{E}_D/\bigcap_{i\geq 0}\mathcal{E}_D^i\right) \geq 0.$$

Moreover we may write:

$$\deg\left(\mathcal{E}_D/\bigcap_{i\geq 0}\mathcal{E}_D^i\right) = \sum_{i\geq 0}\deg\left(\mathcal{E}_D^i/\mathcal{E}_D^{i+1}\right).$$

When D is large enough, the above three inequalities establish that

$$c.\sum_{i\geq 0}(i/D)\,\mathrm{rk}\,(E_D^i/E_D^{i+1}) \leq \deg \mathcal{P}^*\mathcal{L}.\sum_{i\geq 0}\mathrm{rk}\,(E_D^i/E_D^{i+1}),$$

and finally that the ratio (2.3) is at most $(\deg \mathcal{P}^*\mathcal{L})/c$.

The following application of Theorem 3.1 illustrates how the algebrization criteria contained in Proposition 2.1 and Theorem 3.1 lead to non-trivial geometric consequences in spite of the elementary nature of its proof (see also [9], [20], and [32] for applications of similar techniques to algebraic and rationally connected leaves of algebraic foliations).

Theorem 3.2 ([13], Theorem 2.6). *Let C be a smooth projective, geometrically connected curve over a field k of characteristic zero, $K := k(C)$ its function field, and $\pi : \mathcal{G} \to C$ a smooth group scheme over C. Let $G := \mathcal{G}_K$ be its generic fiber, and* $\mathrm{Lie}\,\mathcal{G}$ *its Lie algebra*[4].

If the sub-vector bundle of $\mathrm{Lie}\,\mathcal{G}$ *defined by some Lie subalgebra (over K)* $\mathbf{h}$ *of* $\mathrm{Lie}\,G$ *is ample, then it is an algebraic Lie subalgebra. More specifically, there exists a unipotent linear K-algebraic subgroup H in G such that* $\mathbf{h} = \mathrm{Lie}\,H$.

In the classical analogy between function fields and number fields, this statement may be considered as a counterpart of Diophantine results concerning algebraic groups over number fields such as Theorem 6.3 below.

Observe also that, if G is a semi-abelian variety over K (hence does not admit any non-trivial unipotent algebraic subgroup), Theorem 3.2 asserts the semi-negativity of $\mathrm{Lie}\,\mathcal{G}$. When $\mathcal{G}$ is an abelian scheme over C, this also follows from a classical curvature argument due to Griffiths.

Under the hypotheses of Theorem 3.2, to establish the existence of an algebraic subgroup H of G such that $\mathbf{h} = \mathrm{Lie}\,H$, one applies Theorem 3.1 in the situation where X is G, P is the unit element e of $G(K)$, and $\hat{V}$ is the "formal exponential" of the Lie subalgebra $\mathbf{h}$, namely, the formal subgroup of $\hat{G}_e$ such that $T_P\hat{V} = \mathbf{h}$. Its Zariski closure provides the sought for algebraic subgroup.

When the Lie bracket vanishes on $\mathbf{h}$, one may consider the vector group $\mathrm{Vect}(\mathbf{h})$ defined by $\mathbf{h}$ and the product $G' := \mathrm{Vect}(\mathbf{h}) \times G$. The graph of the injection $\mathbf{h} \hookrightarrow \mathrm{Lie}\,G$ is an algebraic Lie subalgebra of $\mathrm{Lie}\,G'$, and is the Lie algebra of the graph of an isomorphism $\mathrm{Vect}(\mathbf{h}) \simeq H \hookrightarrow G$.

In general, one establishes that H is linear and unipotent by a similar argument, after having deduced that $\mathbf{h}$ is nilpotent from analyzing the compatibility of the Lie bracket on $\mathbf{h}$ with the Harder-Narasimhan filtration of the associated sub-vector bundle in $\mathrm{Lie}\,\mathcal{G}$.

[4]It is defined as the restriction along the zero section of π of the relative tangent bundle T_π. It is a vector bundle over C, equipped with a $\mathcal{O}_C$-bilinear Lie bracket, which coincides at the generic point $\mathrm{Spec}\,K$ of C with the Lie bracket of the Lie algebra $\mathrm{Lie}\,G \simeq (\mathrm{Lie}\,\mathcal{G})_K$ of the K-algebraic group G.

4. Sizes of formal subschemes over p-adic and global fields

This section and the next one are devoted to preliminaries needed for stating our arithmetic algebraization criteria in Sections 6 and 7.

We first describe some constructions introduced in [12], Section 3, and further developed in [13], Section 4.1, and [14], Section 2. We refer to these papers for details and proofs.

Let k be a p-adic field (*i.e.*, a finite extension of $\mathbb{Q}_p$), $\mathcal{O}$ its subring of integers (*i.e.*, the integral closure of $\mathbb{Z}_p$ in k), $|\ |: k \to \mathbb{R}_+$ its absolute value, and $\mathbb{F}$ its residue field[5].

4.1. Groups of formal and analytic automorphisms. If $g := \sum_{I \in \mathbb{N}^d} a_I X^I$ is a formal power series in $k[[X_1, \cdots, X_d]]$ and if $r \in \mathbb{R}_+^*$, we define

$$\|g\|_r := \sup_I |a_I| r^{|I|} \in \mathbb{R}_+ \cup \{+\infty\}.$$

The "norm" $\|g\|_r$ is finite iff the series g is convergent and bounded on the open ball of radius r in $\overline{k}^d$.

Let $\hat{\mathbb{A}}_k^d$ be the formal completion at the origin of the d-dimensional affine space over k. Its group $\operatorname{Aut} \hat{\mathbb{A}}_k^d$ of automorphisms may be identified with the space of d-tuples $f = (f_i)_{1 \leq i \leq d}$ of formal series $f_i \in k[[x_1, \cdots, x_d]]$ such that $f(0) = 0$ and $Df(0) := \left(\frac{\partial f_i}{\partial x_j}(0)\right)_{1 \leq i,j \leq d}$ belongs to $GL_n(k)$.

We shall consider the following subgroups of $\operatorname{Aut} \hat{\mathbb{A}}_k^d$:

- the subgroup G_{for} formed by the formal automorphisms f such that $Df(0)$ belongs to $GL_n(\mathcal{O})$;
- the subgroup G_ω formed by the elements $f := (f_i)_{1 \leq i \leq d}$ of G_{for} such that the series f_i have positive radii of convergence;
- for any $r \in \mathbb{R}_+^*$, the subgroup $G_\omega(r)$ of G_ω formed by the elements $f := (f_i)_{1 \leq i \leq d}$ of G_{for} such that the series f_i satisfy the bounds $\|f_i\|_r \leq r$.

The group $G_\omega(r)$ may be seen as the group of analytic automorphisms, defined over k and preserving the origin, of the open d-dimensional ball of radius r. Moreover we have:

$$r' > r > 0 \Longrightarrow G_\omega(r') \subset G_\omega(r) \quad \text{and} \quad \bigcup_{r>0} G_\omega(r) = G_\omega.$$

[5] Actually we might assume more generally that k is any field equipped with a complete non-Archimedean absolute value $|\ |: k \to \mathbb{R}_+$ and let $\mathcal{O} := \{t \in k \mid |t| \leq 1\}$ be its valuation ring.

4.2. The size $S_{\mathcal{X}}(\hat{V})$ of a formal germ $\hat{V}$. The filtration $(G_\omega(r))_{r>0}$ of the group G_ω may be used to attach a number $S_{\mathcal{X}}(\hat{V})$ in $[0,1]$ to any smooth formal germ $\hat{V}$ in an algebraic variety X over k, depending on the choice of some model $\mathcal{X}$ of X over $\mathcal{O}$. This number shall provide some quantitative measure of the analyticity of $\hat{V}$.

Let $\hat{V}$ be a formal subscheme of $\hat{\mathbb{A}}^d_k$. For any φ in $\operatorname{Aut}\hat{\mathbb{A}}^d_k$, we may consider its inverse image $\varphi^*(\hat{V})$, which also is a formal subscheme of $\hat{\mathbb{A}}^d_k$. Observe that $\hat{V}$ is a smooth formal scheme of dimension v iff there exists φ in $\operatorname{Aut}\hat{\mathbb{A}}^d_k$ such that $\varphi^*(\hat{V})$ is the formal subscheme $\hat{\mathbb{A}}^v_k \times \{0\}$ of $\hat{\mathbb{A}}^d_k$. Moreover, when this holds, φ may be chosen in G_{for}.

Similarly, the formal germ $\hat{V}$ is *analytic and smooth* — namely, it is the formal scheme attached to some germ at 0 of smooth analytic subspace of dimension v of the d-dimensional affine space over k — iff there exists φ in G_ω such that $\varphi^*(\hat{V})$ is the formal subscheme $\hat{\mathbb{A}}^v_k \times \{0\}$ of $\hat{\mathbb{A}}^d_k$.

These observations lead us to introduce the *size of a smooth formal subscheme* $\hat{V}$ of dimension v of $\hat{\mathbb{A}}^d_k$, defined as the supremum $S(\hat{V})$ in $[0,1]$ of the real numbers $r \in]0,1]$ for which there exists φ in $G_\omega(r)$ such that $\varphi^*(\hat{V})$ is the formal subscheme $\hat{\mathbb{A}}^v_k \times \{0\}$ of $\hat{\mathbb{A}}^d_k$.

More generally, if $\mathcal{X}$ is an $\mathcal{O}$-scheme of finite type equipped with a section $\mathcal{P} \in \mathcal{X}(\mathcal{O})$ and if $\hat{V}$ is a smooth formal subscheme of the formal completion $\hat{X}_P$ of $X := \mathcal{X}_k$ at $P := \mathcal{P}_k$, then the *size* $S_{\mathcal{X}}(\hat{V})$ *of* $\hat{V}$ *with respect to the model* $\mathcal{X}$ of X will be defined as the size of $i(\hat{V})$, where $i : U \hookrightarrow \mathbb{A}^d_{\mathcal{O}}$ is an embedding of some open neighbourhood U in $\mathcal{X}$ of the section $\mathcal{P}$ into an affine space of large enough dimension d, which additionally maps $\mathcal{P}$ to the origin $0 \in \mathbb{A}^d_{\mathcal{O}}(\mathcal{O})$.

This definition is independent of the choices of U, d, and i, and extends the previous one. Actually it satisfies the following invariance properties:
I1. If $\mathcal{X}$ is a subscheme of a scheme $\mathcal{X}'$ over $\mathcal{O}$, then $S_{\mathcal{X}'}(\hat{V}) = S_{\mathcal{X}}(\hat{V})$.
I2. If $\mathcal{X}$, X, $\mathcal{P}$, $\hat{V}$ and $\mathcal{X}'$, X', $\mathcal{P}'$, $\hat{V}'$ are as above, and if there exists an $\mathcal{O}$-morphism $\phi : \mathcal{X} \to \mathcal{X}'$ mapping $\mathcal{P}$ to $\mathcal{P}'$, étale along $\mathcal{P}$, such that the formal isomorphism $\hat{\phi}_k : \hat{X}_P \xrightarrow{\sim} \hat{X}'_{P'}$ maps isomorphically $\hat{V}$ onto $\hat{V}'$, then $S_{\mathcal{X}'}(\hat{V}') = S_{\mathcal{X}}(\hat{V})$.

Besides, the size $S_{\mathcal{X}}(\hat{V})$ is invariant by extension of the p-adic base field k (*cf.* [14]).

Finally observe that, with the same notation as above, the size $S_{\mathcal{X}}(\hat{V})$ is positive iff $\hat{V}$ is analytic. Moreover, if $\hat{V}$ extends to a formal subscheme $\hat{\mathcal{V}}$ of the formal completion of $\mathcal{X}$ along $\mathcal{P}$ which is smooth along $\mathcal{P}$, then $S_{\mathcal{X}}(\hat{V}) = 1$.

4.3. Size of formal leaves of algebraic foliations. It is possible to establish lower bounds on the sizes of formal germs of solutions of algebraic ordinary differential equations. These bounds will allow us to apply our arithmetic algebraization criteria below to the solutions of algebraic differential equations — or more generally, to leaves of algebraic foliations — defined over number fields.

Proposition 4.1. *Let* $\mathcal{X}$ *be a smooth scheme over* $\operatorname{Spec}\mathcal{O}$, $\mathcal{P}$ *a section in* $\mathcal{X}(\mathcal{O})$, *and* $\mathcal{F}$ *a sub-vector bundle of rank* f *in* $T_{\mathcal{X}/\mathcal{O}}$. *Let us assume that the subbundle*

$F := \mathcal{F}_k$ of the tangent bundle T_X of the smooth k-variety $X := \mathcal{X}_k$ is involutive, and let $\hat{V}$ be the formal germ of leave of this involutive bundle through $P := \mathcal{P}_k$.

1) The size of $\hat{V}$ with respect to $\mathcal{X}$ satisfies the lower bound:

$$S_{\mathcal{X}}(\hat{V}) \geq |\pi| := |p|^{\frac{1}{p-1}}. \tag{4.1}$$

2) If moreover k is absolutely unramified and if the reduction $\mathcal{F}_{\mathbb{F}} \hookrightarrow T_{\mathcal{X}_{\mathbb{F}}}$ of $\mathcal{F}$ to the closed fiber $\mathcal{X}_{\mathbb{F}}$ of $\mathcal{X}$ is closed under p-th power, then

$$S_{\mathcal{X}}(\hat{V}) \geq |p|^{\frac{1}{p(p-1)}}. \tag{4.2}$$

This is proved in [12], Proposition 3.9, and [13], Proposition 4.1, by first reducing the construction of $\hat{V}$ to the one of the formal flow $\psi_{(t_1,\ldots,t_f)}$ of suitably chosen commuting sections $v_1, \ldots, v_f$ of F. Then one studies the analyticity properties of this flow by expanding the map $\psi^*_{(t_1,\ldots,t_f)}$ (defined by $\psi_{(t_1,\ldots,t_f)}$ acting on functions) *à la* Cauchy:

$$\psi^*_{(t_1,\ldots,t_f)} = \exp\Big(\sum_{1\leq i\leq f} t_i.v_i\Big) := \sum_{(i_1,\ldots,i_f)\in\mathbb{N}^f} \frac{t_1^{i_1}\ldots t_f^{i_f}}{i_1!\ldots i_f!} D_1^{i_1}\circ\ldots\circ D_f^{i_f},$$

where $D_1, \ldots, D_f$ denote the derivations of the sheaf of regular functions on X defined by $v_1, \ldots, v_f$.

4.4. A-germs. Consider an algebraic variety X over some number field K, P a point in $X(K)$, and $\hat{V}$ a smooth formal subscheme in $\hat{X}_P$.

Let N be a positive integer and $(\mathcal{X}, \mathcal{P})$ a model of (X, P) over $\mathcal{O}_K[1/N]$. For any maximal ideal $\mathbf{p}$ in $\mathcal{O}_K$ not dividing N, by base change we get a smooth formal germ $\hat{V}_{K_\mathbf{p}}$ through $P_{K_\mathbf{p}}$ in the algebraic variety $X_{K_\mathbf{p}}$ over the p-adic field $K_\mathbf{p}$, and a model $(\mathcal{X}_{\mathcal{O}_\mathbf{p}}, \mathcal{P}_{\mathcal{O}_\mathbf{p}})$ over $\mathcal{O}_\mathbf{p}$ of the pair $(X_{K_\mathbf{p}}, P_{K_\mathbf{p}})$. Consequently, for any such $\mathbf{p}$, the size $S_{\mathcal{X}_{\mathcal{O}_\mathbf{p}}}(\hat{V}_{K_\mathbf{p}})$ is a well-defined element in $[0, 1]$.

We shall say that the formal germ $\hat{V}$ in X is *A-analytic*, or is an *A-germ*, when the following two conditions are satisfied:

1. *for any place v of K, the formal germ $\hat{V}_{K_v}$ is K_v-analytic*[6];
2. *the infinite product $\prod_{\mathbf{p}\nmid\mathbf{N}} S_{\mathcal{X}_{\mathcal{O}_\mathbf{p}}}(\hat{V}_{K_\mathbf{p}})$ is positive*, or equivalently,

$$\sum_{\mathbf{p}\nmid\mathbf{N}} \log S_{\mathcal{X}_{\mathcal{O}_\mathbf{p}}}(\hat{V}_{K_\mathbf{p}})^{-1} < +\infty.$$

This pair of conditions does not depend on the choices of the integer N and the model $(\mathcal{X}, \mathcal{P})$. Moreover, it is invariant under extension of the base field[7].

[6] or equivalently, if for any maximal ideal $\mathbf{p}$ of $\mathcal{O}_K$, the formal germ $\hat{V}_{K_\mathbf{p}}$ is $K_\mathbf{p}$-analytic in $X_{K_\mathbf{p}}$, and for any complex embedding $\sigma : K \hookrightarrow \mathbb{C}$, $\hat{V}_\sigma$ is $\mathbb{C}$-analytic in X_σ.

[7] Namely, with the above notation, for any finite degree extension L of K, the formal germ $\hat{V}_L$ through P_L in the algebraic variety X_L over the number field L is A-analytic iff $\hat{V}$ is.

Recall that, if X is a variety over a number field K and P is some smooth point in $X(K)$, then the G-functions at the point P of X are the elements f in the completion $\mathcal{O}_{\hat{X}_P}$ of $\mathcal{O}_{X,P}$ defined by similar conditions: the analyticity at every place of K, and the positivity of the infinite product as in Condition 2 above, where $S_{\mathcal{X}_{\mathcal{O}_\mathbf{p}}}(\hat{V}_{K_\mathbf{p}})$ is replaced by $\min(1, R_\mathbf{p})$, $R_\mathbf{p}$ denoting the $\mathbf{p}$-adic radius of convergence of f expressed in some fixed system of local coordinates on X at P (see for instance [2] and [22] for details and references).

It is straightforward that, if the graph $\mathrm{Gr}\, f$ of some f in $\mathcal{O}_{\hat{X}_P}$ — which a smooth formal germ through $P' := (P, f(P))$ in $X' := X \times \mathbb{A}^1$ — is A-analytic, then f is a G-function. Let us emphasize that the converse does *not* hold[8].

Observe also that an algebraic smooth formal germ is always A-analytic. Even more, if $\hat{V}$ is an algebraic smooth formal germ in $\hat{X}_P$, where as above X denotes an algebraic variety over some number field K and P a point in $X(K)$, and if N is a positive integer and $(\mathcal{X}, \mathcal{P})$ some model of (X, P) over $\mathrm{Spec}\, \mathcal{O}_K[1/N]$, then almost all the sizes $S_{\mathcal{X}_{\mathcal{O}_\mathbf{p}}}(\hat{V}_{K_\mathbf{p}})$ are equal to one. Indeed, after "shrinking" $\mathrm{Spec}\, \mathcal{O}_K[1/N]$ to $\mathrm{Spec}\, \mathcal{O}_K[1/N']$, with N' a suitable multiple of N, the formal scheme $\hat{V}$ extends to a formal subscheme $\hat{\mathcal{V}}$ of the formal completion of $\mathcal{X}$ along $\mathcal{P}$ which is smooth along $\mathcal{P}$. (In substance, this observation goes back to the last memoir of Eisenstein [23], which may be considered as the starting point of the arithmetic theory of differential equations.)

Finally, observe that Proposition 4.1 admits the following straightforward consequence:

Corollary 4.2. *If X is a smooth algebraic variety over some number field K and if F is an involutive subbundle of T_X that satisfies the Grothendieck-Katz condition* (see Introduction)*, then the formal germ of leave of the so-defined algebraic foliation through any point P in $X(K)$ is A-analytic.*

5. Condition L and canonical semi-norms

5.1. Consistent sequences of norms. Let k be a local field, X a projective scheme over k, and L a line bundle over X.

We may consider the following natural constructions of sequences of norms on the spaces of sections $\Gamma(X, L^{\otimes n})$:

1. When k is a p-adic field, with ring of integer $\mathcal{O}$, we may choose a pair $(\mathcal{X}, \mathcal{L})$, where $\mathcal{X}$ is a projective flat model of X over $\mathcal{O}$, and $\mathcal{L}$ a line bundle over $\mathcal{X}$ extending L. Then, for any integer n, the $\mathcal{O}$-module $\Gamma(\mathcal{X}, \mathcal{L}^{\otimes n})$ is (torsion-)free of finite rank and may be identified with an $\mathcal{O}$-lattice in the k-vector space $\Gamma(X, L^{\otimes n})$, and consequently defines a norm on the latter — namely, the norm $\|.\|_n$ such that a section $s \in \Gamma(X, L^{\otimes n})$ satisfies $\|s\|_n \leq 1$ iff s extends to a section of $\mathcal{L}^{\otimes n}$ over $\mathcal{X}$.

[8] For instance, the series $\log(1+x) := \sum_{n=1}^{+\infty} x^n/n \in \mathbb{Q}[[x]]$ defines a G-function at the point 0 in $\mathbb{A}^1_\mathbb{Q}$. However, its graph coincides with the transpose of the graph of the series $\exp y - 1 := \sum_{n=1}^{+\infty} y^n/n!$, which is not a G-function, and consequently is not an A-germ.

2. When $k = \mathbb{C}$ and X is reduced, we may consider any continuous norm $\|.\|_L$ on the $\mathbb{C}$-analytic line bundle L_{an} defined by L on the compact and reduced complex analytic space $X(\mathbb{C})$. Then, for any integer n, the space of algebraic regular sections $\Gamma(X, L^{\otimes n})$ may be identified with a subspace of the space of continuous sections of $L_{an}^{\otimes n}$ over $X(\mathbb{C})$. THus it is endowed with the restriction of the L^∞-norm, defined by:

$$\|s\|_{\mathrm{L}^\infty,n} := \sup_{x \in X(\mathbb{C})} \|s(x)\|_{L^{\otimes n}} \quad \text{for any } s \in \Gamma(X, L^{\otimes n}), \tag{5.1}$$

where $\|.\|_{L^{\otimes n}}$ denotes the continuous norm on $L_{an}^{\otimes n}$ deduced from $\|.\|_L$ by taking the n-th tensor power.

This construction admits a variant where, instead of the sup-norms (5.1), we may consider the L^p-norms defined by using some "Lebesgue measure" (*cf.* [12], 4.1.3, and [38], Théorème 3.10).

3. When $k = \mathbb{R}$ and X is reduced, the previous constructions define complex norms on the complex vector spaces

$$\Gamma(X, L^{\otimes n}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \Gamma(X_{\mathbb{C}}, L_{\mathbb{C}}^{\otimes n})$$

and, by restriction, real norms on the real vector spaces $\Gamma(X, L^{\otimes n})$.

For any given k, X, and L as above, we shall say that two sequences $(\|.\|_n)_{n\in\mathbb{N}}$ and $(\|.\|'_n)_{n\in\mathbb{N}}$ of norms on the finite k-dimensional vector spaces $(\Gamma(X, L^{\otimes n}))_{n\in\mathbb{N}}$ are *equivalent* when, for some positive constant C and any positive integer n,

$$C^{-n} \|.\|'_n \leq \|.\|_n \leq C^n \|.\|'_n .$$

One easily checks that the previous constructions provide sequences of norms $(\|.\|_n)_{n\in\mathbb{N}}$ on the spaces $(\Gamma(X, L^{\otimes n}))_{n\in\mathbb{N}}$ which are all equivalent. A sequence of norms on these spaces equivalent to one (or, equivalently, to any) of the sequences thus constructed will be called *consistent*. This notion immediately extends to sequences $(\|.\|_n)_{n\geq n_0}$ of norms on the spaces $\Gamma(X, L^{\otimes n})$ defined for n large enough.

When the line bundle L is ample, consistent sequences of norms are provided by additional constructions. Indeed we have:

Proposition 5.1. *Let k be a local field, X a projective scheme over k, and L an* ample *line bundle over X. Let moreover Y be a closed subscheme of X, and assume X and Y reduced when k is archimedean.*

For any consistent sequence of norms $(\|.\|_n)_{n\in\mathbb{N}}$ on $(\Gamma(X, L^{\otimes n}))_{n\in\mathbb{N}}$, the quotient norms $(\|.\|'_n)_{n\in\mathbb{N}}$ on the spaces $(\Gamma(Y, L_{|Y}^{\otimes n}))_{n\geq n_0}$, deduced from the norms $\|.\|_n$ by means of the restriction maps $\Gamma(X, L^{\otimes n}) \longrightarrow \Gamma(Y, L_{|Y}^{\otimes n})$ — which are surjective for $n \geq n_0$ large enough since L is ample — *constitute a consistent sequence.*

When k is archimedean, this is proved in [13], Appendix, by introducing a positive metric on L, as a consequence of Grauert's finiteness theorem for pseudo-convex domains applied to the unit disk bundle of $\check{L}$ (see also [38]). When k is a p-adic field with ring of integers $\mathcal{O}$, Proposition 5.1 follows from the basic properties of ample line bundles over projective $\mathcal{O}$-schemes.

Let E be a finite dimensional vector space over the local field k, equipped with some norm, supposed to be euclidean or hermitian in the archimedean case. This norm induces similar norms on the tensor powers $E^{\otimes n}$, $n \in \mathbb{N}$, hence — by taking the quotient norms — on the symmetric powers $S^n E$. If X is the projective space $\mathbb{P}(E) := \mathbf{Proj}\,\mathrm{Sym}^{\cdot}(E)$ and L the line bundle $\mathcal{O}(1)$, then the canonical isomorphisms $S^n E \simeq \Gamma(X, L^{\otimes n})$ allow one to see these norms as a sequence of norms on $(\Gamma(X, L^{\otimes n}))_{n\in\mathbb{N}}$. One easily checks that this sequence is consistent[9].

For any closed subvariety Y in $\mathbb{P}(E)$ and any $n \in \mathbb{N}$, we may consider the commutative diagram of k-linear maps:

$$\begin{array}{ccccc} S^n E & \xrightarrow{\sim} & S^n\Gamma(\mathbb{P}(E), \mathcal{O}(1)) & \xrightarrow{\sim} & \Gamma(\mathbb{P}(E), \mathcal{O}(n)) \\ & & \downarrow & & \downarrow \alpha_n \\ & & S^n\Gamma(Y, \mathcal{O}(1)) & \xrightarrow{\beta_n} & \Gamma(Y, \mathcal{O}(n)) \end{array}$$

where the vertical maps are the obvious restriction morphisms. The maps α_n, and consequently β_n, are surjective if n is large enough.

Together with Proposition 5.1, these observations yield the following corollary:

Corollary 5.2. *Let k, E and Y a closed subscheme[10] of $\mathbb{P}(E)$ be as above. Let us choose a norm on E (resp. on $\Gamma(Y, \mathcal{O}(1))$) and let us equip $S^n E$ (resp. $S^n\Gamma(Y, \mathcal{O}(1))$) with the induced norm, for any $n \in \mathbb{N}$.*

Then the sequence of quotient norms on $\Gamma(Y, \mathcal{O}(n))$ defined for n large enough by means of the surjective morphisms $\alpha_n : S^n E \longrightarrow \Gamma(Y, \mathcal{O}(n))$ (resp. $\beta_n : S^n\Gamma(Y, \mathcal{O}(1)) \longrightarrow \Gamma(Y, \mathcal{O}(n))$) is consistent.

5.2. Conditions L and L_v. Let k be a local field, and X a projective integral scheme over k, equipped with an ample line bundle L. Moreover let m $\hat{V} \hookrightarrow \hat{X}_P$ be a smooth formal germ in X through a point $P \in X(k)$, and consider its tangent space $T_P\hat{V}$, the fiber L_P of L at P, and the evaluation maps

$$\gamma_D^i : \Gamma(X, \mathcal{I}_{V_{i-1}} \otimes L^{\otimes D}) \longrightarrow S^i \check{T}_{\hat{V}} \otimes L_P^{\otimes D} \tag{5.2}$$

introduced in Section 2.

Let us choose a consistent sequence of norms $(\|.\|_n)_{n\in\mathbb{N}}$ on the k-vector spaces $(\Gamma(X, L^{\otimes n}))_{n\in\mathbb{N}}$, and arbitrary norms $\|.\|_{T_P\hat{V}}$ on $T_P\hat{V}$ and $\|.\|_{L_P}$ on L_P. Then we may consider the operator norms $\left\|\gamma_D^i\right\|$ of the maps (5.2) and their logarithms $\log\left\|\gamma_D^i\right\|$ in $[-\infty, +\infty[$.

We shall say that $\hat{V}$ *satisfies condition* L when

$$\lim_{i/D\to+\infty} \frac{1}{i} \log\left\|\gamma_D^i\right\| = -\infty. \tag{5.3}$$

Clearly this condition does not depend on the above choices of norms. It is also invariant by extension of the local field k, and is easily seen not to depend on the

[9] This is straightforward in the p-adic case. When k is archimedean, this follows for instance from [15], Lemma 4.3.6.

[10] reduced if k is archimedean.

choice of the ample line bundle L. Proposition 5.1 also implies that it is invariant under "reembedding" of X into some larger projective variety.

Moreover condition L is birationally invariant in the following sense: if $X' \cdots \to X$ is a birational map between projective varieties over k that define an isomorphism $f : U' \xrightarrow{\sim} U$ between non-empty open subvarieties in X' and X, and if P belongs to $U(k)$, then a smooth formal germ $\hat{V}$ through P in X satisfies L iff the smooth formal germ $f^*\hat{V}$ through $P' := f^{-1}_{|U'}(P)$ in X' does. (See [13], 3.2, when k is archimedean and $\dim \hat{V} = 1$; the general case is similar.)

As a consequence, condition L makes sense for a smooth formal germ $\hat{V} \hookrightarrow \hat{X}_P$ through a k-rational point in a general algebraic variety X over k — namely, if U is any quasi-projective open neighbourhood of P in X and if $\tilde{X}$ is a projective completion of U, we shall say that $\hat{V}$ satisfies L when $\hat{V}$ seen as a formal subscheme of $\tilde{X}$ satisfies it. Again, this condition is invariant under extension of k and reembedding of X.

Finally, if K is a number field and v a place of K, we shall say that a smooth formal subscheme $\hat{V} \hookrightarrow \hat{X}_P$ through a rational point P in a variety X over K satisfies *condition* L_v when the formal subscheme $\hat{V}_{K_v}$ through P in X_{K_v}, deduced from $\hat{V}$ by extension of scalars from K to the completion K_v of K at v satisfies condition L over the local field K_v.

5.3. Condition L over $\mathbb{C}$ and Liouville complex manifolds. A connected complex manifold M is said to satisfy the *Liouville property*, or to be a *Liouville complex manifold,* when every bounded plurisubharmonic function on M is constant. In particular, the connected Riemann surfaces satisfying the Liouville property are precisely the ones which are "parabolic" in the sense of Myrberg, or equivalently, have "null-boundary" in the sense of R. Nevanlinna.

The following observations are straightforward consequences of the basic properties of plurisubharmonic functions and algebraic varieties:

1. Let $\pi : M \longrightarrow N$ be a surjective analytic map between connected complex manifolds. If M is a Liouville, then N is Liouville. Conversely, when π has smooth connected fibers, if N and the fibers of π are Liouville, then M also is Liouville.

2. The complement of any closed pluri-polar subset (for instance, a lower dimensional analytic subset) in a Liouville complex manifold is again Liouville.

3. Any compact connected complex manifold is Liouville.

4. The manifold of complex points of any smooth connected complex algebraic variety is Liouville.

5. Any connected complex Lie group is a Liouville complex manifold.

Over archimedean local fields, the property L may be checked in various significant cases by means of the following criterion:

Proposition 5.3 ([12], Proposition 4.12)). *Let X be a complex algebraic variety, P a point in $X(\mathbb{C})$, and $\hat{V} \hookrightarrow \hat{X}_P$ a smooth formal germ through P.*

Let us assume that there exist a connected complex manifold M, a point O in M, and a $\mathbb{C}$-analytic map $\varphi : M \longrightarrow X(\mathbb{C})$ sending O to P that induces an isomorphism of formal germs

$$\hat{\varphi}_O : \hat{M}_O \xrightarrow{\sim} \hat{V}. \tag{5.4}$$

If furthermore M is Liouville, then $\hat{V}$ satisfies L.

5.4. Germs of analytic curves in algebraic varieties over local fields and canonical semi-norms.

In this paragraph, we return to the notation of the beginning of 5.2, and we assume that the smooth formal germ $\hat{V}$ is *one-dimensional* and *k-analytic*. Then, by means of the evaluation maps γ_D^i (see (5.2)) and their operator norms, as in the definition of condition L by (5.3), we may define some *canonical semi-norm* $\|.\|_{P,\hat{V}}^{\mathrm{can}}$ on the k-line $T_P\hat{V}$ as follows. We consider

$$\rho := \limsup_{i/D \to +\infty} \frac{1}{i} \log \left\| \gamma_D^i \right\|.$$

A straightforward application of Cauchy's inequalities shows that it belongs to $[-\infty, +\infty[$, and therefore by setting

$$\|.\|_{P,\hat{V}}^{\mathrm{can}} := e^{\rho} \|.\|_{T_P\hat{V}},$$

we define a semi-norm on $T_P\hat{V}$.

For a given projective variety X containing $\hat{V}$, one easily checks that it depends neither on the auxiliary choices of norms, nor on the ample line bundle L. Actually, like condition L, the canonical semi-norm $\|.\|_{P,\hat{V}}^{\mathrm{can}}$ is invariant under "reembedding" of X in some larger projective variety, and by birational isomorphisms which are isomorphisms in some neighbourhood of P ([13], 3.2-3, and [14]). Consequently, the canonical semi-norm on $T_P\hat{V}$ may be defined for any smooth analytic germ of curve $\hat{V} \hookrightarrow \hat{X}_P$ through a rational point in a algebraic scheme over k.

In the p-adic case, Cauchy's inequalities lead actually to the following upper bound on the canonical semi-norm in terms of the size relative to some model:

Lemma 5.4. *Let k be a p-adic field, $\mathcal{O}$ its ring of integers, and $\mathcal{X}$ a separated scheme of finite type over $\mathcal{O}$ equipped with a section P. Let $\hat{V}$ be a smooth formal subscheme of the formal completion $\hat{X}_{P_k}$ of $X := \mathcal{X}_k$ at P_k. If $\hat{V}$ is one-dimensional and analytic, and if $\|.\|_{T_P\hat{V}}^{\mathcal{X}}$ denote the p-adic norm on the k-line $T_P\hat{V}$ defined by the integral model $\mathcal{X}$[11], then we have:*

$$\|.\|_{P,\hat{V}}^{\mathrm{can}} \leq S_{\mathcal{X}}(\hat{V})^{-1} . \|.\|_{T_P\hat{V}}^{\mathcal{X}}.$$

Finally observe that the construction of $\|.\|_{P,\hat{V}}^{\mathrm{can}}$ is compatible with (finite degree) extensions of the local field k.

[11] By definition, the unit disk in $\check{T}_P\hat{V}$ equipped with the norm dual to $\|.\|_{T_P\hat{V}}^{\mathcal{X}}$ is the $\mathcal{O}$-lattice image of the composite map $P^*\Omega^1_{\mathcal{X}/\mathcal{O}} \to \Omega^1_{X/k|P_k} \twoheadrightarrow \check{T}_P\hat{V}$.

5.5. Canonical semi-norms and capacity. Recall that, if M is a Riemann surface, O a point of M, and Ω an open neighbourhood of O that is relatively compact in M and has a non-empty sufficiently regular boundary[12], we may consider the *Green function of O in Ω*, namely the continuous function $g_{O,\Omega} : \overline{\Omega} \setminus \{O\} \to \mathbb{R}_+$ which vanishes on the boundary of Ω, is harmonic on $\Omega \setminus \{O\}$, and possesses a logarithmic singularity at O. In other words, if z denotes some holomorphic coordinates on some open neighbourhood U of O, we have

$$g_{O,\Omega} = \log |z - z(O)|^{-1} + h \quad \text{on } U \setminus \{O\},$$

where h is a harmonic function on U. From the value of h at O, one defines the *capacitary norm* $\|.\|^{\mathrm{cap}}_{O,\Omega}$ on the complex line $T_O M = \mathbb{C} \frac{\partial}{\partial z}_{|O}$ by

$$\| \frac{\partial}{\partial z}_{|O} \|^{\mathrm{cap}}_{O,\Omega} := e^{-h(O)} = \lim_{Q \to O} \frac{e^{-g_{O\Omega}(Q)}}{|z(Q) - z(O)|}. \tag{5.5}$$

Proposition 5.5 ([13], Proposition 3.6). *With the above notation, if $f : \Omega \to X$ is a holomorphic map with value in some complex algebraic variety X, and if C denotes some germ of smooth analytic curve through $P := f(O)$ in X such that f maps the germ of Ω at O to C, then, for any v in $T_O M$,*

$$\|Df(O)v\|^{\mathrm{can}}_{P,C} \leq \|v\|^{\mathrm{cap}}_{O,\Omega}.$$

The construction of the Green function $g_{P,\Omega}$ admits analogues over p-adic curves, developed in particular by Rumely [39] and Thuillier [41] (see also [14] for a more "algebraic" approach relying on formal geometry). This Green function makes sense for instance when M is a smooth projective geometrically connected algebraic curve over some p-adic field k, O is some point in $M(k)$, and Ω is defined as the complement of some non-empty affinoid subspace of X that does not contain P. It has a logarithmic singularity at O and the equation (5.5) still makes sense and defines the capacitary norm as a p-adic norm on the k-line $T_O M$. Moreover Proposition 5.5 still holds with k instead of $\mathbb{C}$ as a base field, and "rigid analytic" instead of "holomorphic" (*cf.* [14], Sections 6 and 7).

6. An algebraicity criterion for smooth formal germs in varieties over number fields

6.1. An algebraization theorem. The following theorem provides sufficient conditions of algebraicity for a formal subscheme of the formal completion $\hat{X}_P$ of some algebraic variety X over a number field K at a rational point $P \in X(K)$.

[12] say, a domain with differentiable non-empty boundary, or in other terms, the interior of some connected 2-dimensional submanifold with non-empty boundary. See also [11], especially A.8, for weaker conditions.

Theorem 6.1. *Let X be an algebraic variety over a number field K, P a point in $X(K)$, and $\hat{V}$ a smooth formal subscheme of the completion $\hat{X}_P$ of X at P.*

If $\hat{V}$ is A-analytic and satisfies condition L_v for some place v of K, then $\hat{V}$ is algebraic.

This theorem is proved by using the algebraicity criterion (iii) $\Rightarrow$ (i) in Proposition 2.1. Validity of condition (iii) is derived by means of slope inequalities, involving now heights of K-linear maps and arithmetic slopes attached to hermitian vector bundles over $\operatorname{Spec}\mathcal{O}_K$, in the spirit of the proof of Theorem 3.1. See [12], Section 4 when the place v is archimedean; the general case is similar.

6.2. Algebraic leaves of algebraic foliations over number fields.

Combined with Corollary 4.2 and Proposition 5.3, Theorem 6.1 admits the following consequence:

Theorem 6.2 ([12], Theorem 2.1). *Let X be a smooth algebraic variety of a number field K equipped with an involutive subbundle F, and let P be point in $X(K)$. If (i) the algebraic foliation (X, F) satisfies the Grothendieck-Katz condition, and (ii) for some field embedding $\sigma_0 : K \hookrightarrow \mathbb{C}$, the analytic leaf through P of the complex analytic foliation $(X(\mathbb{C}), F_{\mathbb{C}})$ is Liouville, then the leaf of (X, F) through P is algebraic.*

6.3. Algebraic Lie subalgebras of algebraic groups over number fields.

Let G be an algebraic group over a number field K. For any sufficiently divisible integer N, there exists a model $\mathcal{G}$ of G, *i.e.*, a smooth quasi-projective group scheme over $S := \operatorname{Spec}\mathcal{O}_K[1/N]$ whose generic fiber $\mathcal{G}_K$ coincides with G. The restriction to the zero-section of $\mathcal{G}$ of the relative tangent bundle $T_{\mathcal{G}/S}$ defines the Lie algebra $\operatorname{Lie}\mathcal{G}$ of $\mathcal{G}$: it is a finitely generated projective module and a Lie algebra over $\mathcal{O}_K[1/N]$, and the K-Lie algebra $(\operatorname{Lie}\mathcal{G})_K$ is canonically isomorphic to $\operatorname{Lie} G$.

Moreover, for every maximal ideal $\mathbf{p}$ of $\mathcal{O}_K[1/N]$ with residue field $\mathbb{F}_{\mathbf{p}}$ and characteristic p, the $\mathbb{F}_{\mathbf{p}}$-Lie algebra $(\operatorname{Lie}\mathcal{G})_{\mathbb{F}_{\mathbf{p}}}$ is canonically isomorphic to the Lie algebra of the smooth algebraic group $\mathcal{G}_{\mathbb{F}_{\mathbf{p}}}$ over the finite field $\mathbb{F}_{\mathbf{p}}$, and is therefore endowed with a p-th power map, given by the restriction of the p-th power map on global sections of $T_{\mathcal{G}_{\mathbb{F}_{\mathbf{p}}}}$ to the left-invariant ones.

For translation invariant foliations on G, Theorem 6.2 takes the following form which proves a conjecture of Ekedahl, Shepherd-Barron, and Taylor ([24]):

Theorem 6.3 ([12], Theorem 2.3). *For any Lie subalgebra $\mathbf{h}$ of $\operatorname{Lie} G$ (defined over K), the following two conditions are equivalent:*

(i) For almost every maximal ideal $\mathbf{p}$ of $\mathcal{O}_K[1/N]$, the $\mathbb{F}_{\mathbf{p}}$-Lie subalgebra $(\mathbf{h} \cap \operatorname{Lie}\mathcal{G})_{\mathbb{F}_{\mathbf{p}}}$ of $\operatorname{Lie}\mathcal{G}_{\mathbb{F}_{\mathbf{p}}}$ is closed under p-th powers.

(ii) $\mathbf{h}$ is an algebraic Lie subalgebra of $\operatorname{Lie} G$.

6.4. Ogus conjecture on absolute Tate cycles in abelian varieties.

Theorem 6.3 may be extended to the case where the field K is any extension of finite type of $\mathbb{Q}$, in the spirit of the original formulation of the Grothendieck-

Katz conjecture [31]. In this section, we discuss some consequence of this generalization.

Let K be a field of characteristic zero, extension of finite type of $\mathbb{Q}$, and let X be a proper and smooth scheme over K. We can find models of X and K which are smooth over $\operatorname{Spec}\mathbb{Z}$ — namely an integral affine scheme $S = \operatorname{Spec} R$ smooth over $\operatorname{Spec}\mathbb{Z}$ such that K is the function field $\kappa(S)$ of S, and a proper and smooth scheme $\mathcal{X}$ over S such that $X = \mathcal{X}_K$. After possibly shrinking S, we can also assume that the Hodge cohomology groups $H^q(\mathcal{X}, \Omega^p_{\mathcal{X}/S})$ are flat R-modules. This implies that the formation of the (relative) Hodge and de Rham cohomology groups of $\mathcal{X}$ is compatible with any base change $S' \to S$ and the degeneracy of the "Hodge to de Rham" spectral sequence.

Let k be a perfect field of characteristic $p > 0$, W its ring of Witt vectors, $\overline{\sigma}$ a point in $S(k)$, and σ a lift of $\overline{\sigma}$ in $S(W)$. The crystalline cohomology groups $H^i_{\mathrm{cris}}(\mathcal{X}_{\overline{\sigma}}/W)$ are W-modules functorially attached to the k-scheme $\mathcal{X}_{\overline{\sigma}}$. In particular, the absolute Frobenius endomorphism of $\mathcal{X}_{\overline{\sigma}}$ induces a Frobenius-linear endomorphism Φ of the W-module $H^i_{\mathrm{cris}}(\mathcal{X}_{\overline{\sigma}}/W)$. Besides, the comparison theorem of Berthelot ([8]) provides a canonical isomorphism from this W-module onto the de Rham cohomology $H^i_{\mathrm{dR}}(\mathcal{X}_\sigma/W) \simeq H^i_{\mathrm{dR}}(\mathcal{X}/R) \otimes_\sigma W$. Consequently, Φ defines a semi-linear endomorphism Φ_σ of the W-module $H^i_{\mathrm{dR}}(\mathcal{X}/R) \otimes_\sigma W$.

Let r be an integer in $\{0, \ldots, \dim X\}$. Following the terminology of Ogus in [36], a class ξ in the algebraic de Rham cohomology group $H^{2r}_{\mathrm{dR}}(\mathcal{X}/S)$ over R is said to be *absolutely Tate* iff, for any p, k, and σ as above[13], the equality

$$\Phi_\sigma(\xi) = p^r \xi$$

holds in $H^{2r}_{\mathrm{dR}}(\mathcal{X}/S) \otimes_\sigma W$. More generally a class ξ in $H^{2r}_{\mathrm{dR}}(X/K)$ is said to be absolutely Tate if, after possibly replacing S by some non-empty affine open subscheme, it is absolutely Tate in $H^{2r}_{\mathrm{dR}}(\mathcal{X}/S)$.

For instance, any algebraic class, namely any class in the image of the "cycle class" map $Z^r(X)_{\mathbb{Q}} \longrightarrow H^{2r}_{\mathrm{dR}}(X/K)$, is absolutely Tate. Ogus ([36], Section 2) conjectured that the converse holds, that is:

$\mathrm{O}(X, r)$: *every absolutely Tate class in* $H^{2r}_{\mathrm{dR}}(X/K)$ *is algebraic.*

As a consequence of the aforementioned generalization[14] of Theorem 6.3 to algebraic groups over K, we can prove:

Theorem 6.4. *For any field extension K of finite type of $\mathbb{Q}$, the conjecture $\mathrm{O}(X, r)$ holds when X is an abelian variety over K and $r = 1$.*

See [3], Section 7.4, for a related result, which characterizes the K-linear maps between the first de Rham cohomology groups of abelian varieties over a number

[13] Actually, one might consider only some "limited" classes of fields k and points $\overline{\sigma}$ in $\mathcal{X}(k)$ — for instance, closed points, or geometric generic points of the fibres of $S \to \operatorname{Spec}\mathbb{Z}$ — which still define the same condition.

[14] This generalization concerns an algebraic group G over K, and a smooth group scheme $\mathcal{G}$ extending it over S; in condition (i) in Theorem 6.3, one now requires the p-closure condition to hold for every closed point $\mathbf{p}$ in some non-empty open subscheme of S over which $\mathbf{h}$ extends as a subvector bundle of $\operatorname{Lie}\mathcal{G}$. Its proof relies on the fact that a complex manifold is Liouville when it may be fibered over a complex algebraic variety with fibers some complex Lie groups.

field K induced by morphisms of K-varieties as the ones compatible with (almost all) the crystalline Frobenius maps[15].

The derivation of Theorem 6.4 relies on the identification of the Lie algebra of the universal vector extension of the dual abelian variety $\hat{X}$ with the de Rham cohomology group $H^1_{\mathrm{dR}}(X/K)$, and on the fact that the p-th power map on the reduction of this Lie algebra at some closed point $\mathbf{p}$ in S of residue characteristic p coincides with the reduction modulo p of the crystalline Frobenius at $\mathbf{p}$.

7. An algebraicity criterion for smooth formal curves in varieties over number fields

7.1. Normed and semi-normed lines over number fields. We define a *normed line*

$$\overline{L} := (L_K, (\|.\|_{\mathbf{p}}), (\|.\|_\sigma))$$

over a number field K as the data of a rank one K-vector space L_K, of a family $(\|.\|_{\mathbf{p}})$ of $\mathbf{p}$-adic norms on the $K_{\mathbf{p}}$-lines $L_K \otimes_K K_{\mathbf{p}}$ indexed by the non-zero prime ideals $\mathbf{p}$ of $\mathcal{O}_K$, and of a family $(\|.\|_\sigma)$ of hermitian norms on the complex lines $L_K \otimes_{K,\sigma} \mathbb{C}$, indexed by the fields embeddings $\sigma : K \hookrightarrow \mathbb{C}$. Moreover the family $(\|.\|_\sigma)$ is required to be stable under complex conjugation[16].

We shall say that a normed K-line is *summable* if for some (or equivalently, for any) non-zero element l of L_K, the family of real numbers $(\log \|l\|_{\mathbf{p}})_{\mathbf{p}}$ is summable. Then we may define its *Arakelov degree* as the real number

$$\widehat{\deg}\,\overline{L} := \sum_{\mathbf{p}} \log \|l\|_{\mathbf{p}}^{-1} + \sum_{\sigma} \log \|l\|_\sigma^{-1}. \tag{7.1}$$

Indeed, by the product formula, the right-hand side of (7.1) does not depend on the choice of l.

Observe that hermitian line bundles over $\operatorname{Spec} \mathcal{O}_K$, as usually defined in Arakelov geometry, provide examples of normed lines over K: if $\overline{\mathcal{L}} = (\mathcal{L}, (\|.\|_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}})$ is such an hermitian line bundle — so $\mathcal{L}$ is a projective $\mathcal{O}_K$-module of rank 1, and $(\|.\|_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}})$ is a family, invariant under complex conjugation, of norms on the complex lines $\mathcal{L}_\sigma := \mathcal{L} \otimes_{\sigma:\mathcal{O}_K\to\mathbb{C}} \mathbb{C}$ — the corresponding normed K-line is $\mathcal{L}_K$ equipped with the p-adic norms defined by the $\mathcal{O}_{\mathbf{p}}$-lattices $\mathcal{L}\otimes_{\mathcal{O}_K} \mathcal{O}_{\mathbf{p}}$ in $\mathcal{L}\otimes_{\mathcal{O}_K} K_{\mathbf{p}}$ $\simeq L \otimes_K K_{\mathbf{p}}$ and with the hermitian norms $(\|.\|_\sigma)$. The so-defined normed lines are

[15]The author became aware of Ogus conjecture in the above formulation while reading a preliminary version of Y. André's beautiful survey on motives [3], and realized that, when K is a number field, X an abelian variety, and $r = 1$, it would be a consequence of Theorem 6.3. Actually, in [36], Section 2, Ogus formulate more general conjectures, stated in terms of the conjugate filtration on $H^*_{\mathrm{dR}}(X/K)$ and its reductions, and asks explicitly about the validity of $\mathrm{O}(X, r)$ only when K is a number field.

[16]The data of these families of norms is equivalent to the data of a family $(\|.\|_v)_v$, indexed by the set of all places v of K, of v-adic norms on the rank one vector spaces $L_v := L_K \otimes_K K_v$ over the v-adic completions K_v of K.

summable, and their Arakelov degree, as defined by (7.1), coincide with the usual Arakelov degree of hermitian line bundles.

It is convenient to extend the definitions of normed lines and Arakelov degree as follows: we shall define a *semi-normed K-line* $\overline{L}$ as a K-vector space L_K of rank one, equipped with families of *semi-norms* $(\|.\|_{\mathbf{p}})$ and $(\|.\|_\sigma)$, where the latter is assumed to be stable under complex conjugation. In other words, we allow some of the $\|.\|_{\mathbf{p}}$ or $\|.\|_\sigma$ to vanish.

We shall say that the Arakelov degree of a semi-normed K-line $\overline{L}$ is *well-defined* if, for some (or equivalently, for any), non-zero element l of L_K, the family of real numbers $(\log^+ \|l\|_{\mathbf{p}})_{\mathbf{p}}$ is summable. Then we may again define its Arakelov degree by means of (7.1), where we follow the usual convention $\log 0^{-1} = +\infty$. It is an element of $]-\infty, +\infty]$.

7.2. Arithmetic positivity and algebraicity of A-germs of curves.

The following algebraicity criterion is a refined version of Theorem 6.1, concerning formal germs of curves:

Theorem 7.1 ([14]). *Let X be an algebraic variety over a number field K, P a point in $X(K)$, and $\hat{V}$ a smooth formal subscheme of dimension 1 in the completion $\hat{X}_P$ of X at P.*

Assume that the following two conditions are satisfied:
(i) $\hat{V}$ is A-analytic;
(ii) the semi-normed K-line

$$\overline{T}_P\hat{V}^{\mathrm{can}} := (T_P\hat{V}, (\|.\|^{\mathrm{can}}_{P,\hat{V}_{K_{\mathbf{p}}}}), (\|.\|^{\mathrm{can}}_{P,\hat{V}_\sigma})),$$

defined by endowing $T_P\hat{V}$ with its canonical semi-norm at every place of K, satisfies

$$\widehat{\deg}\,\overline{T}_P\hat{V}^{\mathrm{can}} > 0. \tag{7.2}$$

Then $\hat{V}$ is algebraic.

Observe that, as a consequence of Lemma 5.4, the Arakelov degree of $\overline{T}_P\hat{V}^{\mathrm{can}}$ is well defined in $]-\infty, +\infty]$ when Condition (i) is satisfied. Moreover it takes the value $+\infty$ if there exists some place v of K such that Condition L_v is satisfied.

7.3. A rationality criterion for formal germs of functions on algebraic curves over number fields.

Let K be a number field, $\mathcal{C}$ a regular projective arithmetic surface over $\operatorname{Spec}\mathcal{O}_K$ whose generic fiber $C := \mathcal{X}_K$ is geometrically connected, P a point in $X(K)$, and $\mathcal{P}$ in $\mathcal{X}(\mathcal{O}_K)$ extending P.

Let F be a finite set of closed points in $\operatorname{Spec}\mathcal{O}_K$ and, for any $\mathbf{p}$ in F, let $\Omega_{\mathbf{p}}$ be the complement in the rigid curve $X_{K_{\mathbf{p}}}$ of some affinoid not containing $P_{K_{\mathbf{p}}}$. Moreover, for any embedding $\sigma : K \to \mathbb{C}$, let Ω_σ be an open neighbourhood of P_σ in the Riemann surface $C_\sigma(\mathbb{C})$, which for simplicity we suppose to be domains with differentiable non-empty boundaries. We shall assume that the data of the Ω_σ's are compatible with complex conjugation, namely, that for any embedding σ, $\Omega_{\overline{\sigma}}$ is the complex conjugate of Ω_σ.

Let finally $\overline{T}_P^{\mathrm{cap}} C := (T_P C, (\|.\|_{\mathbf{p}}), (\|.\|_\sigma))$ be the semi-normed line over K defined by the tangent line T_C of C at P, equipped with the $\mathbf{p}$-adic norm $\|.\|_{\mathbf{p}} := \|.\|^{\mathrm{cap}}_{P_{K_{\mathbf{p}}},\Omega_{\mathbf{p}}}$ if $\mathbf{p}$ belongs to F, and otherwise with the $\mathbf{p}$-adic norm deduced from the integral structure on $N_{\mathcal{P}}\mathcal{X}$ (through the isomorphism $T_P C \otimes_K K_{\mathbf{p}} \simeq N_{\mathcal{P}}\mathcal{X} \otimes_{\mathcal{O}_K} K_{\mathbf{p}}$). Its Arakelov degree is clearly defined.

The following theorem extends the classical rationality criteria of Borel-Dwork and Polya-Bertrandias (*cf.* [21] and [1], Chapter 5); one recovers them in the special case $C = \mathbb{P}^1_K$.

Theorem 7.2 ([14]). *With the above notation, let $\varphi \in \hat{\mathcal{O}}_{C,P}$ be a formal germ of function on C at P, and assume that the following conditions are satisfied:*
(i) for any $\mathbf{p}$ in F (resp. any embedding $\sigma : k \hookrightarrow \mathbb{C}$), after the base change $K \hookrightarrow K_{\mathbf{p}}$ (resp. σ), φ extends to a rigid meromorphic function on $\Omega_{\mathbf{p}}$ (resp. to a meromorphic function on Ω_σ);
(ii) the formal function φ extends to a formal rational function on the completion $\hat{\mathcal{X}}_{\mathcal{P}}$ over $\operatorname{Spec} \mathcal{O}_K \setminus F$;
(iii) $\widehat{\deg}\, \overline{T}_P^{\mathrm{cap}} C > 0$.
Then φ is rational, i.e.*, is an element of the local ring $\mathcal{O}_{X,P} \subset \hat{\mathcal{O}}_{X,P}$.*

To establish Theorem 7.2, we consider the graph of φ. It is a formal germ of curve through the point $(P, \varphi(P))$ in the algebraic variety $X := C \times \mathbb{P}^1_K$ over K. From Theorem 7.1, Proposition 5.5, and its p-adic analogue, we derive that this graph is algebraic. Finally, we show that it is the germ of graph of some rational function on C by applying a generalization of the connectedness theorems in [11], Section 4.

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