A. JAVANPEYKAR. — Arakelov geometry on arithmetic surfaces

Hironaka's theorem on formal functions

Theorem (Hironaka 1968) $k = \bar{k}$ field

X smooth projective connected surface $/k$

$Y \hookrightarrow X$ smooth projective connected curve

$\hat{X}_Y$ = completion of X along Y

If $Y \cdot Y > 0$, then $K(X) = K(\hat{X})$

function field of X

K sheaf on $|\hat{X}| = Y$ which associates to

$\mathcal{U} \subseteq Y$ open affine the function field

of $\mathcal{O}_{\hat{X}}(\mathcal{U}) = \varprojlim \mathcal{O}_{Y_n}(\mathcal{U})$

any element in $K(X)$ induces an element of $K(\hat{X})$.

Counter-example: $X = Bl_p \mathbb{P}^2$, Y = exceptional divisor

Strategy of the proof

Step 1: $K(X) \hookrightarrow K(\hat{X})$ algebraic

Step 2: $K(X) \hookrightarrow K(\hat{X})$ is "algebraically closed"

"Function field" application

B = smooth proj. connected curve $/_k$ "base curve"

$\mathfrak{X} \xrightarrow{\pi} B$ fibered surface, ρ a section of π

reg. 2-dim scheme

π: flat proj. whose geom. fibers are connected (N.B. $\mathfrak{X} \to \operatorname{Spec} k$ is projective)

$\rho \in \mathfrak{X}(B)$ section

If $f \in \widehat{\mathcal{O}}_{X,P}$ extends to a formal rational function on $\hat{X}_\rho$ and $\deg(\rho^* T_\pi) > 0$,

$X = \mathfrak{X}_{K(B)}$

$\rho_{K(B)} = P$

then f is a formal germ of a rational function on X.

Proof. $\deg(\rho^* T_\pi) = \rho.\rho > 0$ □

Proof of Step 2

Lemma ① "connectedness lemma"

X sm. proj. connected surface $/k$, D effective divisor,

If D is big and nef, then $\mathrm{Supp}(D)$ is connected.

(i.e. $D \cdot D > 0$ and for all $C \hookrightarrow X$ integral curve $D \cdot C \geq 0$)

Proof. We show that D is numerically connected.

$$D = D_1 + D_2 \qquad D_1 \geq 0 \qquad D_2 \geq 0 \qquad D_1 \neq 0 \qquad D_2 \neq 0$$

To show $D_1 \cdot D_2 > 0$

Assume for a contradiction $D_1 \cdot D_2 \leq 0$

$$D \text{ nef} \Rightarrow \begin{array}{l} D \cdot D_1 \geq 0 \\ D \cdot D_2 \geq 0 \end{array} \Rightarrow \begin{array}{l} D_1 \cdot D_1 \geq -D_1 \cdot D_2 \\ D_1 \cdot D_2 \geq -D_1 \cdot D_2 \end{array}$$

$$\Rightarrow (D_1 \cdot D_1)(D_2 \cdot D_2) \geq (D_1 \cdot D_2)^2$$

$$\Rightarrow (D_1 \cdot D_1)(D_2 \cdot D_2) - (D_1 \cdot D_2)^2 \geq 0$$

$$\Rightarrow \det \begin{pmatrix} D_1 D_1 & D_1 D_2 \\ D_1 D_2 & D_2 D_2 \end{pmatrix} \geq 0$$

HIT III however implies since $D^2 > 0$

$$\det \begin{pmatrix} D_1 \cdot D_1 & D_1 D_2 \\ D_1 D_2 & D_2 D_2 \end{pmatrix} \leq 0$$

$\det = 0 \Rightarrow \exists \lambda \in \mathbb{R}_{\geq 0}$

$[D_1] = \lambda [D_2]$ in $\mathrm{Num}(X)_{\mathbb{R}}$

$\lambda D_1 \cdot D_1 = D_1 \cdot D_2 \leq 0$ (by assumption)

$(1+\lambda)^2 D_1 \cdot D_1 \leq 0$

$$\overset{\shortparallel}{((1+\lambda)D_1)\,((1+\lambda)D_1)} = (D_1 + D_2)(D_1 + D_2)$$

$$= D^2 > 0 \quad \text{↯}$$

$\square$

(C.P. Ramanujam)

To conclude the proof of Step 2, let $\varphi \in K(\hat{X})$ which is algebraic over $K(X)$.

$K(X) \subseteq K(X)(\varphi)$ is finite

X' = normalization of X in $k(X)(\varphi)$

$$\begin{array}{c} X' \\ f \big\downarrow \text{finite} \\ X \hookrightarrow Y \end{array} \Rightarrow k(X)(\varphi) = k(X).$$

$$\Rightarrow \varphi \in k(X)$$

$\square$

<u>Lemma ①</u>: f is birational

$\Downarrow$ $\nearrow$

$f^{-1}(Y)$ will be connected

<u>Aim</u>: <u>state</u> and prove "the arithmetic analogue" of Step 2.

- Intersection theory — Arakelov
- HIT

Fibered surfaces

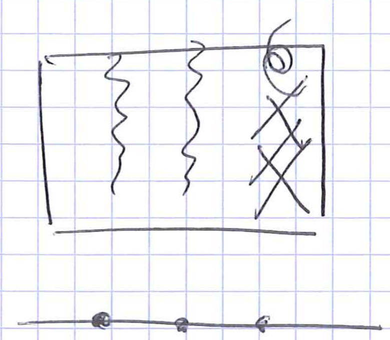
B "base curve" $\mathbb{A}^1_k$, $\mathbb{P}^1_k$, $\operatorname{Spec} \mathbb{Z}$, $\operatorname{Spec} \mathbb{Z}_p$, $\operatorname{Spec} \mathbb{C}[t]$

integral noetherian regular 1-dim scheme

$\mathfrak{X} \xrightarrow{\pi} B$ flat projective (surjective)

$\uparrow$ integral regular scheme of dim 2 with geo. connected fibers

<u>Ex</u>: $\begin{array}{c} \mathbb{P}^1_B \\ \downarrow \\ B \end{array}$ $\qquad \begin{array}{c} \{y^2 z = x(x-z)(x-\lambda z)\} \subseteq \mathbb{P}^2_{\mathbb{A}^1} \\ \downarrow \\ \mathbb{A}^1_{\mathbb{C}} \ni \lambda \end{array}$



$D \hookrightarrow \mathcal{X}$ integral divisor

$$\pi(D) = \begin{cases} pt & D \text{ is } \underline{\text{vertical}} \\ B & D \text{ is } \underline{\text{horizontal}} \end{cases}$$

$b \in B$ closed point

$$\mathrm{Div}_b(\mathcal{X}) = \{ D \in \mathrm{Div}(\mathcal{X}) \mid \pi(D) = \{b\} \}$$
$$\subseteq \mathrm{Div}(\mathcal{X}) \text{ subgroup}$$

<u>Message</u>: $\exists \ \mathrm{Div}(\mathcal{X}) \times \mathrm{Div}_b(\mathcal{X}) \xrightarrow[\text{pairing}]{\text{intersection}} \mathbb{Z}$

<u>Theorem</u>: $\exists!$ bilinear map $\mathrm{Div}(\mathcal{X}) \times \mathrm{Div}_b(\mathcal{X}) \xrightarrow{\text{in}} \mathbb{Z}$

such that

1) $D \in \mathrm{Div}(\mathcal{X})$, $E \in \mathrm{Div}_b(\mathcal{X})$ have no common component

$$i_b(D, E) = \sum_{\substack{x \in \mathcal{X} \\ \pi(x) = b}} i_x(D, E)\,[\kappa(x) : \kappa(b)]$$

$$i_x(D,E) = \text{length}\left(\mathcal{O}_{X,x} / (I_{D,x} + I_{E,x})\right)$$

2) $i_b|_{\mathrm{Div}_b(\mathcal{X}) \times \mathrm{Div}_b(\mathcal{X})}$ symmetric

3) If $D, D' \in \mathrm{Div}(\mathcal{X})$ such that

$$D = D' + \mathrm{div}(f) \qquad f \in K(X)^*$$

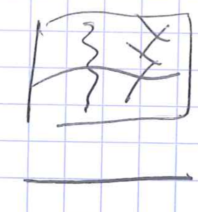
then $i_b(D, E) = i_b(D', E)$.

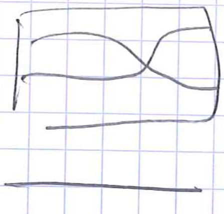
4) If $0 \leq E \leq \mathcal{X}_b$, then $i_b(D, E)$

$$= \deg \mathcal{O}_{\mathcal{X}}(D)|_E$$

5) $\langle , \rangle : \mathrm{Div}_b(\mathcal{X})_{\mathbb{R}} \times \mathrm{Div}_b(\mathcal{X})_{\mathbb{R}} \to \mathbb{R}$

is negative semi-definite.





Ex: $\mathrm{Pic}(B) = 0$ e.g. $B = \mathrm{Spec}\,\mathbb{Z}$ or $\mathbb{A}^1_{\mathbb{C}}$

$b \in B$ is a "principal divisor"

$\mathcal{X}_b = \pi^* b$ is a principal divisor

If E is vertical 3) $\Rightarrow$ $\pi^* b \cdot E = 0$

Remark: If $\rho : B \to \mathcal{X}$, then $\mathcal{X}_b \cdot \rho \neq 0$

so linear equivalence is not preserved.

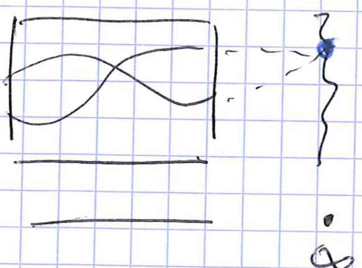
$B = \mathbb{A}^1_{\mathbb{C}}$

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "\pi"'] & \boxed{\overline{\mathcal{X}}} \arrow[d, "\pi"] \\ b \in \mathbb{A}^1_{\mathbb{C}} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd}$$

$\Rightarrow b$ is not principal

int. pairing extends to

$\mathrm{Div}(\overline{\mathcal{X}}) \times \mathrm{Div}(\overline{\mathcal{X}}) \to \mathbb{Z}$



$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d] & \widehat{\mathcal{X}} \arrow[d] \\ \mathrm{Spec}\,\mathbb{Z} \arrow[r, hook] & \widehat{\mathrm{Spec}\,\mathbb{Z}} \end{tikzcd}$$

?

Arakelov's intersection pairing

Use complex analysis on $\mathcal{X}(\mathbb{C})$ Riemann surface

Analytic: X compact connected R-S.

$g(X) \geq 1$ $\quad \omega_1, \dots \omega_g$ basis of $H^0(X, \Omega^1_X)$

orthonormal for

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_X \omega \wedge \bar{\eta}$$

Def. $\mu_{Ar} = \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \bar{\omega}_k$ independent of the choice of orthonormal basis

$$\int \mu_{Ar} = 1$$

There is a unique $\mathcal{C}^\infty$ function

$$gr_X : X \times X \setminus \Delta \to \mathbb{R}$$

Rmk: gr_X are symmetric

i) $\partial \bar{\partial} \, gr_X (x, y) = 2\pi i \, \mu_{Ar}(x)$

ii) $\int_X gr_X (x, y) \mu_{Ar}(x) = 1$

$\mathcal{X} \longrightarrow B = \mathrm{Spec}(\mathbb{Z})$

$\widehat{\mathrm{Div}}(\mathcal{X}) = \mathrm{Div}(\mathcal{X}) \oplus \mathbb{R} \cdot F_\infty$

F_∞ = formal symbol which mimics the fiber at ∞

$\hat{D} \in \widehat{\mathrm{Div}}(\mathcal{X})$ is called Arakelov divisor

$\hat{D} = D + \alpha \cdot F_\infty \quad \alpha \in \mathbb{R}$

$= (D, \alpha)$

Def. $\hat{D}_1, \hat{D}_2 \in \widehat{\mathrm{Div}}(\mathcal{X})$. We define $\hat{D}_1 \cdot \hat{D}_2$:

i) If $\hat{D}_1 = D_1$ and $\hat{D}_2 = D_2$ is vertical, then $\hat{D}_1 \cdot \hat{D}_2$ as before.

$$= \sum_{p \text{ prime}} i_p(D_1, D_2) \log(p)$$

ii) "F_∞ behaves like a fiber" $\quad F_\infty \cdot F_\infty = 0$

iii) D horizontal $\quad D \cdot F_\infty = \deg_\eta D_\eta$

(iv) $D_1, D_2 : \mathcal{B} \to \mathcal{X}$ sections distinct

$$(D_1 \cdot D_2)_{A_2} = (D_1 \cdot D_2)_{\text{usual}} - g_{\mathcal{X}(\mathbb{C})}(D_{1,\mathbb{C}}, D_{2,\mathbb{C}})$$

Def. $f \in K(\mathcal{X})^\times$

$$\widehat{\mathrm{div}}(f) = \mathrm{div}(f) + \underset{\mathbb{R}}{\underset{\Uparrow}{v_\infty(f)}} \cdot F_\infty$$

$$v_\infty(f) = -\int_{\mathcal{X}(\mathbb{C})} \log|f| \, \mu_{A_2}$$

Lemma: $P : \mathcal{B} \to \mathcal{X}$ $\quad f \in K(\mathcal{X})^\times$ $\quad \hat{P} = (P, 0)$

$$\hat{P} \cdot \widehat{\mathrm{div}}(f) = 0$$

Proof. $g_X(\mathrm{div}\, f, P) = \log|f|(P) + v_\infty(f)$

$\square$

Theorem (Arakelov) $\exists!$ bilinear symmetric "intersection" $\widehat{\mathrm{Div}}(\mathcal{X}) \times \widehat{\mathrm{Div}}(\mathcal{X}) \to \mathbb{R}$

with respects linear equivalence:

$$\widehat{\mathrm{Cl}}(\mathcal{X}) \times \widehat{\mathrm{Cl}}(\mathcal{X}) \to \mathbb{R}$$

D divisor on X

gr_X gives a metric on $\mathcal{O}_X(D)$

$1_D \in \mathcal{O}_X(D)$ canonical section

$\log \| 1_D \|^2_{Ar} = gr_X(D, -)$ on $X \setminus \mathrm{Supp}(D)$

Ex: $P : B \to \mathcal{X}$ section, assume $g(\mathcal{X}_\eta) = 1$

$\hat{P} \cdot \hat{P} = -\frac{1}{12} \log | N_{K/\mathbb{Q}} (\Delta_{min} (\mathcal{X}_K)) |$

Hriljac - Faltings Hodge Index Theorem

$\mathcal{X} \xrightarrow{\pi} B = \mathrm{Spec}\, \mathcal{O}_K$

Thm: The signature of $\langle , \rangle_{Ar} : \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{X}) \times \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R}$

is $(+, -, -, \dots -)$

Proof: If D is an Arakelov divisor which is perpendicular to all fibres, then $\deg D_\eta = 0$, and this induces $\mathcal{O}(D) \in \mathrm{Jac}(X)$.

$$-\frac{1}{2[K:\mathbb{Q}]}(D\cdot D)_{A_2} = \hat{h}_{NT}(\mathcal{O}(D))$$

use the arithmetic Riemann-Roch $\square$

Drawbacks of $(\,,\,)_{A_2}$

i) $g(X) \geq 1$

ii) gr_X not the only Green function

iii) μ_{A_2} not compatible with pullback and pushforward

$f_*\mu_{A_2}$ not even $\mathcal{C}^\infty$

1999 Deligne - Rumely

Bost's extension of Arakelov's intersection pairing

$\hat{Z}_1(\mathcal{X}) = \{(D, g) \mid$ D divisor, g is an L_1^2-Green's function for D $\}$

Potential theory on X

D divisor, $\operatorname{supp}(D) \subseteq \Omega \underset{\text{open}}{\subseteq} X$ $\quad X \setminus \Omega$ not polar "equilibrium potential" $g_{D,\Omega}$

$\rightsquigarrow \| \ \|^{cap}_{\Omega}$ "capacity metric"

$\hat{Z}_1(\mathfrak{X}) \times \hat{Z}_1(\mathfrak{X}) \longrightarrow \mathbb{R}$

<u>Theorem</u> (Bost 1999)

analogue of lemma 1

$\hat{D} \in \hat{Z}_1(\mathfrak{X})$ such that $\hat{D}_\Omega \cdot \hat{D}_\Omega > 0$

and $\hat{D}_\Omega$ is "nef", then $\mathrm{Supp}(D)$ is connected.

<u>Proof</u>: Hodge index theorem for $\hat{Z}_1(\mathfrak{X})$ $\square$

<u>Theorem</u> (Bost - Chambert-Loir) $\Omega \subseteq \mathfrak{X}(\mathbb{C})$

$\mathfrak{X}/\mathcal{O}_K$ arithmetic surface, $P \in \mathfrak{X}(\mathcal{O}_K)$

$\varphi \in \hat{\mathcal{O}}_{X,P}$ which is "integral". Suppose

φ is algebraic over $\mathcal{O}_{X,P}$, $\widehat{\deg}(T_P X, \|\cdot\|^{cap}_{\Omega}) > 0$

then φ is a formal germ of a rational function.