THE SCHWARZ LEMMA AND NEVANLINNA THEORY

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Contents

1. From the Schwarz Lemma to Nevanlinna Theory

The most classical formulation – and the easiest to remember – of the Schwarz Lemma is this.

Proposition 1 (Schwarz Lemma)**.** *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.*

If equality holds at some point $z_0 \in \mathbb{D} \setminus \{0\}$, then $f(z) = \omega z$ for some $\omega \in \mathbb{S}^1$.

By rescaling the two disks we obtain the following, obviously equivalent, version:

Proposition 2. *Let $r > 0$, and let $f\colon \mathbb{D}(r) \to \mathbb{C}$ be holomorphic with $f(0) = 0$ and $|f(z)| \leq R$ for all $z \in \mathbb{D}(r)$. Then*

$$|f(z)| \leq \frac{R}{r}|z|$$

for all $z \in \mathbb{D}(r)$.

Proof. To have the easiest possible case, prove Proposition 1 in the case where f extends to a holomorphic function on $\mathbb{D}(1+\varepsilon)$. Then $f(z) = zg(z)$ for some holomorphic function $g\colon \mathbb{D}(1+\varepsilon) \to \mathbb{C}$. By the maximum modulus principle, we obtain the following for all $z \in \mathbb{D}$:

$$\frac{|f(z)|}{|z|} = |g(z)| \leq \max_{w\in\partial\mathbb{D}} |g(w)| = \max_{w\in\partial\mathbb{D}} |f(w)| \leq 1.$$

This is the conclusion of Proposition 1.

Hence also Proposition 2 holds if f extends a little beyond the boundary of $\mathbb{D}(r)$. But then we can apply Proposition 2 to $f|_{\mathbb{D}(r-\varepsilon)}$ and see that

$$|f(z)| \leq \frac{R}{r-\varepsilon}|z|$$

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for all $z \in \mathbb{D}(r-\varepsilon)$, $\varepsilon > 0$. Letting $\varepsilon \to 0$ we obtain Proposition 2 in general, hence also Proposition 1. □

Corollary 3 (Liouville)**.** *Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomorphic and bounded. Then f is constant.*

Proof. Wlog $f(0) = 0$ and $|f| \le R$; we have to show that f is identically zero. Let $z_0 \in \mathbb{C} \smallsetminus \{0\}$; we must show that $f(z_0) = 0$. Now, for any $r > |z_0|$ apply Proposition 2 to $f|_{\mathbb{D}(r)}$ and find that

$$|f(z_0)| \le \frac{R}{r}|z_0|.$$

But this holds for all $r \gg 0$, hence $f(z_0) = 0$. □

Corollary 4 (Picard's Small Theorem)**.** *Let $f\colon \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ be meromorphic, and assume it misses at least three values. Then f is constant.*

Proof. Wlog the values $0, 1, \infty$ are ommitted. The universal cover of $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$ is biholomorphic to $\mathbb{D}$ (use modular functions, or modern methods). Hence f lifts to a holomorphic function $\tilde{f}\colon \mathbb{C} \to \mathbb{D}$. This is bounded, hence constant by Corollary 3. Hence so is f. □

This can be seen as a first and very elementary result in *Nevanlinna theory*, also known as *value distribution theory*, which studies the distribution of values of meromorphic entire functions. Discuss example of exponential function.

Let $f \in \mathscr{M}(\mathbb{C})$. We shall define two families of functions, $m_f(a,r)$ and $N_f(a,r)$, on $(0,\infty)$, which measure how close f comes to assume a on $\partial\mathbb{D}(r)$ and how often it assumes a in $\mathbb{D}(r)$.

The proximity functions. Set first

$$m_f\colon (0,\infty) \to \mathbb{R}, \quad r \mapsto \int_0^{2\pi} \log^+|f(re^{i\vartheta})|\frac{d\vartheta}{2\pi}.$$

For $a \in \mathbb{P}^1(\mathbb{C})$ set then

$$m_f(a,r) = \begin{cases} m_f(r), & a = \infty; \\ m_{1/(f-a)}(r), & a \in \mathbb{C}. \end{cases}$$

This is large iff f often stays close to a on $|z = r|$.

The counting functions. Set

$$N_f(r) = \mathrm{ord}_0^-(f) \cdot \log r + \sum_{0<|z|<r} \mathrm{ord}_z^-(f) \cdot \log \frac{r}{|z|},$$

where $\mathrm{ord}_z^-(f) = d$ if f has a pole of order d at z, $= 0$ if it is holomorphic there. Again set

$$N_f(a,r) = \begin{cases} N_f(r), & a = \infty; \\ N_{1/(f-a)}(r), & a \in \mathbb{C}. \end{cases}$$

This counts, weighted with multiplicities and some simple factors, the number of times f assumes the value a in $|z| < a$.

Theorem 5 (First Main Theorem of Nevanlinna Theory)**.** *There exists a function $T_f\colon (0,\infty)\to\mathbb{R}$, well defined up to addition of a bounded function, such that*

$$m_f(a,r)+N_f(a,r)=T_f(r)+O_{f,a}(1).$$

Usually we take $T_f(r)=m_f(r)+N_f(r)$. This is called the *height* of f.

Remark 6.
(i) f is constant iff $T_f=O(1)$.
(ii) If f is nonconstant, then $\underline{\lim}_{r\to\infty} T_f(r)/\log r>0$.
(iii) f is rational iff $T_f(r)=O(\log r)$.
(iv) For $f(z)=\mathrm{e}^z$ we obtain $T_f(r)=\frac{r}{\pi}$.

The height enjoys some other formal properties similar to those of heights of algebraic numbers and so on. Get a dictionary of analogies, where meromorphic functions correspond to sequences of numbers, the restriction to $|z|=r$ corresponds to an element in the sequence, arguments ϑ correspond to places, etc. There is an analogue of the First Main Theorem whose definitions seem rather artificial. However, the Second Main Theorem closely resembles Roth's Theorem in diophantine approximation. Alas, the proofs are very different.

2. Chow's Theorem Revisited

Theorem 7 (Chow)**.** *Let $M\subseteq\mathbb{P}^N(\mathbb{C})$ be a smooth complex-analytic submanifold. Then M is algebraic.*

We will reprove this, using the Schwarz lemma.

Consider the following setup. $\varphi_1,\dots,\varphi_n\colon\mathbb{B}^d\to\mathbb{P}^N(\mathbb{C})$ are holomorphic embeddings. Let $V\subseteq\mathbb{P}^N(\mathbb{C})$ be the Zariski closure of the union of their images. For $i\geq 0$ and $D\in\mathbb{Z}$ let

$$E_D^i=E_D^i(\varphi_1(0),\dots,\varphi_n(0))=\{s\in\Gamma(V,\mathscr{O}(D))\mid \mathrm{mult}_0(\varphi_\nu^*s)\geq i \text{ for all } \nu\}.$$

Proposition 8. *Let $\varphi_1,\dots,\varphi_n\colon\mathbb{B}^d\to\mathbb{P}^N(\mathbb{C})$ be holomorphic embeddings. Assume that there exists some $c>0$ such that $E_D^i=0$ for all $i,D>0$ with $\frac{i}{D}>c$.*

Then the dimension of V is d, i.e. each $\varphi_\nu(\mathbb{B}^d)$ is algebraisable. □

For $n=1$ this was Proposition 2 in Yohan's talk.

Idea of proof. Essentially the same as for $n=1$: the rank of the evaluation homomorphism on $\Gamma(V,\mathscr{O}(D))$ is $O(D^d)$, hence the dimension of the Zariski closure is d. □

Proposition 9. *Let $X\subseteq\mathbb{P}^N(\mathbb{C})$ be a compact complex manifold of dimension d, and let $\overline{X}\subseteq\mathbb{P}^N(\mathbb{C})$ be its Zariski closure. Find a finite atlas of X, consisting of charts $\varphi_\nu\colon\mathbb{B}^d\xrightarrow{\cong}U_\nu\subset X$, for $1\leq\nu\leq n$. Set $P_\nu=\varphi_\nu(0)$.*

Let furthermore $\mathscr{L}$ be a holomorphic line bundle on $\overline{X}$, and set

$$E_D^i=\{s\in\Gamma(\overline{X},\mathscr{L}^{\otimes D})\mid \mathrm{mult}_{P_\nu}(s)\geq i \text{ for all } \nu=1,\dots,n\}.$$

Then there exists $c>0$ such that $E_D^i=0$ whenever $i,D>0$ and $\frac{i}{D}>c$.

Note that it is clear that these two Propositions together imply Chow's Theorem: choose a finite number of charts $\varphi_\nu\colon\mathbb{B}^d\to X$ and apply Proposition 8 to these.

Yohan proved Proposition 9 for $X\subseteq\mathbb{P}^N(\mathbb{C})$ with $c=\deg(V)$ (topological degree). We go a different path, using a compactness argument and a higher-dimensional variant of the Schwarz Lemma:

Lemma 10. *Let $f\colon \mathbb{B}^d(r) \to \mathbb{C}$ be a holomorphic function having multiplicity at least $m \geq 0$ in 0 and such that $|f(z)| \leq R$ for all $z \in \mathbb{B}^d(r)$. Then*

$$|f(z)| \leq \left(\frac{\|z\|}{r}\right)^m R.$$

Proof. First reduce to $d = 1$ by looking at one-dimensional subspaces. Then apply the same argument as before to $f(z)/z^m$. □

Proof of Proposition 9. We choose the following auxiliary data:

- some $0 < r < 1$ such that the $U_\nu(r) = \varphi_\nu^{-1}(\mathbb{B}^d(r))$ still cover X;
- an auxiliary parameter $r < r' < 1$;
- an hermitian metric $\|\cdot\|_{\mathscr{L}}$ on $\mathscr{L}$, yielding metrics $\|\cdot\|_{\mathscr{L}^{\otimes D}}$ on each $\mathscr{L}^{\otimes D}$ for $D \in \mathbb{Z}$;
- trivialisations of $\mathscr{L}|_{U_\nu}$ by non-vanishing sections $\varepsilon_\nu \in \Gamma(U_\nu, \mathscr{L})$. These exist because any holomorphic line bundle on $\mathbb{B}^d$ is trivial.

Now consider a section $s \in \Gamma(\overline{X}, \mathscr{L}^{\otimes D})$. For each ν there exist holomorphic functions $f_\nu\colon \mathbb{B}^d \to \mathbb{C}$ such that

$$s|_{U_\nu} = (f_\nu \circ \varphi_\nu^{-1}) \cdot \varepsilon_\nu^{\otimes D}.$$

We obtain the following trivial estimates:

- for all $x \in U_\nu(r)$:

$$\|s(x)\|_{\mathscr{L}^{\otimes D}} \leq \sup_{y \in U_\nu(r)} \|\varepsilon_\nu(y)\|_{\mathscr{L}}^D \cdot |f_\nu(\varphi_\nu^{-1}(x))|; \tag{1}$$

- for all $x \in U_\nu(r')$:

$$|f_\nu(\varphi_\nu^{-1}(x))| \leq \sup_{y \in U_\nu(r')} \|\varepsilon_\nu(y)\|_{\mathscr{L}}^{-D} \cdot \|s(x)\|_{\mathscr{L}^{\otimes D}}. \tag{2}$$

Furthermore, if f_ν vanishes to order i at the origin, by the Schwarz Lemma we get

$$|f_\nu(\varphi_\nu^{-1}(x))| \leq \left(\frac{r}{r'}\right)^i \sup_{w \in \mathbb{B}^d(r)} |f_\nu(w)|. \tag{3}$$

Combining these three we find that for all $x \in U_\nu(r)$,

$$\begin{aligned}
\|s(x)\|_{\mathscr{L}^{\otimes D}} &\overset{(1)}{\leq} \sup_{y \in U_\nu(r)} \|\varepsilon_\nu(y)\|_{\mathscr{L}}^D \cdot |f_\nu(\varphi_\nu^{-1}(x))| \\
&\overset{(3)}{\leq} \sup_{y \in U_\nu(r)} \|\varepsilon_\nu(y)\|_{\mathscr{L}}^D \cdot \left(\frac{r}{r'}\right)^i \sup_{w \in \mathbb{B}^d(r)} |f_\nu(w)| \\
&\overset{(2)}{\leq} \sup_{y \in U_\nu(r)} \|\varepsilon_\nu(y)\|_{\mathscr{L}}^D \cdot \left(\frac{r}{r'}\right)^i \sup_{y \in U_\nu(r')} \|\varepsilon_\nu(y)\|_{\mathscr{L}}^{-D} \cdot \sup_{y \in U_\nu(r')} \|s(y)\|_{\mathscr{L}^{\otimes D}}.
\end{aligned}$$

Hence if we set $\lambda = \frac{r}{r'} < 1$ and

$$M = \max_{\nu=1}^{n} \frac{\sup_{y \in U_\nu(r)} \|\varepsilon_\nu(y)\|_{\mathscr{L}}}{\inf_{y \in U_\nu(r')} \|\varepsilon_\nu(y)\|_{\mathscr{L}}},$$

then we obtain

$$\max_{x \in X} \|s(x)\|_{\mathscr{L}^{\otimes D}} \leq \lambda^i M^D \max_{x \in X} \|s(x)\|_{\mathscr{L}^{\otimes D}}. \tag{4}$$

Hence s has to vanish identically on X, *therefore on* $\overline{X}$, as soon as $\lambda^i M^D < 1$. The Proposition is therefore true for

$$c = -\frac{\log M}{\log \lambda}. \qquad \square$$

3. An estimate later used for algebraisation

Let X be a projective complex manifold, $\mathscr{L}$ a line bundle on X, $P \in X$, $\mathfrak{V}$ a germ of complex analytic submanifolds of X at P, of dimension $d > 0$.

For $D \geq 0$ and $i > 0$ let E^i_D be the subspace of $\Gamma(X, \mathscr{L}^{\otimes D})$ consisting of sections whose restriction to $\mathfrak{V}$ vanishes of order at least i at P. Consider the corresponding jet map

$$\varphi^i_D : E^i_D \to \operatorname{Sym}^i \mathrm{T}_P\mathfrak{V} \otimes \mathscr{L}_P^{\otimes D}.$$

We wish to estimate the operator norm of this map, for suitable norms on either spaces. More precisely, choose an hermitian metric on $\mathscr{L}$; this gives an L^∞-, i.e. supremum, norm on its space of global sections, therefore on E^i_D. We also choose a norm on the tangent space $\mathrm{T}_P\mathfrak{V}$, which together with the norm on $\mathscr{L}_P$ induces a norm on each $\operatorname{Sym}^i \mathrm{T}_P\mathfrak{V} \otimes \mathscr{L}_P^{\otimes D}$, the codomain of φ^i_D.

Theorem 11. *Assume that there exist a Liouville manifold M, some $O \in M$, and a holomorphic map $f : M \to X$ with $f(O) = P$ and which sends the germ of M at O biholomorphically to $\mathfrak{V}$.*

Then for every $r > 0$ there exists some $C(r) \in \mathbb{R}$ such that

$$\|\varphi^i_D\| \leq \frac{C(r)^D}{r^i}.$$

For most applications of Theorem 11, $M = \mathbb{C}^d$ will suffice. We will only consider the case $M = \mathbb{C}$, the case $M = \mathbb{C}^d$ being very similar but even more laden with notation.

Proof. As $M = \mathbb{C}$ is one-dimensional, so is $\mathfrak{V}$. We may therefore identify the codomain of φ^i_D with $\mathrm{T}_P\mathfrak{V}^{\otimes i} \otimes \mathscr{L}_P^{\otimes D}$.

The pullback $f^*\mathscr{L}$ is trivial, hence we can fix a trivialising section $\varepsilon \in \Gamma(\mathbb{C}, f^*\mathscr{L})$. For $s \in E^i_D$ write $\Gamma(\mathbb{C}^n, f^*\mathscr{L}^{\otimes D}) \ni f^*s = g \cdot \varepsilon^{\otimes D}$ for some holomorphic function $g : \mathbb{C} \to \mathbb{C}$.

By construction, the operator norm $\|\varphi^i_D\|$ is the supremum of all $\|\varphi^i_D(s)\|$, where s ranges through the elements $s \in E^i_D \subseteq \Gamma(X, \mathscr{L}^{\otimes D})$ with $\|s(x)\| \leq 1$ for all $x \in X$. So choose some such s, and write $s = g \cdot \varepsilon^{\otimes D}$ as above. Then $\varphi^i_D(s)$ is some constant vector $v \in \mathrm{T}_P\mathfrak{V} \otimes \mathscr{L}_P^{\otimes P}$ times $g^{(i)}(0)$. We fix some $r > 0$ and consider the restriction of g to $\mathbb{B}(r)$. Since g vanishes at least to the i-th order at the origin, we find that (by the Schwarz Lemma!)

$$\|\varphi^i_D(s)\| \sim |g^{(i)}(0)| \leq \frac{\sup_{z \in \mathbb{B}(r)} |g(z)|}{r^i}.$$

Now $g = f^*s \cdot \varepsilon^{\otimes -D}$ with s bounded, hence there is some constant $C(r) > 0$, which does not depend on s but on r, such that $|g(z)| \leq C(r)^D$ on $\mathbb{B}(r)$. Combining these observations, we find that

$$\|\varphi^i_D(s)\| \leq \frac{C(r)^D}{r^i}.$$

Since s was arbitrary with $\|s\| \geq 1$, we find that

$$\|\varphi_D^i\| \leq \frac{C(r)^D}{r^i}. \qquad \square$$

Bost proves a more precise estimate:

$$\|\varphi_D^i\| \leq \left(\frac{n\sqrt{\mathrm{e}}}{r}\right)^i \cdot \left(\mathrm{e}^{T_{f,\overline{\mathscr{L}}}(r)}\right)^D,$$

where $T_{f,\overline{\mathscr{L}}}$ is a generalisation of a Nevanlinna height. Just as for the classical case, it is $O(\log r)$ if and only if f is a regular map of algebraic varieties $\mathbb{A}^n \to X$.

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