1 From Schwarz's lemma to Nevanlinna theory

Theorem 1 (Schwarz lemma)

Let $f : \mathbb{D} \to \mathbb{D}$ holomorphic, $f(0) = 0$. Then

$\Updownarrow$ $|f(z)| \leq z$ for all $z \in \mathbb{D}$, $|f'(0)| \leq 1$.

Theorem 1' Let $f : \mathbb{D}(r) \to \mathbb{D}(R)$ holomorphic, $f(0) = 0$. Then $|f(z)| \leq \frac{R}{r} |z|$, $|f'(0)| \leq \frac{R}{r}$.

Proof. Wlog, f extends beyond $\partial \mathbb{D}(r)$. (Then for general f, apply to $f|_{\mathbb{D}(r-\varepsilon)}$, let $\varepsilon \to 0$).

Now $r = R = 1$. Then $f(z) = z\, g(z)$, g holomorphic

$$\text{MMP} \Longrightarrow \max_{|z| \leq 1} |g(z)| = \max_{|z|=1} |g(z)| = \max_{|z|=1} |f(z)| = 1$$

$\square$

N.B. $g(0) = f'(0)$

Generalization: $f : \mathbb{B}^d(r) \to \mathbb{B}^{\#}(R)$ holomorphic, $\mathrm{mult}_0 f \geq i$, then $|f(z)| \leq \left(\frac{\|z\|}{r}\right)^i R$.

Proof: Consider 1-dimensional slice $\Rightarrow$ WLOG $d = 1$, apply above argument to $f(z) = z^i g(z)$ $\square$

Theorem 2 (Liouville) $f : \mathbb{C} \to \mathbb{C}$ holomorphic, bdded. Then f is constant.

Proof. Wlog $f(0) = 0$ $\quad |f| \leq R$

$z_0 \in \mathbb{C}$ want $f(z_0) = 0$

for $r > |z_0|$, apply Thm 1' to $f|_{\mathbb{D}(r)}$

$$\Rightarrow |f(z_0)| \leq \underbrace{\frac{R}{r}}_{\to 0} |z_0| \Rightarrow f(z_0) = 0 \qquad \square$$

Definition. A complex space X is a Liouville space if every bdded plurisubharmonic function on X is constant.

Examples. • $\mathbb{C}^n$, compact, X alg. variety $X(\mathbb{C})$

• $\mathbb{D}$ bdded domain in $\mathbb{C}^n$ are not Liouville.

Theorem 3 (Picard) Let $f \in \mathcal{M}(\mathbb{C})$ missing ≥ 3 values in $\mathbb{P}^1(\mathbb{C})$. Then f is constant.

Proof.

$$\begin{tikzcd} & \mathbb{H} \cong \mathbb{D} \arrow[d, "\lambda"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} \end{tikzcd}$$

universal cover

$\tilde{f} : \mathbb{C} \to \mathbb{D}$ constant

$\Rightarrow f = \text{cst} \quad \square$

Let $f \in \mathcal{M}(\mathbb{C})$.

* Proximity functions :

for $a \in \mathbb{P}^1(\mathbb{C})$, set

$$m_f(a, -) :]0, \infty[\longrightarrow \mathbb{R}$$

max(log, 0)

$$r \longmapsto \begin{cases} \displaystyle\int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} & a = \infty \\ \displaystyle\int_0^{2\pi} \log^+ \left|\frac{1}{f(re^{i\theta}) - a}\right| \frac{d\theta}{2\pi} & a \in \mathbb{C} \end{cases}$$

* Counting function

Let $\operatorname{ord}_z^+ f = \begin{cases} i & \text{if } f \text{ has a zero of order } i \text{ at } z \\ 0 & \text{if } f \text{ has a pole at } z \end{cases}$

$$N_f(a, -) :]0, \infty[\longrightarrow \mathbb{R}$$

$$r \longmapsto \operatorname{ord}_0^+ (f-a) \log r + \sum_{0<|z|<r} \operatorname{ord}_z^+ (f-a) \log(\quad)$$

for $a \in \mathbb{C}$, $N_f(\infty, r) = N$

Theorem 4 (1^{st} main theorem of Nevanlinna Theory)

For $f \in \mathcal{M}(\mathbb{C})$, $\exists\, T_f :]0, \infty[\to \mathbb{R}$, unique up to $\pm\, O(1)$, such that for all $a \in \mathbb{P}^1(\mathbb{C})$

$T_f(r)$ is called the N. height of f.

$$m_f(a, r) + N_f(a, r) = T_f(r) + O_{f,a}(1)$$

② Chow's Theorem revisited

Theorem 5. Every complex submanifold of $\mathbb{P}^N(\mathbb{C})$ is algebraic.

Proof. Break it in two steps:

Step 1: let $\varphi_1, \dots \varphi_n \in \mathbb{B}^d \hookrightarrow \mathbb{P}^N(\mathbb{C})$ be holomorphic embedding. let $V = \overline{\varphi_1(\mathbb{B}^d) \cup \dots \cup \varphi_n(\mathbb{B}^d)}^{Zar}$.

For $i, D > 0$ set

$$E_D^i = \{ s \in \Gamma(V, \mathcal{O}(D)) \mid \operatorname{mult}_0(\varphi_r^* s) \geq i \text{ for all } r \}$$

Assume $\exists\, c$ such that, if $\frac{i}{D} > c$, then $E_D^i = 0$.

Then $\dim V = d \iff$ each $\varphi_r(\mathbb{B}^d)$ is algebraisable.

Step 2: $X \subseteq \mathbb{P}^N(\mathbb{C})$ complex submanifold of dimension d

Find a finite atlas: $\varphi_r : \mathbb{B}^d \xrightarrow{\sim} \mathcal{U}_r \subset X$ $\quad r = 1, \dots n$

Set $P_r = \varphi_r(0)$.

Let L be a holomorphic line bundle on $\overline{X}^{\mathrm{Zar}}$, set

$$E_D^i = \{ s \in \Gamma(\overline{X}, L^{\otimes D}) \mid \mathrm{mult}_{P_r}(s) \geq i \text{ for all } r \}$$

Then $\exists\, c > 0$, $\forall \frac{i}{D} > c$, $E_D^i = 0$.

<u>Proof of Step 2</u> Choose

* $0 < r < r' < 1$ such that the $\mathcal{U}_r(r) = \varphi_r(\mathbb{B}^d(r))$ still cover X.

* hermitian metric on L, hence on $L^{\otimes D}$

* trivialisations $\varepsilon_r \in \Gamma(\mathcal{U}_r, L)$

Let $s \in \Gamma(\overline{X}, L^{\otimes D}) \rightsquigarrow \exists$ holomorphic functions

$f_r : \mathbb{B}^d \to \mathbb{C}$, $\quad s|_{\mathcal{U}_r} = (f_r \circ \varphi_r^{-1}) \cdot \varepsilon_r^{\otimes D}$

If $\mathrm{ord}_0 f_r = \mathrm{ord}_{P_r}(s) \geq i$, then Schwarz lemma

$\Longrightarrow$

$$(*)\quad \left|f_r\left(\varphi_r^{-1}(x)\right)\right| \le \left(\frac{R}{r'}\right)^{i} \sup_{w\in \mathbb{B}^d(r')} |f_r(w)|$$

for all $x \in U_r(r)$.

for all $x \in U_r(r)$

$$\|s(x)\| \le \sup_{y\in U_r(r)} \|\varepsilon_r(y)\|^{0} \underbrace{\left|f_r\left(\varphi_r^{-1}(x)\right)\right|}_{\le \left(\frac{R}{r'}\right)^{i} \underbrace{\sup_{w\in \mathbb{B}^d(r')} |f_r(w)|}_{\le \sup_{y\in U_r(r')} \|\varepsilon_r(y)\|^{-0} \sup_{y\in U_r(r')} \|s(y)\|}}$$

So there are constants

$$0<\lambda<1<M$$

such that

$$\max_{x\in X} \|s(x)\| \le \lambda^{i} M^{0} \max_{x\in X} \|s(x)\|$$

If this is <1, then $s=0$.

$\rightsquigarrow$ statement holds with $c = -\dfrac{\log M}{\log \lambda}$ $\square$

3 An estimate

$X \subseteq \mathbb{P}^N(\mathbb{C})$ closed submanifold, $L \to X$ a line bundle, $P \in X$, V germ of analytic submanifold of X at P, $\dim V > 0$.

$$E_D^i = \{ s \in \Gamma(X, L^{\otimes D}) \mid \operatorname{mult}_P(s|_V) \geq i \}$$

i^{th} jet map ($\sim i^{th}$ partial derivatives)

$$\varphi_D^i : E_D^i \longrightarrow \operatorname{Sym}^i T_P V^\vee \otimes L_P^{\otimes D}$$

hermitian metric on L $\leadsto$ operator norm $\|\varphi_D^i\|$.
(L^∞ on Γ)

Theorem 6 (Bost) Assume $\exists f : M \to X$ holomorphic, M invertible manifold, such that $f(0) = P$, f(germ of M at 0) $= V$. Then, for all $r > 0$, $\exists C(r) \in \mathbb{R}$ such that

$$\|\varphi_D^i\| \leq \frac{C(r)^D}{r^i}$$

Idea of proof for $M = \mathbb{C}^d$

trivialise f^*L, write $f^*s = g \cdot \varepsilon^{\otimes D}$

$$\|\varphi_D^i(s)\| \sim |g^{(i)}(0)| \, \|\varepsilon\|$$

bd this by Schwarz

$\square$

Remark: for $M = \mathbb{C}^n$, more precise estimate

$$\|\varphi_D^i\| \le \left(\frac{n\sqrt{e}}{r}\right)^i \left(e^{T_{f,\bar{L}}(r)}\right)^D$$

$T_{f,\bar{L}}$ generalized Nevanlinna height

About classical T_f:

$T_f(r) \gg \log r$ for $r \gg 0$

$T_f(r) = O(\log r) \Leftrightarrow f$ is rational

$T_{f,\bar{L}}(r) = O(\log r) \Leftrightarrow f : \mathbb{C}^n \to X$ is a regular map of algebraic variety $\mathbb{A}^n \to X$.