P. JOSSEN. – Arithmetic algebraization à la Chudnovsky I. Statements and proofs

Three arithmetic algebraisation theorems

$$\overset{3}{A} \Rightarrow \overset{1}{B} \Rightarrow \overset{2}{C}$$

A: about formal subschemes

B: about foliations

C: about alg. groups and Lie algebras

[Bo 01]

**Theorem B.** Let $X$ be an alg. smooth variety over a number field $K$, $P \in X(K)$, $F \subseteq T_X$ an involutive integrable subvector bundle. The formal leaf of $F$ through $P$ is algebraic if the following conditions hold:

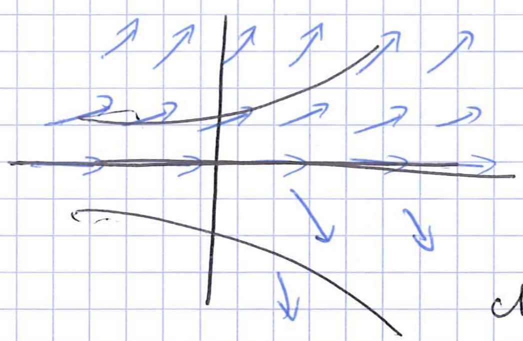
$\boxed{B1}$ Choose a model of $X$, $F$ over some open $U \subseteq \operatorname{Spec} \mathcal{O}_K$. For almost all $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$, the subbundle $(F \bmod \mathfrak{p}) \subseteq T_{X_\mathfrak{p}}$ is stable under $p$-th powers. $(p = \operatorname{char}(\mathcal{O}_K/\mathfrak{p}))$

$\boxed{B2}$ For some $K \overset{\sigma}{\hookrightarrow} \mathbb{C}$ there exists a complex manifold $M$ satisfying the Liouville property

$0 \in M$ and a meromorphic map $M \xrightarrow{u} X_\sigma(\mathbb{C})$ $u(0) = P_\sigma$ mapping a neighborhood of 0 biholomorphically onto the germ of F at P.

Example: k any number field, $X = \mathbb{A}^2_k = \mathrm{Spec}\, k[x,y]$

Vector field $F(x,y) = (1, a \cdot y)$ $\quad a \in k$ fixed



characteristic lines $\quad z \xrightarrow{\gamma_c} (z, c \cdot \exp(az))$

$(\gamma_c)_{c \in \mathbb{C}}$ "is" the foliation associated to F

$P = (0,1)$ complex leaf for $\sigma : k \hookrightarrow \mathbb{C}$ is the image of $\mathbb{C} \xrightarrow[\gamma_1]{} \mathbb{A}^2_\sigma(\mathbb{C})$ $\quad \gamma_1(z) = (z, \exp(\sigma(a) z))$

$\boxed{B2}$ ✓

Check that $F_{\mathrm{mod}\, p}$ is stable under $p^{th}$ power.

As a derivation, F is

$$\partial_F : f(x,y) \longmapsto \frac{\partial f}{\partial x}(x,y) + a \cdot y \frac{\partial f}{\partial y}(x,y)$$

<u>mod $p$</u>: $\partial_F^p : x \mapsto 0$

$y \mapsto a^p \cdot y$

Need $\partial_F^p = \lambda_p \cdot \partial_F$

$(0, a^p y) = \lambda_p (1, ay)$

Need $a^p = 0 \mod p$, $a = 0 \mod p$ for almost all $p$. $\Longleftrightarrow a = 0$.

<u>Theorem C</u>. Let $G$ be an algebraic group over $k$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. There exists an algebraic subgroup $H \subseteq G$ with $\mathrm{Lie}(H) = \mathfrak{h}$ iff the following two conditions hold:

$\boxed{C1}$ $(\mathfrak{h} \bmod p) \subseteq (\mathfrak{g} \bmod p)$ is stable under $p^{th}$ power for almost all $p \in \mathrm{Spec}\, \mathcal{O}_K$.

$\boxed{C2}$ $\emptyset$

<u>Example</u>: $G = \mathbb{G}_m \times \mathbb{G}_m$, $\mathrm{Lie}(G) = \langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y} \rangle$

$\mathfrak{h} \subseteq \mathfrak{g}$ given by $\langle \underbrace{x\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}}_{(u_a)^p} \rangle$, $a \in k$ fixed compute $p$-th power mod $p$

$$(\mu_a)^p = \begin{cases} x \longmapsto x \\ y \longmapsto a^p y \end{cases}$$

$(\mu_a)^p \in$ ~~h~~ mod $p$ $\iff a = a^p$ mod $p$

The Lie subalgebra $\mathfrak{h}_a$ is algebraic

$\iff a^p \equiv a \mod p$ for all $p$

Thm C

$\iff$ the image of

$$\mathbb{C} \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times$$

$$z \longmapsto (\exp(z), \exp(\sigma(a).z))$$

is algebraic

$\iff a \in \mathbb{Q}$

<u>Kronecker</u> (1880) $a^p \equiv a \mod p$ for almost all $p$

$\iff a \in \mathbb{Q}$ ∎

## <u>Towards Theorem A</u>

Sizes of formal solutions

$K/\mathbb{Q}_p$ finite field extension

$X/K$ algebraic variety, $P \in X(K)$.

$\hat{V}$ a formal smooth subscheme of $\hat{X}_p$.

[Suppose $\hat{V}$ is the image of a map

$$\hat{\mathbb{A}}^r_K \xrightarrow[u_1, \dots u_d]{} \hat{\mathbb{A}}^d_K \qquad r \le d$$

power series in $r$ variables

$\hat{V}$ large size $\Leftrightarrow$ $u_1, \dots u_d$ large convergence radius ]

Notation

for $g \in K[[x_1, \dots x_d]]$, say $g = \sum_{I \in \mathbb{N}^d} a_I x^I$, $r \geq 0$

set

$$\| g \|_r = \sup_{I \in \mathbb{N}^d} |a_I| \cdot r^{|I|} \in [0, +\infty]$$

$|I| \leftarrow$ sum

$|a_I|$: p-adic absolute value

Automorphisms of $\widehat{\mathbb{A}}^d_K$ are given by $d$ series

$g = (g_1, \dots g_d)$ in $d$ variables, $g(0) = 0$

$(Dg)(0) \in GL_d(K)$.

For $r \geq 0$ let $G(r) \subseteq \mathrm{Aut}(\hat{\mathbb{A}}^d_K)$ be the subgroup formed by those $g = (g_1, \dots g_d)$

such that $Dg(0) \in GL_d(\mathcal{O}_K)$.

$$\|g_i\|_r \leq r$$

So $G(r)$ sends the disc of radius $r$ in $K^d$

to itself.

$r' \geqslant r \implies G(r) \supseteq G(r')$

$G(0)$ series

$Dg(0) \in GL_d(\mathcal{O}_K)$ positive convergence radius

Let $\hat{V}$ a smooth formal subscheme of $\widehat{\mathbb{A}}^d_K$.

There exists an element $g$ of $G(0)$ such that $g^*(\hat{V}) = \widehat{\mathbb{A}}^r_K \times \{0\} \subseteq \widehat{\mathbb{A}}^d_K$.

<u>Definition</u>. The size of $\hat{V}$ is

$$R(\hat{V}) = \sup \{ r \in [0,1] \mid \exists\, g \in G(r) \text{ with } g^*(\hat{V}) = \widehat{\mathbb{A}}^r_K \times \{0\} \}$$

$r = \dim \hat{V}$

Extend this definition to $\hat{V} \subseteq \hat{X}_P$ by choosing $\mathcal{X}/\mathcal{O}_K$ and an embedding $\mathcal{X} \hookrightarrow \mathbb{A}^d_{\mathcal{O}_K}$.

<u>Theorem A</u>. $K$ number field, $X/_K$ algebraic variety, $P \in X(K)$. Let $\hat{V}$ be a smooth formal subscheme of $\hat{X}_P$. There exists an algebraic subvariety $Y \subseteq X$ containing $P \in Y(K)$ such that $\hat{V}$ is a <u>branch</u> of $Y$ at $P$ if the following two conditions hold:

$\boxed{A1}$ For every $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ the size $R_{\mathfrak{p}}(\hat{V})$ is $> 0$ and

$$\sum_{\substack{\text{almost all} \\ \mathfrak{p}}} -\log(R_{\mathfrak{p}}(\hat{V})) \text{ is finite}$$

$\boxed{A2}$ For every $K \overset{\sigma}{\hookrightarrow} \mathbb{C}$, the formal subscheme $\hat{V} \subseteq X_\sigma(\mathbb{C})$ is the germ of an analytic submanifold, and for one $\sigma_0 : K \hookrightarrow \mathbb{C}$ there exist a complex manifold $M$ with the Liouville property, meromorphic $\mu : M \to X_{\sigma_0}(\mathbb{C})$ biholomorphic into the analytic germ of $\hat{V}$ in $X_{\sigma_0}(\mathbb{C})$.

Elements of the proof in a <u>simplified setting</u>:

* $\hat{V}$ is a germ of a curve (avoids use of symmetric powers)

* $X$ projective, comes with an ample line bundle $L$

$\hat{V}$ is Zariski dense in $X$

* $K = \mathbb{Q}$, $\widehat{\deg} = \widehat{\deg}_n$

<u>Evaluation maps</u>: $D \geq 0$

$$E_D = \Gamma(X, L^{\otimes D})$$

$$\eta_D^i : E_D = \Gamma(X, L^{\otimes D}) \longrightarrow \Gamma(V_i, L^{\otimes D})$$

$$E_D^i = \ker(\eta_D^{i-1})$$

Obtain a filtration on $E_D$.

$$E_D^i \xrightarrow[\gamma_D^i]{} \ker\left(\Gamma(V_i, L^{\otimes D}) \to \Gamma(V_{i-1}, L^{\otimes D})\right)$$

$$\| $$

$$T_V^{\otimes -i} \otimes L_p^{\otimes D}$$

<u>Lemma</u> If $\hat{V}$ is <u>not</u> algebraic, then

$$\lim_{D\to\infty} \sum_{i\geq 0} \frac{i}{D} \frac{\mathrm{rk}\,(E_D^i/E_D^{i-1})}{\mathrm{rk}\, E_D} = +\infty .$$

<u>Choose</u>: • model $\mathcal{X}$, $\mathcal{L}$, $\mathcal{P}$ of $X, L, P$ over $\mathrm{Spec}(\mathcal{O}_K)$. $\left(\mathcal{O}_K = \mathbb{Z}\right)$

and a hermitian metric on $\mathcal{L}$.

• a positive Lebesgue measure on $X(\mathbb{C})$.

• a norm $\|\cdot\|_0$ on the tangent line

$$t = T_{\hat{V},P}, \qquad \bar{t} = (\mathbb{Z}, \|\cdot\|_0).$$

Input from analysis

On $T_{\hat{V},P,\mathbb{C}} = t_{\mathbb{C}}$ define the canonical seminorm

$$\|\cdot\|_{can} = \exp\left(\limsup_{\frac{i}{D}\to\infty} \frac{1}{i} \log \|\gamma_D^i\|\right) \|\cdot\|_0$$

operator norm of $\gamma_D^i$

<u>Aside</u>:

$$\widehat{\deg}\,(t, R_p^{-1}|\cdot|_p, \|\cdot\|_{can}) > 0$$

<u>Hypothesis A2</u> says: $\exists\ \lambda > 0,\ d > 0$ s.t.

for all $(D, i)$, $i > \lambda D$

$$-\frac{1}{i}\log\|\gamma_D^i\| \geq d$$

$$\widehat{\deg}\, t + \sum_{\wp} \log R_p(\hat{\nabla})$$

$$\hat{\mu}(\overline{\mathcal{E}}_D) \leq \frac{1}{\mathrm{rk}\, E_D} \sum_{i \geq 0} \mathrm{rk}\, E_D^i / E_D^{i+1} \left( \overbrace{\widehat{\deg}(\overline{t}^{\otimes -i} \otimes \mathcal{L}_{\mathcal{O}_p}^{\otimes D})}^{-i\,\widehat{\deg}(t) + D\,\widehat{\deg}(\mathcal{L}_p)} + \underline{\text{height}(\gamma_D^i)} \right)$$

$\leq$ with the $L^2$-norm

$-cD$

$$\text{height of } \gamma_D^i = \sum_p \cancel{\deg(p)} \log\left(\overbrace{\|\gamma_D^i \otimes \mathcal{O}_p\|}^{\leq R_p(\hat{\nabla})^{-i}}\right) + \underbrace{\log\left(\|\gamma_D^i \otimes_{\mathbb{Q}} \mathbb{C}\|\right)}_{\leq \alpha i + \beta D \text{ for some } \alpha, \beta \geq 0}$$

Hence
$$\frac{1}{\mathrm{rk}(E_D)} \sum_{0 \leq i \leq \lambda D} \mathrm{rk}\, E_D^i / E_D^{i+1} \left[ -i\,\widehat{\deg}(t) + \text{height}(\gamma_D^i) \right] \leq C(\lambda) \cdot D$$

$$-cD \le D\cdot \widehat{\deg}(\bar{\mathcal{L}}_\varphi) + c(\lambda)\cdot D$$

$$+ \frac{1}{\mathrm{rk}\, E_D} \sum_{\boxed{i > \lambda D}} \mathrm{rk}\, E_D^i/E_D^{i+1} \Big[-i\,\widehat{\deg}(t) + \mathrm{height}(\gamma_D^i)\Big]$$

Our choice of $\lambda$ gives $[\,\ldots\ldots\,] \le -i\cdot d$

$$\underbrace{\frac{1}{\mathrm{rk}\, E_D} \sum_{i \ge \lambda D} \frac{i}{D}\, \mathrm{rk}\left(E_D^i/E_D^{i+1}\right)}_{\ge 0} \le \underbrace{\frac{c + c(\lambda) + \widehat{\deg}(\bar{\mathcal{L}}_p)}{d} + \lambda}_{\text{independent of } D}$$

$\le$ constant $\square$