F. CHARLES — Arithmetic algebraization à la Schneider-Lang II. Diophantine applications

Version of Schneider-Lang $\quad K \hookrightarrow \mathbb{C}$

X/K smooth variety over a number field

$\mathcal{F}$ a 1-dimensional foliation, $P \in X(K)$, $\hat{V}_P$ the leaf through P
complex

Assume we have:

$$\varphi : \mathbb{C} \cancel{\setminus \Delta} \longrightarrow \hat{V}_P$$

meromorphic function of order at most ρ.

Let $F \subset \mathbb{C} \setminus \{\text{poles}\}$ such that $\varphi(F) \subset X(K)$
(locally finite)

If $\dfrac{\# F}{\rho [K:\mathbb{Q}]} \geq C$ (absolute constant), then $\hat{V}_P$ is algebraizable.

Remark. If F is infinite, then $\hat{V}_P$ is algebraizable

Corollary. (Hermite-Lindemann)

If $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, then $\exp(\alpha)$ is transcendental.

Pf. The graph Γ of $\exp : \mathbb{C} \to \mathbb{C}$ is the leaf of a foliation on $\mathbb{C}^2$. If $(\alpha, \exp(\alpha)) \in K^2$ # field

$(n\alpha, \exp(n\alpha)) \in K^2$

So we get infinitely many points in $\Gamma \cap K^2$ which contradicts Schneider-Lang. $\square$

<u>Strategy</u> (F finite) $\dim X = 2$)

consider $\hat{V}$ the formal scheme with support in $\varphi(F)$ defined by F

(formal scheme $/_K$)

Consider $\overline{\mathcal{L}}$ hermitian line bundle on $\mathfrak{X}$ some model of X over $\mathcal{O}_K$; ample.

Consider the evaluation map

$$H^0(\mathfrak{X}, \mathcal{L}^D) \xrightarrow{ev} H^0(\hat{V}, \mathcal{L}^D)$$

limit of H^0 over thickenings (K-vector spaces)

If ev is <u>not</u> injective for some D, then V is not Zariski dense.

Assume $\dim X = 2$. If ev is <u>not</u> injective, then V is algebraizable.

<u>Need</u>: to put metrics + integral structure on the right hand side. Then show that it is <u>not</u> too big.

* <u>Integral structure</u>

need integral structure on $H^0(\hat{V}, \mathcal{L}^D)$ $= \varprojlim H^0(V_i, \mathcal{L}^D)$

such that ev is integral.

put the structure such that,

if f is a section of $\mathcal{L}^D$ that vanishes to order k at a point P

$$k!\, j^k(s) \in S^k N_P \hat{V} \otimes \mathcal{L}^D$$

is integral

* <u>Metric</u>: take the quotient metric (on the image of $H^0(\mathcal{X}, \mathcal{L}^D)$).

E_i^D = hermitian $\mathcal{O}_K$-module corresponding to sections of $\mathcal{L}^D$ on $\hat{V}$ vanishing to the $i-1^{th}$ order on the support

L_i^D the quotient E_i^D / E_{i+1}^D

L_i^D a hermitian line bundle on $\mathcal{O}_K$

can define $h^0_\theta(L_i^D)$.

If $\sum_{i \geq 0} h^0_\theta(L_i^D(\varepsilon)) < +\infty$, then we can define

$$h^0_\theta(H^0_{int}(\hat{V}, \bar{\mathcal{L}}^D))$$

in such a way that

$$h^0_\theta(H^0_{int}(\hat{V}, \bar{\mathcal{L}}^D)) \leq \sum_i h^0_\theta(L_i^D).$$

If ev: $H^0(\mathfrak{X}, \bar{\mathcal{L}}^D) \to H^0_{int}(\hat{V}, \bar{\mathcal{L}}^D)$ were

injective, then

$$h^0_\theta(H^0(\mathfrak{X}, \bar{\mathcal{L}}^D)) \leq h^0_\theta(H^0_{int}(\hat{V}, \bar{\mathcal{L}}^D))$$

$$\vee\!/$$

(arithmetic Hilbert-Samuel) $\quad C \cdot D^3$

	geom. case	Bost-Chambert	Schneider-Lang
upper bnd for $h^0_\theta(L_i^D)$	$(-\alpha i + \beta D)^+$	$\alpha i + \beta D$	$i \log i + \beta D$
Schwarz lemma $i \gg 0$	same	$(-\gamma i + \beta D)^+$	$(i \log i + \beta D - \gamma \frac{\# F}{p} i \log(\frac{i}{D}))^+$ (*)

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$h^0_\theta(L_i^?) \neq 0$ for $i = ?$	$i = 10$	$i = 10$	$i = D^{1+\frac{1}{\gamma^{\frac{\#F}{\rho}}-1}}$
h^0_θ	$\approx D^2$	$\approx D^2$	$\approx D^{2+\frac{1}{\gamma^{\frac{\#F}{\rho}}-1}}$

If $\gamma < 3$, we are done.

<u>Proof of (*)</u>

<u>Setup</u>: $\varphi : \mathbb{C} \longrightarrow X(\mathbb{C})$

$F = \{p_0, \dots p_{\#F}\} \subset \mathbb{C}$, $p_0 = 0$ $\omega = \varphi^*(\text{curvature of } \bar{L})$

$g_r = 2 \log^+ \frac{r}{|z|}$ for some $r > 0$

$dd_c g_r = -\delta_0 + \mu_{\Gamma_r}$ (mass 1) $\Gamma_r = \partial \Omega_r$ (disc of radius 1)

s section of $\varphi^* L^D$ $\operatorname{div} s \geq \sum_{k=1}^{\#F} i\, p_k$

$s = z^i \tilde{s}$ $\tilde{s}(0) \neq 0$.

$$\int_{\mathbb{C}} \log \|\tilde{s}\|^2 \, dd_c g_r = \int_{\mathbb{C}} g_r \, dd_c \log \|\tilde{s}\|^2$$
(Stokes)

$$-\log\|\tilde{s}(0)\|^2 + \int_{\Gamma_r} \underbrace{\log\|\tilde{s}\|^2}_{\substack{\log\|s\|^2 \\ -2i\log r}}\, d\mu = \int g_r \left(\delta_{dir(s)} - Dw\right)$$

$$-\log\|\tilde{s}(0)\|^2 + \log\|s\|_\infty^2 - 2i\log r \geq 2i(\#F-1)\log r - D r^\rho$$

finite order $\leq \rho$

for r big enough

$$r^\rho = \frac{i}{D}$$

$$\log\|s\|_\infty^2 \geq \overbrace{\log\|\tilde{s}(0)\|^2}^{\geq -2i\log i} + \frac{2\#F}{\rho}\, i \log\frac{i}{D} + \text{indep } \rho \qquad \square$$

Easy "higher dimensional" variant

for a leaf V with uniformization by $\mathbb{C}^n \to V$

(finite order)

$F = \mathbb{Z}^n$

Applications to Lie groups

K # field

$G/_K$ commutative algebraic group

$\mathfrak{h} \hookrightarrow \mathrm{Lie}\, G$ sub k-Lie algebra

Assume that $\mathfrak{h}_{\mathbb{C}}$ has a basis B such that

$$\exp(B) \subset G(\overline{\mathbb{Q}})$$

Then $\mathfrak{h}$ is algebraic.

<u>Example</u>: Gelfond-Schneider

apply to $\mathbb{G}_m \times \mathbb{G}_m$

if $\alpha \in \overline{\mathbb{Q}}$, $\beta \in \overline{\mathbb{Q}}$ $\quad \alpha^{\beta} \in \overline{\mathbb{Q}} \Rightarrow \beta \in \mathbb{Q}$

$$\begin{array}{c} \alpha^{\beta} \\ \parallel \\ \exp(\beta \underset{\neq 0}{\log \alpha}) \end{array}$$

<u>Example</u>: G_1, G_2 commutative alg. groups

Assume that

$\mathrm{Lie}\, G_{1,\mathbb{C}}$ is generated as $\mathbb{C}$-vector

space by <u>periods</u> $= \mathrm{Ker}\, \exp_{G_1}$

Consider a morphism $\mathrm{Lie}\, G_1 \xrightarrow{\varphi} \mathrm{Lie}\, G_2$

s.t. $\varphi_{\mathbb{C}}$ (periods of G_1) $\subset$ periods of G_2

Then φ comes from a morphism $G_1 \to G_2$.

<u>Example</u>: $E_1, E_2 / K$ elliptic curves

$$H^1_{dR}(E_1/K) \xrightarrow{\varphi} H^1_{dR}(E_2/K)$$

$$\varphi_{\mathbb{C}} : H^1(E_1, \mathbb{C}) \longrightarrow H^1(E_2, \mathbb{C}) \quad \text{(singular cohomology)}$$

$$\begin{array}{ccc} \cup & & \cup \\ H^1(E_1, \mathbb{Q}) & \longrightarrow & H^1(E_2, \mathbb{C}) \end{array}$$

If $\varphi_{\mathbb{C}}(H^1(E_1, \mathbb{Q})) \subset H^1(E_2, \mathbb{Q})$, then φ comes from an isogeny.