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# The Hodge Index Theorem and Connectedness Theorems.

$X$ sm. proj surface over $k=\bar{k}$. Intersection pairing on divisors $\operatorname{Div}X \times \operatorname{Div}X \to \mathbb{Z}$ extends to real coefficients, and when we quotient by the kernel of num. trivial elems, get non-degenerate pairing

$$NS(X)_{\mathbb{R}} \times NS(X)_{\mathbb{R}} \to \mathbb{R}$$

Note: Theorem of the base gives $\rho := \operatorname{rk} NSX < \infty$

If $H \in NS(X)$ ample, and $D$ irred divisor, then $H.D>0$. Also $H^2>0$

Thm (HIT I) For $D \in NS(X)$ s.t. $D.H=0$, we have $D^2<0$

Proof Note $H^2>0 \Rightarrow NS(X) \cong \mathbb{R}[H] \oplus H^\perp$. So wts $D \in H^\perp \setminus 0 \Rightarrow D^2<0$.

Assume $D^2>0$. RR gives $\chi(nD) = \frac{n^2D^2 - nDK_X}{2} + \chi(\mathcal{O}_S)$

so $\chi(nD) \to \infty$ as $n \to \infty$. By Serre duality $h^0(nD) + h^0(K-nD) \geq \chi(nD)$

so one must go to $\infty$ as $n \to \infty$.

i) If $h^0(nD) \to \infty$, then $nD$ effective, so $nD.H>0 \Rightarrow D.H>0$

Also $(K-nD).H<0$ for $n \gg 0$, so $h^0(K-nD)=0$

ii) If $h^0(K-nD) \to \infty$, then $K-nD$ effective so $(K-nD).H>0$. Hence $nD.H<0$ and so $D.H<0$. Also $h^0(nD)=0$

These are hence mutually exclusive and give a contradiction $\square$

Equivalently

(HIT II): The signature on $NS(X)$ is $(1, \rho-1)$

(HIT III): $D_1, \dots, D_n$ divisors st. $\exists\, \alpha_1, \dots, \alpha_n \in \mathbb{Z}$ with $D = \sum \alpha_i D_i$ and $D^2>0$

Then $(-1)^{n-1}\det(D_i.D_j) \geq 0$

(I) $\Rightarrow$ (II) trivial and (III) $\Rightarrow$ (I) follows by setting $D_1 = H$, $D_2 = D$

(II) $\Rightarrow$ III: The map $\varphi: \mathbb{R}^n \to NS(X)$ with $\varphi(x_1, \dots, x_n) = \sum x_i D_i$ is injective if $\det(D_i.D_j) \neq 0$ and the induced pairing on $\mathbb{R}^n$ has signature $(1, n-1)$ and is pos. def. on $\mathbb{R}\left[\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}\right]$. The determinant of this pairing is $\det(D_i.D_j)$ with sign $(-1)^{n-1}$ $\square$

## II Applications

(i) Let $C$ be a smooth proj curve over $\overline{\mathbb{F}}_p \rightsquigarrow \exists q = p^r$ with $C/\mathbb{F}_q$.
Let $F: C \to C$ the absolute Frobenius $f \mapsto f^q$ on functions.
Let $X = C \times C$ and take $p \in C(\overline{\mathbb{F}}_p)$ and denote $V = \{p\} \times C$, $H = C \times \{p\}$
$\Delta$ the diagonal and $G$ the graph of Frobenius
Note: that $G = (F, id)^{-1}(\Delta)$ $(*)$

If $N := \# C(\mathbb{F}_p)$, then

| | $V$ | $H$ | $\Delta$ | $G$ |
|---|---|---|---|---|
| $V$ | $0$ | $1$ | $1$ | $1$ |
| $H$ | $1$ | $0$ | $1$ | $q$ |
| $\Delta$ | $1$ | $1$ | $2-2g$ | $N$ |
| $G$ | $1$ | $q$ | $N$ | $q(2-2g)$ |

where
- $H.G$ = number of $q$-th roots $= q = |Fr^{-1}(x)|$
- $\Delta^2 = 2-2g$ by adjunction
- $\Delta . G$ = fixed pts of Frob $= N$
- $G^2 = \deg(F) \cdot \Delta^2$ by $(*)$.

By setting $D_1 = V$, $D_2 = H$, $D_3 = \Delta$, $D_4 = G$, see that for $(\alpha_1, \dots, \alpha_4) = (1,1,0,0)$ have $D^2 := (\sum \alpha_i D_i)^2 = (V+H)^2 = 2 > 0$ so by HIT III have that det of above matrix has sign $(-1)^{4-1} = -1$.
But compute $\det = (N-(q+1))^2 - 4gq$ $\Rightarrow$ Thm(Weil) $|N - (q+1)| \leq 2g\sqrt{q}$

(ii) Let $X$ be a sm. proj srf and $H$ an ample divisor which is effective
Then $H$ is connected: if not, $H = D_1 + D_2$ $D_i$ eff and $D_1 \cdot D_2 = 0$
But then $D_1 D_1 = D_1 \cdot (D_1 + D_2) = D_1 H > 0$ & $D_2^2$ also $> 0$. Contradiction to HIT III.
Propn $D$ effective on $X$ s.t. $D$ nef and $D^2 > 0$. Then $D$ connected.
Thm $f: X' \to X$ dominant morph of sm. proj surfaces. Assume $\exists C \subset X$ sm. proj curve and $C'$ sm. proj in $X'$ s.t.
(i) $f|_{C'}: C' \to C$ is an iso (ii) $f$ is étale in a nbhd of $C$ (iii) $C^2 > 0$
Then $f$ birational.
Proof Wts $\deg(f) = 1$. Let $D' = f^* C$, eff. div on $X'$. Then $C$ nef $\Rightarrow D'$ nef and $D'^2 = \deg(f) . C^2 > 0$ so Propn $\Rightarrow D'$ connected.
Write $D' := f^* C = C' + R$ where $R$ = ramification. Then (ii) $\Rightarrow R \cap C' = \emptyset$ so $R = 0$ since $D'$ connected. Hence $(C^2 =) C'^2 = D'^2 = \deg(f) C^2 \Rightarrow \deg f = 1$. $\square$

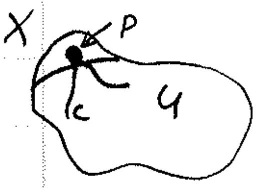
# III Chow Connected and Density and Bertini Theorems

$X$ quasi-proj complex scheme. WLOG assume reduced & irreducible.

**Prop 1** If $U \subset X$ Zariski open (dense), then $U(\mathbb{C}) \subset X(\mathbb{C})$ analytically dense

**Proof** First, assume $X = C$ a curve. Take $\nu : \tilde{C} \to C$ normalisation

Then $U' = \nu^{-1}(U)$ Zar dense in a Riemann surface, so $U'(\mathbb{C}) \subset \tilde{C}(\mathbb{C})$ complement of fin. many pts, so dense. Hence also $U(\mathbb{C}) \subset C(\mathbb{C})$ dense since $\nu$ continuous




To reduce to curves, take $p \in (X \setminus U)(\mathbb{C})$

Claim $\exists$ curve (integral) $C$ in $X$ s.t. $p \in C(\mathbb{C})$ and $C \setminus \{p\} \subset U$

Then, by curve case $p \in \overline{(C \setminus \{p\})(\mathbb{C})}^{an} \subset \overline{U(\mathbb{C})}^{an}$ so Prop 1 follows.

Proof of Claim: Take hyperplane $H$ of $X \subset \mathbb{P}^N$, through $p$, transversal to $X \setminus U$.

Take cpt $Y_0 \subset X \cap H$ through $p$. Show $U \cap Y_0 \subset Y_0$ open. Proceed by induction $\square$.

**Prop 2** $X$ (Zariski) connected $\Rightarrow$ $X(\mathbb{C})$ (analytically) connected.

**Proof** First, assume $X = C$ an integral curve. $\nu : \tilde{C} \to C$ as before & $\tilde{C} \subset \bar{\tilde{C}}$ compactification

Now $\bar{\tilde{C}}(\mathbb{C})$ conn $\Rightarrow$ $C(\mathbb{C})$ conn. so assume $C$ sm. proj curve.

Take $p, q \in C(\mathbb{C})$ and $f \in |\mathcal{O}_C(Np - q)|$, $f \neq 0$. Then $\tilde{f} = f|_{C(\mathbb{C}) \setminus \{p\}}$ holomorphic and $\tilde{f}(q) = 0$. If $C = C_1 \sqcup C_2$ and $p \in C_1$, $q \in C_2$, then $f|_{C_2}$ holom so constant hence zero $\Rightarrow$ $\tilde{f} \in \hat{\mathcal{O}}_{C,q}$ zero contradicting $f \neq 0$.

For any $X$: Take any $p, q \in X(\mathbb{C})$ and $\pi : \tilde{X} \to X$ the blowup at $p, q$.

$E_p, E_q$ exc div. Let $i : \tilde{X} \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$. By Bertini, $\exists H_1, \dots, H_{\dim X - 1}$ so that $X \cap (\cap H_i) = C$ irreducible of dim 1. Also $C.E_p$ and $C.E_q > 0$

Then $\pi(C)$ through $p, q$ integral curve, so use curve case. $\square$

Thm (Bertini) $X$ irred variety and $f: X \to \mathbb{P}^r$ gen-fin onto its image
For $d < \dim X$, if $L \subset \mathbb{P}^r$ general $(r-d)$-plane, $f^{-1}(L)$ irred.

Proof (See Lazarsfeld)

Remark: Take $\tilde{X} \xrightarrow{\gamma} X \xrightarrow{f} \mathbb{P}^r$ resln by Hironaka. Then $D = \gamma^* \mathcal{O}(1)$ is big and nef
Then $0 \to \mathcal{O}_{\tilde{X}}(-D) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{D}} \to 0$ and by Kawamata-Viehweg
$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(-D)) = 0 \ \forall\, i < n$. So $H^0(\mathcal{O}_{\tilde{X}}) \simeq H^0(\mathcal{O}_{\tilde{D}})$ so $\tilde{D}$ connected.
What about irreducible??

Suffices to show for $d = n-1$. Fix a general $(r-n)$-plane $\Lambda$, meeting $X$ in finitely many smooth points. Consider

$$V := \{ (x,t) \mid f(x) \in L_t \}$$

$$\begin{array}{ccc} & V & \\ {}^{pr_1}\swarrow & & \searrow{}^{pr_2} \\ X & & T = \mathbb{P}^{n-1} = \{ L : n\text{-d plane through } \Lambda \} \end{array}$$

wts $pr_2$ has irred fibres. $V$ irred since $pr_1$ is the blowup along $f^{-1}(\Lambda) = \{p_1, \dots, p_s\}$. Fix pt $p_1 \in f^{-1}(\Lambda)$, then $s: T \to V$ $t \mapsto (p_1, t)$ gives a section meeting smooth locus of $V$.
After shrinking $V$ and $T$ assume smooth loc trivial fibration by generic smoothness, and ets connected, but this follows by existence of section.

Thm (Fulton-Hansen) $X$ irred proj variety and $f: X \to \mathbb{P}^r \times \mathbb{P}^r$ s.t. $\dim f(X) > r$. Then $f^{-1}(\text{diagonal}) \subseteq X$ is connected.

Cor 1 $X$ proj irred variety of $\dim X = n$ and $f: X \to \mathbb{P}^r$ unramified. If $2n > r$ then $f$ is a closed embedding.

Proof: $f: X \to \mathbb{P}^r$ unramified $\Rightarrow \Delta_X \subset X \times X$ is a clopen of $X \times_{\mathbb{P}^r} X = (f \times f)^{-1}(\Delta_{\mathbb{P}^r})$
$\Rightarrow \Delta_X$ a conn cpt of $X \times_{\mathbb{P}^r} X$. Now FH applied to $X \times X \xrightarrow{(f,f)} \mathbb{P}^r \times \mathbb{P}^r$ gives $X \times_{\mathbb{P}^r} X$ connected $\Rightarrow f$ is one to one, but $f$ proper + unramified $\Rightarrow$ closed embedding. $\square$

Cor 2 $X \subset \mathbb{P}^r$ irred of $\dim X = n$ with $2n > r$. Then $X$ alg simply conn. i.e. no non-triv conn ét covers.

Pf: $Y \to X$ conn. ét cover and $Y' \subseteq Y$ irred cpt. Then $Y' \hookrightarrow Y \to X \hookrightarrow \mathbb{P}^r$ unram so closed embedding by Cor 1 $\Rightarrow Y' \simeq X$ so $Y$ trivial. $\square$