F. GOUNELAS – The Hodge Index Theorem on projective surfaces. Connectivity theorems

X smooth projective surface $/_{k=\bar{k}}$

Intersection pairing

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \longrightarrow \mathbb{Z}$$

extends $\otimes \mathbb{R}$ to real coefficients, and after quotient by numerically trivial divisors, we get a symmetric non-degenerate pairing

$$\begin{array}{ccccc} NS(X) & \times & NS(X) & \longrightarrow & \mathbb{R} \\ D_1 & & D_2 & & D_1 \cdot D_2 \end{array}$$

Observations.

i) Theorem of the base says that $NS(X)$ is finitely generated, call rank $\rho = r_k$.

(ii) If $H \in NS(X)$ is ample and D is an irreducible divisor, then $H \cdot D > 0$

integral curve

Also, $H^2 > 0$.

Fix H an ample divisor on X

Theorem (HIT I)

$0 \neq D \in NS(X)$ and $D \cdot H = 0$, then $D^2 < 0$

Proof: Note that $H^2 > 0 \Rightarrow$

$$NS(X) = \mathbb{R}[X] \oplus H^{\perp}$$

where $D \in H^{\perp}$ if $D \cdot H = 0$. So what we want to show is that $D \in H^{\perp} \Rightarrow D^2 < 0$.

Assume for a contradiction that $D^2 > 0$.

Riemann-Roch for surfaces:

$$\chi(nD) = \frac{n^2 D^2 - n D \cdot K_X}{2} + \chi(\mathcal{O}_X)$$

where K_X = canonical divisor of X

As a polynomial in n this satisfies

$$\chi(nD) \longrightarrow \infty \quad \text{as } n \to \infty$$

By Serre duality: $h^0(nD) + h^0(K_X - nD)$

$$\geq \chi(nD)$$

Since $\chi(nD) \to \infty$, at least one of $h^0(nD)$ or $h^0(K_X - D)$ is unbounded as $n \to \infty$.

Two cases:

(a) If $h^0(nD) \to \infty$ then nD effective, so $nD \cdot H > 0 \implies D \cdot H > 0$

Also $(K_X - nD) \cdot H < 0$ for $n \gg 0$

so $h^0(K_X - nD) = 0$.

(b) If $h^0(K_X - nD) \to \infty$, then $K_X - nD$ effective, so $(K_X - nD) \cdot H > 0$. Hence $nD \cdot H \leq 0$

$\Rightarrow D \cdot H \leq 0. \implies h^0(nD) = 0.$

<u>Upshot</u>: these are mutually exclusive and, in particular, they give a contradiction □

<u>Theorem</u> (HIT II) The signature of the pairing is $(1, \rho - 1)$.

Theorem (HIT III)

$D_1, \dots D_n$ any divisors such that $\exists\, \alpha_1, \dots \alpha_n \in \mathbb{Z}$ with $D = \sum \alpha_i D_i$ and $D^2 > 0$. Then

$$(-1)^{n-1} \det (D_i \cdot D_j) \geq 0$$

The three version are equivalent.

(I) $\Rightarrow$ (II) is trivial

(III) $\Rightarrow$ (I) $D_1 = D$, $D_2 = H$ and apply III

(II) $\Rightarrow$ (III) Define the map $\varphi : \mathbb{R}^n \to NS(X)$

$$(x_1, \dots x_n) \longmapsto \sum_{i=1}^{n} x_i D_i$$

If $\det (D_i \cdot D_j) \neq 0$, $\{D_i\}$ are linearly independent hence φ is injective. Have an induced pairing $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ given by

$$\langle (x_1 \dots x_n), (y_1 \dots y_n) \rangle = \left(\sum x_i D_i\right) \cdot \left(\sum y_i D_i\right)$$

This pairing has signature $(1, n-1)$ and the positive definite part is given by $\mathbb{R}\left[\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}\right]$.

The determinant of this pairing is $\det(D_i \cdot D_j)$ with sign $(-1)^{n-1}$. $\Rightarrow (-1)^{n-1}\det(D_i \cdot D_j) \geq 0$

$\square$

II. Application

Let C be a smooth projective curve over $\overline{\mathbb{F}_p}$.

$\exists\, q = p^2$ so that C is defined over $\mathbb{F}_q$. Let $F: C \to C$ be the absolute Frobenius ($f \mapsto f^q$ on functions on C). Fix $p \in C(\overline{\mathbb{F}_p})$

$V = \{p\} \times C \qquad H = C \times \{p\}$

$\Delta =$ diag $\qquad G =$ graph of Frobenius

Observe that $G = (F, \mathrm{id})^{-1}(\Delta)$ (*)

If $N = \# C(\mathbb{F}_q)$, intersection matrix:

	V	H	Δ	G
V	0	1	1	1
H	1	0	1	q
Δ	1	1	$2-2g$	N
G	1	q	N	$q(2-2g)$

where $g = \text{genus}(C)$.

(diff. of Frobenius $= 0$)

$$H \cdot G = \deg(F) = q$$

$$\Delta^2 = 2 - 2g \quad \text{by adjunction}$$

$$\Delta \cdot G = \text{fixed points of Frobenius} = N$$

$$G^2 = \deg(F) \cdot \Delta^2 \quad \text{by observation } (*)$$

By setting $D_1 = V$, $D_2 = H$, $D_3 = \Delta$, $D_4 = G$

See that $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 0, 0)$

$$D = \sum \alpha_i D_i = \quad + H \quad \text{satisfies } D^2 = 2 > 0.$$

So by HIT III, have the determinant of the above intersection matrix has sign -1.

Explicitly:

$$\det = (N - (q+1))^2 - 4g^2 q$$

<u>Theorem</u> (Weil) $|N - (q+1)| \leq 2g\sqrt{q}$

③ Chow connected & density and Bertini (Fulton-Hansen) theorems

X quasi-proj. scheme$/\mathbb{C}$, reduced irreducible

<u>Prop. 1</u>:

If $U \subset X$ Zariski open (dense), then $U(\mathbb{C}) \subset X(\mathbb{C})$

analytically dense.

<u>Proof</u>: reduce by hyperplanes to the case of curves and, in the curve case, normalise and use the same result for Riemann surfaces □

<u>Prop 2</u>: If X is Zariski connected then $X(\mathbb{C})$ is analytically connected.

<u>Proof</u>: If $\underset{C}{\overset{X}{\shortparallel}}$ is an integral curve, then

$\nu : \tilde{C} \to C$ normalisation, and compactify

$\hat{\tilde{C}} \to \tilde{C}$. observe $\hat{\tilde{C}}(\mathbb{C})$ connected $\Rightarrow$

$C(\mathbb{C})$ connected

Hence might as well assume C smooth projective.

Take $p, q \in C(\mathbb{C})$ and $0 \neq f \in H^0(\mathcal{O}(Np - q))$.

for $N >> 0$. Then $\underset{\tilde{f}}{\overset{f|_{C(\mathbb{C}) \setminus \{p\}}}{\shortparallel}}$ holomorphic

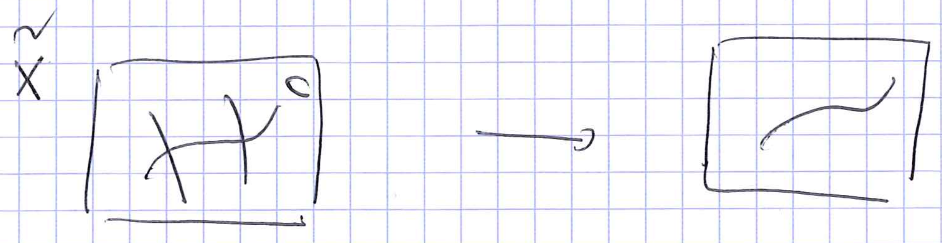
and $\tilde{f}(q) = 0$. If $C = C_1 \amalg C_2$ disconnected union such that $p \in C_1$ and $q \in C_2$. Then $f|_{C_2}$ is holomorphic and since $f|_{C_2}(q) = 0$
$\Rightarrow f = 0$ on C_2 contradiction!

<u>Any X</u>: Take $p, q \in X(\mathbb{C})$. Let $\pi : \tilde{X} \to X$ be the blow-up at p, q and exceptional divisors $E_p = \pi^{-1}(p)$, $E_q = \pi^{-1}(q)$.

$i : \tilde{X} \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$

By Bertini: $\exists H_1, \dots H_{n-1}$ $(n = \dim X)$ general hyperplanes, so that $X \cap (\cap H_i) = C$ irreducible curve.

Moreover, $C \cdot E_p > 0$ and $C \cdot E_q > 0$.



reduce to the case of curves

<u>Theorem</u> (Bertini)

X irreducible variety and $f: X \to \mathbb{P}^r$ generically finite onto its image. For $d < \dim X$, if $L \subset \mathbb{P}^r$ a general $(r-d)$-plane, then $f^{-1}(L)$ is irreducible.

<u>Theorem</u> (Fulton-Hansen)

X irreducible projective, $f: X \to \mathbb{P}^r \times \mathbb{P}^r$ such that $\dim f(X) > r$. Then

$$f^{-1}(\text{diagonal } \Delta_{\mathbb{P}^r}) \subseteq X$$

is connected.

<u>Corollary</u>: $X \subseteq \mathbb{P}^r$ irreducible of $\dim X = n$ such that $2n > r$. Then X is algebraically simply connected, i.e. no non-trivial connected étale covers.