N. MAZZARI – Formal geometry

① Basic definitions

All schemes (also formal) locally noetherian.

local picture: $A$ is an adic ring if it is complete and separated for the $I$-adic topology, $I \subseteq A$ ideal

i.e. $A = \lim_n A/I^{n+1}$ e.g. $\mathbb{Z}_p$, $k[[t]]$

$\rightsquigarrow$ topological ringed space $\mathrm{Spf}(A)$

top. space: $|\mathrm{Spec}\, A/I|$

sheaf of top. $\mathcal{O}_{\mathrm{Spf}(A)} = \lim_n \mathcal{O}_{\mathrm{Spec}(A/I^{n+1})}$

What we have:

$$\mathrm{Spf}(A) = \operatorname{colim}_n \underbrace{\mathrm{Spec}(A/I^{n+1})}_{\text{ringed space}}$$

affine formal scheme

colimit of topologically ringed spaces

Examples: i) $A$ any ring, $I = (0)$ $\rightsquigarrow$ affine schemes

ii) $A = \mathbb{Z}_p$, $I = p\mathbb{Z}_p$, $|\mathrm{Spf}(\mathbb{Z}_p)| = |\mathrm{Spec}\, \mathbb{F}_p|$

+ a sheaf $\mathbb{Z}_p$ with its topology

$$\begin{array}{ccc} \operatorname{Spec} \mathbb{Z}_p & \neq & \operatorname{Spf} \mathbb{Z}_p \\ \| & & \| \\ \{s, \eta\} & & \{s\} \end{array}$$

Global picture: want to glue $\operatorname{Spf}(A)$

A (locally noetherian) formal scheme is a top. ringed space $(X, \mathcal{O}_X)$ locally isomorphic to some $\operatorname{Spf}(A)$.

They form a category with morphisms of ringed spaces + continuous + local.

Basic constructions

$X$ = locally noetherian scheme

$Z = V(I) \subseteq X$ closed subscheme defined by $I \subseteq \mathcal{O}_X$ sheaf of ideals

$$\hat{X}_Z = \operatorname{colim} X_n \qquad \text{nilpotent thickening}$$

$$X_n = \operatorname{Spec}\left(\mathcal{O}_X / I^{n+1}\right) \hookrightarrow X_{n+1}$$

the formal completion of $X$ along $Z$

e.g. $X = \operatorname{Spec} R$, $Z = V(I)$, $\hat{X} = \operatorname{Spf}(\hat{R})$

Formal germs

$k$ = field of char 0 as before, $\hat{X} = \hat{X}_Z$

with $X, Z$ defined over $k$

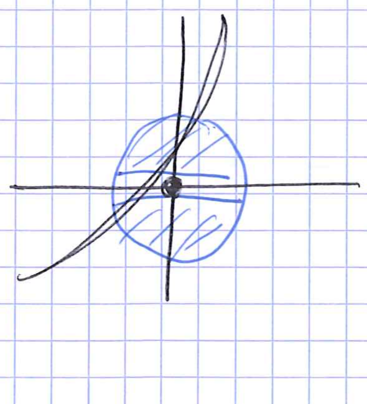
A smooth formal germ of $X$ through $Z$ is $\mathcal{V} \subseteq \hat{X}$ closed formal subscheme such that

i) $\mathcal{V}$ is formally smooth

ii) $Z \subset \mathcal{V}$ + same topological space

e.g. $X = \mathbb{A}^2_k$, $Z = (0,0)$

$$\hat{X} = \operatorname{Spf} k[[x,y]] \supset \mathcal{V} = \operatorname{Spf} k[[x]]$$



$$\mathcal{V}' = \operatorname{Spf} \frac{k[[x,y]]}{(y+1-\exp(x))}$$

Group schemes

$G$ = smooth alg. group scheme $/k$

$\mathfrak{g}$ = Lie $G$ Lie algebra whose functor of points is

$$\mathfrak{g}(R) = \ker\left( G\left( R[\varepsilon]/(\varepsilon^2) \right) \longrightarrow G(R) \right)$$

any ring $\qquad\qquad \varepsilon \longmapsto 0$

$\hat{\mathfrak{g}}$ = completion of $\mathfrak{g}$ along $0$

$\hat{G}$ = completion of $G$ along $e$ (unit of $G$)

Proposition: $k$ char $0$

$\exists$ a canonical isomorphism of formal schemes

$$\widehat{\exp}_G : \hat{\mathfrak{g}} \longrightarrow \hat{G}$$

such that i) the diff. at zero is the identity

ii) for any

$$\begin{tikzcd} \hat{\mathbb{G}}_a \arrow[r, "\varphi"] \arrow[dr] & \hat{\mathfrak{g}} \arrow[d, "\widehat{\exp}_G"] \\ & \hat{G} \end{tikzcd}$$

is a group homomorphism for all vector grp. morphisms

$\hat{\mathbb{G}}_a$ = completion of $\mathbb{G}_a$ along zero

$\hat{\mathbb{G}}_a(R) = \underline{\mathrm{Nilp}}(R)$ nilpotent elements

<u>Example</u>: $G = GL_n$, $\mathfrak{g}(R) \simeq M_n(R)$ matrices

$$\begin{array}{c} \hat{\mathfrak{g}}(R) \simeq M_n(\mathrm{Nilp}(R)) \\ \big\downarrow \exp \\ \hat{G}(R) = \mathrm{Id} + M_n(\mathrm{Nilp}(R)) \end{array}$$

## <u>The Grothendieck comparison theorem</u>

<u>Mittag-Leffler</u> $(A_n, \varphi_{n',n})$ inverse system of abelian groups. It satisfies ML if $\forall n \in \mathbb{N}$, $\exists n_0 \geq n$ such that $\forall n', n'' \geq n_0$

$$\begin{tikzcd} A_{n'} \arrow[r, "\varphi_{n',n}"] & A_n \\ A_{n''} \arrow[ur, "\varphi_{n'',n}"'] & \end{tikzcd}$$

$\varphi_{n',n}$ and $\varphi_{n'',n}$ have same images

<u>Lemma</u>: If $(A_n, \varphi_{n',n})$ satisfies Mittag-Leffler, then $R^q \lim_n A_n = 0 \quad \forall q > 0$.

Artin-Rees lemma:

$A =$ noetherian ring

$I \subseteq A$ ideal, $M =$ fin. gen. $A$-module and

$M' \subseteq M$ sub $A$-module.

$\Rightarrow \exists\, n_0 > 0$ such that $\forall\, n \geq n_0$

$$M' \cap I^n M = I^{n-n_0}(M' \cap I^{n_0} M)$$

In particular:

1) the $I$-adic topology on $M$ induces the $I$-adic topology on $M'$

2) $I$-completion is exact on fin. gen. $A$-modules

3) canonical isomorphism $\boxed{\hat{A} \otimes_A M \simeq \hat{M}}$.

Setting:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"'] \\ Y_0 \arrow[r, hook] & Y \end{tikzcd}$$

morphism of (locally noeth.) schemes

$Y_0 =$ closed subscheme

completions $\rightsquigarrow$

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & \hat{X} = \operatorname{colim} X_n \arrow[d, "\hat{f}"] \\ Y \arrow[r, hook, "j"] & \hat{Y} = \operatorname{colim} Y_n \end{tikzcd}$$

$\mathcal{F}$ $\mathcal{O}_X$-module

base change map

$$\gamma : j^* \mathbb{R}^q f_* (\mathcal{F}) \longrightarrow \mathbb{R}^q \hat{f}_* (i^* \mathcal{F})$$

Theorem: Assume i) $\mathcal{F}$ is coherent.

$$i^* \mathcal{F} = \lim_n \mathcal{F} \otimes \mathcal{O}_{X_n} = \hat{\mathcal{F}}$$

ii) $f$ of finite type & supp $\mathcal{F}$ is proper over $Y$

Then base change

$$\gamma : \widehat{\mathbb{R}^q f_* \mathcal{F}} \longrightarrow \mathbb{R}^q \hat{f}_* \hat{\mathcal{F}}$$

is an $\mathcal{O}_{\hat{Y}}$-isomorphism.

Sketch of the proof: $f$ proper, $Y = \operatorname{Spec}(A)$

$\hat{Y} = \operatorname{Spf}(\hat{A})$ (ring $A \hookrightarrow \hat{A}$)

We are reduced to prove

the following two isomorphisms:

$$H^q(\hat{X}, \hat{\mathcal{F}}) \xrightarrow[\sim]{a} \lim_n H^q(X, \mathcal{F}/I^{n+1})$$

$$\lim_n H^q(X, \mathcal{F})/I^{n+1} \xrightarrow[\sim]{b} \lim_n H^q(X, \mathcal{F}/I^{n+1}), \qquad \lim_n H^q(X, \mathcal{F})/I^{n+1} \overset{\gamma}{\dashrightarrow} H^q(\hat{X}, \hat{\mathcal{F}})$$

a = continuity

b = completion

<u>Application</u>: $f$ proper, as in the theorem,

$Y_0 = y$ a point. Then

$$\widehat{(\mathbb{R}^q f_* \mathcal{O}_X)_y} = \lim_n H^q(X_y, \mathcal{O}_X/m_y^{n+1})$$

<u>Corollary</u>: if $\mathcal{O}_Y = f_* \mathcal{O}_X$, then the fibers are non-empty and connected

<u>Algebraization problem</u>

$A$ = adic noetherian ring with ideal $I$

$$\begin{tikzcd} Y = \mathrm{Spec}(A) & & Y_0 = \mathrm{Spec}(A/I) \arrow[ll, hook'] \arrow[dl] \\ & \hat{Y} = \mathrm{Spf}(A) \arrow[ul] & \end{tikzcd}$$

Let $Z$ = adic noeth. formal scheme $/\hat{Y}$

$$= \operatorname{colim}_n \underbrace{(Z \times_{\hat{Y}} Y_n)}_{\text{alg. scheme}}$$

(Grothendieck's Existence Theorem)

Theorem : same notation for $Y$

$$\begin{array}{c} X \\ f\big\downarrow \\ Y \end{array}$$ of finite type $\Longrightarrow$ We have an equivalence of categories

$$\left\{ \begin{array}{c} \mathcal{F} \in \mathrm{Coh}(\mathcal{O}_X) \\ \mathrm{supp}\,\mathcal{F} \text{ proper}/Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{G} \in \mathrm{Coh}(\hat{X}) \\ \mathrm{supp}\,\mathcal{G} \text{ proper}/\hat{Y} \end{array} \right\}$$