F. MARTIN - The theorems of Andreotti and Hartshorne

① An algebraization result

V complex manifold, $M \subset V$ submanifold

normal bundle to M in V

= vector bundle on M defined by

$$0 \to TM \to TV_{|M} \to N_V M \to 0$$

rank = dim V − dim M

If M is a closed submanifold, $I \subset \mathcal{O}_V$ ideal sheaf of M, $N_V M \simeq \left(I/I^2{}_{|M} \right)^*$.

$V \subset \mathbb{P}^N(\mathbb{C})$ submanifold, V is algebraizable

if $\dim\left(\overline{V}^{zar}\right) = \dim V$

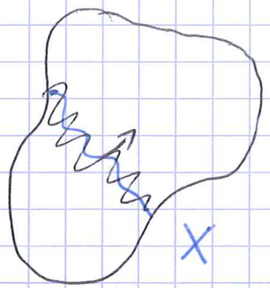
Example 1) $V = \{(z, z\sqrt{1+z}),\ z \in \mathbb{C},\ |z| < 1\}$

$V^{zar} = \{y^2 = x^3 + x^2\}$

2) $V' = \{(z, e^z) \mid z \in \mathbb{C}\}$ $\quad \overline{V'}^{zar} = \mathbb{P}^2(\mathbb{C})$

Theorem 1: $X \subseteq \mathbb{P}^N(\mathbb{C})$ smooth projective connected subvariety of dim ≥ 1. Let $V \subseteq \mathbb{P}^N(\mathbb{C})$ a complex analytic submanifold (not necessarily closed) such that $X \subseteq V$, V connected.

If $N_X V$ is ample, V is algebraizable.



② Ample vector bundles

X proj. scheme over k

Rk: $H \subseteq X$ hypersurface $\mathcal{O}(H) \simeq N_H X$

If $Y \subset X$ subvariety "Y ample subvariety"

$\Leftrightarrow$ "$N_Y X$ is ample"

Recall: L line bundle $/X$. The following are equivalent:

1) $\exists\, m > 0$, L^m is very ample $(X \xhookrightarrow{L^m} \mathbb{P}^N)$

2) for all $\mathcal{F} \in Coh(X)$, $\exists n_0 \ \forall n \geq n_0$ $\mathcal{F} \otimes L^n$ is globally generated

3) ——— $\forall q > 0$, $\exists n_0 \ \forall n \geq n_0$ $H^q(X, \mathcal{F} \otimes L^n) = 0$.

$E \to X$ vector bundle of rank $r+1$

associated projective bundle $\mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}\, \mathcal{E})$

$$\mathrm{Sym}\, \mathcal{E} = \bigoplus_{n \geq 0} S^n \mathcal{E}$$

(E vector space $S^n E = \bigotimes^n E / S_n$

$e_0, \dots e_r$ basis

$k[e_0 \dots e_r]_n$)

$\pi : \mathbb{P}(E) \to X$

$\mathcal{O}_{\mathbb{P}(E)}(1)$ Tautological line bundle

(if E is trivial, $E \simeq X \times \mathbb{A}^{r+1}$, $\mathcal{E} = \bigoplus \mathcal{O}_X$

$\mathbb{P}(E) = \mathbb{P}^r_X$, $\mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_{\mathbb{P}^r_X}(1)$)

__Definition__. $E \to X$ is ample if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample on $\mathbb{P}(E)$.

The two definitions agree for line bundles.

Proposition 1 $E \to X$ vector bundle, $n \in \mathbb{N}$

1) $\pi_*(\mathcal{O}_{\mathbb{P}(E)}(n)) \simeq S^n \mathcal{E}$

2) $\forall q > 0 \quad R^q \pi_*(\mathcal{O}_{\mathbb{P}(E)}(n)) = 0$

3) $\forall q \geqslant 0 \quad H^q(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(n)) \simeq H^q(X, S^n\mathcal{E})$

Proof: $X = \operatorname{Spec} A$, $\mathcal{E} = \mathcal{O}_X^{\oplus n+1}$

1) $H^0(\mathbb{P}^2_X, \mathcal{O}_{\mathbb{P}^2_X}(n)) \simeq A[T_0, \dots T_2]_n$

2) $H^q(\mathbb{P}^2_X, \mathcal{O}_{\mathbb{P}^2_X}(n)) = 0 \quad q > 0, \; n \geqslant 0$

3) $\Leftarrow$ 1) 2) by spectral sequence $\square$

Proposition 2. $E \to X$ vector bundle

The following are equivalent:

1) E ample

2) $\forall \mathcal{F} \in \mathrm{Coh}(X)$, $\exists n_0 \; \forall n \geqslant n_0 \quad S^n\mathcal{E} \otimes \mathcal{F}$ is globally generated

③

3) $\forall \mathcal{F} \in \mathrm{Coh}(X) \quad \forall q > 0 \quad \exists n_0 \quad \forall n \geq n_0$

$$H^q(X, S^n\mathcal{E} \otimes \mathcal{F}) = 0$$

vector bundle on a curve is ample $\Leftrightarrow$ all slopes are positive

③ Proof of Theorem 1

$X \subset \mathbb{P}^N(\mathbb{C})$ smooth projective subvariety

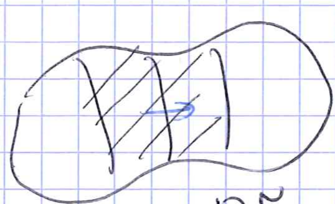
$\dim X > 0$, $X \subset V$

V submanifold, connected in $\mathbb{P}^N(\mathbb{C})$

$N_X V$ ample $\Rightarrow$ V is algebraizable

Remark: GAGA $\quad N_X V$ is analytic $\Rightarrow$ algebraic

$L := \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(1)|_V \qquad D \in \mathbb{N}$



$\mathbb{P}^N$

$$\eta_D : H^0(\overline{V}^{zar}, \mathcal{O}(D)) \hookrightarrow H^0(V, L^D)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathcal{O}(1)^D_{|V}$$

injective: otherwise

$$s \in \mathrm{Ker}(\eta_D) \setminus \{0\}$$

$$V \subseteq \overline{V}^{zar} \cap \{s = 0\} \subsetneq \overline{V}^{zar}$$

Claim 1: $\dim H^0(V, L^D) = O(D^{\dim V})$

$$\dim H^0(\bar{V}^{zar}, \mathcal{O}(D)) \leq \dim H^0(V, L^D)$$

$$C \cdot D^{\dim \bar{V}^{zar}} \leq \qquad = O(D^{\dim V})$$

Hilbert polynomial

$\exists C > 0$

$$\Rightarrow \dim V \geq \dim \bar{V}^{zar}$$

Proof of Claim 1

$I \subset \mathcal{O}_V$ the ideal sheaf of X

$\mathcal{O}_V \supset I \supset I^2 \supset \dots \supset I^i \supset \dots \qquad D \in \mathbb{N}$

$L^D \supset I \otimes L^D \supset I^2 \otimes L^D \supset \dots$

$$E_D^i = H^0(V, L^D \otimes I^i)$$
$$= \{ s \in H^0(V, L^D) \mid s \text{ vanish along } X \text{ with order} \geq i \}$$

$H^0(V, L^D) = E_D^0 \supset E_D^1 \supset \dots \qquad \bigcap_{i \geq 0} E_D^i = \{0\}$

$$\dim H^0(V, L^D) = \sum_{i\geq 0} \dim \left(E_D^i / E_D^{i+1} \right)$$

exact sequence

$$0 \to I^{i+1} \to I^i \to I^i / I^{i+1} \to 0$$

$$0 \to H^0(V, L^D \otimes I^{i+1}) \to \underset{\substack{\| \\ E_D^i}}{H^0(V, L^D \otimes I^i)}$$

$$\to H^0(V, L \otimes I^i / I^{i+1}) \to \cdots$$

$$E_D^i / E_D^{i+1} \subset H^0(V, L^D \otimes I^i / I^{i+1})$$

$$\begin{array}{c} \wr\!\wr \\ H^0(X, L^D \otimes I^i / I^{i+1}) \end{array} \longleftarrow \operatorname{supp}(I^i / I^{i+1}) \subset X$$

$$\cancel{E}\, I^i / I^{i+1} \big|_{X} \simeq j_* S^i(N_X V^*)$$

$$\simeq H^0(X, L^D \otimes S^i(N_X V^*))$$

$$= H^0(X, L^D \otimes S^i(N_X V)^*)$$

chm 0

$$= H^d(X, L^{-D} \otimes S^i(N_X V) \otimes \omega)$$

$d = \dim X > 0$

$$\simeq H^d\left(\mathbb{P}(N_X V), \mathcal{O}_{\mathbb{P}(N_X V)}(i) \otimes \pi^* L^{-D} \otimes \pi^* \omega\right)$$

<u>Claim 2</u>: A ample line bundle $/X$

M line bundle on X, $q > 0$

$F \in Coh(X)$ ① $\exists e > 0$ s.t. $i \geq eD$ $i >> 0$

$$H^q(X, A^i \otimes M^D \otimes F) = 0$$

$\exists \alpha, \beta$

② $\forall i, D \geq 0$ $H^q(X, A^i \otimes M^D \otimes F)$

has dimension $\leq \alpha (i+D)^{\dim X} + \beta$

<u>Proof</u>: $\exists m$ such that $A^m \otimes M$ ample

$e = m+1 \rightsquigarrow i \geq mD + D$

$$H^q(X, (A^m \otimes M)^D \otimes \tilde{A} \otimes \mathcal{F}) \qquad \square$$

<u>Conclusion</u>

$$\dim H^0(V, L^D) \leq \sum_{i \geq 0} \dim H^1(X, L^D \otimes I^i / I^{i+1})$$

$$= \sum_{i \geq 0} \dim H^d\left(\mathbb{P}(N_X V), \mathcal{O}_{\mathbb{P}(XX)}(i) \otimes \pi^* L^D \otimes \pi^* \omega\right)$$

$$\overset{\text{Claim}}{\leq} \sum_{i=0}^{[eD]} \alpha (i+D)^{\dim \mathbb{P}(N_X V)} + \beta = O(D^{\dim V}). \qquad \square$$

4 Hartshorne's theorem

Theorem 2: $X \subseteq \mathbb{P}^N_k$ smooth proj. sub variety

$\dim X > 0$

$\hat{V} \hookrightarrow \widehat{\mathbb{P}^N_k/X}$ such that $|\hat{V}| = |X|$.

$N_X \hat{V}$ is ample $\Rightarrow$ $\hat{V}$ algebraizable formal scheme

Theorem (Hartshorne)

$K(\hat{V})$ has transcendence degree $\leq \dim \hat{V}$

$$H^0(X, \underbrace{L^{\otimes d} \otimes S^i N^\vee}_{d \cdot \deg L + i \mu_{max}(N)})$$

$\mu_{max} < 0 \Rightarrow$ no sections

$\mu_{max}(L \otimes V) = \deg L + \mu_{max}(V)$

$\mu_{max}(S^i V) = i \mu_{max}(V)$