O. BENOIST. – The theorems of Grauert and Grothendieck (SGA 2)

I Setting

k a field

$X \subseteq \mathbb{P}^N_k$ smooth projective of pure dimension d

$t \in H^0(\mathbb{P}^N_k, \mathcal{O}(1))$ that intersects X properly $\Rightarrow \{t=0\} = Y \subseteq X$ has dimension $d-1$

We wish to compare Y and X.

Introduce $Y_n \subseteq X \rightsquigarrow \widehat{X}_Y = \varinjlim_n Y_n$

$$Y_n = \{t^{n+1} = 0\}$$

3 steps

$$Y \underset{\text{deformation}}{\rightsquigarrow} \widehat{X}_Y \underset{\text{algebraization}}{\rightsquigarrow} \text{some open subset } Y \subseteq \mathcal{U} \subseteq X \underset{\text{extension}}{\rightsquigarrow} X$$

II Main theorems

Theorem 1 (Comparison) Let $E \to X$ be a vector bundle. If $d \geq 2$, $H^0(X, E) \simeq H^0(\widehat{X}_Y, E|_{\widehat{X}_Y})$.

Theorem 2 (Existence) Let $\mathcal{E}$ be a vector bundle on $\hat{X}_Y$, (i.e. $E_n \to Y_n$ vector bundle, $E_n|_{Y_{n-1}} \simeq E_n$.)

If $d \geq 3$, there exists a coherent sheaf E on X such that $E|_{\hat{X}_Y} = \mathcal{E}$.

Remarks: Theorem 1 fails if $d = 1$

$X = \mathbb{P}^1_k$, $Y =$ point, $E = \mathcal{O}_X$

$H^0(X, \mathcal{O}_X) = k$

$H^0(\hat{X}_Y, \mathcal{O}_{\hat{X}_Y}) = k[[t]]$

Remark: Theorem 2 fails in $d = 2$

$X = \mathbb{P}^2_k$ $\quad Y = \mathbb{P}^1_k \underset{\text{line}}{\hookrightarrow} \mathbb{P}^2_k$, there are line bundles on $\hat{X}_Y$ that do not algebraize.

Remark: In SGA 2, there are

* weaker hypothesis on singularities of X
* variants over a base
* local variants

III Comparison theorem

<u>Proof</u>: $$0 \to \mathcal{O}_X(-n-1) \to \mathcal{O}_X \to \mathcal{O}_{Y_n} \to 0$$

$\otimes E$ $$0 \to E(-n-1) \xrightarrow[j]{} E \to E_{|Y_n} \to 0$$

$$0 \to H^0(X, E(-n-1)) \to H^0(X,E) \to H^0(Y_n, E)$$

$$\to H^1(X, E(-n-1)) \qquad H^0(X, E(-n-1)) \overset{\text{Serre}}{=} H^d(X, E^\vee \otimes \omega_X(n+1))^\vee \overset{n>>0}{=} 0$$

$$H^1(X, E(-n-1)) \overset{\text{Serre}}{=} H^{d-1}(X, E^\vee \otimes \omega_X(n+1))^\vee \overset{n>>0}{=} 0$$

because $d \geq 2$!

Thus $$H^0(X,E) \Rightarrow \varprojlim_n H^0(Y_n, E) = H^0(\hat{X}_Y, E_{|\hat{X}_Y})$$ since all transition maps are surjections for $n>>0$

$\square$

IV Existence ($d \geq 3$)

<u>Proof</u>: We rely on the following lemma:

<u>lemma</u>: let $\mathcal{E}$ be a vector bundle on $\hat{X}_Y$. Then

(i) $H^i(\hat{X}_Y, \mathcal{E})$ is finite dimensional for $i=0,1$

(ii) If $\ell >> 0$

$$H^0(\hat{X}_Y, \mathcal{E}(\ell)) \otimes \mathcal{O}_{\hat{X}_Y} \twoheadrightarrow \mathcal{E}(\ell)$$

i.e. $\mathcal{E}(\ell)$ is globally generated

End <u>of the</u> proof

By the lemma, there exists ℓ, r such that

$$\mathcal{O}_{\hat{X}_Y}(-\ell)^{\oplus r} \twoheadrightarrow \mathcal{E} \to 0$$

Applying this again to the kernel, we obtain an exact sequence

$$\mathcal{O}_{\hat{X}_Y}(-\ell')^{\oplus r'} \xrightarrow[\hat{\varphi}]{} \mathcal{O}_{\hat{X}_Y}(-\ell)^{\oplus r} \to \mathcal{E} \to 0$$

By Theorem 1, $\hat{\varphi} \in H^0(\hat{X}_Y, \mathcal{H}om(\mathcal{O}(-\ell')^{\oplus r'}, \mathcal{O}(-\ell)^{\oplus r}))$

comes from $\varphi \in H^0(X, \mathcal{H}om(\mathcal{O}(-\ell')^{\oplus r'}, \mathcal{O}(-\ell)^{\oplus r}))$.

Define $E = \operatorname{coker}(\varphi)$.

Proof of the lemma

(i)

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathcal{O}_X(-n-1) \arrow[r, "t"] \arrow[d, "t^{n+1}"] & \mathcal{O}_X(-n) \arrow[r] \arrow[d, "t^{n}"] & \mathcal{O}_Y(-n) \arrow[r] & 0 \\
 & & \mathcal{O}_X \arrow[r, equal] \arrow[d] & \mathcal{O}_X \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_Y(-n) \arrow[r] & \mathcal{O}_{Y_n} \arrow[r] \arrow[d] & \mathcal{O}_{Y_{n-1}} \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}
$$

$\otimes\, \mathcal{E}$ + cohomology

$$0 \to H^0(Y, \mathcal{E}_0(-n)) \to H^0(Y_n, \mathcal{E}|_{Y_n}) \to H^0(Y_{n-1}, \mathcal{E}|_{Y_{n-1}})$$

$$\to H^1(Y, \mathcal{E}_0(-n)) \to H^1(Y_n, \mathcal{E}_n) \to H^1(Y_{n-1}, \mathcal{E}_{n-1})$$

$$H^{d-1}(Y, \mathcal{E}_0^\vee \otimes \omega_Y(n))^\vee \underset{n \gg 0}{=} 0$$

$$H^{d-2}(Y, \mathcal{E}_0^\vee \otimes \omega_Y(n))^\vee$$

$\overset{\|}{0}$ $n \gg 0$ because $d \geq 3$

Then $H^0(\hat{X}_Y, \mathcal{E}) = \lim_n H^0(Y_n, \mathcal{E}_n)$

$$\downarrow \wr \quad n_0 \gg 0$$

$$H^0(Y_{n_0}, \mathcal{E}_{n_0})$$

finite dimension.

$$H^1(\hat{X}_Y, \mathcal{E}) \xrightarrow{\sim} \lim_{\overleftarrow{n}} H^1(Y_n, \mathcal{E}_n)$$

because $(H^1(Y_n, \mathcal{E}_n))_n$ satisfy Mittag-Leffler by finite dimensionality of each of them.

$$\downarrow \quad n_0 \gg 0$$

$H^1(Y_{n_0}, \mathcal{E}_{n_0})$ of finite dimension

Now part (ii)

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

$\otimes\, \mathcal{O}_{\hat{X}_Y} \otimes \mathcal{E}(l)$ + cohomology:

$$H^0(\hat{X}_Y, \mathcal{E}(l)) \to H^0(Y, \mathcal{E}(l)) \to H^1(\hat{X}_Y, \mathcal{E}(l-1))$$

$$\xrightarrow{t} H^1(\hat{X}_Y, \mathcal{E}(l)) \to H^1(Y, \mathcal{E}(l))$$

Trick: Instead of working with $\ell \gg 0$, we consider all of them together.

$$M = \bigoplus_{\ell \geq 0} H^1(\hat{X}_Y, \mathcal{E}(\ell))$$

viewed as a graded $k[t]$-module.

$$\underbrace{H^1(\hat{X}_Y, \mathcal{E}(-1))}_{\substack{\text{finite dimensional} \\ \text{by part (i)}}} \oplus M \longrightarrow M \longrightarrow \bigoplus_{\ell \geq 0} \underbrace{H^1(Y, \mathcal{E}(\ell))}_{\substack{\text{finite dim} \\ \text{by Serre} \\ \text{vanishing}}}$$

Thus, M/tM is finite dimensional.

This implies that M is a $k[t]$-module of finite type (lift a basis of M/tM to M).

$N = \{ x \in M \mid tx = 0 \} \subseteq M$ is a $k[t]$-module of finite type. Thus N has finite dimension, thus N vanishes in higher degree (for the grading).

This $\cdot t$ is injective when $\ell \gg 0$

N.B. Better: notice that $\cdot t$ are onto, for $l \gg 0$, hence $h^1(\hat{X}_Y, \mathcal{E}(l))$ eventually decreases, hence becomes stationary, hence $\cdot t$ becomes injective.

If $l \gg 0$, then: $\cdot\ H^0(\hat{X}_Y, \mathcal{E}(l)) \twoheadrightarrow H^0(Y, \mathcal{E}(l))$

$\cdot\ \mathcal{E}(l)$ globally generated on Y

$$H^0(\hat{X}_Y, \mathcal{E}(l)) \otimes \mathcal{O}_{\hat{X}_Y} \twoheadrightarrow \mathcal{E}_0(l)$$

By Nakayama: $H^0(\hat{X}_Y, \mathcal{E}(l)) \otimes \mathcal{O}_{\hat{X}_Y} \twoheadrightarrow \mathcal{E}(l)$ $\square$

V Applications to π_1

Theorem: If $d \geq 3$, $\pi_1(Y) \xrightarrow{\sim} \pi_1(X)$.

Proof. We need to show that

$$\acute{E}t(X) \to \acute{E}t(Y)$$ finite étale covers

is an equivalence.

$$\acute{E}t(X) \xrightarrow{(a)} \lim_{\substack{Y \subseteq U \subseteq X \\ \text{open}}} \acute{E}t(U) \xrightarrow{(b)} \acute{E}t(\hat{X}_Y) \xrightarrow{(c)} \acute{E}t(Y)$$

(a) is an equivalence because a finite étale cover of $\mathcal{U}$ extends uniquely to a normal finite cover of X. By Zariski–Nagata, the locus where it is not étale is a divisor that cannot meet Y, hence is empty because Y is ample.

(b) A finite étale cover is the data of a vector bundle E + $\mathcal{O}_X \to E$ + $E \otimes E \to E$ satisfying compatibilities.

By Theorem 1 & 2: we have an equivalence of categories

$$\varinjlim_{Y \subseteq \mathcal{U} \subseteq X} \text{vector bundles on } \mathcal{U} \xrightarrow{\sim} \text{vector bundles on } \hat{X}_Y$$

This formally implies that

$$\varinjlim_{\substack{Y \subseteq \mathcal{U} \subseteq X \\ \text{open}}} \bar{E}t(\mathcal{U}) \xrightarrow{\sim} \acute{E}t(\hat{X}_Y)$$

(together with the fact that being étale is an open cond.)

(c) is an equivalence because

$$\acute{E}t(Y_n) \xrightarrow{\sim} \acute{E}t(Y_{n-1})$$

because finite étale covers are insensitive to nilpotents $\Rightarrow$ $\acute{E}t(\hat{X}_Y) \xrightarrow{\sim} \acute{E}t(Y)$ $\square$

<u>Remarks</u>: If $d \geq 2$, this proves that

$$\pi_1(Y) \twoheadrightarrow \pi_1(X)$$

(VI) <u>Applications to Pic</u>

<u>Theorem</u>. If $d \geq 3$ and $H^i(Y, \mathcal{O}(-n)) = 0$ for $i = 1, 2$, $n > 0$. Then

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y)$$

<u>Proof</u>: $$\mathrm{Pic}(X) \xrightarrow{(a)} \varinjlim_{\substack{Y \subseteq U \subseteq X \\ \text{open}}} \mathrm{Pic}(U) \xrightarrow{(b)} \mathrm{Pic}(\hat{X}_Y) \xrightarrow{(c)} \mathrm{Pic}(Y)$$

(a) is an surjection because a line bundle on U

extends to X by regularity of X uniquely because $X \setminus U$ has codimension > 1 (it is a closed subset of X along the ample divisor Y).

(b) is an isomorphism by Thm 1 & Thm 2

(c)
$$0 \to \mathcal{O}(-n)|_{Y_0} \longrightarrow \mathcal{O}^*_{Y_n} \to \mathcal{O}^*_{Y_{n-1}} \longrightarrow 0$$
$$f \longmapsto 1+f$$

morphism of sheaves of groups

$\rightsquigarrow$ cohomology

$$\underbrace{\cancel{H^1(Y, \mathcal{O}(-n))}}_{=0} \to H^1(Y_n, \mathcal{O}^*_{Y_n}) \to H^1(Y_{n-1}, \mathcal{O}^*_{Y_{n-1}}) \to \underbrace{\cancel{H^2(Y, \mathcal{O}(-n))}}_{=0}$$

by assumption

$$\operatorname{Pic}(\hat{X}_Y) = \varprojlim_n \operatorname{Pic}(Y_n) \xrightarrow{\sim} \operatorname{Pic}(Y)$$

$\square$

Ex: $X = \mathbb{P}^2_k$, $Y \underset{\text{line}}{\hookrightarrow} X$

$$\begin{array}{ccc} \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}\,\hat{X}_Y \\ \parallel & & \parallel \\ \mathbb{Z}\cdot\mathcal{O}(1) & \xleftarrow{\ \ } & \varprojlim \mathrm{Pic}(Y_n) \end{array}$$

very big!