S. PEPIN-LEHALLEUR — Some geometric applications of algebraization theorems

(I) Algebraicity of certain foliations in char 0

Idea: • use a relative version of Frobenius integrability / formal exponential to construct formal schemes along a subscheme.

• use Hartshorne's theorem

Lie subalgebras of smooth algebraic groups over function fields

k field of char 0

C smooth projective connected curve over k

$(\mathcal{G}, \varepsilon)$ smooth quasi-proj. scheme over C

Ex: • $Gl_n/_C$, $V/_C$ vector bundle $Aut(V)/_C$

abelian schemes, semi-abelian schemes, Néron models ...

$(G, e) = \mathcal{G}_K$ smooth alg. group $/K$

$K = k(C)$

Lie $\mathfrak{g}$ vector bundle over C

$$\left\{\begin{array}{c}\text{sub vector bundles}\\ \text{of Lie } \mathfrak{g}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{sub vector space}\\ \text{of Lie } G\end{array}\right\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\left\{\begin{array}{c}\text{sub Lie algebras}\\ \text{of Lie } \mathfrak{g}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{sub Lie algebras}\\ \text{of Lie } G\end{array}\right\}$$

<u>Theorem</u> (Bost) Let F be a sub Lie algebra of G, write $\mathcal{F}$ for the corresponding sub Lie algebra of $\mathfrak{g}$.

Assume $\mathcal{F}$ is ample.

Then F is an algebraic sub Lie algebra of Lie G with associated smooth connected subgroup H. Moreover, H is unipotent.

<u>Remark</u>: • fails if char $k \neq 0$

• If G is an abelian variety, then G has no non-trivial unipotent subgroup. Then Lie $\mathfrak{g}$ has no ample subbundles (Griffiths)

There exists a relative formal exponential map

$$\hat{\mathcal{L}} = \mathbb{V}(\mathrm{Lie}\, \mathcal{G}^\vee)^\wedge_\varepsilon \xrightarrow[\widehat{\exp}_{\mathcal{G}}]{\sim} \hat{\mathcal{G}}_\varepsilon$$

isomorphism of formal schemes

(not of formal group schemes unless $\mathcal{G}$ is commutative)

$$d\,\widehat{\exp}_{\mathcal{G}} : T_0 \hat{\mathcal{L}} \longrightarrow T_\varepsilon \hat{\mathcal{G}}_\varepsilon$$

$$\begin{array}{ccc} \wr\!\wr & & \wr\!\wr \\ \mathrm{Lie}\,\mathcal{G} & = & \mathrm{Lie}\,\mathcal{G} \end{array}$$

$$\widehat{\exp}_G : \hat{L} \xrightarrow{\sim} \hat{G}_e$$

Put $\hat{\mathcal{V}} = \widehat{\exp}_{\mathcal{G}}(\mathcal{F})$ $\qquad \varepsilon^* N_\varepsilon \hat{\mathcal{V}} \simeq \mathcal{F}$

$\hat{V} = \widehat{\exp}_G(F)$ $\qquad$ ample vector bundle on C

By Hartshorne's theorem, $\hat{\mathcal{V}}$ is algebraizable i.e. its Zariski closure in $\mathcal{G}$ is of dimension $\dim F$.

then $\overline{\hat{\mathcal{V}}}^{\mathrm{Zar}}$ is a commutative subgroup of $\mathcal{G}$.

<u>$\mathcal{F}$ abelian</u> $\Leftrightarrow$ F abelian $\qquad \mathcal{F} \hookrightarrow \mathrm{Lie}\,\mathcal{G}$

The graph Γ of this embedding gives a sub Lie algebra of $\mathrm{Lie}(\mathbb{W}(\mathcal{F}^\vee) \times \mathcal{G})$.

As a vector bundle on C, it is similar to $\mathcal{F}$ ample.

$\Rightarrow \widehat{\exp}(\Gamma)$ is algebraizable

$\Rightarrow \mathcal{H} \subset \mathbb{W}(\mathcal{F}^\vee) \times \mathcal{G}$

At the generic fiber

$$\begin{tikzcd}
\tilde{H} \arrow[r] \arrow[d, "\wr"', "\text{étale morphism}"] & \mathcal{G} \\
\mathbb{W}(F^\vee) \arrow[ur, "\text{iso onto its image}"'] &
\end{tikzcd}$$

in fact finite étale surjective, hence an isomorphism

$\Rightarrow \mathbb{W}(F^\vee) \simeq H$.

<u>In general</u>:

write the Harder-Narasimhan filtration of $\mathrm{Lie}\,\mathcal{G}$

$$\{0\} \subseteq E_1 \subseteq E_2 \subseteq \dots \subseteq E_n = \mathrm{Lie}\,\mathcal{G}$$

$\mu_1 > \mu_2 > \dots > \mu_n$ slopes of semistable quotients E_i / E_{i-1}

$E_+ \subset E$ the maximal E_i with $\mu_j \geq 0 \ \forall j < i$

Then any ample subbundle of Lie $\mathfrak{g}$ is contained in E_+

Any subgroup of a unipotent alg. group is unipotent.

We show that $(E_+)_k$ is the Lie algebra of a unipotent subgroup of Lie G.

<u>Lemma</u>: • $\mu_i > 0 \implies E_{i,k}$ is a sub Lie algebra of Lie G

• $\forall j \leq i$ $E_{j,k}$ is a Lie ideal in $E_{i,k}$ $([E_{i,k}, E_{j,k}] \subset E_{j,k})$

• $\mu_i > 0 \implies E_{i,k} / E_{i-1,k}$ is abelian.

The proof uses that $\mu_{min}(E) > \mu_{max}(F)$ $\implies$ Hom$(E, F) = 0$.

By the previous case, $E_{i,k}/E_{i-1,k}$ is algebraic.

$\overline{H}_{i,k}$ vector group with $\mathrm{Lie}(\overline{H}_{i,k}) = E_{i,k}/E_{i-1,k}$

$\square$

$\widehat{\exp}_G(F)$ is the formal flow for the foliation on G associated to F.

(II) Kodaira vanishing for H^1

<u>Theorem</u>. X smooth projective variety of dimension $d \geq 2$ over k, char $k = 0$

L ample line bundle on X

Then $H^1(X, L^\vee) = 0$.

<u>Remark</u>: general form of Kodaira vanishing

$$H^i(X, L^\vee) = 0 \qquad i < d \qquad (1)$$

Serre
$$\Longleftrightarrow \quad H^j(X, L \otimes \omega_X) = 0 \qquad j > 0. \qquad (2)$$

- Kodaira (1953) proved (2) by analytic techniques.
- Deligne - Illusie proved (1) $\Rightarrow$ (2) algebraically
- Algebraic proofs for H^1: Mumford ('67) using connectedness for ample divisors and reducedness of the Picard variety in char 0.

Mumford ('77) using results of Bogomolov on rank 2 vector bundles.

Proof (Bost)

reduction to the case of surfaces

for $N \gg 0$, Y general member of $|L^{\otimes N}|$

Y smooth (Bertini)

$H^1(X, L^{\otimes -(N-1)}) = 0$ (Enriques - Severi - Zariski lemma $d \geq 2$)

$H^1(X, L^\vee) \hookrightarrow H^1(Y, L^\vee)$

So we can assume that $\dim X = 2$.

let $\alpha \in H^1(X, L^\vee)$ represented by

$\alpha_{ij} \in Z^1(\mathcal{U}, L^\vee)$ for $\mathcal{U} = (\mathcal{U}_i)_{i \in I}$

open covering

$$\alpha_{ij} \in \Gamma(\mathcal{U}_i \cap \mathcal{U}_j, L^\vee)$$
$$\cap\!\!|$$
$$\mathrm{Hom}(\mathcal{U}_i \cap \mathcal{U}_j, \mathbb{V}(L^\vee))$$

$V = \mathbb{V}(L^\vee)$

$\hat{V} = \hat{V}_0$

$$\exp(\alpha_{ij}) = \sum_{n \geq 0} \frac{1}{n!} \alpha_{ij}^n \in \Gamma(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}_{\hat{V}})$$

$$\alpha_{ij} \in Z^1(\mathcal{U}, L^\vee) \implies \exp(\alpha_{ij}) \in Z^1(\mathcal{U}, \mathcal{O}_{\hat{V}}^\times)$$
$$\exp(\alpha_{ij}) = \exp(\alpha) \in H^1(\hat{V}, \mathcal{O}_{\hat{V}}^\times)$$

$\rightsquigarrow$ $\hat{E}$ formal line bundle on $\hat{V}$

In fact: $\exp(\alpha_{ij}) \in \Gamma(\mathcal{U}_i \cap \mathcal{U}_j, \mathrm{Ker}(\mathcal{O}_{\hat{V}}^\times \to \mathcal{O}_X^*))$

$\rightsquigarrow \hat{E}|_X \simeq \mathcal{O}$

X smooth $\Rightarrow \varepsilon^*: \mathrm{Pic}(V) \to \mathrm{Pic}(X)$

Assume that $\hat{E}$ is algebraizable, i.e. $\exists E$ line bundle in V with $E|_{\hat{V}} = \hat{E}$.

Then: $\hat{E} = 0$.

Hence $\exp(\alpha) = 0 \underset{\text{restriction to } V_1}{\Longrightarrow} \alpha = 0$.

Algebraization of $\hat{E}$:

- $\dim(X) \geq 2 \Rightarrow \dim(V) \geq 3$
- We embedd V into a projective cone V^+ over X such that $N_{V^+}X$ is ample (Grauert). Then we apply the existence theorem. $\Rightarrow \hat{E}$ algebraizable

$\square$

III Lefschetz theorem for F-divided sheaf

k field of char $p > 0$

$X/_k$ smooth quasi-projective variety

An F-divided sheaf $E_\bullet$ on X is a collection of coherent sheaves E_n together with isomorphisms $F^* E_{i+1} \xrightarrow{\sim} E_i$ with F the absolute Frobenius of X.

$F^{-1} E_{i+1} \otimes \mathcal{O}_X$

$, F^* \mathcal{O}_X$

Theorem: X smooth

(i) $E_\bullet \in FDiv(X) \implies \forall n \geq 0$ E_n locally free

(ii) $FDiv(X)$ equipped with the monoidal structure $(E_\bullet \otimes F_\bullet)_n = E_n \otimes F_n$

Tannakian category

(iii) (Katz) $FDiv(X)$ is equivalent to the cat. of integrable connections.

$x \in X(k) \Rightarrow x^*: FDiv(X) \to Vec_k$

fiber functor

$$\pi_1^{alg}(X, x) = \pi(FDiv(X), x^*)$$

<u>Question</u>: (Gieseker)

If $\pi_1^{et}(X) = 0$, does $\pi_1^{alg}(X) = 0$?

<u>Theorem</u> (Biswas, Esnault, Srinivas)

X smooth projective of pure dim d

Y smooth ample divisor, $x \in Y(k)$

Then $i_*: \pi_1^{alg}(Y, x) \to \pi_1^{alg}(X, x)$

is an isomorphism if $\dim(X) \geq 3$ and a

surjection if $\dim(X) = 2$.

<u>$d = 2$</u>: Tannakian formulation

i_* surjective $\Leftrightarrow$ $i^*: FDiv(X) \to FDiv(Y)$ fully faithful

For $E_\bullet \in FDiv(X)$, $V'_\bullet \subset i^* E_\bullet$ sub-F-divided sheaf, then there exists $V_\bullet \subset E$ with $i^* V = V'$.

$$FDiv(X) \xrightarrow{\text{extension}} \operatorname*{colim}_{Y \subset \mathcal{U} \subset X} FDiv(\mathcal{U}) \underset{i_1^*}{\xrightarrow{\text{algebraization}}} FDiv(\hat{X}_Y) \underset{\text{definition}}{\xrightarrow{i_2^*}} FDiv(Y)$$

Full faithfulness

- i_2^* is an eq. of categories

  (For I nilp. ideal in an $\mathbb{F}_p$-algebra

  $F^n I = 0$ for $n \gg 0$)

- A morphism of F-divided sheaves is a collection of morphisms between vector bundles, here i_1^* is fully faithful by the comparison theorem.

Theorem. with the hypothesis of the theorem

E coherent sheaf on X, $\hat{E} = E|_{\hat{X}_Y}$ vector bundle

Thus any subquotient bundle of $\hat{E}$ is algebraizable.