# Cycles in $H^{\otimes}_{dR}(A/K)$



**Notation**: $[K:\mathbb{Q}]<\infty$. $A/\operatorname{Spec} K$ (sometimes may write $/K$) polarized abelian variety. $v$ a place of $K$, $k_v$ = res. fld of ~~…~~ $v$.

$H^i_{dR}(A/K) := \mathbb{H}^i(\Omega^\bullet_{A/K})$ $\quad H^{\otimes}_{dR}(A/K)$ = the tensor alg of $\bigoplus_{i=0}^{2\dim A} H^i_{dR}(A/K)$

## I. An application of algebraization results

**Thm** (Bost, see Peter's talk):

$G/K$ commutative algebraic group. $W \subseteq \operatorname{Lie} G$ $K$-vector space.

Assume: for all but finitely many places $v$ of $K$, ~~the~~ $W\otimes k_v$ is closed under the $p$-th power map (of derivatives).

Then $\exists\, H \subseteq G$ algebraic subgp s.t. $\operatorname{Lie} H = W$

**Rmk**: Gasbarri's & Herblot's results ("$\alpha=0$", $\rho\le 2$), To ~~we~~ obtain the same conclusion, we only need to assume:

$\exists\, M$ a set of rational primes of natural density 1 s.t. $\forall p\in M$ $\forall v|p$, $W\otimes k_v$ is closed under $p$-th power map.

From now on, I'll use $\forall\!\!\forall$ for ~~either~~ all but finitely many ~~a natural density~~ 1

**Cor** (Bost + Rmk): Assume that $s\in \operatorname{End}_K(H^1_{dR}(A/K))$ is fixed by (for ~~$\forall p$~~ $\forall p\in M$ $\forall v|p$) the crystalline Frobenius $\varphi_v$ (well-defined for all but finitely many $v$ by the canonical comparison b/w crys. & dR. coh., it is a $\sigma$-linear map on $H^1_{dR}\otimes K_v$)

Then $s$ comes from $\operatorname{End}^\circ_K(A)$.

**Pf** (Bost): $H^1_{dR}(A/K)\simeq \operatorname{Lie} E(A^\vee)$. Here $E(-)$ denotes the universal vector ext.

Notice that (see e.g. Mumford's book on AV) $\varphi_v\otimes k_v$ = $p$-th power map.

Then one concludes by applying Thm + Rmk to the sub Lie alg of $E(A^\vee)\times E(A^\vee)$ defined by the graph of $s$. ∎

## II. Ogus Conjecture (say that he conjectured for all smooth projective varieties)

$L/K$ ~~$\Leftrightarrow$~~ $\forall\!\!\forall\, v$, (use $v$ for ~~any~~ places of both $L$ & $K$ --) $\varphi_v$ crystalline Frobenius $\curvearrowright H^1_{dR}(A_L/L)\otimes L_v$

**Conj** (Ogus): Let $s\in H^{\otimes}_{dR}(A_L/L)$. If $\varphi_v(s)=s$ ~~$\forall$~~ for almost all $v$, then $s$ is a Hodge cycle.

**Hodge cycles**: (Fix $\sigma: K\hookrightarrow\mathbb{C}$) $s\in H^{\otimes}_B(A_\sigma(\mathbb{C}),\mathbb{Q})$ s.t. $s\in H^{\otimes}_B(-)^{0,0}$ ($\Leftrightarrow$ $s\in \operatorname{Fil}^0(—)$)

from Hodge theory $\qquad$ the filtration on de Rham coh.

**Rmk**:

1) the converse statement: Hodge cycles are fixed by $\varphi_v$ $\forall\!\!\forall\, v$. (Ogus / Deligne + Blasius)
2) the above Cor deals w/ $s\in \operatorname{End}(H^1_{dR}(-))$.
3) Ogus: conj holds for $A$ has CM. Serre–Tate theory: ell. curves. (see André's book)

## III. de Rham - Tate cycles + main result

<u>de Rham - Tate (dRT) cycles</u>: Ogus cycles ($s \in H^{\otimes}_{dR}(A_L/L)$ $\varphi_v(s) = s$ $\forall v$)

$+ \ \forall \sigma : K \hookrightarrow \mathbb{C}$, via ~~the~~ $H^{\otimes}_{dR}(A/K) \otimes_K \mathbb{C} \simeq H^{\otimes}_B(A_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$, (*)

$s \in H^{\otimes}_B(A_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ ~~[illegible]~~

<u>Thm (T.)</u>: via (*), {dRT cycles} = {Hodge cycles} if either

1) $A \otimes_K \bar{K}$ simple (ab. var.), dim $A$ is a prime, $\mathrm{End}_{\bar{K}}(A) \neq \mathbb{Z}$, or

2) $K = \mathbb{Q}$, for some $\ell$ (eq'v all $\ell$), all $\ell$-adic Tate cycles are fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and assume that the Mumford - Tate conj holds

~~[illegible] $\ell$-adic Tate cycles are Hodge cycles.~~

Here <u>$\ell$-adic Tate cycles</u>: for $A/K$, $\mathrm{Gal}(\bar{K}/K) \curvearrowright H^{\otimes}_{et}(A_{\bar{K}}, \mathbb{Q}_\ell)$.

prime $\quad s \in H^{\otimes}_{et}$ is called $\ell$-adic Tate if $\exists\, L/K$ fin. s.t. $s$ is fixed by $\mathrm{Gal}(\bar{K}/L)$.

<u>Conj</u> (Mumford - Tate): via $H^{\otimes}_B(A_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq H^{\otimes}_{et}(A_{\bar{K}}, \mathbb{Q}_\ell)$

{$\ell$-adic Tate cycles} = {$\mathbb{Q}_\ell$ - lin. comb. of Hodge cycles}

<u>rmk</u>: 1) Ogus conjecture is a crystalline analogue of MT conj.

2) Bost's theorem $\longleftrightarrow$ Faltings theorem for $T_\ell(A)$.

3) for $K \neq \mathbb{Q}$, MT "$\Rightarrow$" Ogus w/ some technical conditions

4) product case is fine w/ only one non-CM type IV. use D. Lombardo's result on MT.

## IV. Outline of proof

$G_{MT}$ = the Mumford - Tate gp = the largest alg. subgp of $GL(H^1_B(\ ,\mathbb{Q}))$ fixing all Hodge cycles

$G_{dR}$ = the de Rham - Tate gp = ———— $GL(H^1_{dR}(A/K))$ — dRT —

<u>Fact</u>: {dRT cycles} $\otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow[\text{via comparison}]{} H^{\otimes}_B(A_\sigma(\mathbb{C}), \mathbb{C})$ is injective.

$\Rightarrow$ Enough to show $G_{dR} \otimes_K \mathbb{C} \underset{\text{via comparison isom.}}{=} G_{MT} \otimes_{\mathbb{Q}} \mathbb{C}$.

<u>Lemma</u> (Zarhin): $G_1 \subseteq G_2 \subseteq GL(V)$ conn. reductive gps. $V$ v.s. / char 0 fld.

If $\mathrm{rk}\, G_1 = \mathrm{rk}\, G_2$ and $\mathrm{Cent}_{\mathrm{End}(V)} G_1 = \mathrm{Cent}_{\mathrm{End}(V)} G_2$, then $G_1 = G_2$.

So <u>we want</u>: ⓪ reductivity

① $\mathrm{rk}\, G^\circ_{dR} = \mathrm{rk}\, G_{MT}$

② $\mathrm{Cent}_{\mathrm{End}(H^1_{dR})} G^\circ_{dR} = \mathrm{Cent}_{\mathrm{End}(H^1_{dR})} G_{MT}$.

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For ①: MT conj holds for $A$ in (1) by Serre and Chi.

MT conj $\Rightarrow$ $\mathrm{rk}\, G^{\circ}_{dR} = \mathrm{rk}\, G_{MT}$ b/c $G^{\circ}_{dR}$ contains ∀ Frobenius tori

For ①, ②: we have

w/c End(M) s.s. + Jansen's theorem

<u>Prop</u>: One can construct a Tannakian cat. $\mathcal{M}_{dRT}$ following Deligne's construction of $\mathcal{M}_{AH}$ (AH = absolute Hodge) w/ morphisms = dRT cycles. $\forall M \in \mathcal{M}_{dRT}$ End(M) is semi-simple. (This follows from our assumption that dRT cycles are inv. under cpx conj.) and $\mathrm{Aut}^{\otimes}$ (fiber functor to de Rham coh.) of $\mathcal{M}_{dRT} = G_{dR}$.

In particular, $G^{\circ}_{dR}$ is reductive and $\forall s \in H^{\otimes}_{dR}(A_L/L)$,

$s$ is a linear combination of dRT cycles $\Leftrightarrow$ $s$ is fixed by $G_{dR}$.

Bost's theorem + Prop. $\Rightarrow$ $\mathrm{Cent}\, G_{dR} = \mathrm{Cent}\, G_{MT}$.

<u>We want</u>: $\mathrm{Cent}\, G^{\circ}_{dR} = \mathrm{Cent}\, G_{dR}$.

## <u>IV Centralizers of $G_{dR}$ & $G^{\circ}_{dR}$</u>

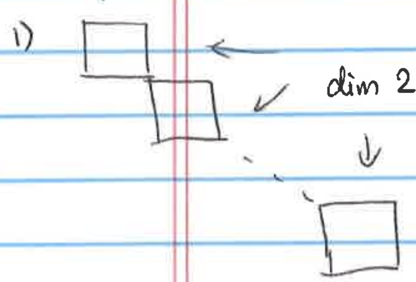
<u>Case (1)</u>: WMA $A$ doesn't have CM. $\dim A = p$

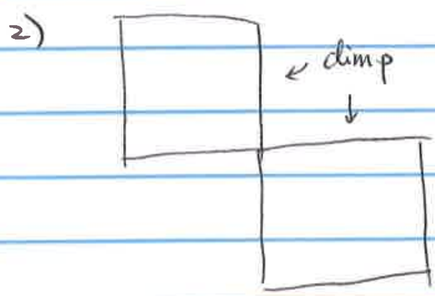
then 1) $\mathrm{End}^{\circ}(A) = F$ tot. real and $[F:\mathbb{Q}] =$ ~~[illegible]~~ $p$ prime

2) $p$ odd $\mathrm{End}^{\circ}(A) = F$ $[F:\mathbb{Q}] = 2$

$p = 2$ $\mathrm{End}^{\circ}(A) =$ quaternion alg.

$H^1_{dR}(A/K) \otimes \bar{K}$ as a rep'n of $G_{dR}$ (resp. $H^1_B(A_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ as rep'n of $G_{MT}$) decomposes into irreducibles (by Bost's thm)




we write

irred $G_{dR}$-rep'n

$H^1{}_{/\bar{K}} = \bigoplus V_i$

$= \bigoplus \bigoplus V_{i,j}$

irred $G^{\circ}_{dR}$-rep'n

<u>Lemma 1</u>: All $V_{i,j}$ are of same dim

<u>Lemma 2</u>: If $\dim V_{i,j} = 1$, i.e. $G^{\circ}_{dR}$ is a torus, then $A$ has CM.

<u>Lemma 3</u>: $V_i \simeq V_j$ as $G^{\circ}_{dR}$-rep'n $\Leftrightarrow$ $V_i \simeq V_j$ as $G_{dR}$-rep'n.

In our case. Lem 1, 2 $\Rightarrow$ $V_i$ irred as $G^{\circ}_{dR}$-replns. $\Rightarrow$ $\mathrm{Cent}\, G^{\circ}_{dR} = \mathrm{Cent}\, G_{dR}$

+ Lem 3

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rmk: 1) Lem 1 uses the fact that $\varphi_v$ "normalizes" $G^{\circ}_{dR}$ and Bost's theorem

2) Lem 2 uses Serre-Tate theory + Noot's result on formal defm. at ordinary good reduction places.

3) Lem 3 is proved by studying Frobenius actions

Case (2): Thm (Serre): Extend $K$ s.t. $G_\ell$ is conn. ($\ell$-adic algebraic monodromy gp = Zariski closure of the image of $Gal(\bar{K}/K) \subseteq H^{1}_{et}$) Then $\exists$ $M$ a set of primes of $K$ of natural density 1 s.t. $\forall v \in M$, the commutative alg. subgp $T_v$ generated by $Frob_v$ ($\in$ Conj Classes of $Im(Gal(\bar{K}/K))$) is of max'l rank. In particular, $T_v$ is connected.

Now $K = \mathbb{Q}$. Katz-Messing: $\varphi_p$ and $Frob_p$ has the same char. poly

$\Downarrow$

One may view $T_p$ as a subgp of $G_{dR, \mathbb{Q}_p}$.

By def'n, $\varphi_p \in G_{dR}(\mathbb{Q}_p)$ $\forall\forall p$. Then $T_p(\mathbb{Q}_p) \subseteq G_{dR}(\mathbb{Q}_p)$

Serre's thm $\Rightarrow$ $T_p(\mathbb{Q}_p) \subseteq G^{\circ}_{dR}(\mathbb{Q}_p)$ $\Rightarrow$ $\varphi_p \in G^{\circ}_{dR}(\mathbb{Q}_p)$ $\forall\forall p \in M$.

i.e. $s \in Cent(G^{\circ}_{dR})$ is fixed by $\forall\forall \varphi_p \in M$.

Cor of (Bost's thm + rmk) $\Rightarrow$ $s$ is alg. i.e. $Cent(G^{\circ}_{dR}) = Cont(G_{MT})$.

## VI. A relative version of Bost's thm

Ques: $s \in End(H^1_{dR}(A_L/L))$, $\varphi_v^{m_v}(s) = s$ $\forall\forall v$ where $m_v := [K_v : \mathbb{Q}_p]$.

Is $s$ an $L$-lin. comb. of dRT cycles?

If yes, then $\forall A/L$, MT $\Rightarrow$ Ogus by a result of Chi on connectedness of $T_v$.

Known cases: $A$ = ell. curve and more generally, if the dim of the smallest Shimura subvariety of $\mathcal{A}_g$ containg the pt $[A]$ is one.

(pf uses Serre-Tate theory)

A related Ques: $s \in End(H^1_{dR}(A_L/L))$, $s$ is $\alpha$-good i.e. $\varphi_v(s) = s$ for $v$ over a set of rat'l primes of density $(1-\alpha)$. If $\alpha < \frac{1}{2}$, is $s$ alg?

Known case: Yes for $\alpha < \frac{1}{4}$ (pf is a direct application of Gasbarri's & Herblot's results).