Y. TANG — Cycles in the de Rham cohomology of abelian varieties over number fields (1)

Cycles in $H^{\otimes}_{dR}(A/K)$

$[K:\mathbb{Q}] < \infty$

A/K polarized abelian variety

$H^i_{dR}(A/K)$

$H^{\otimes}_{dR}(A/K)$ = tensor algebra of $\bigoplus_{i=0}^{2\dim A} H^i_{dR}(A/K)$

(I) An application of algebraization theorem

<u>Thm</u>: G/K commutative alg. group

$W \subseteq \mathrm{Lie}\, G$ K-vector space

Assume: for almost all v of K s.t. $W \otimes k_v$ (residue field) is closed under p^{th} power map.

Then $\exists H \subseteq G$ alg. such that $\mathrm{Lie}\, H = W$

<u>Remark</u>: Gasbarri & Herblot ("$\alpha = 0$") $p=2$, one point = id $\in$ formal leaf)

W is alg. if for a density one.

$\exists M$ a set of prime numbers s.t. $\forall p \in M, \forall v | p$

$W \otimes K_v$ is closed under p^{th} power

$\Rightarrow W$ is alg.

Corollary (Bost)

Assume $s \in \mathrm{End}_K(H^1_{dR}(A/K))$ is fixed by the crystalline Frobenius φ_v for almost all v. (a density is enough). Then s is algebraic

$\mathrm{End}^\circ_K(A)$

Proof. $H^1_{dR}(A/K) = \mathrm{Lie}\, E(A^\vee)$

$\varphi_v \otimes K_v = p^{th}$ power map

$\varphi_v(s) = s \Leftrightarrow \Gamma_s$ is closed under p^{th} power $\square$

(II) Ogus conjecture

L/K fin. ext. $\varphi_v \circlearrowright H^i_{dR}(A_L/L) \otimes L_v$

Conjecture (Ogus) Let $s \in H^\otimes_{dR}(A_L/L)$

If $\varphi_v(s) = s$ for almost all v, then s is a Hodge cycle.

Hodge cycles Fix $\sigma : K \hookrightarrow \mathbb{C}$

$$s \in H_B^{\otimes}(A_\sigma(\mathbb{C}), \mathbb{Q}) \text{ and } s \in (\quad)^{0,0}$$

$$\Leftrightarrow s \in \mathrm{Fil}^0$$

Remark: 1) the converse statement

Hodge is fixed by almost all φ_v

(Ogus / Deligne + Blasius)

2) Corollary verifies the conjecture for

$s \in \mathrm{End}(H^1_{dR}(-))$.

3) Ogus: conjecture for A has CM

Serre-Tate A = elliptic curves
(see André's book)

III de Rham-Tate cycles + main result

dRT: $s \in H^{\otimes}_{dR}(A/L)$, $\varphi_v(s) = s$ for almost all

$\forall \sigma : K \hookrightarrow \mathbb{C}$ via de R-Betti

$$s \in H_B^{\otimes}(A_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}.$$

Theorem (T.)

$\{$ dRT cycles $\}$ = $\{$ Hodge cycles $\}$

in the following cases:

1) $A \otimes_K \bar{K}$ simple, dim A a prime number

$\mathrm{End}_{\bar{K}}(A) \neq \mathbb{Z}$

2) $K = \mathbb{Q}$ for some l ($\Leftrightarrow \forall l$), the l-adic

Tate cycles are fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Assume Mumford-Tate conjecture holds for A.

l-adic Tate cycles $\in H^{\otimes}_{\bar{et}}(A_{\bar{K}}, \mathbb{Q}_l)$ fixed

by $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$

Mumford-Tate: $H^i_B(\ ) \otimes \mathbb{Q}_l = H^i_{et}(\ )$

$\{l$-adic Tate cycles$\} = \{\mathbb{Q}_l$-linear combinations of Hodge cycles$\}$

IV outline of proof

G_{MT} = the largest subgroup of $GL(H^1_B(-, \mathbb{Q}))$

fixing all Hodge cycles

$G_{dR} = \ldots\ldots \quad GL(H^1_{dR}(-,k))$ fixing all dRT cycles

$G_{dR} \subseteq G_{MT}$

<u>Lemma</u> (Zarhin)

$G_1 \subseteq G_2 \subseteq GL(V)$

V vector space / field of char 0

⓪ G_1, G_2 connected reductive groups

① $\operatorname{rank} G_1 = \operatorname{rank} G_2$

② $\operatorname{Cent}_{\operatorname{End}(V)} G_1 = \operatorname{Cent}_{\operatorname{End}(V)} G_2$

<u>Then</u> $G_1 = G_2$.

<u>Fact</u>: $\{\text{dRT cycles}\} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_B^{\otimes}(-)$

$\Rightarrow$ Enough to show $G_{dR} = G_{MT}$ $(\otimes \mathbb{C})$.

<u>For ①</u>: MT holds $\Rightarrow \operatorname{rk} G^{\circ}_{dR} = \operatorname{rk} G_{MT}$

$\Downarrow$

$\operatorname{rk} G^{\circ}_{dR} = \operatorname{rk} G_{MT}$

For ⓪ & ②

<u>Prop.</u> One can construct a Tannakian category

$\mathcal{M}_{dRT}$ mimicking $\mathcal{M}_{AH}$ (by Deligne)

with morphisms = dRT cycles

$$\underline{\mathrm{Aut}}^{\otimes} = G_{dR}$$

Remark: we need complex conjugation

$$dRT \subseteq H_B^{\otimes}(-, \mathbb{R})$$

In particular

① G_{dR} is reductive

② $s \in H_{dR}^{\otimes(/L)}$ is fixed by G_{dR}

$\Leftrightarrow$ s is an L-linear combination of dRT cycles

$\hookrightarrow$ $\mathrm{Cent}_{\mathrm{End}(H^1_{dR})}(G_{dR})$ = linear comb. of algebraic cycles

$= \mathrm{Cent}_{\mathrm{End}(H^1_{dR})} G_{MT}$

Want ? $\|$

$\mathrm{Cent}_{\mathrm{End}(H^1_{dR})} G^{\circ}_{dR}$

prime dimensional case:

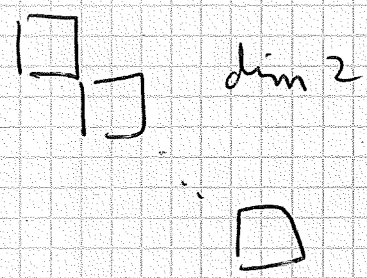
We may assume A is <u>not</u> CM

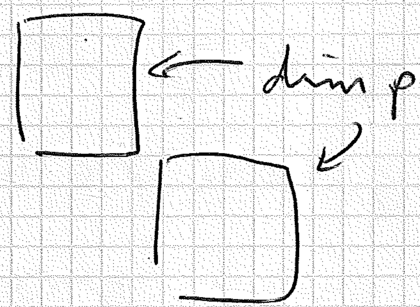
1) $\mathrm{End}^\circ(A) = F$ totally real field of degree $[F:\mathbb{Q}] = p$

2) p odd $\quad \mathrm{End}^\circ(A) = F$ CM field of deg 2

$p = 2 \quad \mathrm{End}^\circ(A) =$ quat. alg.

look at representations of G_{MT} (resp G_{dR})

1) 

2) 

$$H^1_{dR} = \bigoplus_i V_i$$

$$= \bigoplus_i \bigoplus_{i,j} V_{i,j},$$

irred of G°_{dR}

<u>Lemma 1</u>: all $V_{i,j}$ are of same dimension

<u>Lemma 2</u>: If all $V_{i,j}$ have dim 1 i.e. G°_{dR} is a torus $\Rightarrow$ A has CM

<u>Lemma 3</u>: $V_i \simeq V_j$ as G°_{dR} representations

---- as G_{dR}-representations

For lemma 1: $G_{dR}^\circ \lhd G_{dR}$

$\rightsquigarrow$ φ_v fixes the same dim flat

Case where everything $/\mathbb{Q}$

Theorem (Serre) Extend K such that G_ℓ is connected. Then $\exists$ M set of primes of K of natural density 1 such that for all $v \in M$ the commutative algebraic group generated by ~~Frob~~$_v$'s of maximal rank is connected.

everything$/\mathbb{Q}$ $\Rightarrow$ $\exists M$ s.t. $\forall p \in M$
$\varphi_p \in G_{dR}^\circ$.

Apply Corollary of Borel + E

$\Rightarrow$ $\mathrm{Cent}\, G_{dR}^\circ = \mathrm{Cent}\, G_{dR}$

VI A relative version of Borel's theorem

Question: $s \in \mathrm{End}(H^1_{dR})$ $\varphi_v^{m_v}(s) = s$

$m_v = [K_v : \mathbb{Q}_p]$. Is s necessarily an

L-linear combination of dRT cycles?

relative =
relative Frobenius

Claim: If yes, then MT $\Rightarrow$ (dRT = H)

Known cases:

A = elliptic curve or more generally Sh(A)

is a curve, Serre-Tate, Noot $\Rightarrow$ yes.

A related question

$s \in \mathrm{End}(H^1_{dR})$ $\varphi_v(s) = s$ for a density β

If $\beta > \frac{1}{2}$, is s algebraic? $1-\alpha$

A naive application of Gasbarri-Herblot

yes for $\beta > \frac{3}{4}$ (A absolutely simple)

$\rightsquigarrow$ some cases for $\dim A = 2$

$\beta > \frac{3}{4} - \varepsilon$ $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$