MHM refers to the MHM project:
`http://www.cmls.polytechnique.fr/perso/sabbah/MHMProject/mhm.html`

The series "Training on D-modules" aims at defining the environment where mixed Hodge modules live. Its purpose is to justify for algebraic geometers the need of considering filtered D-modules for treating variations of Hodge structures with singularities.

1. Monday

1.1. Training on D-modules 1 (CS). The ring of differential operators $\mathcal{D}$, its filtration $F_\bullet\mathcal{D}$, side-changing, $\mathcal{O}_X$ and ω_X as $\mathcal{D}$-modules, filtrations.

Equivalence: Left $\mathcal{D}$-modules $\iff$ $\mathcal{O}_X$-modules with integrable connection.

Source: MHM §§A.1–A.6.

1.2. Direct image of D-modules and Gauss-Manin, Filtered direct image. The case of a closed immersion and that of a projection. This talk should explain why the direct image of a D-module is more natural and more flexible than the definition by Katz-Oda of the Gauss-Manin connection.

Source: MHM, §A.8 and [**Pop16**, §3]

For the comparison with Katz-Oda definition of G-M, Only give a sketch of the proof in MHM, §A11.c.

1.3. Statement of the decomposition theorem for filtered D-modules underlying a polarized Hodge module. [**Pop16**, Th. 9].

Application to the decomposition of the first level of the Hodge filtration [**Pop16**, Ex. 11 &12], and give details. See also the overview in Section 25 of [**Sch14**]. Compare with the proof of Kollár [**Kol86**]. Slogan: whenever ω_X occurs in a formula, one should explain it by considering right $\mathcal{D}_X$-modules.

1.4. Overview on polarizable and mixed Hodge modules [**Pop16**, §4] and [**Sch14**, Introduction] (one does not expect yet a precise definition of the category, but motivating examples).

Example 1: $\mathcal{O}_X$ with its trivial filtration underlies a pure Hodge module of weight $\dim X$. The proof of this theorem is not given now. Explain why the weight is shifted (see the decomposition theorem).

Example 2: A pVHS of weight w underlies a pure HM of weight $w + \dim X$.

Example 3: HM with punctual support of weight w $\iff$ pHS of weight w

Example 4: pVHS of weight w on a closed submanifold $Y \subset X$ $\iff$ pure HM of weight $w + \dim Y$ on X supported on Y.

2. Tuesday

2.1. Training on $\mathcal{D}$-modules 2 (CS). The graded sheaf of rings $\mathrm{gr}^F\mathcal{D}_X$ and its relation with the ring of functions on the cotangent space T^*X. Definition of the characteristic variety. Notion of Inverse image of a $\mathcal{D}$-module, specifically non-characteristic inverse image.

2.2. Statement and sketch of proof of the vanishing theorem of Saito [**Pop16**, §1 §8], see also [**PS13**, Th. 2.1]. The notion of MHM is introduced in a vague sense as well as the notion of localization of a filtered D-module (the results concerning localization on pages 72-73 will thus be taken for granted). One should insist more on the

Consequence: Proof of other vanishing theorems [**Pop16**, §9].

2.3. Application to positivity [**Pop16**, §§10 & 11]. The lowest piece of the Hodge filtration of a mixed Hodge module is a coherent sheaf that satisfies positivity properties. The proof uses vanishing theorems for mixed Hodge modules. Other properties satisfied by this remarkable sheaf, in particular precise vanishing theorems.

2.4. Application to Abelian varieties 1: Theorem 1.1 of [**PS13**] and its proof in Section 2.3 of loc. cit. For a mixed Hodge module on an Abelian variety, each graded piece of the Hodge filtration is a coherent sheaf whose jumping loci are not too big.

3. Wednesday

Application to Abelian varieties 2: This morning session intends to explaining the proof of Theorem 1.2 in [**PS13**]. It concerns the jumping loci of the sheaf of q-differential forms on a projective variety: they are not too big, in a precise way. The idea is to use the Albanese map to reduce to results on Abelian varieties.

3.1. Goal: To explain the notion of defect of semi-smallness with the example in Section 2.4 of [**dM05**].

Proof of Proposition 2.9 of [**PS13**].

3.2. Explain Laumon's theorem 2.4 and Section 2.5 of [**PS13**]. A celebrated result of Kashiwara estimates the characteristic variety of the pushforward of a holonomic D-module. For mixed Hodge modules, the degeneration at E_1 of the Hodge-to-de Rham spectral sequence makes this result much more precise. This is applied to the pushforward by the Albanese map.

3.3. Section 4.3 of [**PS13**] and proof of Theorem 1.2 on jumping loci.

Lunch + Free afternoon.

3.4. Training on $\mathcal{D}$-modules 3 (CS). (Before dinner) Nearby/Vanishing cycles and specialization of filtered D-modules. Localization of filtered D-modules. These are fundamental functors for defining pure and mixed Hodge modules, although the first ones do not belong to the "six functors".

4. Thursday

This session makes precise the notion of polarizable Hodge module.

4.1. Theory of Griffiths, Schmid and Zucker on curves: degeneration of pVHS and the Hodge-Zucker theorem. Source: [**Sch73**] and [**Zuc79**]. See also MHM Chapter 5. This talk aims at explaining classical results on curves, existing before MHM.

4.2. Polarizable Hodge modules on curves. MHM Chapter 6. This talks interprets the above classical results in terms of pure Hodge modules, giving the first and simplest non trivial example of a Hodge module and the Hodge theorem in this setting.

4.3. Category of polarizable Hodge modules. Notion of nearby cycles, definition of the category by induction. Main properties (abelianity, etc.) and examples. Brief sketch of proof of the decomposition and structure theorems.

Source: Sections B and C in [**Sch14**] and [**Sai17**, §§1 & 2].

4.4. Training on $\mathscr{D}$-modules 4 (CS). As an introduction to mixed Hodge modules, definition of the relative monodromy filtration and its properties. The need for an admissibility condition.

5. Friday

5.1. Mixed Hodge modules: Sections 14.4.1–14.4.3 of [**PS08**], together with Section 3 in [**SZ85**], mainly explaining Conditions (3.13) in loc. cit. and examples (3.15) and (3.16).

5.2. Final talk by Christian Schnell: Overview of other applications of MHM in complex geometry (e.g., [**PS14**]).

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