# UNLIKELY INTERSECTIONS, O-MINIMAL GEOMETRY, AND TORSION POINTS ON A FAMILY OF ABELIAN VARIETIES JEUNES EN ARITHMÉTIQUE ET VARIÉTÉS ALGÉBRIQUES 2025

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PHILIPP HABEGGER

## Abstract

In this series of talks we cover topics from the conjectures on unlikely intersections, as formulated by Zilber [41], Pink [30], and Bombieri–Masser–Zannier [6]. This conjecture subsumes many known results in diophantine geometry. This includes the Mordell Conjecture, proved by Faltings; the André–Oort Conjecture, proved by Pila–Shankar–Tsimerman; the Manin–Mumford Conjecture, proved by Raynaud; and others. Later, we focus on *special points* problems in a family of abelian varieties.

Our goal is to give an overview of the tools required to attack several cases of this conjecture. The emphasis is on the use of o-minimal geometry, developed by van den Dries, Pillay, and Steinhorn in the context of mathematical logic. We will also need arithmetic input derived from height theory on projective varieties.

The main application we have in mind is towards the relative Manin–Mumford Conjecture. Roughly speaking, this conjecture concerns the distribution of points of finite order in a family of abelian varieties. We first illustrate the Pila–Zannier counting strategy on the example of the Manin–Mumford Conjecture. We then focus on the relative case of this conjecture in a family of abelian varieties. The first results towards the relative case were obtained by Masser–Zannier and Corvaja–Masser–Zannier. Later work is due to Gao–Habegger and Corvaja–Tsimerman–Zannier. If time permits, we will also cover applications of this circle of ideas to billiards on an elliptic table.

## General Information

The content mentioned under *supplementary material* can be presented if time permits. A *key (non)-example* is a suggested example.

There are in total 16 talks and they are split into four parts: A, B, C, and D. Section titles ending in a * are not entirely necessary for the program. They can be omitted if there are insufficient speakers or there is not enough time.

As a disclaimer: The list of references is not exhaustive and presents only a cross-section of the work done in this area.

## 1. Unlikely Intersection and its Predecessors (A)

The first talk is an introduction to the conjectures on unlikely intersection. A rather general version is due to Pink [30]; it generalizes the Manin–Mumford Conjecture as well as the Mordell–Lang and the André–Oort Conjectures. The Manin–Mumford Conjecture

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has a rich history, going back to work of Ihara–Serre–Tate [17]. Introduce and discuss the Manin–Mumford Conjecture and its relative version for families of abelian varieties. If time permits, also introduce the André–Oort Conjecture and present its connection to the Manin–Mumford Conjecture. Zannier's book [40] contains an overview from around 2012.

*Supplementary material*: Zilber's work [41] on his conjectures is informed by mathematical logic and the Schanuel Conjecture. Another motivation was the work of Bombieri–Masser–Zannier [5].

Additional care is required for families of semi-abelian varieties after an example of Bertrand [3].

## 2. Height Functions (A)

Discuss the Weil height on algebraic points of projective space, see Chapter 1.5 [4]. This talk should cover Northcott's Theorem. Mention Lehmer's Problem on the affine line, see Chapter 1.6.15 [4] and the work of Dobrowolski.

Weil's *height machine* provides height functions on projective varieties equipped with a line bundle over number fields. Explain the construction in Chapter 2.3 [4], culminating in Theorem 2.3.8, for projective varieties and line bundles. The resulting height function is only defined up to a bounded function. An alternative approach is given in the book of Hindry–Silverman [16], Chapter B.3.

*Supplementary material*: A canonical representative for a height function requires a metric on the line bundle at all places of the base field, see Chapter 2.7 [4].

## 3. Abelian Varieties, Complex Uniformizations and the Néron–Tate Height (A)

Recall the basic properties of abelian varieties, which are projective group varieties. Mention the universal cover of an abelian variety considered as a complex analytic space. Describe the structure of the torsion group over an algebraically closed field (in characteristic 0).

Consider an abelian variety defined over a number field. Introduce the Néron–Tate height associated to a symmetric and ample line bundle; it is a key tool. See Chapter 9, and especially Theorem 9.2.7, in [4]. Describe Kronecker's Theorem characterizing the points of Néron–Tate height 0.

*Supplementary material*: When the abelian variety $A$ is defined over a number field $K$ and $N \geq 1$ is an integer, there is a natural action of $\mathrm{Gal}(\overline{K}/K)$ on the $N$-torsion points $A[N]$ of $A$. Present Serre's result (cours au Collège de France 85-86): there is an integer $c \geq 1$ such that for all $N \geq 1$ the $c$-th power of any homothety in $\mathrm{Aut}(A[N])$ is in the image of the Galois representation. See also Wintenbergers [38, Théoréme 3]. This statement can be used in a proof of the Manin–Mumford Conjecture for abelian varieties, see Talk 9.

## 4. SILVERMAN'S SPECIALIZATION THEOREM (A)

Present Silverman's Height Limit Theorem, [31, Theorem B] in a family of abelian varieties over a 1-dimensional base. By family of abelian varieties we mean an abelian scheme whose base is a quasi-projective variety defined over a field. Silverman's Theorem requires the notion of the Néron–Tate height on an abelian variety defined over the function field of a curve.

An important application is the Specialization Theorem [31, Theorem C]. It uses the fact that the Néron–Tate height is a quadratic form. Here would be a good time to recall the statement of the Mordell–Weil Theorem over number fields and function fields. In the latter case, the trace of the abelian variety is required; see Chapter VIII, §3 [18] or [8].

*Supplementary material*: Section 4 [31] contains a generalization of work on Zimmer [42]: by how much does the Néron–Tate differ from the Weil height in a family of abelian varieties?

## 5. INTRODUCTION TO O-MINIMAL GEOMETRY (B)

This talk should serve as an introduction to *tame* or *o-minimal* geometry as introduced by van den Dries, Pillay, and Steinhorn. Many good survey papers, lecture notes, and recordings are available on the internet. Our main reference is the book of van den Dries [33].

This talk should cover a cross-section of Chapters 1 and 2 *loc.cit.* It should exhibit the two approaches to o-minimal geometry.

The first approach is rooted in the language of mathematical logic. Explain *model-theoretic structures* as in Section 5, Chapter 1 [33] and *o-minimal structures* as in Definition 5.7.

The second approach is a point-set definition. See Section 2 [35] to the concept of a *structure on the real field.*

*Key example*: We will almost exclusively work with expansions of $\mathbb{R}_{\mathrm{alg}} = (\mathbb{R}, <, 0, 1, +, \cdot)$ the ordered field of real numbers. The definable sets of this structure are precisely the real semi-algebraic sets by the Tarski–Seidenberg Theorem, [33, Chapter 2, Corollary 2.11]. In particular, $\mathbb{R}_{\mathrm{alg}}$ is o-minimal.

*Key non-example*: The sine map $\mathbb{R} \ni x \mapsto \sin x$ is not definable in any o-minimal expansion of $\mathbb{R}_{\mathrm{alg}}$. Equivalently, the graph $\{(x, \sin x) : x \in \mathbb{R}\}$ is not a definable subset of $\mathbb{R}^2$.

*Supplementary material*: For a coordinate-free formalism, see the notion of *definable space* in Chapter 10 [33].

## 6. THE CELL DECOMPOSITION THEOREM (B)

The notion of *cell* is the key to powerful uniformity results in o-minimal structures and the further development of the general theory. This includes the development of a dimension theory.

A set definable in an o-minimal structure is a disjoint finite union of particularly simple sets called *cells*. This is the content of the important Cell Decomposition Theorem as covered in Chapter 3 [33]. The presenter may assume that the ambient o-minimal structure is an expansion of $\mathbb{R}_{\mathrm{alg}}$. The audience should know about *definable maps* and their basic properties at this point. The Monotonicity Theorem in Section 1 of Chapter 3 *loc.cit.* is an elementary preliminary step towards the Cell Decomposition Theorem. This theorem gives a feeling for o-minimal structures and I recommend working through (and presenting) a proof of one of the lemmas; all proofs are elementary. The proof of the Cell Decomposition Theorem is also elementary, but a bit more technical, see Section 2 *loc.cit.* Discuss the definition of a cell and provide the statement of the Cell Decomposition Theorem. Section 3 in Chapter 3, *loc.cit.*, covers the important concept of a definable family. Mention Corollary 3.6 in Chapter 3 *loc.cit.* Give an overview of dimension theory, Section 1, Chapter 4 *loc.cit.*

*Key example*: Let $F \in \mathbb{R}[X_1, \ldots, X_n]$ have degree at most $d$. The number of connected components of

$$\{x \in \mathbb{R}^n : F(x) = 0\}$$

is bounded above as a function of $n$ and $d$.

*Supplementary material*: There is a $C^p$ variant of the Cell Decomposition Theorem for $p \geq 1$, see Section 3, Chapter 7 [33] for the case $p = 1$.

## 7. EXAMPLES OF O-MINIMAL STRUCTURES (B)

Until now, our only example of an o-minimal structure is $\mathbb{R}_{\mathrm{alg}}$. For many applications, we need more definable sets. This talk will cover examples containing non-algebraic sets.

1. The o-minimal structure of restricted analytic functions $\mathbb{R}_{\mathrm{an}}$. In [32], van den Dries "points out some remarkable facts implicit in the results of Łojasiewicz $[\cdots]$ and Gabrielov $[\cdots]$". Explain finitely subanalytic subsets of $\mathbb{R}^n$ and the structure $\mathbb{R}_{\mathrm{an}}$ they generate. The hard part in proving o-minimality of $\mathbb{R}_{\mathrm{an}}$ is Gabrielov's Theorem on the Complement. See also Example 2.5(4) [35].
2. The expansion $\mathbb{R}_{\mathrm{exp}}$ of $\mathbb{R}_{\mathrm{alg}}$ obtained by adjoining the graph of the unrestricted real exponential function $\mathbb{R} \ni x \mapsto \exp(x)$. Wilkie [36] proved that this structure on the real field is o-minimal.
3. The structure $\mathbb{R}_{\mathrm{an,exp}}$ generated by $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\mathrm{exp}}$ is o-minimal by work of van den Dries and Miller [34].

The proofs that the structures involved are indeed o-minimal are quite involved. It will suffice to give an overview of the results.

*Key examples*: One identifies $\mathbb{C}$ with $\mathbb{R}^2$ by sending a complex number to its real and imaginary parts. Show that the map $[0,1] \ni t \mapsto \exp(2\pi i t)$ is definable in $\mathbb{R}_{\mathrm{an}}$. Let $E$ be an elliptic curve defined over $\mathbb{C}$. Let $u\colon \mathbb{C}^n \to E^n(\mathbb{C})$ be a complex analytic uniformizing map and a group homomorphism. Let $F \subseteq \mathbb{C}$ be a (closed) fundamental domain, *i.e.*, a parallelogram spanned by a period basis. Show that $u|_{F^n}^{-1}(Z(\mathbb{C}))$ is definable in $\mathbb{C}^n = \mathbb{R}^{2n}$ for all algebraic subsets $Z \subseteq E^n$.

*Supplementary material*: For a beautiful application of the o-minimality of $\mathbb{R}_{\exp}$ and the *Trivialization Theorem* see Chapter 9, Proposition 3.2 [33].

## 8. The Pila–Wilkie Theorem (B)

A key tool when applying o-minimal geometry to problems in number theory is the Pila–Wilkie Theorem [28]. Roughly speaking, the theorem provides an upper bound for the number of rationals points of bounded (exponential) height on a definable set. The proof combines two ideas. First, the reparametrization theorem (after Yomdin and Gromov), Theorem 2.3 *loc.cit.* This allows definable sets to be covered by the images of well-behaved definable functions. Second, an application of the determinant method. It was explored earlier in work of Bombieri–Pila [7] for certain real curves.

In addition, the proof of the Pila–Wilkie makes heavy use of uniformity aspects of o-minimal geometry. I suggest to sketch the proof of the counting theorem taking the reparametrization theorem as a black box.

*Key example*: The graph of the real exponential function $\{(x, \exp(x)) : x \in \mathbb{R}\}$ contains a single rational point $(0, 1)$; this follows from the Lindemann–Weierstrass Theorem. But $X = \{(x, y, \exp(xy)) : (x, y) \in \mathbb{R}^2\}$ contains $\gg T^2$ rational points of exponential height $\leq T$. They are on the semi-algebraic curves $\mathbb{R} \times \{(0, 1)\} \subseteq X$ and $\{0\} \times \mathbb{R} \times \{1\} \subseteq X$.

*Supplementary material*: Pila generalized the Pila–Wilkie Theorem from rational points to algebraic points in a fixed number field, see [26]. Wilkie gave a somewhat different approach to the counting theorem in Chapter 2 of [37].

## 9. The Pila–Zannier Counting Strategy (C)

The Manin–Mumford Conjecture was originally proved by Raynaud. This talk covers a proof of the Manin–Mumford Conjecture for abelian varieties using the Pila–Zannier strategy. The source material is the two authors' paper [29]. Theorem 1.1 *loc.cit.* is our goal; the starred theorem can also be mentioned. The method requires various inputs, such as lower bounds for the size of Galois orbit of a point of finite order. For this see Masser's [20], which can be presented as a black box. An alternative approach is via Serre's result on homotheties in the Galois representation mentioned in Talk 3.

Zannier's book [40] contains an overview of the proof of the Manin–Mumford Conjecture in Section 1 of Chapter 3, and a lot more.

*Supplementary Material*: Pila gave a first unconditional proof of the André–Oort Conjecture for powers of a modular curve [27].

## 10. The Manin–Mumford Conjecture in a Family (C)

The general conjecture on unlikely intersection also covers the distribution of torsion points in a family of abelian varieties. The first result towards the relative Manin–Mumford Conjecture is due to Masser–Zannier [23]. See also their paper [19], which considers the case when the base is a curve over $\mathbb{C}$. The authors wrote several additional papers, some in collaboration with Corvaja.

This talk should cover the Theorem in [19]. For simplicity we can assume that the family of abelian varieties is the fibered square of the Legendre family of elliptic curves.

There are new difficulties when compared to the (non-relative) Manin–Mumford Conjecture. As the abelian variety is varying in a family, extra care is required when estimating the size of the Galois orbit of a point of finite order. Thus we need Silverman's Height Limit Theorem from Talk 4. The paper [19] does not formally require o-minimal geometry or the Pila–Wilkie Theorem; rather, it relies on a point counting theorem of Pila for subanalytic surfaces which predates [28]. However, the speaker can adapt the method to use the Pila–Wilkie Theorem applied to $\mathbb{R}_{\mathrm{an}}$. The speaker may also consult Chapter 3, Sections 2 and 3, of [40] for an overview of the argument.

*Key example*: A basic example for a family of abelian varieties is the Legendre family of elliptic curves and its fibered powers.

*Supplementary material*: The authors use the language of subanalytic sets, which corresponds to the o-minimal structure $\mathbb{R}_{\mathrm{an}}$. Thus, they must address the fact that the modular curve $Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, the base of the Legendre family, is not complete. Masser and Zannier approach the subtle definability question using Silverman's Theorem from Talk 4. See Section 6 [19] and Lemma 8.2, both [19], for details. Silverman's Theorem can be avoided by using $\mathbb{R}_{\mathrm{an,exp}}$ instead of $\mathbb{R}_{\mathrm{an}}$ as the ambient o-minimal structure. We refer to the definability result by Peterzil–Starchenko [25].

## 11. THE AX–LINDEMANN–WEIERSTRASS THEOREM FOR ABELIAN VARIETIES* (C)

Present the Ax–Lindemann–Weierstrass Theorem for abelian varieties. As a reference, one can consult Orr's treatment [24], see Theorem 7.1. The proof relies on the Pila–Wilkie Counting Theorem and is presented in Section 9 of *loc.cit.*

The theorem's name refers to Ax's proof of the functional Schanuel conjecture [2] and the classical Lindemann–Weierstrass Theorem on transcendence of values of the exponential function.

*Supplementary material*: Explain the classical (and still open) Schanuel Conjecture from transcendence theory.

## 12. THE BETTI MAP (D)

The Betti map appeared implicitly in the work of Masser–Zannier [23, 19]. We follow the work of André–Corvaja–Zannier [1] in the more general setting of a family of abelian varieties over $\mathbb{C}$. See Section 3 *loc.cit.* for a definition of the Betti map. It is a real analytic map, locally defined on the abelian schemes, and has complex analytic fibers. Present Proposition 2.1.1 *loc.cit.*, which is an illustrative example. Also present Theorem 6.1.1 and its application, Theorem 2.3.1 *loc.cit.*

*Supplementary material*: Cover the appendix by Gao in [1]

## 13. Billiards, Ellipses, and Jacobians* (D)

As an application to the notions of the previous talk, we consider a game of billiards on a table shaped like an ellipse. The goal of this talk is to understand how periodic ball trajectories connect to families of abelian varieties and torsion points. Specifically, our aim is Theorem 1.2 [9], which is proved in Appendix A.1 by Corvaja, Demeio, and Zannier. Section 2 of the mentioned paper addresses the construction of the *billiard section.*

*Supplementary material*: If time permits, the speaker can also cover one of the finiteness results, Theorems 1.4 and 1.5 *loc.cit.*

## 14. Arithmetic Lower Bounds for the Néron–Tate Height (D)

Masser [20] explored lower bounds for the Néron–Tate height in the spirit of Lehmer's problem. For a version that takes into account families of abelian varieties see [21]. Present the Theorem in *loc.cit.* For our purposes we will also need to track the dependency on the degree of the field of definition. This is achieved in work of David [10]. This talk should cover Théorème 1.4 in *loc.cit.*

Another approach is via the Masser–Wüstholz Isogeny Estimates [22], albeit with a non-explicit numerical constant C. To apply the Pila–Zannier strategy, it is important to know that C depends polynomially on the degree $d$ (using the notation of the Theorem in *loc.cit.*). Gaudron–Rémond proved a completely explicit version in Corollaire 1.7 [15]. These estimates require the concept of *Faltings height* of an abelian variety.

*Supplementary Material*: Explain Silverman's Conjecture, Conjecture 1.3 [10].

## 15. A Height Bound in a Family of Abelian Varieties (D)

The purpose of this talk is to discuss the height upper bound, Theorem 1.6 in [11], in a family of abelian varieties. The inequality holds generically on a non-degenerate subvariety of a family of abelian varieties. The inequality relates a (suitable) height of the abelian variety with the Néron–Tate height on the fiber. The concept of non-degenerate subvariety relies on the notion of the Betti map from Talk 12. For several applications, this height inequality can serve as a replacement for Silverman's Height Limit Theorem from Talk 4.

If time permits, the speaker can also explain the application of the height upper bound to questions regarding uniform bounds for the number of rational points on a curve of genus at least 2.

*Supplementary material*: For a generalization of the height upper bound see the foundational work of Yuan and Zhang in Arakelov theory. In particular, Theorem 6.2.2 [39].

## 16. Application to the Relative Manin–Mumford Conjecture (D)

The final talk discusses a proof of the relative Manin–Mumford Conjecture for a family of abelian varieties over a general base variety defined over a number field. Silverman's Height Limit Theorem is sufficient when the base of the family of abelian varieties is a curve. For a higher-dimensional base, we require the height upper bound from Talk 15.

In addition to the height bound, the proof relies on the Pila–Zannier counting strategy. The ambient o-minimal structure is $\mathbb{R}_{\mathrm{an,exp}}$. The height upper bound is combined with the results considered in Talk 14 to obtain a lower bound for the size of the Galois orbit of a point of finite order.

Discuss Theorem 1.1 [14] in the special case where the family of abelian varieties is defined over a number field. The proof of this case is detailed in Sections 2 through 7 *loc.cit.* The required non-degeneracy result is due to Gao [12] and is based on his Ax–Schanuel Theorem [13] for the universal family of abelian varieties.

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