# Elliptic Fibrations on K3 surfaces

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# Introduction

Consider the surface defined by the polynomial

$$P = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - 2$$

There are singular points. After solving these singularities we get a K3-surface $\mathcal{Y}$.

We can write the polynomial in a projective form as

$$P = xyz(x + y + z) + t^2(xy + xz + yz) - 2xyzt$$

So the line x+y+z=0, t=0 is on the surface $\mathcal{Y}$.
Cutting now $\mathcal{Y}$ by the family of hyperplanes x+y+z=st it follows

$$Y_s \qquad (x + y)(x + z)(y + z) + (s - 1)^2 xyz = 0$$

So we can view $\mathcal{Y}$ as the collection of $Y_s$. For all but a finite number of $s$, $Y_s$ defines an elliptic curve with Weierstrass form

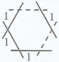
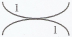
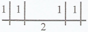
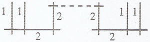
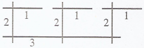
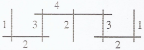
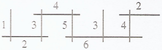
$$Y^2 - (s - 1)^2(X + 1)Y = X(X + 1)(X + (s - 1)^2).$$

Thus is realized the elliptic fibration of $\mathcal{Y}$

$$\mathcal{Y} \longrightarrow \mathbb{P}^1_s.$$

$Y_s$ is a fiber of this elliptic fibration
From this Weierstrass equation, one can deduce the rank and torsion over $\mathbb{C}(s)$ of the Mordell-Weil group of $Y_s$.

For bad $s$, $Y_s$ is no longer an elliptic curve and is called a singular fiber.
Singular fibers have been classified by Kodaira.
To know the type of singular fibers of an elliptic fibration we use in general Tate's algorithm.
Thus $Y_s$ has fibers $I_{12}$ over $\infty$, $I_6$ over 1, $I_2$ over 0 and 2, $I_1$ over $s = \alpha$ and $s = \beta$ roots of $(s-1)^2 = -8$.

| Reduction Type | Number of Components | Configuration (with multiplicity) |
|---|---|---|
| $I_0$ | 1 |  |
| $I_1$ | 1 |  |
| $I_n$ | n |  |
| II | 1 |  |
| III | 2 |  |
| IV | 3 |  |
| $I_0^*$ | 5 |  |
| $I_n^*$ | n + 5 |  |
| $IV^*$ | 7 |  |
| $III^*$ | 8 |  |
| $II^*$ | 9 |  |

The Kodaira-Néron Classification of Special Fibers

# Another elliptic fibration

The line z=0, t=0 is also on the surface $\mathcal{Y}$.
Cutting $\mathcal{Y}$ by the family of hyperplanes z=vt it follows

$$Y_v \qquad (x+y)(xy+t^2)+((v-1)^2/v)xyt=0.$$

So we can view $\mathcal{Y}$ as the collection of $Y_v$, i.e. a new elliptic fibration of $\mathcal{Y}$ with another Weierstrass equation

$$Y^2-(v-1)^2XY=X(X-v^2).$$

Now, $Y_v$ has type $I_8$ over 0 and $\infty$, $I_4$ over 1, $I_2$ over $-1$ and $I_1$ over the roots of $v^2-6v+1$.

# Question and Theorem

Question:how many elliptic fibrations does possess $\mathcal{Y}$?

Answer: the following theorem (B. - Lecacheux)

**Theorem**

*There are* 30 *elliptic fibrations with section, all distinct up to isomorphism, on the elliptic surface*

$$X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = 2.$$

*They are listed in the following table with the rank and torsion of their Mordell-Weil group. The list consists of* 14 *fibrations of rank* 0, 13 *fibrations of rank* 1 *and* 3 *fibrations of rank* 2.

The proof uses the connection between the singular fibers and root lattices.

# Root lattices

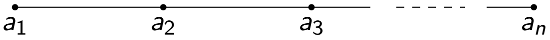
Let $L$ be a negative-definite even lattice.
$e \in L$ is a **root** if $q_L(e) = -2$. Define the root lattice

$$L_{\text{root}} := \langle \Delta(L) := \{e \in L / q_L(e) = -2\} \rangle \subset L.$$

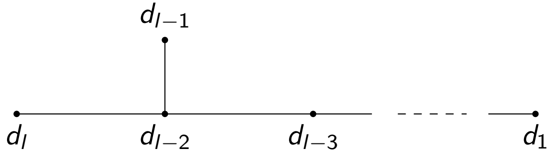
Every root lattice is a direct sum of irreducible ones $A_n$, $D_l$, $E_p$.
The root lattices $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ $(n \geq 1)$, $D_l = \langle d_1, d_2, \ldots, d_l \rangle$ $(l \geq 4)$, $E_p = \langle e_1, e_2, \ldots, e_p \rangle$ $(p = 6, 7, 8)$ are defined by the following Dynkin diagrams. All the vertices $a_j$, $d_k$, $e_l$ are roots and two vertices $a_j$ and $a'_j$ are joined by a line if and only if $b(a_j, a'_j) = 1$.

# Dynkin diagrams of root lattices

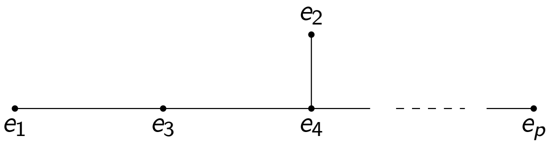
$A_n = \langle a_1, a_2, \ldots, a_n \rangle$




$D_l = \langle d_1, d_2, \ldots, d_l \rangle$



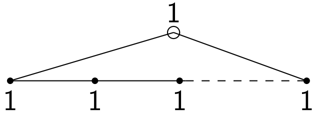

$E_p = \langle e_1, e_2, \ldots, e_p \rangle$



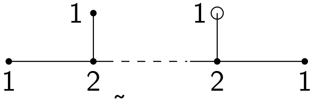
# Singular fibers and Dynkin diagrams

Considering the dual of each model of singular fibers (One rational component corresponding to a point; two points in the dual joined by a line if the corresponding components intersect in one point), we get the following correspondance between the Kodaira type of singular fibers and the extended Dynkin diagrams $\tilde{A}_n$, $\tilde{D}_l$, $\tilde{E}_p$.
Deleting the zero component, i.e. the component meeting the zero section, gives the Dynkin diagram graph $A_n$, $D_l$, $E_p$.

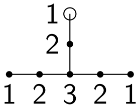
# Extended Dynkin diagrams



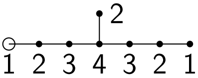

$\tilde{A}_n(I_{n+1})$



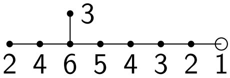

$\tilde{D}_l(I^*_{l-4})$




$\tilde{E}_6(IV^*)$




$\tilde{E}_7(III^*)$




$\tilde{E}_8(II^*)$

# The elliptic fibrations of $\mathcal{Y}$

| Fibers | Rank | Torsion |
|---|---|---|
| $E_7 A_3 E_8$ | 0 | (0) |
| $A_1 E_8 E_8$ | 1 | (0) |
| $E_7 D_{11}$ | 0 | (0) |
| $A_1 D_{16}$ | 1 | $\mathbb{Z}/2\mathbb{Z}$ |
| $A_3 A_1 D_{14}$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ |
| $E_8 A_1 D_9$ | 0 | (0) |
| $E_7 D_6 D_5$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ |
| $D_6 A_1 D_{10}$ | 1 | (0) |
| $E_7 D_{10}$ | 1 | $\mathbb{Z}/2\mathbb{Z}$ |
| $E_7 E_7 A_1 A_3$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ |
| $A_1 A_1 D_8 E_7$ | 1 | $\mathbb{Z}/2\mathbb{Z}$ |
| $A_{17}$ | 1 | $\mathbb{Z}/3\mathbb{Z}$ |
| $A_1 A_{15}$ | 2 | (0) |
| $A_1 D_{17}$ | 0 | (0) |
| $A_1 D_{10} D_7$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ |

# The elliptic fibrations of $\mathcal{Y}$ (continued)

| Fibers | Rank | Torsion |
|---|---|---|
| $A_1 D_5 D_{12}$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ |
| $A_1 D_6 A_3 D_8$ | 0 | $(\mathbb{Z}/2)^2$ |
| $A_1 D_8 D_8$ | 1 | $\mathbb{Z}/2\mathbb{Z}$ |
| $A_1 A_1 A_1 A_{15}$ | 0 | $\mathbb{Z}/4\mathbb{Z}$ |
| $D_4 A_{13}$ | 1 | $(0)$ |
| $A_5 E_6 E_6$ | 1 | $\mathbb{Z}/3\mathbb{Z}$ |
| $A_5 A_1 A_1 A_{11}$ | 0 | $\mathbb{Z}/6\mathbb{Z}$ |
| $A_9 A_1 A_1 E_6$ | 1 | $(0)$ |
| $A_{11} E_6 A_1$ | 0 | $\mathbb{Z}/3\mathbb{Z}$ |
| $A_9 D_7$ | 2 | $(0)$ |
| $A_{11} A_1 D_5$ | 1 | $\mathbb{Z}/4\mathbb{Z}$ |
| $A_1 D_4 D_6 D_6$ | 1 | $(\mathbb{Z}/2)^2$ |
| $A_7 A_9$ | 2 | $(0)$ |
| $A_1 A_3 A_7 A_7$ | 0 | $\mathbb{Z}/8\mathbb{Z}$ |
| $D_5 A_5 A_7$ | 1 | $(0)$ |

# Basic facts on K3-surfaces

A $K3$-surface is a smooth surface $X$ satisfying

- $H^1(X, \mathcal{O}_X) = 0$ i.e. $X$ simply connected
- $K_X = 0$ i.e. the canonical bundle is trivial i.e. there exists a unique, up to scalars, holomorphic 2-form $\omega$ on $X$.

# Example and main properties

- A double covering branched along a plane sextic for example defines a K3-surface X.
  In our previous example

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

**Main properties**

- If $X$ is $K3$, $H^2(X, \mathbb{Z})$ is a free group of rank 22.
- With the cup product, $H^2(X, \mathbb{Z})$ has a structure of even lattice of signature $(3, 19)$ (by Hodge index theorem).
- By Poincaré duality, $H^2(X, \mathbb{Z})$ is unimodular and

$$H^2(X, \mathbb{Z}) = U^3 \oplus E_8[-1]^2 := \mathcal{L}$$

$\mathcal{L}$ is the $K3$-lattice, $U$ the hyperbolic lattice of rank 2, $E_8$ the unimodular lattice of rank 8.

# Main properties (continued)

- The Néron-Severi group $NS(X)$, i.e. the group of line bundles modulo algebraic equivalence, with the intersection pairing, is a lattice of signature $(1, \rho(X) - 1)$.
- $$NS(X) \simeq \mathbb{Z}^{\rho(X)}$$
  $$\rho(X) := \text{Picard number of } X$$
  $$1 \leq \rho(X) \leq 20$$
- The natural embedding
  $$NS(X) \hookrightarrow H^2(X, \mathbb{Z})$$
  is a primitive embedding of lattices.

# More definitions

- The trivial lattice $T(X)$ inside $NS(X)$ is the orthogonal sum

$$T(X) =< \bar{O}, F > \oplus_{v \in S} T_v$$

where $\bar{O}$ denotes the zero section, $F$ the general fiber, $S$ the points of $\mathbb{P}^1$ corresponding to the reducible singular fibers and $T_v$ the lattice generated by the fiber components except the zero component.

- From Shioda's results on height pairing, we can define a positive-definite lattice structure on the Mordell-Weil lattice

$$MWL(X) := E(K(\mathbb{P}^1))/E(K(\mathbb{P}^1))_{tor}$$

- The frame $W(X)$

$$W(X) = \langle \bar{O}, F \rangle^{\perp} \subset NS(X).$$

is a negative-definite even lattice of rank $\rho(X) - 2$.

# The frame and the singular fibers

- One can read off, the Mordell-Weil lattice, the torsion in the Mordell-Weil group MW and the type of singular fibers from $W(X)$ by

$$MWL(X) = W(X)/\overline{W(X)}_{\mathrm{root}} \quad (MW)_{\mathrm{tors}} = \overline{W(X)}_{\mathrm{root}}/W(X)_{\mathrm{root}}$$

$$T(X) = U \oplus W(X)_{\mathrm{root}}.$$

# Nishiyama's method

A $K3$ surface may admit more than one elliptic fibration, but up to isomorphism, there is only a finite number of elliptic fibrations.
Nishiyama embeds the frames of all elliptic fibrations into a Niemeier lattice, one of 24 positive definite even unimodular lattices of rank 24. For this purpose, he determines an even negative-definite lattice $M$ such that

$$q_M = -q_{NS(X)}, \quad \text{rank}(M) + \rho(X) = 26.$$

By Nikulin's results, $M \oplus W(X)$ has a Niemeier lattice $L$ as an overlattice for each frame $W(X)$ of an elliptic fibration on $X$.
Thus one is bound to determine the (inequivalent) primitive embeddings of $M$ into Niemeier lattices $L$.
An embedding $i : S \to S'$ is called primitive if $S'/i(S)$ is a free group.
The orthogonal complement of $M$ into $L$ gives the corresponding frame $W(X)$.

# Niemeier lattices

The 24 Niemeier lattices are listed below.

| $L_{\text{root}}$ | $L/L_{\text{root}}$ | $L_{\text{root}}$ | $L/L_{\text{root}}$ |
|---|---:|---|---:|
| $E_8^3$ | $(0)$ | $D_5^{\oplus 2} \oplus A_7^{\oplus 2}$ | $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ |
| $E_8 \oplus D_{16}$ | $\mathbb{Z}/2\mathbb{Z}$ | $A_8^{\oplus 3}$ | $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$ |
| $E_7^{\oplus 2} \oplus D_{10}$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $A_{24}$ | $\mathbb{Z}/5\mathbb{Z}$ |
| $E_7 \oplus A_{17}$ | $\mathbb{Z}/6\mathbb{Z}$ | $A_{12}^{\oplus 2}$ | $\mathbb{Z}/13\mathbb{Z}$ |
| $D_{24}$ | $\mathbb{Z}/2\mathbb{Z}$ | $D_4^{\oplus 6}$ | $(\mathbb{Z}/2\mathbb{Z})^6$ |
| $D_{12}^{\oplus 2}$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $D_4 \oplus A_5^{\oplus 4}$ | $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^2$ |
| $D_8^{\oplus 3}$ | $(\mathbb{Z}/2\mathbb{Z})^3$ | $A_6^{\oplus 4}$ | $(\mathbb{Z}/7\mathbb{Z})^2$ |
| $D_9 \oplus A_{15}$ | $\mathbb{Z}/8\mathbb{Z}$ | $A_4^{\oplus 6}$ | $(\mathbb{Z}/5\mathbb{Z})^3$ |
| $E_6^{\oplus 4}$ | $(\mathbb{Z}/3\mathbb{Z})^2$ | $A_3^{\oplus 8}$ | $(\mathbb{Z}/4\mathbb{Z})^4$ |
| $E_6 \oplus D_7 \oplus A_{11}$ | $\mathbb{Z}/12\mathbb{Z}$ | $A_2^{\oplus 12}$ | $(\mathbb{Z}/3\mathbb{Z})^6$ |
| $D_6^{\oplus 4}$ | $(\mathbb{Z}/2\mathbb{Z})^4$ | $A_1^{\oplus 24}$ | $(\mathbb{Z}/2\mathbb{Z})^{12}$ |
| $D_6 \oplus A_9^{\oplus 2}$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ | $0$ | $\Lambda_{24}$ |

Table: Niemeier lattices

# How to find $M$: the trancendental lattice

- The transcendental lattice of $X$, $\mathbb{T}(X)$, is the orthogonal complement of $NS(X)$ in $H^2(X,\mathbb{Z})$ with respect to the cup-product,

$$\mathbb{T}(X) = \mathrm{NS}(X)^{\perp} \subset H^2(X,\mathbb{Z}).$$

- In general, $\mathbb{T}(X)$ is an even lattice of rank $r = 22 - \rho(X)$ and signature $(2, 20 - \rho(X))$.
- Let $t = r - 2$. By Nikulin's results, $\mathbb{T}(X)[-1]$ admits a primitive embedding into the following indefinite unimodular lattice:

$$\mathbb{T}(X)[-1] \hookrightarrow U^t \oplus E_8.$$

- Then define $M$ as the orthogonal complement of $\mathbb{T}(X)[-1]$ in $U^t \oplus E_8$. By construction, $M$ is a negative-definite lattice of rank $2t + 8 - r = r + 4 = 26 - \rho(X)$. One can prove that $M$ takes exactly the shape required for Nishiyama's technique.