# Mahler measure from Number Theory To Algebraic Geometry（ICPAMS2022） 

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## Introduction

Introduced by Mahler in 1962, the logarithmic Mahler measure of a polynomial $P$ is

$$
m(P):=\frac{1}{(2 \pi i)^{n}} \int_{\mathbb{T}^{n}} \log \left|P\left(x_{1}, \cdots, x_{n}\right)\right| \frac{d x_{1}}{x_{1}} \cdots \frac{d x_{n}}{x_{n}}
$$

and its Mahler measure

$$
M(P)=\exp (m(P))
$$

where

$$
\mathbb{T}^{n}=\left\{\left(x_{1}, \cdots, x_{n}\right) \in \mathbb{C}^{n} /\left|x_{1}\right|=\cdots=\left|x_{n}\right|=1\right\}
$$

## Remarks

- $n=1$

By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$
M(P)=\prod_{P(\alpha)=0} \max (|\alpha|, 1)
$$

So it is related to Lehmer's question (1933)
Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

$$
1<M(P)<M\left(P_{0}\right)=1.1762 \cdots ?
$$

The polynomial

$$
P_{0}(X)=X^{10}+X^{9}-X^{7}-X^{6}-X^{5}-X^{4}-X^{3}+X+1
$$

is the Lehmer polynomial, in fact a Salem polynomial.

Lehmer's problem is still open.
A partial answer by Smyth (1971)

$$
M(P) \geq 1.32 \cdots
$$

if $P$ is non reciprocal.

## First explicit Mahler measures

- $m\left(x_{0}+x_{1}\right)=0$ (by Jensen's formula)

$$
\begin{gathered}
m\left(x_{0}+x_{1}+x_{2}\right)=\frac{3 \sqrt{3}}{4 \pi} L\left(\chi_{-3}, 2\right)=: L^{\prime}\left(\chi_{-3},-1\right) \text { Smyth (1980) } \\
m\left(x_{0}+x_{1}+x_{2}+x_{3}\right)=\frac{7}{2 \pi^{2}} \zeta(3) \text { Smyth (1980) }
\end{gathered}
$$

## Deninger (1996) conjectured

$$
m\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right) \stackrel{?}{=} \frac{15}{4 \pi^{2}} L(E, 2)=: L^{\prime}(E, 0)
$$

E elliptic curve of conductor 15 defined by the polynomial This conjecture was proved (May 2011) by Rogers and Zudilin thanks to a previous result due to Lalin.
Deninger's guess comes from Beilinson's Conjectures.

## Villegas's results

$$
m(x+1 / x+y+1 / y-k)=\frac{1}{2} \Re\left[-2 \pi i \tau+4 \sum_{n=1}^{\infty} \sum_{d \mid n} \chi(d) d^{2} \frac{q^{n}}{n}\right]
$$

or in terms of Eisenstein's series

$$
\Re\left[\frac{16 \Im(\tau)}{\pi^{2}} \sum_{m, n \in \mathbb{Z}} \chi(n) \frac{1}{(m 4 \tau+n)^{2}(m 4 \bar{\tau}+n)}\right]
$$

where $q=\exp 2 \pi i \tau$ and $\chi(n)=\left(\frac{n}{4}\right)$

$$
k^{2}=1 / \mu(\tau) \quad \mu=q-8 q^{2}+44 q^{3}-192 q^{4}+\ldots
$$

When $k$ defines a CM elliptic curve, namely $k=4 \sqrt{2}$ defining

$$
A: y^{2}=x^{3}-44 x+112 \quad \text { with conductor }
$$

it follows

$$
m(x+1 / x+y+1 / y-4 \sqrt{2})=\frac{64}{4 \pi^{2}} L(A, 2)
$$

Also, if $k=4 / \sqrt{2}$ defining

$$
B: y^{2}=x^{3}+4 x \quad \text { with conductor } \quad 32
$$

it follows

$$
m(x+1 / x+y+1 / y-4 / \sqrt{2})=\frac{32}{4 \pi^{2}} L(B, 2)
$$

Finally for $k=3 \sqrt{2}$ we get the modular elliptic curve $X_{0}(24)$ and using Beilinson's theorem it is possible to get a formula of the same type for the Mahler measure.

A similar result was proved by Benferhat (2009) (one of my former students) concerning the family

$$
x+1 / x+y+1 / y+x / y+y / x-k=0
$$

written as

$$
1 / x y[(x+y+1)(x y+y+x)-(k+3) x y]=0
$$

Hints of proof
From Verrill we know that putting $k+3=1 / t$, it defines an elliptic modular surface for the congruence group $\Gamma_{1}(6)$ with Picard-Fuchs equation near 0 (satisfied by the periods)

$$
t(t-1)(9 t-1) f^{\prime \prime}+\left(27 t^{2}-20 t+1\right) f^{\prime}+3(3 t-1) f=0
$$

with two properties

- For the Hauptmodul
$t=\frac{\eta(6 \tau)^{8} \eta(\tau)^{4}}{\eta(3 \tau)^{4} \eta(2 \tau)^{8}}=q-4 q^{2}+10 q^{3}-20 q^{4}+39 q^{5}+\ldots$
- the solution near 0 is expressed as

$$
f=\frac{\eta(2 \tau)^{6} \eta(3 \tau)}{\eta(\tau)^{3} \eta(6 \tau)^{2}}
$$

- With $k+3=1 / t$ it follows that

$$
\tilde{m}^{\prime}(k)=\frac{1}{2 i(\pi)^{2}} \int_{(\mathbb{T})^{2}} \frac{t}{-1+\frac{(x+y+1)(x y+y+x)}{x y}} \frac{d x}{x} \frac{d y}{y}
$$

is a period of the elliptic curve. Hence it satisfies the Picard-Fuchs equation; moreover it can be identified with the solution near 0 . Thus

$$
\begin{gathered}
\tilde{m}^{\prime}(k)=-t f \quad d \tilde{m}^{\prime}=-f \frac{d t}{t}=-f \frac{t^{\prime}(q) d q}{t} \\
-f(t) \frac{q \frac{d t}{d q}}{t}=1+L(q)+8 L\left(q^{2}\right) \quad L(q)=\sum_{n \geq 1}\left(\sum_{d \mid n} \chi(d) d^{2}\right) q^{n}
\end{gathered}
$$

Finally by integration we get

$$
\begin{aligned}
m(k)= & \Re\left(-2 i \pi \tau+\sum_{n \geq 1}\left(\sum_{d \mid n} \chi(d) d^{2}\right) \frac{\exp 2 i \pi n \tau}{n}\right) \\
& +8\left(\Re \sum_{n \geq 1}\left(\sum_{d \mid n} \chi(d) d^{2}\right) \frac{\exp 4 i \pi n \tau}{2 n}\right)
\end{aligned}
$$

and in terms of Eisenstein-Kronecker series

$$
\begin{aligned}
m(k)= & \Re\left(\frac{9 \sqrt{3} \Im \tau}{4 \pi^{2}} \sum_{(m, n) \neq(0,0)} \frac{\chi(n)}{(3 m \tau+n)^{2}(3 m \bar{\tau}+n)}\right) \\
& +8 \Re\left(\frac{9 \sqrt{3} \Im \tau}{4 \pi^{2}} \sum_{(m, n) \neq(0,0)} \frac{\chi(n)}{(6 m \tau+n)^{2}(6 m \bar{\tau}+n)}\right)
\end{aligned}
$$

For $k=0$ the elliptic curve is CM with conductor 36 more precisely $36 a 1$ with $j=0, \tau$ is imaginary quadratic and we can recover $m(0)=2 L^{\prime}\left(E_{36}, 2\right)$.

CM elliptic curves and elliptic modular curves are rare in these families. Other people Mellit, Zudilin, Brunault used other techniques. For example Mellit obtained results on the same modular surface, that is

$$
m(1)=b_{14} \quad m(-5)=6 b_{14} \quad m(10)=10 b_{14}
$$

(all these conjectured by Boyd.)
A new technique was elaborated by Zudilin and Brunault parametrizing the elliptic curves with modular units. Based on regulators and modular units I obtained (August 2015, unpublished)

$$
m(4)=3 b_{20} \quad m(-2)=2 b_{20}
$$

thus solving Touafek's conjectures on regulators.

Finally with similar techniques Brunault considered the family

$$
y^{2}+k x+y-x^{3}
$$

and proved (arXiv 2015)

$$
m(-1)=2 b_{14} \quad m(-2)=b_{35} \quad m(-3)=b_{54}
$$

Remark that this defines an elliptic surface with 4 singular fibers $[9,1,1,1]$ which is precisely one of Beauville modular elliptic surface for the congruence group $\Gamma_{0}(9) \cap \Gamma_{1}(3)$.
While preparing this talk I noticed that one of my former students Rémi Trannoy studied experimentally this family and conjectured $m(-5) \stackrel{?}{=} 7 b_{20}$, $m(6) \stackrel{?}{=} 3 b_{27}$. Combining all the previous methods can we prove these conjectures?

## From elliptic curves to K3 surfaces

So replace $E$ by a surface $X$ which is also a Calabi-Yau variety, i.e. a $K 3$-surface and try to answer the questions:
What are the analog of Deninger, Boyd, R-Villegas 's results and conjectures?
Which type of Eisenstein-Kronecker series corresponds to $L(X, 3)$ ?

## Basic facts on K3-surfaces

Our results concern polynomials of the family

$$
P_{k}=x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}-k
$$

defining K3-surfaces $Y_{k}$. What's a K3-surface?
It is a smooth surface $X$ satisfying

- $H^{1}\left(X, \mathcal{O}_{X}\right)=0$ i.e. $X$ simply connected
- $K_{X}=0$ i.e. the canonical bundle is trivial i.e. there exists a unique, up to scalars, holomorphic 2-form $\omega$ on $X$.


## Example and main properties

- A double covering branched along a plane sextic for example defines a K3-surface $X$.
In our case

$$
\left(2 z+x+\frac{1}{x}+y+\frac{1}{y}-k\right)^{2}=\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right)^{2}-4
$$

## Main properties

- $H_{2}(X, \mathbb{Z})$ is a free group of rank 22 .


## Main properties (continued)

- With the intersection pairing, $H_{2}(X, \mathbb{Z})$ is a lattice and

$$
H_{2}(X, \mathbb{Z}) \simeq U_{2}^{3} \perp\left(-E_{8}\right)^{2}:=\mathcal{L}
$$

$\mathcal{L}$ is the $K 3$-lattice, $U_{2}$ the hyperbolic lattice of rank $2, E_{8}$ the unimodular lattice of rank 8.

$$
\operatorname{Pic}(X) \subset H_{2}(X, \mathbb{Z}) \simeq \operatorname{Hom}\left(H^{2}(X, \mathbb{Z}), \mathbb{Z}\right)
$$

where $\operatorname{Pic}(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles (since for $K 3$ surfaces linear and algebraic equivalence are the same).

$$
\begin{gathered}
\operatorname{Pic}(X) \simeq \mathbb{Z}^{\rho(X)} \\
\rho(X):=\text { Picard number of } X \\
1 \leq \rho(X) \leq 20
\end{gathered}
$$

$$
T(X):=(\operatorname{Pic}(X))^{\perp}
$$

is the transcendental lattice of dimension $22-\rho(X)$

- If $\left\{\gamma_{1}, \cdots, \gamma_{22}\right\}$ is a $\mathbb{Z}$-basis of $H_{2}(X, \mathbb{Z})$ and $\omega$ the holomorphic 2-form,

$$
\int_{\gamma_{i}} \omega
$$

is called a period of $X$ and

$$
\int_{\gamma} \omega=0 \text { for } \gamma \in \operatorname{Pic}(X)
$$

- If $\left\{X_{z}\right\}$ is a family of $K 3$ surfaces, $z \in \mathbb{P}^{1}$ with generic Picard number $\rho$ and $\omega_{z}$ the corresponding holomorphic 2-form, then the periods of $X_{z}$ satisfy a Picard-Fuchs differential equation of order $k=22-\rho$. For our family $k=3$.
- In fact, by Morrison, a $\mathcal{M}$-polarized K3-surface, with Picard number 19 has a Shioda-Inose structure, that means

- If the Picard number $\rho=20$, then the elliptic curve is CM .


## Mahler measure of $P_{k}$

## Theorem

(B. 2005) Let $k=t+\frac{1}{t}$ and

$$
t=\left(\frac{\eta(\tau) \eta(6 \tau)}{\eta(2 \tau) \eta(3 \tau)}\right)^{6}, \eta(\tau)=e^{\frac{\pi i \tau}{12}} \prod_{n \geq 1}\left(1-e^{2 \pi i n \tau}\right), q=\exp 2 \pi i \tau
$$

$$
\begin{aligned}
m\left(P_{k}\right)= & \frac{\Im \tau}{8 \pi^{3}}\left\{\sum _ { m , \kappa } ^ { \prime } \left(-4\left(2 \Re \frac{1}{(m \tau+\kappa)^{3}(m \bar{\tau}+\kappa)}+\frac{1}{(m \tau+\kappa)^{2}(m \bar{\tau}+\kappa)^{2}}\right)\right.\right. \\
& +16\left(2 \Re \frac{1}{(2 m \tau+\kappa)^{3}(2 m \bar{\tau}+\kappa)}+\frac{1}{(2 m \tau+\kappa)^{2}(2 m \bar{\tau}+\kappa)^{2}}\right) \\
& -36\left(2 \Re \frac{1}{(3 m \tau+\kappa)^{3}(3 m \bar{\tau}+\kappa)}+\frac{1}{(3 m \tau+\kappa)^{2}(3 m \bar{\tau}+\kappa)^{2}}\right) \\
& \left.\left.+144\left(2 \Re \frac{1}{(6 m \tau+\kappa)^{3}(6 m \bar{\tau}+\kappa)}+\frac{1}{(6 m \tau+\kappa)^{2}(6 m \bar{\tau}+\kappa)^{2}}\right)\right)\right\}
\end{aligned}
$$

## Sketch of proof

Let

$$
P_{k}=x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}-k
$$

defining the family $\left(X_{k}\right)$ of $K 3$-surfaces.

- For $k \in \mathbb{P}^{1}$, generically $\rho=19$.
- The family is $\mathcal{M}_{k}$-polarized with

$$
\mathcal{M}_{k} \simeq U_{2} \perp\left(-E_{8}\right)^{2} \perp\langle-12\rangle
$$

- Its transcendental lattice satisfies

$$
T_{k} \simeq U_{2} \perp\langle 12\rangle
$$

- The Picard-Fuchs differential equation is

$$
\left(k^{2}-4\right)\left(k^{2}-36\right) y^{\prime \prime \prime}+6 k\left(k^{2}-20\right) y^{\prime \prime}+\left(7 k^{2}-48\right) y^{\prime}+k y=0
$$

- The family is modular in the following sense if $k=t+\frac{1}{t}, \tau \in \mathcal{H}$ and $\tau$ as in the theorem

$$
t\left(\frac{a \tau+b}{c \tau+d}\right)=t(\tau) \forall\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \Gamma_{1}(6,2)^{*} \subset \Gamma_{0}(12)^{*}+12
$$

where

$$
\begin{gathered}
\Gamma_{1}(6)=\left\{\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in S I_{2}(\mathbb{Z}) / a \equiv d \equiv 1(6) c \equiv 0(6)\right\} \\
\left.\Gamma_{1}(6,2)=\left\{\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \Gamma_{1}(6) c \equiv 6 b(12)\right\}
\end{gathered}
$$

and

$$
\Gamma_{1}(6,2)^{*}=\left\langle\Gamma_{1}(6,2), w_{6}\right\rangle
$$

- The P-F equation has a basis of solutions $G(\tau), \tau G(\tau), \tau^{2} G(\tau)$ with

$$
G(\tau)=\eta(\tau) \eta(2 \tau) \eta(3 \tau) \eta(6 \tau)
$$

satisfying

$$
G(\tau)=F(t(\tau)), \quad F(t)=\sum_{n \geq 0} v_{n} t^{2 n+1}, \quad v_{n}=\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{n+k}{k}^{2}
$$

- $\frac{d m\left(P_{k}\right)}{d k}$ is a period, hence satisfies the P-F equation

$$
\begin{gathered}
\frac{d m\left(P_{k}\right)}{d k}=G(\tau) \\
d m\left(P_{k}\right)=-G(\tau) \frac{d t}{t} \frac{1-t^{2}}{t}
\end{gathered}
$$

is a weight 4 modular form for $\Gamma_{1}(6,2)^{*}$

- so can be expressed as a combination of $E_{4}(n \tau)$ for $n=1,2,3,6$
- By integration you get

$$
m\left(P_{k}\right)=\Re\left(-\pi i \tau+\sum_{n \geq 1}\left(\sum_{d \mid n} d^{3}\right)\left(4 \frac{q^{n}}{n}-8 \frac{q^{2 n}}{2 n}+12 \frac{q^{3 n}}{3 n}-24 \frac{q^{6 n}}{6 n}\right)\right)
$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

The singular $K 3$ surfaces of the Apéry-Fermi's family $\left(Y_{k}\right)$ correspond to imaginary quadratic $\tau$ such that

$$
t=\left(\frac{\eta(\tau) \eta(6 \tau)}{\eta(2 \tau \eta(3 \tau)}\right)^{6}, \quad k=t+\frac{1}{t}
$$

They have been computed by Boyd.

| k | $\tau$ | Equation of $\tau$ |
| :--- | :--- | :--- |
| 0 | $\frac{-3+\sqrt{-3}}{6}$ | $3 \tau^{2}+3 \tau+1=0$ |
| 2 | $\frac{-2+\sqrt{-2}}{6}$ | $6 \tau^{2}+4 \tau+1=0$ |
| 3 | $\frac{-3+\sqrt{-15}}{12}$ | $6 \tau^{2}+3 \tau+1=0$ |
| 6 | $\frac{\sqrt{-6}}{6}$ | $6 \tau^{2}+1=0$ |
| 10 | $\frac{\sqrt{-2}}{2}$ | $2 \tau^{2}+1=0$ |
| 18 | $\frac{\sqrt{-30}}{6}$ | $6 \tau^{2}+5=0$ |
| 102 | $\frac{\sqrt{-6 \times 13}}{6}$ | $6 \tau^{2}+13=0$ |
| 198 | $\frac{\sqrt{-17 \times 6}}{6}$ | $6 \tau^{2}+17=0$ |
| $2 \sqrt{5}$ | $\frac{-1+\sqrt{-5}}{6}$ | $6 \tau^{2}+2 \tau+1=0$ |
| $3 \sqrt{6}$ | $\frac{\sqrt{-3}}{3}$ | $3 \tau^{2}+1=0$ |
| $2 \sqrt{-3}$ | $\frac{-1+\sqrt{-1}}{2}$ | $2 \tau^{2}+2 \tau+1=0$ |
| $3 \sqrt{-5}$ | $\frac{-3+\sqrt{-15}}{6}$ | $3 \tau^{2}+3 \tau+2=0$ |

## Livné's modularity theorem

## Theorem

Let $S$ be a K3-surface defined over $\mathbb{Q}$, with Picard number 20 and discriminant $N$. Its transcendental lattice $T(S)$ is a dimension 2 $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$-module thus defines a $L$ series, $L(T(S), s)$.
There exists a weight 3 modular form , $f, C M$ over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$
L(T(S), s) \doteq L(f, s)=\sum_{n \geq 1} \frac{A_{n}}{n^{s}}
$$

## How to compute the $A_{n}$ of the $L$-series

## Lemma

Let $Y$ an elliptic K3-surface defined over $\mathbb{Q}$ by a Weierstrass equation $Y(t)$. If rank $(Y(t))=r$ and the $r$ infinite sections generating the Mordell-Weil lattice are defined respectively over $\mathbb{Q}\left(\sqrt{d_{i}}\right), i=1, \ldots, r$, then

$$
A_{p}=-\sum_{t \in \mathbb{P}^{1}\left(\mathbb{F}_{p}\right),} \sum_{Y(t)} a_{p}(t)-\sum_{t \in \mathbb{P}^{1}\left(\mathbb{F}_{p}\right),} \sum_{Y(t)} \epsilon_{p}(t)-\sum_{i=1}^{r}\left(\frac{d_{i}}{p}\right)
$$

where

$$
a_{p}(t)=p+1-\# Y(t)\left(\mathbb{F}_{p}\right)
$$

and $\epsilon_{p}(x)$ defined by

$$
\epsilon_{p}(t)=\left\{\begin{array}{ll}
0, & \text { if the reduction of } Y(t) \text { is additive } \\
1, & \text { if the reduction of } Y(t) \text { is split multiplicative } \\
-1, & \text { if the reduction of } Y(t) \text { is non split multiplicative }
\end{array} .\right.
$$

## Theorem

(Schütt's classification) Consider the following classification of singular K3-surfaces over $\mathbb{Q}$
(1) by the discriminant $d$ of the transcendental lattice of the surface up to squares,
(2) by the associated newform up to twisting,
(3) by the level of the associated newform up to squares,
(9) by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.

Then, all these classifications are equivalent. In particuliar, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

Let

$$
P_{k}=x^{2} y z+x y^{2} z+x y z^{2}+t^{2}(x y+x z+y z)-k x y z t .
$$

$Y_{k}$ is the desingularization of the set of zeroes of $P_{k}$.

## Computation of the discriminant $N$

Different methods allow us to obtain the transcendental lattice, hence its determinant equal to $N$, of the singular K3 lattice. (see Bertin and Lecacheux,arxiv 2022)

| $Y_{0}$ | $\left[\begin{array}{lll}4 & 2 & 4\end{array}\right]$ |
| :---: | :---: |
| $Y_{2}$ | $\left[\begin{array}{lll}{[2} & 0 & 4\end{array}\right]$ |
| $Y_{3}$ | $\left[\begin{array}{lll}2 & 1 & 8\end{array}\right]$ |
| $Y_{6}$ | $\left.\begin{array}{ccc}{[2} & 0 & 12\end{array}\right]$ |
| $Y_{10}$ | $\left[\begin{array}{lll}{[6} & 0 & 12\end{array}\right]$ |
| $Y_{18}$ | $\left.\begin{array}{ccc}{[10} & 0 & 12\end{array}\right]$ |
| $Y_{102}$ | $\left.\begin{array}{cccc}{[12} & 0 & 26\end{array}\right]$ |
| $Y_{198}$ | $\left.\begin{array}{ccc}{[12} & 0 & 34\end{array}\right]$ |
| $Y_{k^{2}=20}$ | $\left[\begin{array}{ccc}2 & 0 & 10\end{array}\right]$ |
| $Y_{k^{2}=54}$ | $\left.\begin{array}{ccc}{[4} & 0 & 12\end{array}\right]$ |
| $Y_{k^{2}=-12}$ | $\left[\begin{array}{lll}6 & 0 & 6\end{array}\right]$ |
| $Y_{k^{2}=-45}$ | $\left[\begin{array}{lll}8 & 2 & 8\end{array}\right]$ |

- From the expression of the Mahler measure in terms of Eisenstein Kronecker series depending of quadratic imaginary $\tau$ it follows the Mahler measure as the $L$-series of a modular form (modular part) plus eventually a Dirichlet $L$-series.

| $m\left(P_{0}\right)=$ | $d_{3}$ Boyd, Bertin (2005) |
| :--- | :--- |
| $m\left(P_{2}\right)=$ | $\frac{8 \sqrt{8}}{\pi^{3}} L\left(f_{8}, 3\right)$ B 2009 |
| $m\left(P_{3}\right)=$ | $\frac{15 \sqrt{15}}{2 \pi^{3}} L\left(f_{15}, 3\right)($ BFFLM $)$ |
| $m\left(P_{6}\right)=$ | $\frac{24 \sqrt{24}}{2 \pi^{3}} L\left(f_{24}, 3\right)($ BFFLM $)$ |
| $m\left(P_{10}\right)=$ | $\frac{72 \sqrt{72}}{9 \pi^{3}} L\left(f_{8}, 3\right)+2 d_{3}(B)$ |
| $m\left(P_{18}\right)=$ | $\frac{120 \sqrt{120}}{9 \pi^{3}} L\left(f_{120}, 3\right)+\frac{14}{5} d_{3}($ BFFLM $)$ |
| $m\left(P_{k^{2}=20}\right)=$ | $2 \frac{20 \sqrt{20}}{4 \pi^{3}} L\left(f_{20} \otimes \chi_{5}, 3\right)(B)$ |
| $m\left(P_{k^{2}=-45}\right)=$ | $\frac{6}{5} \frac{15 \sqrt{15}}{2 \pi^{3}} L\left(f_{15} \otimes \chi_{5}, 3\right)+\frac{d_{15}(B)}{10}(\mathrm{~B})$ |
| $m\left(P_{k^{2}=-12}\right)=$ | $\frac{36}{p i^{3}} L\left(f_{36}, 3\right)+\frac{4}{3} d_{3}(\mathrm{~B})$ |

## The $L$-series of $Y_{k^{2}=-45}$ and $Y_{k^{2}=-12}$

(1) $L\left(Y_{k^{2}=-45}, 3\right)=L\left(f_{15} \otimes \chi_{5}, 3\right)$
(2) $L\left(Y_{k^{2}=-12}, 3\right)=L\left(f_{36} \otimes \chi_{3}, 3\right)$

To compute these $L$-series we apply the lemma. Thus we need

- an elliptic fibration with Weierstrass equation defined over $\mathbb{Q}$;
- the $r$ infinite sections generating the Mordell-Weil lattice For both Weierstrass equations defining $Y_{k^{2}=-45}$ and $Y_{k^{2}=-12}$ we get $r=2$. For both, from results of Bertin and Lecacheux, we obtain one infinite section.
In the first case, $Y_{k^{2}=-45}$ is the Kummer surface of another surface $Z_{-3}$ since $T_{Z_{-3}}=\left[\begin{array}{lll}4 & 1 & 4\end{array}\right]$. Thus there exists a 2-isogeny between the surface and its Kummer.
Since $Z_{-3}$ has an elliptic fibration with $r=0$ its $L$-series can be easily computed and gives (1).


## Proof of (2)

$Y_{k^{2}=-12}$ has an elliptic fibration with Weierstrass equation

$$
y^{2}=x^{3}-\left(t^{3}+3 t^{2}-6 t+4\right) x^{2}+t^{3} x
$$

with two infinite sections

$$
\left.(1,(t-1) \sqrt{-3}) \text { from(B-L), }\left(\frac{t-4}{t+2}\right)^{2}, \frac{3\left(t^{2}-16\right) t(t-1)}{(t+2)^{3}}\right)(\text { Sage })
$$

One infinite section defined over $\mathbb{Q}(\sqrt{-3})$ and the other over $\mathbb{Q}$. The $A(p)$ are computed using the Pari order

$$
A(p)=-\operatorname{sum}(t=2, p-1, \operatorname{ellak}(e(t), p))-\operatorname{kronecker}(-3, p) p
$$

-p-kronecker(-1,p)

## Proof of (2)

Now we must compare to the $\alpha(p)$ given by the CM newform of level 36 and weight 3 (36.3.d.a in LMFDB)

$$
f_{36}(q)=q-2 q^{2}+4 q^{4}+8 q^{5}-8 q^{8}-16 q^{10}-10 q^{13}+16 q^{16}+32 q^{20}+39 q^{25}+\ldots
$$

